Chapter 3 Exercises

(1) A fair coin is tossed three times. Let $X$ be the total number of heads that appeared in the three tosses.
   a. Write down the probability distribution of $X$.
   b. Find $P(X = 3|X \geq 2)$.

(2) An urn contains five balls numbered 1 to 5. Two balls are drawn simultaneously. Let $X$ be the larger of the two numbers drawn.
   a. Write down the probability distribution of $X$.
   b. Find $P(X < 5)$.
   c. Find $P(X = 4|X \geq 3)$.

(3) Let $X$ be a random variable with distribution:

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{2}{5}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{5}$</td>
</tr>
</tbody>
</table>

(a) Let $Y$ be a random variable defined by $Y = X - 2$. Find the distribution of $Y$.
(b) Let $Z$ be a random variable defined by the $Z = X^2$ if $X \leq 3$ and $Z = 3^2$ if $X > 3$. Find the distribution of $Z$.

(4) A resort has three chalets. On any given day, the number of chalets that will be rented out is a random variable $X$ with the probability distribution function $P(X = k) = \frac{1}{3}, k = 1, 2, 3$.
Each chalets costs $200 per day and the resort has a fixed daily running cost of $100.
   a. Find the probability distribution of $Y$, the daily revenue of the resort.
   b. Is $Y$ a random variable and why?

(5) Suppose $X$ is a continuous random variable with probability density function (PDF):

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

a. Find the cumulative distribution function (CDF) of $X$ and write it down carefully.
   b. Find $P(0.5 < X < 0.75)$.
   c. Find $P(X > 0.7)(0.5 < X < 0.75)$.
   d. Find $P(-1 < X < 0.5)$.
   e. Find $P(0.5 < X < 1.5)$.

(6) Let $F(x) = \begin{cases} \frac{x^2}{4}, & 0 < x < 2 \\ 1, & x \geq 2 \end{cases}$ be the CDF for a continuous random variable $X$.
   a. Find $P(X > 1)$.
   b. What is the value of $P(X = 1)$?
   c. Find $P(X < 1.5|X > 1)$.

(7) Let $f(x) = 3x^2, 0 \leq x \leq 1$. Furthermore, let the area under $f(x)$ from $x = 0$ to 1 be one. Can $f(x)$ be used to define the PDF of a variable $0 < X < 1$?

(8) Let the CDF for a continuous random variable $Y$ be

$$F(y) = \begin{cases} y^2, & 0 < y < 1 \\ 1, & y \geq 1 \end{cases}$$

a. Find $P(0 \leq Y \leq 0.5)$.
   b. Find $P(Y > 0.5|Y > 0.25)$.
   c. Are the events $A=\{Y > 0.5\}$ and $B=\{Y > 0.25\}$ independent?
(9) The length of time (in hours), $Y$, that passengers have to wait for a ferry is a random variable with CDF

$$F(y) = \begin{cases} 
1 - e^{-0.2y}, & y > 0 \\
0, & \text{otherwise}
\end{cases}.$$ 

Find the probability that passengers will have to wait more than 10 minutes?

(10) Suppose the CDF of a random variable $Y$ is

$$F(y) = \begin{cases} 
\frac{1}{12}(y^2 + y^3), & 0 < y < 2 \\
1, & y \geq 2
\end{cases}.$$ 

a. Write down $P(Y < 1.5)$
b. Find $P(0.5 < Y < 1.5)$

(11) Suppose $Z$ is a continuous random variable with PDF:

$$f(z) = \begin{cases} 
2(1 - z), & 0 < z < 1 \\
0, & \text{otherwise}
\end{cases}.$$ 

a. Find the cumulative distribution function (CDF) of $Z$ and write it down carefully 
b. Find $P(1/3 < Z < 2/3)$ 
c. Find $P(1/9 < Z^2 < 4/9)$
Answers

(1)
a. The probability distribution of $X$ is:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{8}$</td>
</tr>
<tr>
<td>1</td>
<td>$3(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{3}{8}$</td>
</tr>
<tr>
<td>2</td>
<td>$3(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{3}{8}$</td>
</tr>
<tr>
<td>3</td>
<td>$(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{8}$</td>
</tr>
</tbody>
</table>

b. $P(X = 3 | X \geq 2) = \frac{P(X = 3, X \geq 2)}{P(X \geq 2)} = \frac{P(X = 3)}{P(X \geq 2)} = \frac{1/8}{3/8 + 1/8} = \frac{1}{4}$.

(2) This problem can be solved by first forming a table of outcomes of two marbles drawn simultaneously. The numbers within the table are the possible values of $X$, the maximum of the number on the two marbles.

<table>
<thead>
<tr>
<th>Marble 2</th>
<th>Marble 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2 3 4 5</td>
</tr>
<tr>
<td>2</td>
<td>2 - 3 4 5</td>
</tr>
<tr>
<td>3</td>
<td>3 3 - 4 5</td>
</tr>
<tr>
<td>4</td>
<td>4 4 4 - 5</td>
</tr>
<tr>
<td>5</td>
<td>5 5 5 5 -</td>
</tr>
</tbody>
</table>

Note that “-” denotes an impossible outcome because two marbles cannot have the same number.

a. Based on this table, we can deduce the probability distribution of $X$ by finding the proportions of 1, 2, 3, 4, 5 in the table, which gives:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{10}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{2}{10}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{3}{10}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{4}{10}$</td>
</tr>
</tbody>
</table>

Note that $X = 1$ is not possible.

b. $P(X < 5) = 1 - P(X = 5) = 1 - \frac{4}{10} = \frac{6}{10}$.

c. $P(X = 4 | X \geq 3) = \frac{P(X = 4, X \geq 3)}{P(X \geq 3)} = \frac{P(X = 4)}{P(X \geq 3)} = \frac{3/10}{1 - 1/10} = \frac{3}{3}$.

(3)
a. $Y = X - 2$

<table>
<thead>
<tr>
<th>$Y$</th>
<th>$P(Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{2}{5}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{5}$</td>
</tr>
</tbody>
</table>

b. $Z = 1 4 = 2^2, 9 = 3^2, 9 = 3^2$

which can be simplified as

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$P(Z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2}{5}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{2}{5}$</td>
</tr>
</tbody>
</table>

(4)
a. The probability distribution of $Y$ is:
\[
\begin{array}{c|ccc}
X & 1 & 2 & 3 \\
Y = 200X - 100 & 200 - 100 = 100 & 200(2) - 100 = 300 & 200(3) - 100 = 500 \\
P(X) = P(Y) & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{array}
\]

b. Yes, \(Y\) is a random variable because the revenue \(Y\) depends on \(X\) and \(X\) is random.

(5)

a. Since \(\int_0^x 2x \, dx = \sqrt{x^5} = x^2\), hence

\[
F(x) = \begin{cases} 
  x^2, & 0 < x < 1 \\
  1, & x \geq 1 
\end{cases}
\]

b. \(P(0.5 < X < 0.75) = F(0.75) - F(0.5) = 0.75^2 - 0.5^2 = 0.3125\).

c.

\[
P(X > 0.7|0.5 < X < 0.75) = \frac{P(X > 0.7, 0.5 < X < 0.75)}{P(0.5 < X < 0.75)}
\]

\[
= \frac{P(0.7 < X < 0.75)}{P(0.5 < X < 0.75)}
\]

\[
= \frac{F(0.75) - F(0.7)}{F(0.75) - F(0.5)}
\]

\[
= \frac{0.75^2 - 0.7^2}{0.3125} = 0.232.
\]

d. \(P(-1 < X < 0.5) = F(0.5) - F(-1) = 0.5^2 - 0 = 0.25\)

e. \(P(0.5 < X < 1.5) = F(1.5) - F(0.5) = 1 - 0.5^2 = 0.75\).

(6)

a. \(P(X > 1) = 1 - F(1) = 1 - 0.25(1)^2 = 0.75\).

b. \(P(X = 1) \equiv 0\) since \(X\) is continuous.

c. \(P(X < 1.5|X > 1) = \frac{P(X < 1.5, X > 1)}{P(X > 1)} = \frac{P(1 < X < 1.5)}{P(X > 1)}\). Since \(P(1 < X < 1.5) = F(1.5) - F(1) = [0.25(1.5)^2] - [0.25(1)^2] = 0.3125\), therefore, \(P(X < 1.5|X > 1) = \frac{0.3125}{0.75} \approx 0.417\).

(7) For \(f(x) = 3x^2\) to be a density, it must satisfy:

1. \(f(x) \geq 0, 0 \leq x \leq 1\).
2. \(P(0 \leq X \leq 1) = 1\).

Condition (1) is satisfied because \(3x^2 \geq 0, 0 \leq x \leq 1\).

Condition (2) is also satisfied because the area under \(f(x)\) from 0 to 1 is precisely the same as \(P(0 \leq x \leq 1)\).

(8) a. \(P(0 \leq Y \leq 0.5) = F(0.5) = 0.5^4 = \frac{1}{16}\).

b. First of all, \(P(Y > 0.25) = 1 - F(0.25) = 1 - 0.25^4 = \frac{255}{256}\).

\[
P(Y > 0.5|Y > 0.25) = \frac{P(Y > 0.5, Y > 0.25)}{P(Y > 0.25)} = \frac{P(Y > 0.5)}{P(Y > 0.25)} = \frac{1 - 1/16}{255/256} = \frac{240}{255}.
\]
c. Since \( P(Y > 0.5|Y > 0.25) = \frac{240}{255} \neq P(Y > 0.5) = \frac{15}{16} \), therefore, the events \( \{Y > 0.5\} \) and \( \{Y > 0.25\} \) are not independent.

\[ (9) \ P(Y \geq \frac{1}{6}) = 1 - F\left(\frac{1}{6}\right) = 1 - (1 - e^{-0.2/6}) = e^{-0.2/6} = 0.967. \]

(10) a. \( P(Y < 1.5) = F(1.5) = \frac{1}{12}(1.5^2 + 1.5^3) \approx 0.469. \)

b. \( P(0.5 < Y < 1.5) = F(1.5) - F(0.5) = \frac{1}{12}(1.5^2 + 1.5^3) - \frac{1}{12}(0.5^2 + 0.5^3) = 0.4375. \)

(11) a. Since \( \int_0^z 2(1-z)dz = 2(z - z^2/2)|_0^z = z(2-z) \), hence

\[ F(z) = \begin{cases} 
  z(2-z), & 0 < z < 1 \\
  1, & z \geq 1 
\end{cases}. \]

b. \( P\left(\frac{1}{3} < Z < \frac{2}{3}\right) = F\left(\frac{2}{3}\right) - F\left(\frac{1}{3}\right) = \frac{2}{3} \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{3} \left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{3}. \)

c. Since \( Z \) is positive and hence \( 1/9 < Z^2 < 4/9 \Leftrightarrow \sqrt{1/9} < \sqrt{Z^2} < \sqrt{4/9} \Leftrightarrow 1/3 < Z < 2/3. \) Therefore \( P(1/9 < Z^2 < 4/9) = P(1/3 < Z < 2/3) = 1/3. \)