Integration and Stochastic Calculus

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Definition

If \( f(x) \) is a function of one variable, an **indefinite integral** of \( f(x) \) is a function \( F(x) \) whose first derivative is equal to \( f(x) \):

\[
\frac{d}{dx} F(x) = f(x).
\]

An indefinite integral \( F(x) \) is denoted by

\[
F(x) = \int f(x) \, dx.
\]

Indefinite integrals are also called **antiderivatives**.
Definite Integrals

Let $f(x)$ be a function of one variable and $[a, b]$ an interval of real numbers. The definite integral or, simply, the integral from $a$ to $b$ of $f(x)$ is the area of the region in the $x$-$y$ plane bounded by the graph of $f(x)$, the $x$-axis and the vertical lines $x = a$ and $x = b$, where regions below the $x$-axis have negative sign and regions above the $x$-axis have positive sign.

The integral from $a$ to $b$ of $f(x)$ is denoted by

$$
\int_{a}^{b} f(x) \, dx.
$$

$f(x)$ is called the integrand function and $a$ and $b$ are called, respectively, upper and lower bound of integration.
The fundamental theorem of calculus provides the link between definite and indefinite integrals. It has two parts.

On the one hand, if one defines

$$F(x) = \int_{a}^{x} f(t) \, dt,$$

then, the first derivative of $F(x)$ is equal to $f(x)$, i.e.,

$$\frac{d}{dx}F(x) = f(x).$$

In other words, if one differentiates a definite integral with respect to its upper bound of integration, then one obtains the integrand function.
On the other hand, if $F(x)$ is an indefinite integral (an antiderivative) of $f(x)$, then
\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]
In other words, one can use the indefinite integral to compute the definite integral.

The Fundamental Theorem of Calculus Part II is
\[
\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a).
\]

We can also write the fundamental theorem of calculus as
\[
\int_a^b \frac{df}{dx} \, dx = f(b) - f(a).
\]
Integration: Area under the Graph

\[ y = x^2 \]

\[ \int_{0}^{10} x^2 \, dx = \frac{0}{3000} \]

\[ \int_{0}^{10} x^2 \, dx = \frac{0}{3000} \]
Tail Function

For real numbers $x$, $F(x) := \int_{-\infty}^{x} dF(u)$.

Tail Function

$F(x) := 1 - \int_{-\infty}^{x} dF(u) = \int_{-\infty}^{+\infty} dF(u) + \int_{x}^{-\infty} dF(u) = \int_{x}^{\infty} dF(u)$.

For $u \geq 0$,

$\int_{0}^{\infty} u dF(u) = \int_{0}^{\infty} \int_{0}^{u} dx \ dF(u)$

$= \int_{0}^{\infty} \int_{u}^{\infty} dF(x)dx$

$= \int_{0}^{\infty} F(x)dx$. 
Let \( f(x) \) and \( g(x) \) be two functions and \( F(x) \) and \( G(x) \) their indefinite integrals. The following **integration by parts** formula holds

\[
\int_a^b f(x)G(x) \, dx = F(x)G(x) \bigg|_a^b - \int_a^b F(x)g(x) \, dx,
\]

or

\[
\int_x^a f(u)g(u) \, du = \int_x^a f(u) \, du \, g(x) - \int_x^u f(s) \, ds \, \frac{dg}{du} \, du. \quad (1)
\]
An Example of Integration by Parts

\[ \int_{0}^{1} \exp(x) \, x \, dx \] can be integrated by parts, by setting

\[ f(x) = \exp(x), \quad G(x) = x. \]

An indefinite integral of \( f(x) \) is

\[ F(x) = \exp(x), \]

and \( G(x) \) is an indefinite integral of

\[ g(x) = 1, \]

or, said differently, \( g(x) = 1 \) is the derivative of \( G(x) = x \).
Therefore

\[
\int_0^1 \exp(x) \, x \, dx = \exp(x)\, x \Big|_0^1 - \int_0^1 \exp(x) \times 1 \, dx \\
= \exp(1) - 0 - \int_0^1 \exp(x) \, dx \\
= \exp(1) - \exp(x) \Big|_0^1 \\
= 1.
\]
Consider the interval \([a, b]\), and for any arbitrarily tiny \(\epsilon\), which is positive. A function \(f(x)\) is said to be integrable over the interval \(a + \epsilon \leq x \leq b\), and we write

\[
\int_{a}^{b} f(x) \, dx := \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} f(x) \, dx.
\]

In this case, we say that the integral converges at \(a\), even though \(f(a)\) is unbounded (i.e., \(\infty\) or \(-\infty\)).

The same can be defined for \(a \leq x \leq b + \epsilon\):

\[
\int_{a}^{b} f(x) \, dx := \lim_{\epsilon \to 0} \int_{a}^{b+\epsilon} f(x) \, dx.
\]
The function $f(x) = \ln(1/x)$ is unbounded at 0. But its integral
\[ \int_0^1 \ln \left( \frac{1}{x} \right) \, dx \] is bounded at 0.

Performing integration by parts, we obtain
\[
\int_0^1 \ln \left( \frac{1}{x} \right) \, dx = x \left( \ln \left( \frac{1}{x} \right) + 1 \right) \bigg|_0^1 = 1 - \lim_{x \to 0} x \ln \left( \frac{1}{x} \right).
\]

Applying L’Hôpital’s rule, we find that
\[
\lim_{x \to 0} x \ln \left( \frac{1}{x} \right) = \lim_{x \to 0} \frac{\ln \left( \frac{1}{x} \right)}{\frac{1}{x}} = \lim_{x \to 0} \frac{-x^{-1} \cdot \frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x \to 0} \frac{x}{x} = 0.
\]
Show that $I := \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$.

Let $I := \int_0^\infty e^{-x^2} \, dx$. Since we are dealing with definite integral, it doesn’t matter whether it is $x$ or $y$. So we write $I := \int_0^\infty e^{-y^2} \, dy$.

Then

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy.$$

Move to the polar coordinate system:

$$x = r \cos(\theta), \quad y = r \sin(\theta).$$
The radius \( r \) satisfies \( x^2 + y^2 = r^2 \) by the Pythagoras theorem.

The infinitesimal square \( dx \times dy \) is equivalent to \( dr \times rd\theta \).

Since \( \sin^2 \theta + \cos^2 \theta = 1 \), changing the coordinate system leads to

\[
I^2 = \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} d\theta \, rdr = \int_0^{\pi/2} d\theta \times \int_0^{\infty} e^{-r^2} rdr
\]

\[
= \frac{\pi}{2} \times \int_0^{\infty} e^{-r^2} \frac{dr^2}{2} = \frac{\pi}{4} \int_0^{\infty} e^{-z} dz \quad (z := r^2)
\]

\[
= \frac{\pi}{4}.
\]

Therefore, \( I = \frac{\sqrt{\pi}}{2} \).
A Problem

Suppose $\alpha$ is a positive real number and $f(x)$ is a nonnegative function on $[0, 1]$ such that for all $0 \leq x \leq 1$,

$$\int_0^1 (f(t))^\alpha dt \geq \int_0^1 t^\alpha dt.$$

Show that for every positive real $\beta$,

$$\int_0^1 (f(t))^\alpha t^\beta dt \geq \int_0^1 t^{\alpha + \beta} dt.$$
Solution

\[
\int_0^1 (f(t))^\alpha t^\beta \, dt = \int_0^1 (f(t))^\alpha \left( \beta \int_0^t u^{\beta-1} \, du \right) \, dt
\]

\[
= \beta \int_0^1 u^{\beta-1} \left( \left( \int_u^1 (f(t))^\alpha \, dt \right) \right) \, du
\]

\[
\geq \beta \int_0^1 u^{\beta-1} \left( \int_u^1 t^\alpha \, dt \right) \, du
\]

\[
= \int_0^1 t^\alpha \left( \beta \int_0^t u^{\beta-1} \, du \right) \, dt
\]

\[
= \int_0^1 t^{\alpha + \beta} \, dt.
\]
A Mathematical Identity

Consider two real numbers $a$ and $A$, and a twice-differentiable function $f(x)$.

Obviously, by applying the particular case of the integration by parts for the Riemann-Stieltjes integral, and the fundamental theorem of calculus, we obtain

$$f(A) = f(a) + f(A) - f(a) = f(a) + 1_{A>a} \int_a^A f'(x) \, dx - 1_{a>A} \int_a^A f'(x) \, dx.$$

This is because the Riemann-Stieltjes integral has the property that

$$\int_m^n f'(x) \, dx = \int_m^n \frac{df(x)}{dx} \, dx = \int_m^n df(x) = f(x)\bigg|_{x=m}^{x=n} = f(n) - f(m).$$
Moreover, applying the same idea on $f'$, we write for $x > a$,

$$f'(x) = f'(a) + f'(x) - f'(a) = f'(a) + \int_a^x f''(v) \, dv.$$ 

For $x < a$, we have

$$f'(x) = f'(a) + f'(x) - f'(a) = f'(a) - \int_x^a f''(v) \, dv.$$ 

Substitute the first one into \( 1_{A>a} \int_{a}^{A} f'(x) \, dx \), we obtain

\[
1_{A>a} \int_{a}^{A} f'(x) \, dx = 1_{A>a} \int_{a}^{A} \left( f'(a) + \int_{a}^{x} f''(v) \, dv \right) \, dx \\
= 1_{A>a} \int_{a}^{A} f'(a) \, dx + 1_{A>a} \int_{a}^{A} \int_{a}^{x} f''(v) \, dv \, dx \\
= 1_{A>a} f'(a)(A-a) + 1_{A>a} \int_{a}^{A} f''(v)(A-v) \, dv.
\]

Substitute the second one into \( 1_{a>A} \int_{A}^{a} f'(x) \, dx \), we obtain

\[
1_{a>A} \int_{A}^{a} f'(x) \, dx = 1_{a>A} \int_{A}^{a} \left( f'(a) - \int_{x}^{a} f''(v) \, dv \right) \, dx \\
= 1_{a>A} f'(a)(a-A) - 1_{a>A} \int_{A}^{a} \int_{x}^{a} f''(v) \, dv \, dx \\
= -1_{a>A} f'(a)(A-a) - 1_{a>A} \int_{A}^{a} f''(v)(v-A) \, dv.
\]
Accordingly, when combined,

\[ f(A) = f(a) + (1_{A>a} + 1_{a<A}) f'(a) (A-a) + 1_{A>a} \int_a^A f''(v) (A-v) \, dv + 1_{a>A} \int_A^a f''(v) (v-A) \, dv \]

\[ = f(a) + f'(a) (A-a) + \int_a^\infty f''(v) (A-v)^+ \, dv + \int_0^a f''(v) (v-A)^+ \, dv. \]

We have used the fact that

\[ 1_{A>a} + 1_{a<A} + 1_{a=A} = 1, \]

and the notation that

\[ (S - K)^+ := \max(0, S - K). \]
So the final result is

\[
f(A) = f(a) + f'(a)(A - a) + \int_{a}^{\infty} f''(v)(A - v)^+ \, dv + \int_{0}^{a} f''(v)(v - A)^+ \, dv.
\]

This interesting mathematical identity shows that the function of any future value (payoff) of an asset \( A \), up to a constant \( f(a) - af'(a) \), can be replicated by

1. a position of size \( f'(a) \) in the asset \( A \)
2. a portfolio of call options whose strike prices \( v \) are above \( a \)
3. a portfolio of put options whose strike prices \( v \) are below \( a \)

The number of contracts for each call and each put struck at \( v \) is \( f''(v) \).
Suppose the function $f(x)$ is a natural log function, i.e., $f(x) = \ln(x)$. Show that

$$\frac{A - a}{a} = \ln \left( \frac{A}{a} \right) + \int_{a}^{\infty} \frac{(A - v)^+}{v^2} dv + \int_{0}^{a} \frac{(v - A)^+}{v^2} dv.$$ (2)
Definition of a Brownian Motion

Let $t \geq 0$ denote time. A stochastic process $B(t)$ is said to be Brownian when

1. $B(0) = 0$;
2. $\{B(t), t \geq 0\}$ has stationary independent increments;
3. for every $t > 0$, $B(t)$ is normally distributed with mean 0 and variance proportional to $t$.

The first condition sets the origin of the coordinate system.

The third condition is the scaling law of diffusion enunciated by Einstein in 1905. It says that the mean squared displacement of the Brownian motion from the origin is directly proportional to time.
But Bachelier had already discovered this scaling law in 1900, in the context of stock prices!
Intuition Underlying the Second Condition

The second condition of stationarity is a statement about the invariance of the system with respect to a translation in the time coordinate.

Namely, having the same statistical distribution, the increment \( \Delta B(t) := B(t + dt) - B(t) \), as a random variable, is the same as \( \Delta B(t + s) \) for all \( t \).

Also, the system has zero memory since each \( \Delta B(t) \) is independently distributed. Consequently, the correlation of \( \Delta B(t) \) and \( \Delta B(t + s) \) vanishes when \( s \neq 0 \).
Suppose we start the Brownian motion process at time $t = 0$ and consider an infinitesimal increment $\epsilon$ defined as

$$\epsilon := \Delta B = B(\Delta t) - B(0) = B(\Delta t)$$

after an infinitesimal time $\Delta t$ has elapsed.

From the third defining condition, the variance of the infinitesimal increment is normally distributed with mean 0, and with an appropriate choice of scale, the variance of $\epsilon$ is $\sigma^2 \Delta t$:

$$\forall (\epsilon) \equiv \sigma^2 \Delta t,$$

which is the same as saying that

$$\mathbb{E}(\epsilon^2) = \sigma^2 \Delta t. \quad (3)$$
Variance of $(\Delta B)^2$

- Equation (3) suggests that the expected value of $(\Delta B)^2$ is $\sigma^2 \Delta t$. We want to show that, up to the first order in $\Delta t$,

  $$(dB)^2 \approx dt,$$  \hspace{1cm} (4)

  when $\sigma = 1$.

- This is an important component of Itô's formula in differential form. Equation (4) says that the square of the random variable $dB$ is deterministic!

- To prove equation (4), we shall show that the variance of $(\Delta B)^2$ is indeed very small, and vanishes when $\Delta t$ is an infinitesimal.
We start from Equation (3). Knowing that the probability distribution function is normal, specifically \( N(0, \sigma^2 \Delta t) \), we can write it as

\[
\sigma^2 \Delta t = \frac{1}{\sqrt{2\pi \Delta t \sigma}} \int_{-\infty}^{\infty} x^2 \exp \left( -\frac{x^2}{2\sigma^2 \Delta t} \right) \, dx.
\]

Regarding the integrand as made up of two functions, \( x \) and \( x \exp \left( -\frac{x^2}{2\sigma^2 \Delta t} \right) \), we perform the integration by parts (1).
We thus obtain

\[
\sigma^2 \Delta t = \frac{1}{\sqrt{2\pi \Delta t}} \sigma \left( \frac{x^2}{2} x \exp \left( - \frac{x^2}{2\sigma^2 \Delta t} \right) \right) \bigg|_{-\infty}^{\infty} 
- \int_{-\infty}^{\infty} \frac{x^2}{2} \left( \exp \left( - \frac{x^2}{2\sigma^2 \Delta t} \right) - \frac{x^2}{\sigma^2 \Delta t} \exp \left( - \frac{x^2}{2\sigma^2 \Delta t} \right) \right) \, dx
\] .

The first term vanishes because its exponential term vanishes at a much faster rate than \( x^3 \) going towards infinity. Therefore, we have

\[
\sigma^2 \Delta t = -\frac{1}{\sqrt{2\pi \Delta t}} \sigma \int_{-\infty}^{\infty} \frac{x^2}{2} \exp \left( - \frac{x^2}{2\sigma^2 \Delta t} \right) \, dx 
+ \frac{1}{\sqrt{2\pi \Delta t}} \sigma \int_{-\infty}^{\infty} \frac{x^4}{2\sigma^2 \Delta t} \exp \left( - \frac{x^2}{2\sigma^2 \Delta t} \right) \, dx .
\]  

(5)
Note that the first term of Equation (5) is equal to $-\sigma^2 \Delta t / 2$. Rearranging the terms, we obtain

$$\frac{3}{2} \sigma^2 \Delta t = \frac{1}{\sqrt{2\pi \Delta t \sigma}} \int_{-\infty}^{\infty} \frac{x^4}{2\sigma^2 \Delta t} \exp \left( -\frac{x^2}{2\sigma^2 \Delta t} \right) dx .$$

In other words, we have obtained an analytical expression for the integral

$$\frac{1}{\sqrt{2\pi \Delta t \sigma}} \int_{-\infty}^{\infty} x^4 \exp \left( -\frac{x^2}{2\sigma^2 \Delta t} \right) dx = 3\sigma^4 (\Delta t)^2 .$$

With this result, it becomes straightforward to compute the variance of $\epsilon^2$:

$$\operatorname{Var}(\epsilon^2) = \mathbb{E} \left( (\epsilon^2 - \sigma^2 \Delta t)^2 \right) = \mathbb{E}(\epsilon^4) - \sigma^4 (\Delta t)^2 .$$
From equation (6), we obtain the result* $\mathbb{E}(\epsilon^4) = 3\sigma^4(\Delta t)^2$. Thus, we gather that the variance of $\epsilon^2$ as defined by equation (7) is equal to $2\sigma^4(\Delta t)^2$, i.e.,

$$\mathbb{V}(\epsilon^2) = 2\sigma^4(\Delta t)^2.$$ 

Since $\epsilon = \Delta B$, we see that the variance of $(\Delta B)^2$ is $2\sigma^4(\Delta t)^2$, which is negligibly small compared to $\Delta t$ when $\Delta t \to 0$.

It is tantamount to saying that $\epsilon^2$ does not vary stochastically, for its variance is almost zero.

* As a side note, the derivation we have carried out so far amounts to proving the stylized fact that the excess kurtosis of a normal distribution is zero, namely, $\mathbb{E}(\epsilon^4)/\mathbb{E}(\epsilon^2)^2 - 3 = 0$. 

Christopher Ting, Math for QF
Therefore, we can conclude that, with $\sigma$ set equal to 1 and in the ultimate limit when $\Delta t \to 0$, and from equation (3),

$$\mathbb{E} \left( (\Delta B)^2 \right) \longrightarrow (\Delta B)^2 \longrightarrow \Delta t.$$  

Accordingly, while $\Delta B \sim N(0, \Delta t)$ is a random variable, its square is informally written as

$$(dB)^2 \approx dt.$$
Symbolic Rules of Stochastic Calculus

† Heuristically speaking, 

\[ dB \approx \sqrt{dt}. \]

So with respect to \( dt \), the product \( dB \times dt \) is of a higher order and hence equals to 0.

† So from the perspective of \( \Delta t \to 0 \), we have the following symbolic rules to perform stochastic calculus:

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Consider a stochastic process $X(t)$ consisting of two parts—a deterministic part and a stochastic part:

$$dX(t) = g(t) \, dt + f(t) \, dB(t).$$

(8)

This is the stochastic differential equation (SDE) for $X(t)$, and is named the Itô process.

Suppose $\theta(t, x)$ is a continuous function with continuous partial derivatives $\frac{\partial \theta}{\partial t}$, $\frac{\partial \theta}{\partial x}$, and $\frac{\partial^2 \theta}{\partial x^2}$. 
Then $\theta(t, X(t))$ is also an Itô process. Moreover, its SDE is

$$d\theta(t, X(t)) = \left( \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial x} g(t) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2} f(t)^2 \right) dt + \frac{\partial \theta}{\partial x} f(t) dB(t).$$

(9)
Symbolic “Proof”

Apply the Taylor expansion to obtain, up to $dt$,

$$d\theta \approx \frac{\partial \theta}{\partial t} dt + \frac{\partial \theta}{\partial x} dX(t) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2} (dX(t))^2. \quad (10)$$

Then use the symbolic table to get $(dX(t))^2 \approx f(t)^2 dt$. Therefore,

$$d\theta \approx \frac{\partial \theta}{\partial t} dt + \frac{\partial \theta}{\partial x} \left( g(t) dt + f(t) dB(t) \right) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2} f(t)^2 dt$$

$$\approx \left( \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial x} g(t) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2} f(t)^2 \right) dt + \frac{\partial \theta}{\partial x} f(t) dB(t).$$
What Kiyoshi Itô did was to show that “≈” is really “=”. This is indeed a great achievement.
Differential Equations Determining a Markov Process


Source of the original paper: Zenkoku Sizyo Sugaku Danwakai, Osaka University

Source of photographs: http://emuseum.kyotoprize.org/en/KiyosiIto
Let $f(x)$ be a twice differentiable function. The Third Example is the rudimentary form of the Itô formula:

$$
\int_0^t f(X_t)dX_t = \int_0^X f(\lambda)d\lambda - \frac{1}{2} \int_0^t f'(X_t)dt.
$$

We write

$$
\int_0^X f(\lambda)d\lambda = \frac{1}{2} \int_0^t f'(X_t)dt + \int_0^t f(X_t)dX_t.
$$

Differentiate with respect to $x_t$ gives rise to

$$
df(x_t)|_{x_t=X_t} = \frac{1}{2} f''|_{x_t=X_t}dt + f'(x_t)|_{x_t=X_t}dX_t.
$$
Abusing the mathematical symbols, we write it as

\[ df(X_t) = \frac{1}{2} f'''(X_t) dt + f'(X_t) dX_t. \]  

(11)

Remember, you can only integrate a random variable but not to differentiate it because mathematically it is not differentiable.
In § 7 (Page 1374), as the first example of stochastic integration,

\[ \int_0^t B(s) \, dB(s) = \frac{1}{2} B(t)^2 - \frac{1}{2} t. \]

**Proof:** Let \( \theta(t, x) = \frac{1}{2} x^2 \). Then we have

\[ \frac{\partial \theta}{\partial x} = x, \quad \frac{\partial^2 \theta}{\partial x^2} = 1. \]

By applying the Itô formula, equation (11), we obtain

\[ d\theta(t) = \frac{1}{2} dt + B(t)dB(t). \]
Taking the integral, the result is

$$\int_0^t d\theta(s) = \frac{1}{2} \int_0^t ds + \int_0^t B(s) dB(s),$$

which is

$$\frac{1}{2} B(t)^2 = \frac{1}{2} t + \int_0^t B(s) dB(s).$$
Geometric Brownian motion \( S(t) \), a positive definite stochastic process
\[
dS(t) = \mu(t)S(t)\,dt + \sigma(t)S(t)\,dB(t),
\]
which is more appropriately written instead as
\[
\frac{dS(t)}{S(t)} = \mu(t)\,dt + \sigma(t)\,dB(t).
\]
(12)

to be compatible with the Itô process, equation (8).

Log price as a function of \( S(t) \).
\[
\theta(t, x) = \ln x.
\]
We have
\[
\frac{\partial \theta}{\partial x} = \frac{1}{x} \quad \text{and} \quad \frac{\partial^2 \theta}{\partial x^2} = -\frac{1}{x^2}.
\]

Applying the Itô 1942 formula (11), we get
\[
d \left( \ln S(t) \right) = \frac{1}{S(t)} \, dS(t) - \frac{1}{2} \frac{1}{S(t)^2} (dS(t))^2
\]
\[
= \frac{dS(t)}{S(t)} - \frac{1}{2} \left( \frac{dS(t)}{S(t)} \right)^2
\]
Substitute in equation (12) and apply the symbolic rules,

\[
d\left(\ln S(t)\right) = \mu(t) \, dt + \sigma(t) \, dB(t) - \frac{1}{2} \sigma^2(t) \, dt \\
  = \left(\mu(t) - \frac{1}{2} \sigma^2(t)\right) \, dt + \sigma(t) \, dB(t).
\]

Integrate over time,

\[
S(t) = S(0) \exp \left(\int_0^t \left(\mu(t) - \frac{1}{2} \sigma^2(t)\right) \, dt + \int_0^t \sigma(t) \, dB(t)\right).
\]

If \(\mu(t)\) and \(\sigma(t)\) are constants, then since \(B(0) = 0\),

\[
S(t) = S(0) \exp \left(\left(\mu - \frac{1}{2} \sigma^2\right) t + \sigma B(t)\right).
\]
1. Derive the SDE for $S(t)$ when $\theta(t, x) = \sqrt{x}$ using Itô’s formula (11), and the symbolic table.

2. Itô proved in the 1942 seminal paper that, as the Third Example,

$$\int_0^t a(X(s)) \, dX(s) = \int_0^{X(t)} a(\lambda) \, d\lambda - \int_0^t \frac{a'(X(s))}{2} \, ds,$$

where $a'(\lambda)$ is a continuous function of $\lambda$.

Apply this formula to show that

$$\int_0^t X(s)^2 \, dX(s) = \frac{1}{3} X(t)^2 - \int_0^t X(s) \, ds.$$
Difference between Simple Return and Log Return

Earlier we have

\[
\frac{dS(t)}{S(t)} = \mu(t) \, dt + \sigma(t) \, dB(t)
\]

and by the Itô formula

\[
d(\ln S(t)) = \left(\mu(t) - \frac{1}{2} \sigma^2(t)\right) \, dt + \sigma(t) \, dB(t).
\]

Consequently,

\[
\frac{dS(t)}{S(t)} - d(\ln S(t)) = \frac{1}{2} \sigma^2(t) \, dt.
\]
The instantaneous variance can be measured by twice the difference of two returns!

\[ \sigma^2(t) \, dt = 2 \frac{dS(t)}{S(t)} - 2d\left( \ln S(t) \right). \]

The first term is the simple return:

\[ \frac{dS(t)}{S(t)} = \frac{S(t + dt) - S(t)}{S(t)}. \]

The second term is the log return:

\[ d\left( \ln S(t) \right) = \ln \left( \ln S(t + dt) \right) - \ln S(t). \]
Integrated Variance

The integrated variance $V(0, T)$ is defined as

$$V(0, T) := \int_0^T \sigma^2(t) \, dt.$$ 

The variance $V(0, T)$ is the sum of instantaneous variances $\sigma^2(t)$ realized over time 0 to time $T$. By equation (13), we obtain

$$V(0, T) = 2 \left( \int_0^T \frac{1}{S(t)} dS(t) - \ln \left( \frac{S(T)}{S(0)} \right) \right).$$
Log Return and an Integral

Now, for convenience, we write $X(t)$ as $X_t$. Consider a quantity $F_0$ known at time $t = 0$, and we express $\ln \left( \frac{S_T}{F_0} \right)$ as

$$\ln \left( \frac{S_T}{F_0} \right) = \ln S_T - \ln F_0 - S_T \left( \frac{1}{F_0} - \frac{1}{S_T} \right) + \frac{S_T}{F_0} - 1$$

$$= \int_{F_0}^{S_T} \frac{1}{x} \, dx - S_T \int_{F_0}^{S_T} \frac{1}{x^2} \, dx + \frac{S_T}{F_0} - 1$$

$$= - \int_{F_0}^{S_T} \frac{S_T - x}{x^2} \, dx + \frac{S_T}{F_0} - 1. \quad (14)$$

For any $z > -1$, $\ln(1 + z)$ is a strictly concave function and $\ln(1 + z) < z$. The left side of equation (14) is term $\ln(1 + z)$ with

$$z := \frac{S_T}{F_0} - 1.$$
It follows that the integral

$$\int_{F_0}^{S_T} \frac{S_T - x}{x^2} \, dx = -\left( \ln(1 + z) - z \right) > 0,$$

i.e., strictly positive.
We can then rewrite the integral as

\[
\int_{F_0}^{S_T} \frac{S_T - x}{x^2} \, dx = 1_{S_T > F_0} \int_{F_0}^{S_T} \frac{S_T - x}{x^2} \, dx - 1_{S_T < F_0} \int_{S_T}^{F_0} \frac{S_T - x}{x^2} \, dx
\]

\[
= 1_{S_T > F_0} \int_{F_0}^{S_T} \frac{S_T - x}{x^2} \, dx + 1_{S_T < F_0} \int_{S_T}^{F_0} \frac{x - S_T}{x^2} \, dx
\]

\[
= \int_{F_0}^{H} \frac{(S_T - x)^+}{x^2} \, dx + \int_{F_0}^{L} \frac{(x - S_T)^+}{x^2} \, dx. \tag{15}
\]

In the last step, we have used the fact that the asset price $S_T$, which is unknown at time $t = 0$, can potentially attain a low value denoted by $L$, or appreciate substantially to a high value $H$ by time $T$. 
Prices of European Options

The asset pricing equation \( p_t = \mathbb{E}_t(mx) \) is based on the notion that the payoff \( x \) in the future carries a price equal to the expectation of \( x \) discounted by a discount factor \( m \).

In Equation (15), if \( x \) is the strike price of a European option, then we regard the following as the future payoff

\[
(S_T - x)^+ := \max(S_T - x, 0), \quad (x - S_T)^+ := \max(x - S_T, 0),
\]

Suppose the discount factor is \( e^{-rT} \) where \( r \) is the risk-free rate of tenor \( T \). Then under the so-called risk-neutral measure \( \mathbb{Q} \),

European call and put option prices at time 0:

\[
c_0 = \mathbb{E}_0^\mathbb{Q} \left( e^{-rT} (S_T - x)^+ \right), \quad p_0 = \mathbb{E}_0^\mathbb{Q} \left( e^{-rT} (x - S_T)^+ \right).
\]
The risk-neutral measure is a convenient concept. The intuitive meaning is that any portfolio earns only the risk-free rate.

\[ \mathbb{E}_0^Q \left( \int_0^T \frac{1}{S_t} dS_t \right) = r T, \tag{16} \]

Applying the expectation on both sides of the integral (15), we obtain

\[ \mathbb{E}_0^Q \left( \int_{F_0}^{S_T} \frac{S_T - x}{x^2} dx \right) = e^{rT} \int_{F_0}^{H} \frac{c(S_0, x, T)}{x^2} dx + e^{rT} \int_{L}^{F_0} \frac{p(S_0, x, T)}{x^2} dx, \]

where \( c(S_0, x, T) \) and \( p(S_0, x, T) \) are, respectively, the European call and put prices, with strike price \( x \), underlying asset price \( S_0 \), and maturity \( T \).
Application of Risk-Neutral Measure

Under the risk-neutral measure

\[ \mathbb{E}_0^Q (S_T) = e^{rT} S_0 \]

Suppose we let

\[ F_0 = e^{rT} S_0. \]

Since

\[ \ln \frac{S_T}{S_0} = \ln \frac{S_T}{F_0} + \ln \frac{F_0}{S_0}, \]

the expected value of the integrated variance is, in conjunction
with equation (14)

\[ E_0^Q (V(0, T)) = 2rT - 2E_0^Q \left( \ln \left( \frac{S_T}{F_0} \right) \right) - 2E_0^Q \left( \ln \left( \frac{F_0}{S_0} \right) \right). \]

\[ = -2E_0^Q \left( \ln \left( \frac{S_T}{F_0} \right) \right) \]

\[ = 2E_0^Q \left( \int_{F_0}^{S_T} \frac{S_T - x}{x^2} \, dx \right) - 2E_0^Q \left( \frac{S_T}{F_0} \right) + 2. \]
Consequently,

\[
\mathbb{E}_0^Q (V(0, T)) = 2e^{rT} \left( \int_{F_0}^{H} \frac{c(S_0, x, T)}{x^2} \, dx + \int_{L}^{F_0} \frac{p(S_0, x, T)}{x^2} \, dx \right).
\]

So under the risk-neutral measure, the expected value of the integrated variance can be replicated by European option prices.

Define \( \sigma_{MF}^2 T := \mathbb{E}_0^Q (V(0, T)) \) and hence

\[
\sigma_{MF}^2 T = 2e^{rT} \left( \int_{F_0}^{H} \frac{c(S_0, x, T)}{x^2} \, dx + \int_{L}^{F_0} \frac{p(S_0, x, T)}{x^2} \, dx \right).
\]
When $T = 1$ year, $\sigma^2_{MF}$ is the risk-neutral variance to price the variance swap and $\sigma_{MF}$ is the model-free volatility (e.g. VIX).
A Simpler Proof

Let \( A = F_T \) and \( a = F_0 \) for the mathematical identity (2). The futures contract price is assumed to follow the following process:

\[
\frac{dF_t}{F_t} = \sigma_t F_t dB_t.
\]

Apply Itô’s formula to prove that

\[
\mathbb{E}_0 \left( \int_0^T \sigma_t^2 dt \right) = 2 \left( \int_0^{F_0} \frac{p_F(x, T)}{x^2} dx + \int_{F_0}^\infty \frac{c_F(x, T)}{x^2} dx \right),
\]

where \( p_F(x, T) \) and \( c_F(x, T) \) are the put and call options on the European futures contract \( F_t \).
Mean-Reverting Process

Stochastic Differential Equation

\[ dX_t = \alpha (\mu - X_t) \, dt + \sigma dB_t. \]

Solution

\[ X_t = e^{-\alpha t} X_0 + \mu (1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha (t-s)} dB_s. \]
Define $Y_t := G(t, X_t) = e^{\alpha t} X_t$.

Apply Itô’s formula

$$dY_t = \left( \frac{\partial G}{\partial t} + \alpha (\mu - X_t) \frac{\partial G}{\partial X_t} + \frac{1}{2} \sigma^2 \frac{\partial^2 G}{\partial X_t^2} \right) dt + \sigma \frac{\partial G}{\partial X_t} dB_t$$

$$= \left( \alpha e^{\alpha t} X_t + \alpha (\mu - X_t) e^{\alpha t} \right) dt + \sigma e^{\alpha t} dB_t$$

$$= \alpha \mu e^{\alpha t} dt + \sigma e^{\alpha t} dB_t$$

Integrating,

$$Y_t = Y_0 + \int_0^t \alpha \mu e^{\alpha s} ds + \int_0^t \sigma e^{\alpha s} dB_s$$

$$= Y_0 + \mu (e^{\alpha t} - 1) + \int_0^t \sigma e^{\alpha s} dB_s$$
The Cox-Ingersoll-Ross (CIR) Model

⚠️ Stochastic Differential Equation for modeling the dynamics of the instantaneous interest rate:

\[ dX_t = \alpha(\mu - X_t)dt + \sigma \sqrt{X_t}dB_t. \]

⚠️ No explicit solution, in contrast to the Vasicek model.
The Constant Elasticity of Variance (CEV) Model

@ Stochastic Differential Equation for modeling the leverage effect:

\[ dX_t = \mu X_t \, dt + \sigma X_t^{\beta+1} \, dB_t. \]

@ For \( \beta > 0 \) (\( \beta < 0 \)) the local volatility, defined as

\[ \mathbb{S} \left( \frac{dX_t}{X_t} \right) = \sigma X_t^\beta, \]

increases (decreases) monotonically as the asset price increases.

@ The elasticity is defined as

\[ \frac{df(x)}{f(x)} = \frac{df(x)}{dx} \cdot \frac{x}{f(x)} \]

@ For \( f(x) = \sigma x^\beta \), the elasticity is \( \beta \), a constant.
Summary

Integration is a deeper concept than differentiation.

In applying Itô’s formula, differentiation is done according to the usual Leibniz-Newton calculus.

But stochastic process per se cannot be differentiated at any point of the sample path.

Mathematical identity is model-free.

Risk-neutrality means that the market participant can replicate the future payoffs for all possible outcomes and thereby “neutralize” the current risk. So the market participant becomes indifferent to the future outcomes and thus neutral to risk.