Chapter 5

Ruin Theory
Learning Objectives

1. Surplus function, premium rate and loss process
2. Probability of ultimate ruin
3. Probability of ruin before a finite time
4. Adjustment coefficient and Lundberg’s inequality
5. Poisson process and continuous-time ruin theory
5.1 Discrete-Time Surplus and Ruin

- An insurance company establishes its business with a start-up capital of $u$ at time 0, called the **initial surplus**.

- It receives premiums of one unit per period at the end of each period. Loss claim of amount $X_i$ is paid out at the end of period $i$ for $i = 1, 2, \cdots$.

- $X_i$ are independently and identically distributed as the loss random variable $X$.

- The **surplus** at time $n$ with initial capital $u$, denoted by $U(n; u)$, is given by
\[ U(n; u) = u + n - \sum_{i=1}^{n} X_i, \quad \text{for } n = 1, 2, \cdots. \] (5.1)

- The numeraire of the above equation is the amount of premium per period, or the premium rate. All other variables are measured as multiples of the premium rate.

- Thus, the initial surplus \( u \) may take values of 0, 1, \( \cdots \), times the premium rate. Likewise, \( X_i \) may take values of \( j \) times the premium rate with \( pf f_X(j) \) for \( j = 0, 1, \cdots \).

- We denote the mean of \( X \) by \( \mu_X \) and its variance by \( \sigma_X^2 \).

- We assume \( X \) is of finite support, although in notation we allow \( j \) to run to infinity.
• If we denote the premium loading by $\theta$, then we have

$$1 = (1 + \theta)\mu_X,$$

(5.2)

which implies

$$\mu_X = \frac{1}{1 + \theta}.$$

(5.3)

We shall assume positive loading so that $\mu_X < 1$.

• The business is said to be in ruin if the surplus function $U(n; u)$ falls to or below zero sometime after the business started, i.e., at a point $n \geq 1$.

**Definition 5.1:** Ruin occurs at time $n$ if $U(n; u) \leq 0$ for the first time at $n$, for $n \geq 1$. 
**Definition 5.2:** The time-of-ruin random variable $T(u)$ is defined as

$$T(u) = \min \{n \geq 1 : U(n; u) \leq 0\}. \quad (5.4)$$

**Definition 5.3:** Given an initial surplus $u$, the probability of ultimate ruin, denoted by $\psi(u)$, is

$$\psi(u) = \Pr(T(u) < \infty). \quad (5.5)$$

**Definition 5.4:** Given an initial surplus $u$, the probability of ruin by time $t$, denoted by $\psi(t; u)$, is

$$\psi(t; u) = \Pr(T(u) \leq t), \quad \text{for } t = 1, 2, \cdots. \quad (5.6)$$
5.2 Discrete-Time Ruin Theory

5.2.1 Ultimate Ruin in Discrete Time

• We now derive recursive formulas for $\psi(u)$.

• First, for $u = 0$, we have

$$\psi(0) = f_X(0)\psi(1) + S_X(0). \quad (5.7)$$

• Similarly, for $u = 1$, we have

$$\psi(1) = f_X(0)\psi(2) + f_X(1)\psi(1) + S_X(1). \quad (5.8)$$

• The above equations can be generalized to larger values of $u$ as
follows
\[ \psi(u) = f_X(0)\psi(u+1) + \sum_{j=1}^{u} f_X(j)\psi(u+1-j) + S_X(u), \quad \text{for } u \geq 1. \]

(5.9)

• Re-arranging equation (5.9), we obtain the following recursive formula for the probability of ultimate ruin

\[ \psi(u+1) = \frac{1}{f_X(0)} \left[ \psi(u) - \sum_{j=1}^{u} f_X(j)\psi(u+1-j) - S_X(u) \right], \quad \text{for } u \geq 1. \]

(5.10)

• To apply the above equation we need the starting value \( \psi(0) \), which is given by the following theorem.

**Theorem 5.1:** For the discrete-time surplus model, \( \psi(0) = \mu_X \).
Proof: See NAM.

Example 5.1: The claim variable $X$ has the following distribution: $f_X(0) = 0.5$, $f_X(1) = f_X(2) = 0.2$ and $f_X(3) = 0.1$. Calculate the probability of ultimate ruin $\psi(u)$ for $u \geq 0$.

Solution: The survival function of $X$ is $S_X(0) = 0.2 + 0.2 + 0.1 = 0.5$, $S_X(1) = 0.2 + 0.1 = 0.3$, $S_X(2) = 0.1$ and $S_X(u) = 0$ for $u \geq 3$. The mean of $X$ is

$$\mu_X = (0)(0.5) + (1)(0.2) + (2)(0.2) + (3)(0.1) = 0.9,$$

which can also be calculated as

$$\mu_X = \sum_{u=0}^{\infty} S_X(u) = 0.5 + 0.3 + 0.1 = 0.9.$$

Thus, from Theorem 5.1 $\psi(0) = 0.9$, and from equation (5.7), $\psi(1)$ is given
by
\[ \psi(1) = \frac{\psi(0) - S_X(0)}{f_X(0)} = \frac{0.9 - 0.5}{0.5} = 0.8. \]

From equation (5.8), we have
\[ \psi(2) = \frac{\psi(1) - f_X(1)\psi(1) - S_X(1)}{f_X(0)} = \frac{0.8 - (0.2)(0.8) - 0.3}{0.5} = 0.68, \]

and applying equation (5.10) for \( u = 3 \), we have
\[ \psi(3) = \frac{\psi(2) - f_X(1)\psi(2) - f_X(2)\psi(1) - S_X(2)}{f_X(0)} = 0.568. \]

As \( S_X(u) = 0 \) for \( u \geq 3 \), using equation (5.10) we have, for \( u \geq 4 \),
\[ \psi(u) = \frac{\psi(u) - f_X(1)\psi(u) - f_X(2)\psi(u - 1) - f_X(3)\psi(u - 2)}{f_X(0)}. \]
Initial surplus $u$

Prob of ultimate ruin $\psi(u)$
5.2.2 Finite-Time Ruin in Discrete Time

- We now consider the probability of ruin at or before a finite time point $t$ given an initial surplus $u$.

- First we consider $t = 1$ given initial surplus $u$.

- As defined in equation (5.6), $\psi(t; u) = \Pr(T(u) \leq t)$. If $u = 0$, the ruin event occurs at time $t = 1$ when $X_1 \geq 1$. Thus,

  $$\psi(1; 0) = 1 - f_X(0) = S_X(0). \quad (5.20)$$

- Likewise, for $u > 0$, we have

  $$\psi(1; u) = \Pr(X_1 > u) = S_X(u). \quad (5.21)$$

- We now consider $\psi(t; u)$ for $t \geq 2$ and $u \geq 0$. 

• The event of ruin occurring at or before time $t \geq 2$ may be due to (a) ruin at time 1, or (b) loss of $j$ at time 1 for $j = 0, 1, \cdots, u$, followed by ruin occurring within the next $t - 1$ periods.

• When there is a loss of $j$ at time 1, the surplus becomes $u + 1 - j$, so that the probability of ruin within the next $t - 1$ periods is $\psi(t - 1; u + 1 - j)$.

• Thus, we conclude that

$$
\psi(t; u) = \psi(1; u) + \sum_{j=0}^{u} f_X(j) \psi(t - 1; u + 1 - j). \tag{5.22}
$$

Hence, $\psi(t; u)$ can be computed as follows.

1. Construct a table with time $t$ running down the rows for $t = 1, 2, \cdots$, and $u$ running across the columns for $u = 0, 1, \cdots$. 

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2. Initialize the first row of the table for \( t = 1 \) with \( \psi(1; u) = S_X(u) \). Note that if \( M \) is the maximum loss in each period, then \( \psi(1; u) = 0 \) for \( u \geq M \).

3. Increase the value of \( t \) by 1 and calculate \( \psi(t; u) \) for \( u = 0, 1, \cdots \), using equation (5.22). Note that the computation requires the corresponding entry in the first row of the table, i.e., \( \psi(1; u) \), as well as some entries in the \((t - 1)\)th row. In particular, the \( u + 1 \) entries \( \psi(t - 1; 1), \cdots, \psi(t - 1; u + 1) \) in the \((t - 1)\)th row are required.

4. Re-do Step 3 until the desired time point.

**Example 5.3:** As in Example 5.1, the claim variable \( X \) has the following distribution: \( f_X(0) = 0.5, f_X(1) = f_X(2) = 0.2 \) and \( f_X(3) = 0.1 \). Calculate the probability of ruin at or before a finite time \( t \) given initial surplus \( u \), \( \psi(t; u) \), for \( u \geq 0 \).
Solution: The results are summarized in Table 5.1 for \( t = 1, 2 \) and 3, and \( u = 0, 1, \cdots, 6 \).

**Table 5.1:** Results of Example 5.3

<table>
<thead>
<tr>
<th>Time ( t )</th>
<th>Initial surplus ( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.500</td>
</tr>
<tr>
<td>2</td>
<td>0.650</td>
</tr>
<tr>
<td>3</td>
<td>0.705</td>
</tr>
</tbody>
</table>

The first row of the table is \( S_X(u) \). Note that \( \psi(1; u) = 0 \) for \( u \geq 3 \), as the maximum loss in each period is 3. For the second row, the details of the computation is as follows. First, \( \psi(2; 0) \) is computed as

\[
\psi(2; 0) = \psi(1; 0) + f_X(0)\psi(1; 1) = 0.5 + (0.5)(0.3) = 0.65.
\]

Similarly,

\[
\psi(2; 1) = \psi(1; 1) + f_X(0)\psi(1; 2) + f_X(1)\psi(1; 1) = 0.3 + (0.5)(0.1) + (0.2)(0.3) = 0.41,
\]
and

\[ \psi(2; 2) = \psi(1; 2) + f_X(0)\psi(1; 3) + f_X(1)\psi(1; 2) + f_X(2)\psi(1; 1) = 0.18. \]

We use \( \psi(3; 3) \) to illustrate the computation of the third row as follows

\[
\begin{align*}
\psi(3; 3) &= \psi(1; 3) + f_X(0)\psi(2; 4) + f_X(1)\psi(2; 3) + f_X(2)\psi(2; 2) + f_X(3)\psi(2; 1) \\
&= 0 + (0.5)(0.01) + (0.2)(0.05) + (0.2)(0.18) + (0.1)(0.41) \\
&= 0.092.
\end{align*}
\]
Time $t$

Prob of ruin at or before time $t$

- Initial surplus $u = 0$
- Initial surplus $u = 5$
- Initial surplus $u = 10$
5.2.3 Lundberg’s inequality in Discrete Time

**Definition 5.5:** Suppose $X$ is the loss random variable. The adjustment coefficient, denoted by $r^*$, is the value of $r$ that satisfies the following equation

\[
E[\exp \{ r(X - 1) \}] = 1.
\]  

(5.23)

**Example 5.4:** Assume the loss random variable $X$ follows the distribution given in Examples 5.1 and 5.3. Calculate the adjustment coefficient $r^*$.

**Solution:** Equation (5.23) is set up as follows

\[
0.5e^{-r} + 0.2 + 0.2e^{r} + 0.1e^{2r} = 1,
\]
which is equivalent to

\[ 0.1w^3 + 0.2w^2 - 0.8w + 0.5 = 0, \]

for \( w = e^r \). We solve the above equation numerically to obtain \( w = 1.1901 \), so that \( r^* = \log(1.1901) = 0.1740 \).

**Theorem 5.2 (Lundberg’s Theorem):** For the discrete-time surplus function, the probability of ultimate ruin satisfies the following inequality

\[ \psi(u) \leq \exp(-r^*u), \quad (5.28) \]

where \( r^* \) is the adjustment coefficient.

**Proof:** By induction, see NAM.

**Example 5.5:** Assume the loss random variable \( X \) follows the distribution given in Examples 5.1 and 5.4. Calculate the Lundberg upper bound for the probability of ultimate ruin for \( u = 0, 1, 2 \) and 3.
**Solution:** From Example 5.4, the adjustment coefficient is \( r^* = 0.1740 \). The Lundberg upper bound for \( u = 0 \) is 1, and for \( u = 1, 2 \) and 3, we have \( e^{-0.174} = 0.8403 \), \( e^{-(2)(0.174)} = 0.7061 \) and \( e^{-(3)(0.174)} = 0.5933 \), respectively. These figures may be compared against the exact values computed in Example 5.1, namely, 0.8, 0.68 and 0.568, respectively.
5.3 Continuous-Time Surplus Function

• In a continuous-time model the insurance company receives premiums continuously, while claim losses may occur at any time.

• We assume that the initial surplus of the insurance company is $u$ and the amount of premium received per unit time is $c$.

• We denote the number of claims (described as the number of occurrences of events) in the interval $(0, t]$ by $N(t)$, with claim amounts $X_1, \cdots, X_{N(t)}$, which are assumed to be independently and identically distributed as $X$.

• We denote the aggregate losses up to (and including) time $t$ by $S(t)$,
which is given by

\[ S(t) = \sum_{i=1}^{N(t)} X_i, \]  

(5.39)

with the convention that if \( N(t) = 0 \), \( S(t) = 0 \).

- Thus, the surplus at time \( t \), denoted by \( U(t; u) \), is defined as

\[ U(t; u) = u + ct - S(t). \]  

(5.40)

- Figure 5.4 illustrates an example of a realization of the surplus function \( U(t; u) \).

- To analyze the behavior of \( U(t; u) \) we make some assumptions about the claim process \( S(t) \).

- In particular, we assume that the number of occurrences of (claim) events up to (and including) time \( t \), \( N(t) \), follows a **Poisson process**.
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Definition 5.6: $N(t)$ is a Poisson process with parameter $\lambda$, which is the rate of occurrences of events per unit time, if (a) in any interval $(t_1, t_2]$, the number of occurrences of events, i.e., $N(t_2) - N(t_1)$, has a Poisson distribution with mean $\lambda(t_2 - t_1)$, and (b) over any non-overlapping intervals, the numbers of occurrences of events are independently distributed.

- For a fixed $t$, $N(t)$ is distributed as a Poisson variable with parameter $\lambda t$, i.e., $N(t) \sim \mathcal{PN}(\lambda t)$, and $S(t)$ follows a compound Poisson distribution.

- As a function of time $t$, $S(t)$ is a compound Poisson process and the corresponding surplus process $U(t; u)$ is a compound Poisson surplus process. We assume that the claim random variable $X$ has a mgf $M_X(r)$ for $r \in [0, \gamma)$. 


5.4 Continuous-Time Ruin Theory

5.4.1 Lundberg’s Inequality in Continuous Time

- We first define the adjustment coefficient in continuous time. Analogous to the discrete-time case, in which the adjustment coefficient is the solution of

\[
1 + (1 + \theta) r \mu_X = M_X(r).
\]  \hspace{1cm} (5.47)

**Theorem 5.3:** If the surplus function follows a compound Poisson process defined in equation (5.40), the probability of ultimate ruin given initial surplus \( u \), \( \psi(u) \), satisfies the inequality

\[
\psi(u) \leq \exp(-r^* u),
\]  \hspace{1cm} (5.48)
where \( r^* \) is the adjustment coefficient satisfying equation (5.47).

**Example 5.6:** Let \( U(t; u) \) be a compound Poisson surplus function with \( X \sim \mathcal{G}(3, 0.5) \). Compute the adjustment coefficient and its approximate value using equation (5.52), for \( \theta = 0.1 \) and 0.2. Calculate the upper bounds for the probability of ultimate ruin for \( u = 5 \) and \( u = 10 \).

**Solution:** The mgf of \( X \) is, from equation (2.32),

\[
M_X(r) = \frac{1}{(1 - \beta r)^\alpha} = \frac{1}{(1 - 0.5r)^3},
\]

and its mean and variance are, respectively, \( \mu_X = \alpha \beta = 1.5 \) and \( \sigma^2_X = \alpha \beta^2 = 0.75 \). From equation (5.47), the adjustment coefficient is the solution of \( r \) in the equation

\[
\frac{1}{(1 - 0.5r)^3} = 1 + (1 + \theta)(1.5)r,
\]
from which we solve numerically to obtain \( r^* = 0.0924 \) when \( \theta = 0.1 \). The upper bounds for the probability of ultimate ruin are

\[
\exp(-r^* u) = \begin{cases} 
0.6300, & \text{for } u = 5, \\
0.3969, & \text{for } u = 10.
\end{cases}
\]

When the loading is increased to 0.2, \( r^* = 0.1718 \), so that the upper bounds for the probability of ruin are

\[
\exp(-r^* u) = \begin{cases} 
0.4236, & \text{for } u = 5, \\
0.1794, & \text{for } u = 10.
\end{cases}
\]

To compute the approximate values of \( r^* \), we use equation (5.52) to obtain, for \( \theta = 0.1 \),

\[
r^* \simeq \frac{(2)(0.1)(1.5)}{0.75 + (1.1)^2(1.5)^2} = 0.0864,
\]

and, for \( \theta = 0.2 \),

\[
r^* \simeq \frac{(2)(0.2)(1.5)}{0.75 + (1.2)^2(1.5)^2} = 0.1504.
\]
Based on these approximate values, the upper bounds for the probability of ultimate ruin are, for $\theta = 0.1$,

$$\exp(-r^* u) = \begin{cases} 
0.6492, & \text{for } u = 5, \\
0.4215, & \text{for } u = 10.
\end{cases}$$

and, for $\theta = 0.2$,

$$\exp(-r^* u) = \begin{cases} 
0.4714, & \text{for } u = 5, \\
0.2222, & \text{for } u = 10.
\end{cases}$$

Thus, we can see that the adjustment coefficient increases with the premium loading $\theta$. Also, the upper bound for the probability of ultimate ruin decreases with $\theta$ and $u$. We also observe that the approximation of $r^*$ works reasonably well.