Matrix Algebra Notes

Section 1    About these notes

These are notes on matrix algebra that I have written up for use in different courses that I teach, to be prescribed either as refreshers, main reading, supplements, or background readings. These courses are primarily

    Intermediate Math for Economics
    Economic Forecasting
    Advanced Mathematical Methods
    various Econometrics courses

Perhaps others will find them useful too. They were designed for self-study, and intended as a quick and dirty introduction to the essentials in matrix algebra.

If you are using this, please note:

- The exercises are an essential part of the notes. Many of the important concepts and results are in the exercises. Do the exercises.

- These notes are not meant to replace any textbooks on matrix algebra. Please use these only as a supplement to the ‘real’ textbooks.

- The individual sections are not stand-alone. Not complete in its exposition; I have made a number of assumptions regarding what you already know.

- I call these notes to “matrix algebra” as opposed to “linear algebra”, which gets into vector spaces. The coverage in these notes is terribly naive in comparison. Please see Strang (2009) or take a real Linear Algebra course.

- These notes are incomplete. There are many topics I plan to add, and will do so from time to time, with no particular sense of urgency.

Anthony Tay
Singapore Management University
Section 2 Systems of Equations

Solving economic models often requires solving systems of equations. Here we discuss solving systems of linear equations, meaning that the equations in the system take the form

\[ a_1x_1 + a_2x_2 + \ldots + a_nx_n = c. \]

Solution Possibilities Consider the system

(i) \[ \begin{align*}
2x_1 - x_2 &= 4 \\
x_1 + 2x_2 &= 2
\end{align*} \]

A solution of this system is a pair of numbers \((x_1, x_2)\) that satisfy both equations at the same time. In this example, the solution is easily found to be \((x_1, x_2) = (2,0)\).

The system (i) has exactly one solution, but this is not true for all systems. Some have an infinite number of solutions, others have none. Consider

(ii) \[ \begin{align*}
3x_1 + 5x_2 &= 6 \\
6x_1 + 10x_2 &= 12
\end{align*} \]

The second equation is simply two times the first – they are not independent equations. There is effectively only one equation in two unknowns, and any \((x_1, x_2)\) that satisfies the first equation will satisfy the second. The system has infinitely many solutions – any \((x_1, x_2)\) satisfying

\[ x_1 = 2 - (5/3)x_2 \quad \text{or} \quad x_2 = (6/5) - (3/5)x_1 \]

is a solution. The set of solutions is usually written as

\[ (x_1, x_2) = \left( 2 - \frac{5s}{3}, s \right) \quad \text{or} \quad (x_1, x_2) = \left( \frac{6}{5} - \frac{3s}{5}, s \right). \]

The symbol ‘\( s \)’ is sometimes called a ‘parameter’.

The fact that a system may have no solution is illustrated by the next example:

(iii) \[ \begin{align*}
3x_1 + 5x_2 &= 6 \\
3x_1 + 5x_2 &= 7
\end{align*} \]

If the first equation is satisfied, the second cannot be satisfied. This system has no solution. The two equations are conflicting equations.

Drawing the equations on the \(x-y\) plane (more appropriately here, the \(x_1-x_2\) plane) shows clearly what happens in all three situations. The two lines intersect in (i), coincide in (ii), and are parallel to each other in (iii).
A system with at least one solution is called a **consistent** system. A system with no solution is said to be **inconsistent**. Whether a system has one solution or many solutions depends on whether there are as many independent equations as there are variables to be solved.

What if there are three equations in two variables? Here we have all three possibilities. Consider

\[
\begin{align*}
2x_1 + 3x_2 &= 18 \\
3x_1 - 4x_2 &= -7 \\
x_1 + 2x_2 &= 10
\end{align*}
\]

This system has no solution. Solving the last two only gives the unique solution

\[x_1 = \frac{13}{5}, \quad x_2 = \frac{37}{10}.
\]

But this solution cannot be reconciled with the first equation:

\[2x_1 + 3x_2 = 2\left(\frac{13}{5}\right) + 3\left(\frac{37}{10}\right) = \frac{163}{10} \neq 18.
\]

Drawing the three equations in the \(x_1 - x_2\) plane shows clearly what is going on here. For there to be a solution, these three equations must intersect at the same point. In this example, they don’t.

Here is an example where the three equations do intersect at a single point.

\[
\begin{align*}
4x_1 - 2x_2 &= 3 \\
3x_1 - 4x_2 &= -7 \\
x_1 + 2x_2 &= 10
\end{align*}
\]

System (v) has exactly one solution. Solving the last two gives

\[x_1 = \frac{13}{5}, \quad x_2 = \frac{37}{10}
\]

as in the previous case. Substituting this solution into the first equation gives

\[4x_1 - 2x_2 = 4\left(\frac{13}{5}\right) - 2\left(\frac{37}{10}\right) = \frac{15}{5} = 3 \quad \text{so the first equation is also satisfied}.
\]

Another way of looking at this system is to observe that although there are three equations in two unknowns, there are actually only two **independent** equations. Any one of the three equations can be written as a combination of the other two (e.g., 3rd equation is 1st minus 2nd; or 2nd equation is 1st minus 3rd, etc.) Solving any two therefore solves the third. It is the number of independent equations that matter in determining the number of solutions: two independent equations in two unknowns produces a single solution.
If a three-equation two-variable system contains only one independent equation, then effectively we will have only one equation in two unknowns, leading to infinitely many solutions. This is the case with the following system:

\[
\begin{align*}
2x_1 + 4x_2 &= 20 \\
3x_1 + 6x_2 &= 30 \\
x_1 + 2x_2 &= 10
\end{align*}
\]

(vi) Here the 1st equation is twice that of the 3rd, and the second is three times that of the 3rd. Therefore there is only one independent equation, in two unknowns. There are infinitely many solutions \((x_1, x_2) = (10 - 2s, s)\).

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A linear equation involving three variables \(x_1, x_2,\) and \(x_3,\) represents a plane in the 3-dimensional space. This visualization helps you to understand what can happen in linear systems with three variables.

If you have two planes, then either the planes are parallel, are coincident, or intersect. If they intersect, the intersection produces a line in three-dimensional space. Therefore two linear equations involving three variables will either have no solutions (if the two planes are parallel), or an infinite number of solutions represented by the entire plane (if the two planes coincide) or an infinite number of solutions represented by the line of intersection of the two planes.

The following system has infinitely many solutions represented by a single line.

\[
\begin{align*}
2x_1 - 3x_2 + x_3 &= 0 \\
x_1 + x_2 + x_3 &= 1
\end{align*}
\]

(vii) Subtracting the second from the first eliminates \(x_3,\) and gives \(x_1 - 4x_2 = -1.\) Thus for any \(x_2 = s,\) we have \(x_1 = 4s - 1.\) Substituting into the second equation gives

\[
x_3 = 1 - x_1 - x_2 = 1 - 4s + 1 - s = -5s.
\]

We have infinitely many solutions given by \((x_1, x_2, x_3) = (4s - 1, s, -5s).\)

The system

\[
\begin{align*}
2x_1 - 3x_2 + x_3 &= 0 \\
4x_1 - 6x_2 + 2x_3 &= 1
\end{align*}
\]

(viii) however, has no solution. Dividing the second equation by 2 gives \(2x_1 - 3x_2 + x_3 = 1/2.\) Thus if \((x_1, x_2, x_3)\) satisfies the first, it cannot satisfy the second. These two planes are parallel.
The system
\[
\begin{align*}
2x_1 - 3x_2 + x_3 &= 1 \\
4x_1 - 6x_2 + 2x_3 &= 2
\end{align*}
\]
has infinitely many solutions. It is easy to see that the two equations are identical. Every point on the plane is a solution. For any \(x_1 = s, \ x_2 = t\), we have \(x_3 = 1 - 2s + 3t\). The solution is
\[
(x_1, \ x_2, \ x_3) = (s, \ t, \ 1 - 2s + 3t).
\]

In order to get a unique solution to a three-equation system, there must be three independent non-conflicting equations. The three planes must intersect at a single point. If there are any dependence among the three equations, then we will end up with infinitely many solutions (either a line or a plane), or if there are conflicting equations, no solutions. We will explore this further in later sections.

Solving larger systems of equations may seem a more daunting task, but the process of solving small systems is easily systematized. This process is called ‘elimination’ using “elementary row operations”. Take the system:
\[
\begin{align*}
2x_1 + 2x_2 + 4x_3 &= 4 & \text{eq1} \\
3x_1 + x_2 + 2x_3 &= 2 & \text{eq2} \\
5x_1 + 2x_2 + 1x_3 &= 7 & \text{eq3}
\end{align*}
\]
You would generally proceed by using equation 1 to eliminate \(x_1\) from eq2 and eq3:
\[
\begin{align*}
2x_1 + 2x_2 + 4x_3 &= 4 & \text{eq1} \\
\text{eq2 minus } 3/2 \times \text{eq1} & 0x_1 - 2x_2 - 4x_3 = -4 & \text{eq2'} \\
\text{eq3 minus } 5/2 \times \text{eq1} & 0x_1 - 3x_2 - 9x_3 = -3 & \text{eq3''}
\end{align*}
\]
Then use eq2 to eliminate \(x_2\) from eq3, to get the “triangular” system
\[
\begin{align*}
2x_1 + 2x_2 + 4x_3 &= 4 & \text{eq1} \\
0x_1 - 2x_2 - 4x_3 &= -4 & \text{eq2'} \\
\text{eq3 minus } 3/2 \times \text{eq2'} & 0x_1 + 0x_2 - 3x_3 = 3 & \text{eq3''}
\end{align*}
\]
Finally, multiply the third equation by \(-\frac{1}{3}\) to get \(x_3 = -1\). Substituting \(x_3 = -1\) upwards gives \(x_2 = 4\), and \(x_1 = 0\).

You may have to switch the positions of your equations first. If the coefficient on the first variable in the first equation is zero, then switch that equation with another:

\[
\begin{align*}
0x_1 + 2x_2 + 4x_3 &= 4 \\
3x_1 + 1x_2 + 2x_3 &= 2 \\
5x_1 + 2x_2 + 1x_3 &= 7
\end{align*}
\]

switch eq 1 with eq2

\[
\begin{align*}
0x_1 + 2x_2 + 4x_3 &= 4 \\
0x_1 + 2x_2 + 4x_3 &= 4 \\
5x_1 + 2x_2 + 1x_3 &= 7
\end{align*}
\]

and proceed as usual.

Notice the important role played by the boxed numbers. These numbers are called the **pivots** of the coefficient matrix. If a system of equations has a unique solution it will have non-zero pivots. Notice also that we only used three “types” of operations in solving our system of equations: switching equations, multiplying by a (non-zero) constant, and using one equation to eliminate another. These operations are called “row-operations” and underlie much of matrix algebra.

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**Systems of Nonlinear Equations**

We often also have to solve systems of non-linear equations, such as

\[
\begin{align*}
(x) & \quad xy = 4 \\
& \quad x^2 + y^2 = 8
\end{align*}
\]

The ‘method’ for solving systems of non-linear equations is effectively also to eliminate variables, although how best to do this depends of the system; it is a case-by-case situation.

For the system above, we might approach it in the following way: the first equation gives \(x = 4 / y\). Thus

\[
16 / y^2 + y^2 = 8 \quad \Rightarrow \quad 16 + y^4 = 8y^2
\]

which gives

\[
(y^2 - 4)^2 = 0.
\]

The solutions are, therefore, \((x, y) = (2, 2)\) and \((x, y) = (-2, -2)\).
It is easy to see that solving larger systems of non-linear equations can get really tricky. Even small systems can be tricky, and you’ll have to be careful of extraneous solutions and missing solutions. For example, take

\[(xi) \quad y = \sqrt{x} \\
y = 2 - x\]

Perhaps the obvious thing to do here is

\[\sqrt{x} = 2 - x \implies x = (2 - x)^2 = 4 - 4x + x^2 \implies 0 = (x - 4)(x - 1).\]

So you might conclude that \(x = 1\) or \(x = 4\), with \(y = 1\) and \(y = 2\) respectively. However, \((x, y) = (4, 2)\) is not a solution, since this does not lie on the second equation. \([Qn: why is (x, y) = (4, -2) not a solution?]\)

**Exercises**

1. Plot systems (i) to (vi), and (x) and (xi), each in their own diagrams.

2. Solve the system

\[y = x^2 \\
x^2 + y^2 = a,\]

for \(x > 0, \ y > 0, \ a > 0\). Plot the system on the \(x\)-\(y\) plane for different values of \(a\).
Section 3 The Summation Notation

You will frequently deal with complicated expressions involving a large number of additions. Often, these expressions are simplified using the ‘summation’ notation. Many students find difficulty in manipulating such expressions. The purpose of this section is to introduce the notation to you, and to get you comfortable with it.

3.1 Definition and Rules for Sums

The uppercase sigma “\(\Sigma\)” is used to denote summation. For an arbitrary set of numbers \(\{x_1, x_2, ..., x_n\}\) define
\[
\sum_{i=1}^{n} x_i = x_1 + x_2 + ... + x_n.
\]

Example 3.1.1 The average of a set of numbers \(x_1, x_2, ..., x_n\) can be written
\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.
\]

Example 3.1.2 Write the sum 4 + 8 + 12 + 16 + 20 + 24 in summation notation:
Ans: \(\sum_{i=1}^{6} 4i\).

Example 3.1.3 Suppose the following payments are to be made: \(a_1\) in the first period, \(a_2\) in the second period, and so on until \(a_n\) in the \(n\)th period. At a fixed interest rate of \(r\) per period, the present value of the payments is
\[
\frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + ... + \frac{a_n}{(1+r)^n} = \sum_{i=1}^{n} \frac{a_i}{(1+r)^i}.
\]

In Example 3.1.1, the “index of summation” \(i\) enter as subscripts, but otherwise do not enter into the computation of the terms of the summation. In Example 3.1.2, the index is actually part of the computation of the terms. In Example 3.1.3, the index is used both ways.
Example 3.1.4  Economists often use an aggregate price index to track the overall price level in an economy relative to some base year. This is usually done by tracking a weighted average of prices of a certain set of commodities. Let

\[ i = 1, \ldots, n \quad \text{represent} \quad n \quad \text{commodities} \]

\[ q_{0i} \quad \text{be the quantity of good} \quad i \quad \text{purchased in period} \quad 0 \quad (\text{the base year}) \]

\[ p_{0i} \quad \text{be the price of good} \quad i \quad \text{in period} \quad 0 \]

\[ q_{ti} \quad \text{be the quantity of good} \quad i \quad \text{purchased in period} \quad t \]

\[ p_{ti} \quad \text{be the price of good} \quad i \quad \text{in period} \quad t \]

The Laspeyres Price Index is

\[
\frac{\sum_{i=1}^{n} p_{ti} q_{0i}}{\sum_{i=1}^{n} p_{0i} q_{0i}}
\]

The Paasche Price Index is

\[
\frac{\sum_{i=1}^{n} p_{ti} q_{ti}}{\sum_{i=1}^{n} p_{0i} q_{ti}}
\]

Expressions using summation notation are not unique; more than one expression can be used to represent a given sum.

Example 3.1.5  Write \(1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11\) in summation notation:

Ans: \(\sum_{i=1}^{6} (-1)^{i-1} \frac{1}{2i-1}\). Alternate answer: \(\sum_{i=0}^{5} (-1)^{i} \frac{1}{2i+1}\)

3.2  Rules for Working with the Summation Notation

The summation notation greatly simplifies notation (once you get used to it), but this is only helpful then you know how to manipulate expressions written in it. There are only two rules to learn

(i) \(\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\),  
(ii) \(\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i\), where \(c\) is a constant.

Example 3.2.1  \(\sum_{i=1}^{n} c = nc\)

Example 3.2.2  \(\sum_{i=1}^{n} (x_i - \bar{x}) = 0\), where \(\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i\).

\text{Proof:}  \(\sum_{i=1}^{n} (x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} = n \bar{x} - n \bar{x} = 0\)
Example 3.2.3 \[ \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} (x_i - \bar{x})y_i = \sum_{i=1}^{n} (y_i - \bar{y})x_i, \]

where \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \), and \( \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \).

**Proof** We have \[ \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} (x_i - \bar{x})y_i - \sum_{i=1}^{n} (x_i - \bar{x})\bar{y} \]

But the second term is zero: \[ \sum_{i=1}^{n} (x_i - \bar{x})\bar{y} = \bar{y} \sum_{i=1}^{n} (x_i - \bar{x}) = 0, \]

which gives the desired result.

The proof of \[ \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} (y_i - \bar{y})x_i \] is similar.

### 3.3 Some useful formulas involving summations

For every integer \( n \),

\[ \sum_{i=1}^{n} i = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} \]

\[ \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} \]

\[ \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left( \sum_{i=1}^{n} i \right)^2 \]

**Arithmetic Series**

\[ \sum_{i=0}^{n-1} (a + id) = \sum_{i=1}^{n} (a + (i-1)d) \]

\[ \begin{align*}
&= a + (a + d) + (a + 2d) + ... + (a + (n-1)d) \\
&= na + \frac{n(n-1)d}{2}
\end{align*} \]

where \( a \) and \( d \) are real numbers.

**Geometric Series**

\[ \sum_{i=0}^{n-1} ar^i = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + ... + ar^{n-1} = a \frac{1-r^n}{1-r} \]

where \( a \) and \( r \) are real numbers.
3.4 Double summations

Suppose we have a rectangular array of numbers

\[
\begin{array}{cccc}
    & a_{11} & a_{12} & \cdots & a_{1n} \\
    & a_{21} & a_{22} & \cdots & a_{2n} \\
    & \vdots & \vdots & \ddots & \vdots \\
    & a_{m1} & a_{m2} & \cdots & a_{mn}
\end{array}
\]

Let the total sum of these numbers be \( S \). To get \( S \) we can first add up the rows and then add the results,

i.e., \( S = \sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj} = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} \right) \),

or first add up the columns,

i.e., \( S = \sum_{i=1}^{m} a_{i1} + \sum_{i=1}^{m} a_{i2} + \cdots + \sum_{i=1}^{m} a_{in} = \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} \right) \).

The parenthesis makes it clear which summation is to be done first, but it is conventional to leave out the parenthesis and write

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \quad \text{or} \quad \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}
\]

with the understanding that the summations are carried out from right to left, i.e., from the inner summation to the outer.

**Example 3.4.1** Expand \( \sum_{i=1}^{m} \sum_{j=1}^{n} i^2 \).

One way to do this is

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} i^2 = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} i^2 \right) = (1+2+\cdots+m)(1^2 + 2^2 + \cdots + n^2)
\]

A more explicit argument...

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} i^2 = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} i^2 \right) = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \right) = \left( \sum_{i=1}^{m} i \right) \left( \sum_{j=1}^{n} j^2 \right)
\]

\[
= (1+2+\cdots+m)(1^2 + 2^2 + \cdots + n^2)
\]
In the examples so far, we could interchange the summation signs, i.e.,

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}.
\]

We can do this only if the limits of the outer summation do not depend on the limit of any of the inner summations.

**Example 3.4.2** Suppose we have a triangular array of numbers to be expressed in summation notation.

\[
\begin{array}{cccc}
  & a_{11} & & \\
a_{21} & a_{22} & & \\
   &   & & \ddots \\
a_{m1} & a_{m2} & \cdots & a_{mm}
\end{array}
\]

We can write \( \sum_{i=1}^{m} \sum_{j=1}^{i} a_{ij} \). We cannot interchange the summation symbols because the upper limit in the inner summation depends on the index of the outer summation.

**Example 3.4.3** Write the sum of the following triangular array using summation notation.

\[
\begin{array}{cccc}
  & a_{11} & & \\
a_{21} & a_{22} & & \\
   &   & & \ddots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{array}
\]

We can write \( \sum_{j=1}^{n} \sum_{i=j}^{m} a_{ij} \). We cannot interchange the summation symbols because the upper limit in the inner summation depends on the index of the outer summation.

**Solution:** Let \( a_{i,j} \) be the typical element in the sum. Then the first column has \( j=1 \), and \( i \) running from 1 to \( m \); the second column has \( j=2 \), and \( i \) running from 2 to \( m \). In general we have \( j \) running from 1 to \( n \), for the \( j \)th column, \( i \) running from \( j \) to \( m \). Thus the sum is \( \sum_{j=1}^{n} \sum_{i=j}^{m} a_{ij} \).
Exercises

1. Write out in full, then evaluate or simplify
   a. \( \sum_{i=1}^{4} 2i \)
   b. \( \sum_{i=0}^{3} ix_i \)
   c. \( \sum_{i=1}^{4} (i-1)x_{i-1} \)
   d. \( \sum_{i=1}^{10} 2 \)
   e. \( \sum_{i=1}^{10} (2i-1) \)
   f. \( \sum_{i=1}^{10} (-1)^i \)
   g. \( \sum_{j=1}^{i} j \)
   h. \( \sum_{j=1}^{i} i \)

2. Write in summation notation
   a. \( 1 - 3 + 5 - 7 + 9 \)
   b. \( a^n + \left( \frac{m}{1} \right) a^{m-1} b + \left( \frac{m}{2} \right) a^{m-2} b^2 + \ldots + \left( \frac{m}{m-1} \right) ab^{m-1} + \left( \frac{m}{m} \right) b^n \)
   c. \( 2 + 3/2 + 4/3 + 5/4 + 6/5 \)
   d. \( a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \)

3. Prove by writing out in full
   a. \( \sum_{k=1}^{10} [(k+1)^3 - k^3] = 11^3 - 1 \)
   b. \( \sum_{j=1}^{5} x_j x_j = x_{\bar{y}} \sum_{j=1}^{5} x_j \)
   c. \( \sum_{i=1}^{3} \sum_{j=1}^{2} x_j x_i = \sum_{j=1}^{2} \sum_{i=1}^{3} x_j x_i \)
   d. \( \sum_{i=1}^{3} \sum_{j=1}^{2} x_j x_i = \left( \sum_{j=1}^{3} x_j \right) \left( \sum_{j=1}^{2} x_j \right) \)
   e. \( \sum_{i=1}^{3} x_i \bar{x} = \bar{x} \sum_{i=1}^{3} x_i \) where \( \bar{x} = \frac{1}{3} \sum_{i=1}^{3} x_i \)

4. Prove using the rules of summation:
   a. \( \sum_{i=1}^{n} \sum_{j=1}^{i} i = \sum_{j=1}^{n} j^2 \)
   b. \( \sum_{i=1}^{n} j = \sum_{i=1}^{n} i^2 \)

5. Prove all the equality relations in the following:
   \[
   \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} (x_i - \bar{x})x_i = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2
   \]

6. Show that \( \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} x_i y_i - n \bar{x} \bar{y} \).
7. Evaluate by first simplifying, then applying the appropriate formulas
   a. \( \sum_{k=1}^{10} k(k - 2) \)  
   b. \( \sum_{k=1}^{n} k(k - 2)(k + 2) \)

8. Let \( \{ x_1, x_2, ..., x_{10} \} = \{ 1, 2, 6, 4, \pi, e, 0, -1, 2, 4.2 \} \). Verify using excel or otherwise (calculator, manual computation, mental computation, as you please) that
   a. \( \sum_{i=1}^{10} (x_i - \bar{x}) = 0 \)
   b. \( \sum_{i=1}^{10} (x_i - \bar{x})^2 = \sum_{i=1}^{10} x_i^2 - 10\bar{x}^2 \)
   c. \( \sum_{i=1}^{10} (x_i - \bar{x})^2 = \sum_{i=1}^{10} (x_i - \bar{x})x_i \)
   If in (i) you get an answer a little different from 0, explain why this occurs.

9. Express the following in the form \( a x_1 + bx_2 + cx_3 + dx_4 \):
   (i) \( \sum_{i=1}^{4} \sum_{j=1}^{2} i x_j \)
   (ii) \( \sum_{i=1}^{4} \sum_{j=1}^{4} i x_j \)
   (i.e., you have to tell me what \( a, b, c, \) and \( d \) are in each case.)

10. Let \( \{ x_1, x_2, ..., x_{n} \} \) be an arbitrary set of \( n \) real numbers, and let
    \[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i. \]
    Prove that
    \[ \sum_{i=1}^{n} (x_i - \bar{x})(x_i - 1) = \sum_{i=1}^{n} (x_i - \bar{x})(x_i - 10000). \]
Section 4  Matrix Algebra: Definitions and Basic Operations

Definitions

Analyzing economic models often involve working with large sets of linear equations. Matrix algebra provides a set of tools for dealing with such objects.

A **matrix** is a rectangular collection of numbers

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}.
\]

The number of rows \(m\) need not be equal to the number of columns \(n\). A matrix with \(m\) rows and \(n\) columns is said to have **order** \((m, n)\) or **dimension** \((m, n)\), or we simply call it a \((m \times n)\) matrix. The number that appears in the \((i, j)\)th position is called the \((i, j)\)th **element** or the \((i, j)\)th **entry** of the matrix. If \(m = n\), the matrix is a **square matrix**. If \(m = 1\) and \(n > 1\), it is called a **row vector**. If \(m > 1\) and \(n = 1\), we have a **column vector**. If \(m = n = 1\), then we have a **scalar**. The elements of a vector are often called the **components** of the vector.

**Example**  
A row vector \(\mathbf{c} = [c_1, c_2, \cdots, c_n]\), a column vector \(\mathbf{b} = [\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}]\).

By simply stating that \(\mathbf{b}\) is a vector we will usually mean that \(\mathbf{b}\) is a column vector, but you need to be aware whether a row or column vector is being referred to. Matrices are often written in unitalicized bold uppercase letters, vectors in unitalicized bold lowercase letters.

It is often convenient to write a matrix as

\[
A = (a_{ij})_{m \times n},
\]

and often convenient to refer to the \((i, j)\)th element of \(A\) using

\[
[A]_{ij}.
\]

There is variation in notation from author to author, so be careful in your reading.

Two matrices of the same dimension \(m \times n\) are said to be equal if all of their corresponding elements are equal, i.e.,

\[
A = B \iff [A]_{ij} = [B]_{ij} \ \forall i = 1, 2, \ldots, m, \forall j = 1, 2, \ldots, n.
\]

Matrices of different dimensions cannot be equal.
**Basic Operations (Addition, Scalar Multiplication, Subtraction, Transpose)**

**Addition**

Let \( A = (a_{ij})_{m \times n} \) and \( B = (b_{ij})_{m \times n} \) be two arbitrary \((m \times n)\) matrices. Define

\[
A + B = (a_{ij} + b_{ij})_{m \times n},
\]

i.e., addition of matrices is defined to be element by element addition.

**Example**

\[
\begin{bmatrix}
1 & 4 \\
3 & 2
\end{bmatrix}
+ 
\begin{bmatrix}
6 & 9 \\
1 & 2
\end{bmatrix} = 
\begin{bmatrix}
7 & 13 \\
4 & 4
\end{bmatrix}
\]

Matrices being added together obviously must have the same dimensions. It should also be obvious that

\[
A + B = B + A
\]

\[
(A + B) + C = A + (B + C)
\]

This means that as far as addition is concerned, we can manipulate matrices in the same way we manipulate ordinary numbers (as long as they have the same dimensions)

**Scalar Multiplication**

Let \( A \) be a \((m \times n)\) matrix, and \( \alpha \) be a scalar. Then define

\[
\alpha A = (\alpha a_{ij})_{m \times n}
\]

i.e., the product of a scalar and a matrix is defined to be the multiplication of each element of the matrix by the scalar.

**Example**

\[
b \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{bmatrix} = 
\begin{bmatrix}
ba_{11} & ba_{12} \\
ba_{21} & ba_{22} \\
ba_{31} & ba_{32}
\end{bmatrix}
\]

We can use scalar multiplication to define matrix subtraction. Let \( A \) and \( B \) be \((m \times n)\) matrices. Then

\[
A - B = A + (-1)B
\]

**Example**

\[
\begin{bmatrix}
1 & 4 \\
3 & 2
\end{bmatrix}
- 
\begin{bmatrix}
6 & 9 \\
1 & 2
\end{bmatrix} = 
\begin{bmatrix}
-5 & -5 \\
2 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
6 & 5 \\
1 & 10
\end{bmatrix}
+ 
\begin{bmatrix}
6 & 5 \\
1 & 10
\end{bmatrix} = 
\begin{bmatrix}
6-1 & 5-10 \\
2 & 0
\end{bmatrix} = 
\begin{bmatrix}
-5 & -5 \\
5 & -5
\end{bmatrix}
\]

\[\]
Transpose

An important operator is the transpose. When we transpose a matrix, we write its rows as its column, and its columns as its rows. For example, denoting the transpose of $\mathbf{A}$ by $\mathbf{A}^T$ we have

$$
\begin{pmatrix}
1 & 4 \\
3 & 2 \\
6 & 5
\end{pmatrix}^T
= 
\begin{pmatrix}
1 & 3 & 6 \\
4 & 2 & 5
\end{pmatrix}
$$

Put more succinctly, $[\mathbf{A}^T]_{ij} = [\mathbf{A}]_{ji}$. Note that transposes are often denoted using $\mathbf{A}'$ instead of $\mathbf{A}^T$.

One application of the transpose operator is in defining symmetric matrices. A symmetric matrix is defined as one where $\mathbf{A}^T = \mathbf{A}$.

Exercises

1. Let $\mathbf{A} = 
\begin{pmatrix}
7 & 13 \\
4 & 4 \\
7 & 15
\end{pmatrix}$.

What is the dimension of $\mathbf{A}$? What is $[\mathbf{A}]_{12}$? What is $[\mathbf{A}]_{31}$?

2. Suppose $\mathbf{A} = (a_{ij})_{2 \times 4}$ where $a_{ij} = i + j$. Write out the matrix in full.

3. Write out in full the matrix

(i) $(a_{ij})_{4 \times 4}$ where $a_{ij} = 1$ when $i = j$, 0 otherwise.

(ii) $(a_{ij})_{4 \times 4}$ where $a_{ij} = 0$ if $i \neq j$. (Fill in the rest of the entries ‘*’)

(iii) $(a_{ij})_{5 \times 5}$ where $a_{ij} = 0$ when $i < j$. (Fill in the rest of the entries with ‘*’)

(iv) $(a_{ij})_{5 \times 5}$ where $a_{ij} = 0$ when $i > j$. (Fill in the rest of the entries with ‘*’)

These are all square matrices. Matrices (i) and (ii) are called diagonal matrices. Matrix (iii) is a “lower triangular matrix”, and (iv) is an “upper triangular matrix” (so we have in (iii) and (iv) matrices that are square and triangular!)

4. Give an example of a $(4 \times 4)$ matrix such that $[\mathbf{A}]_{ij} = [\mathbf{A}]_{ji}$.

5. If

$$
\begin{pmatrix}
u + 2v & 1 & 3 \\
9 & 0 & 4 \\
3 & 4 & 7
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & 3 \\
9 & 0 & u + v \\
3 & 4 & 7
\end{pmatrix}
$$

What is $u$ and $v$?
6. Let \( v_1, v_2, v_3, v_4 \) represent cities, and suppose there are one-way flights from \( v_1 \) to \( v_2 \) and \( v_3 \), from \( v_2 \) to \( v_3 \) and \( v_4 \), and two-way flights between \( v_1 \) and \( v_4 \). Write out a matrix \( A \) such that \( A_{ij} = 1 \) if there is a flight from \( v_i \) to \( v_j \), and zero otherwise.

7. What is the dimension of the matrix \[
\begin{pmatrix}
1 & 8 & 3 \\
9 & 1 & 9 \\
0 & 0 & 0
\end{pmatrix}
\]?

8. Let \( A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \). Is \( A = B \)?

Matrices with all zero entries are called zero matrices, and written \( 0_{m,n} \), or \( 0_n \) if square, or simply \( 0 \) if the dimensions can be easily obtained from context.

9. If \[
\begin{pmatrix} 1 & u \\ 3 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ w & 7 \end{pmatrix},
\]
what is \( u, v \), and \( w \)?

10. If \( 2A = \begin{pmatrix} 3 & 4 \\ 2 & 8 \\ 1 & 5 \end{pmatrix} \), what is \( A \)? If \( B - \frac{1}{2} \begin{pmatrix} 3 & 4 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 2 & 5 \\ 3 & 1 \end{pmatrix} \), what is \( B \)?

11. Which of the following matrices are symmetric?

(a) \[
\begin{pmatrix}
1 & 2 & 3 & 5 \\
2 & 5 & 4 & b \\
3 & 4 & 3 & 3 \\
5 & b & 3 & 1
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 2 & 3 & 5 \\
0 & 5 & 4 & b \\
0 & 0 & 3 & 3 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(d) \[
\begin{pmatrix}
1 & 1 & 3 & 5 \\
2 & 5 & 4 & b \\
3 & 4 & 3 & 3 \\
5 & b & 3 & 1
\end{pmatrix}
\]

(e) \[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\]
12. True or False?
   (a) Symmetric matrices must be square.
   (b) A scalar is symmetric.
   (c) If $A$ is symmetric, then $\alpha A$ is symmetric.
   (d) The sum of symmetric matrices is symmetric.
   (e) If $(A^T)^T = A$, then $A$ is symmetric.

13. (a) Find $A$ and $B$ if they satisfy

\[
2A + B = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 3 & 0 \end{bmatrix},
\]

\[
A + 2B = \begin{bmatrix} 4 & 2 & 3 \\ 5 & 1 & 1 \end{bmatrix}
\]

simultaneously.

(b) If $A + B = C$ and $3A - 2B = 0$ simultaneously, find $A$ and $B$ in terms of $C$. 
Section 5  Matrix Algebra: Multiplication

Let $A$ be an $(m \times n)$ matrix and $B$ be $(n \times p)$. These dimensions can be arbitrary, but the number of columns of $A$ and the number of rows of $B$ must be the same. The product $AB$ is defined as the $(m \times p)$ matrix whose $(i, j)^{th}$ element is defined by

$$[AB]_{i,j} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$  

That is, the $(i, j)^{th}$ element of the product $AB$ is defined as the sum of the product of the elements of the $i^{th}$ row of $A$ with the corresponding elements in the $j^{th}$ column of $B$. For example, the $(1,1)^{th}$ element of $AB$ is

$$[AB]_{1,1} = \sum_{k=1}^{n} a_{1k}b_{k1} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \ldots + a_{1n}b_{n1}$$

The $(2,3)^{th}$ element is

$$[AB]_{2,3} = \sum_{k=1}^{n} a_{2k}b_{k3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} + \ldots + a_{2n}b_{n3}$$

Visually, for a product of a $(3 \times 3)$ matrix into a $(3 \times 2)$ matrix, we have

$$\begin{bmatrix}
        a_{11} & a_{12} & a_{13} \\
        a_{21} & a_{22} & a_{23} \\
        a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
        b_{11} & b_{12} \\
        b_{21} & b_{22} \\
        b_{31} & b_{32}
\end{bmatrix} =
\begin{bmatrix}
        a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & \cdots & \cdots \\
        a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & \cdots & \cdots \\
        a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & \cdots & \cdots
\end{bmatrix}
\text{ and so on...}$$

**Example**  Let $A = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 \\ 6 & 9 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 6 & 9 \end{bmatrix} =
\begin{bmatrix}
(2)(4) + (8)(6) & (2)(7) + (8)(9) \\
(3)(4) + (0)(6) & (3)(7) + (0)(9) \\
(5)(4) + (1)(6) & (5)(7) + (1)(9)
\end{bmatrix} =
\begin{bmatrix} 56 & 86 \\ 12 & 21 \\ 26 & 44 \end{bmatrix}$$

**Example**  Let $A = \begin{bmatrix} 6 & 5 & -1 \\ 1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 0 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 6 & 5 & -1 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 0 & 1 \end{bmatrix} =
\begin{bmatrix}
(6)(4) + (5)(5) + (-1)(0) & (6)(-1) + (5)(2) + (-1)(1) \\
(1)(4) + (0)(5) + (4)(0) & (1)(-1) + (0)(2) + (4)(1)
\end{bmatrix} =
\begin{bmatrix} 49 & 3 \\ 4 & 3 \end{bmatrix}$$
Example  

The simultaneous equations

\[ 2x_1 - x_2 = 4 \]
\[ x_1 + 2x_2 = 2 \]

can be written in matrix form by defining \( A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \) and \( b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}. \) Then we can write the system as

\[ \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \]

or simply \( Ax = b. \)

Exercises

Many of these exercises provide more than just practice with matrix multiplication; many of the exercises illustrate properties of matrix multiplication which differ from multiplication among ordinary numbers. As you do these exercises, ask yourself what the lesson is.

1. Find \( AB \) when

   (a) \( A = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 8 & 0 \\ 0 & 1 \\ 3 & 5 \end{bmatrix} \)

   (b) \( A = \begin{bmatrix} 2 & 5 & -1 \\ 1 & 0 & 4 \end{bmatrix} \) and \( B = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} \)

2. Let \( A = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}, \) and \( C = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix} \)

   (i) Compute \( BC; \)  (ii) Compute \( CB; \)  (iii) Can \( BA \) be computed?

Remark: This exercise shows that for any two matrices \( A \) and \( B, AB \neq BA. \) We distinguish between premultiplication and postmultiplication. In the product \( AB, \) we say that \( B \) is premultiplied by \( A, \) or \( A \) is postmultiplied by \( B. \)

3. Let \( d = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \) and \( f = [4 \ 2 \ 1 \ 6]. \) Compute (i) \( fd \)  (ii) \( df \)  (iii) \( d^T d. \)
4. Show that for any vector \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \), the product \( \mathbf{x}^T \mathbf{x} \geq 0 \). When will \( \mathbf{x}^T \mathbf{x} = 0 \)?

5. (i) Compute \( \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \). 

(ii) Let \( A = \begin{bmatrix} 1 & b \\ 1 & -1 \end{bmatrix} \). Compute \( A^2 = AA = \begin{bmatrix} 1 & b \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & b \\ -1 & -1 \end{bmatrix} \).

Remark A matrix with all elements equal zero is called the zero matrix \( 0 \). Obviously, \( A + 0 = A \) and \( A0 = 0A = 0 \). Note however that \( AB = 0 \) does not imply \( A = 0 \) or \( B = 0 \). In fact, the square of a non-zero matrix can be a zero matrix!

Matrix multiplication therefore does not behave like the usual multiplication of numbers: the order of multiplication is important, and \( AB = 0 \) does not imply that either \( A = 0 \) or \( B = 0 \). Matrix multiplication, however, does follow associative and distributive laws:

\[
(AB)C = A(BC) \\
A(B + C) = AB + AC \\
(A + B)C = AC + BC
\]

These are easy to prove.

6. Compute

(i) \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \)

(ii) \( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

Remark The square matrix

\[
I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
\]
is called the identity matrix. It behaves like the number ‘1’ in regular multiplication: for any matrix $A$, $AI = IA = A$. To emphasize the dimension of an identity matrix, we sometimes write $I_n$ if it is $(n \times n)$.

7. Compute

(i) \[
\begin{bmatrix}
 b_{11} & 0 & 0 \\
 0 & b_{22} & 0 \\
 0 & 0 & b_{33}
\end{bmatrix}
\begin{bmatrix}
 a_{11} & a_{12} \\
 a_{21} & a_{22} \\
 a_{31} & a_{32}
\end{bmatrix}
\]
(ii) \[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
 b_{11} & 0 & 0 \\
 0 & b_{22} & 0 \\
 0 & 0 & b_{33}
\end{bmatrix}
\]

Remark: Matrices $B$ where $B_{ij} = 0$ for all $i \neq j$ are called diagonal matrices.

(iii) \[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
 b_1 \\
 b_2 \\
 b_3
\end{bmatrix}
\]
(iv) \[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
 b_1 & b_2 & b_3 & b_4
\end{bmatrix}
\]

8. Show that

\[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
 b_1 \\
 b_2 \\
 b_3
\end{bmatrix}
= b_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + b_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + b_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}
\]

9. Show that

\[
\begin{bmatrix}
 b_1 & b_2 & b_3 & b_4
\end{bmatrix}
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

\[
= b_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + b_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + b_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} + b_4 \begin{bmatrix} a_{41} \\ a_{42} \\ a_{43} \end{bmatrix}
\]

10. Write the simultaneous equations

\[
\begin{align*}
4x + z &= 4 \\
19x + y - 3z &= 3 \\
7x + y &= 1
\end{align*}
\]

in matrix notation.
For $\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix}$, prove that $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$ by multiplying out the matrices.

Remark: This result holds generally. For any $(m \times n)$ matrix $\mathbf{A}$ and any $(n \times p)$ matrix $\mathbf{B}$, we have $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$. The proof is not difficult, and provides a very good exercise in using the notation we developed earlier. We want to show that the $(i,j)$th element of $(\mathbf{AB})^T$ is equal to the $(i,j)$th element of $\mathbf{B}^T \mathbf{A}^T$. By the definition of the transpose, $(i,j)$th element of $(\mathbf{AB})^T$ is the $(j,i)$th element of $\mathbf{AB}$, therefore

$$[(\mathbf{AB})^T]_{ij} = [\mathbf{AB}]_{ji}$$

$$= \sum_{k=1}^n a_{jk} b_{ki}$$

$$= \sum_{k=1}^n b_{ki} a_{jk}$$

$$= \sum_{k=1}^n \left[ \mathbf{B}^T \right]_{ik} \left[ \mathbf{A}^T \right]_{kj}$$

$$= \left[ \mathbf{B}^T \mathbf{A}^T \right]_{ij}.$$

12. Prove that $(\mathbf{ABC})^T = \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T$.

13. Let $\mathbf{X}$ be a general $(n \times k)$ matrix. Explain why $\mathbf{X}^T \mathbf{X}$ is square. Explain why $\mathbf{X}^T \mathbf{X}$ is symmetric.

14. Given a $(n \times n)$ square matrix $\mathbf{A}$, define the trace of the matrix to be

$$\text{trace}(\mathbf{A}) = \sum_{i=1}^n a_{ii}$$

That is, the trace of a square matrix is simply the sum of its diagonal elements. Show that

$$\text{trace}(\mathbf{A} + \mathbf{B}) = \text{trace}(\mathbf{A}) + \text{trace}(\mathbf{B}) \quad \text{(of course, both matrices must be the same size)}$$

$$\text{trace}(\mathbf{A}) = \text{trace}(\mathbf{A}^T)$$

$$\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA}) \quad \text{(here $\mathbf{A}$ and $\mathbf{B}$ need not be of the same size)}.$$  

Hint for the last one: look at the proof given in question 11 and adapt it.
Section 6  Introduction to the Inverse Matrix

The inverse of a square matrix $A$ is the matrix, denoted by $A^{-1}$, such that

$$A^{-1}A = I.$$  

**Example**  

The inverse of the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ is

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$$

since

$$A^{-1}A = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

One application of matrix inverses is in solving simultaneous equations. Take for example

$$2x_1 - x_2 = 4$$
$$x_1 + 2x_2 = 2$$

which can be written in matrix form as $Ax = b$, where

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$  

Since we know $A^{-1}$, we can simply (pre)multiply $Ax = b$ with $A^{-1}$ to get

$$A^{-1}Ax = A^{-1}b.$$  

Since $A^{-1}A = I$, and $Ix = x$, we have $x = A^{-1}b$. This is the solution to the system.

$$A^{-1}b = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$  

You can verify on your own that $x_1 = 2$ and $x_2 = 0$ solves the equations.

The formula for the inverse of an arbitrary $(2 \times 2)$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is
\[
A^{-1} = \frac{1}{|A|} \begin{bmatrix}
    a_{22} & -a_{12} \\
    -a_{21} & a_{11}
\end{bmatrix}
\text{ where } |A| = a_{11}a_{22} - a_{12}a_{21}.
\]

To show this, multiply the two together:

\[
\frac{1}{|A|} \begin{bmatrix}
    a_{22} & -a_{12} \\
    -a_{21} & a_{11}
\end{bmatrix} \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix} = \frac{1}{|A|} \begin{bmatrix}
    a_{11}a_{22} - a_{12}a_{21} & a_{12}a_{22} - a_{12}a_{22} \\
    a_{12}a_{21} - a_{12}a_{21} & a_{11}a_{22} - a_{12}a_{21}
\end{bmatrix}
\]

\[
= \frac{1}{|A|} \begin{bmatrix}
    |A| & 0 \\
    0 & |A|
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}
\]

The formula for the inverse of a \((2 \times 2)\) matrix is worth committing to memory.

The expression \(|A|\) is called the ‘determinant’ of \(A\), something we will discuss in detail in later sections. Note that if \(|A|=0\), then the inverse will not exist (we say that \(A\) is ‘singular’). If \(A\) is singular, the system will not have a unique solution.

**Exercises**

1. Find the inverse of the following matrices
   (i) \(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\)
   (ii) \(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\)
   (iii) \(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\)
   (iv) \(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\)
   (v) \(\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}\)
   (vi) \(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\)
   (vii) \(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\)
   (viii) \(\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\)
   (ix) \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\)
   (x) \(\begin{bmatrix} 7 & 1 \\ 2 & 4 \end{bmatrix}\)
   (xi) \(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\)
   (xii) \(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\)

2. Find the inverse of the matrix
   \(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\).

Verify by directly multiplication.
3. Make a guess as to the inverse of the \((n \times n)\) matrix
\[
\begin{bmatrix}
a_{11} & 0 & 0 & \cdots & 0 \\
0 & a_{22} & 0 & \cdots & 0 \\
0 & 0 & a_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}
\end{bmatrix}
\]
Verify your conjecture by directly multiplication.

4. Find the inverse of the matrix \[
\begin{bmatrix}
0 & b \\
c & 0
\end{bmatrix}.
\]

5. Solve the following systems of equations by computing the inverse of the coefficient matrix:

\begin{align*}
\text{(i)} & \quad 2x_1 - x_2 = 4 \\
& \quad x_1 + 2x_2 = 2 \\
\text{(ii)} & \quad 3x_1 + 5x_2 = 6 \\
& \quad 6x_1 + 10x_2 = 12 \\
\text{(iii)} & \quad 3x_1 + 5x_2 = 6 \\
& \quad 6x_1 + 10x_2 = 10 \\
\text{(iv)} & \quad 2y - x = 4 + a \\
& \quad y + 2x = 2 \\
\text{(v)} & \quad 2x_1 - 3x_2 = 0 \\
& \quad x_1 + 2x_2 = 0 \\
\text{(vi)} & \quad 3x_1 + 5x_2 = 0 \\
& \quad 6x_1 + 10x_2 = 0
\end{align*}

6. Let \(A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 3 \\ 1.5 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.\)

\begin{align*}
\text{(i)} & \quad \text{Show that } AB = AC. \\
\text{(ii)} & \quad \text{Show that } A^{-1} \text{ does not exist.}
\end{align*}

\text{Remark} \quad \text{The example in (i) shows that } AB = AC \text{ does not imply that } B = C \text{ in general. However, if } A^{-1} \text{ exists, then it must be that } B = C, \text{ since}
\[
AB = AC \Rightarrow A^{-1}AB = A^{-1}AC \Rightarrow B = C.
\]

7. We defined the inverse matrix of \(A\) as the matrix \(A^{-1}\) such that \(A^{-1}A = I\).

Show that this implies that \(AA^{-1} = I\). That is, it doesn’t matter whether you premultiplying or postmultiplying \(A\) with \(A^{-1}\), you still get \(I\) as a result.

8. Let \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\). Find the inverse of \(A^T\) (assume \(a,b,c,d\) are such that the inverse exists), and show that \((A^T)^{-1} = (A^{-1})^T\).
9. Let $A$ be an $(n \times n)$ matrix whose inverse exists. Show that $(A^T)^{-1} = (A^{-1})^T$. (You don’t need to know how to compute an $(n \times n)$ for this. Start with the fact that $A^{-1}A = I$ and take transposes.)

10. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, and assume that their inverses exist. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

11. Let $A$ and $B$ be $(n \times n)$ matrices whose inverses exist. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

12. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$. Find the inverse of $AB$. Does the relationship $(AB)^{-1} = B^{-1}A^{-1}$ hold for these two matrices? Why?

13. Is it true that $(A + B)^{-1} = A^{-1} + B^{-1}$? Give a counterexample.
Section 7  Finding an Inverse using Elementary Row Operations

The formula for the inverse of \((3 \times 3)\) and larger square matrices is much more complicated. We will see them in a later section. For now, we show a practical (but tedious) way to find the inverse of a matrix using “elementary row operations”. These operations are exactly the steps used in the “elimination” process for solving systems of equations.

The idea here is that the three elementary row operators:

- Switching rows
- Subtracting a constant times one row from another row
- Multiplying an entire row by some number

can all be replicated by premultiplication by certain matrices. For example, to switch rows 1 and 2 in the matrix

\[
A = \begin{bmatrix}
0 & 2 & 4 \\
3 & 1 & 2 \\
6 & 2 & 1 \\
\end{bmatrix}
\]

we can premultiply by

\[
E = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

which is obtained by switching the 1st and 2nd rows of the \((3 \times 3)\) identity matrix. You can verify that premultiplying \(A\) by \(E\) switches rows 1 and 2 of \(A\):

\[
EA = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & 2 & 4 \\
3 & 1 & 2 \\
6 & 2 & 1 \\
\end{bmatrix} = \begin{bmatrix}
3 & 1 & 2 \\
0 & 2 & 4 \\
5 & 2 & 1 \\
\end{bmatrix}
\]

Similarly, premultiplying a matrix by

\[
E_{(2)\leftrightarrow(3)} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix},
\]

which is obtained by switching the 2nd and 3rd rows of the \((3 \times 3)\) identity matrix, switches the first and third rows of the matrix:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
3 & 1 & 2 \\
0 & 2 & 4 \\
5 & 2 & 1 \\
\end{bmatrix} = \begin{bmatrix}
3 & 1 & 2 \\
5 & 2 & 1 \\
0 & 2 & 4 \\
\end{bmatrix}
\]

This generalizes to switching other pairs of rows, and to larger matrices.
Another type of elementary row operation is to add/subtract a constant times one row to another row. For example, you can use the ‘3’ in row 1 as a pivot to eliminate the ‘5’ below it using this operation.

\[
\text{e.g. } \text{row}_2 = \text{row}_2 - \frac{5}{3} \text{row}_1
\]

\[
\begin{bmatrix}
3 & 1 & 2 \\
5 & 2 & 1 \\
0 & 2 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
3 & 1 & 2 \\
0 & \frac{1}{3} & -\frac{7}{3} \\
0 & 2 & 4
\end{bmatrix}
\]

and use the resulting \( \frac{1}{3} \) in row 2 as a pivot to eliminate the 2 in row 3:

\[
\text{e.g. } \text{row}_3 = \text{row}_3 - 6 \times \text{row}_2
\]

\[
\begin{bmatrix}
3 & 1 & 2 \\
0 & \frac{1}{3} & -\frac{7}{3} \\
0 & 2 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
3 & 1 & 2 \\
0 & \frac{1}{3} & -\frac{7}{3} \\
0 & 0 & 18
\end{bmatrix}
\]

These operations can also be done by premultiplying with appropriate matrices. To subtract \( \frac{5}{3} \) of row 1 from row 2, premultiply the matrix by

\[
\begin{bmatrix}
1 & 0 & 0 \\
-\frac{5}{3} & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

which is obtained by taking an identity matrix and subtracting \( \frac{5}{3} \) of row 1 from row 2. Verify that

\[
\begin{bmatrix}
1 & 0 & 0 \\
-\frac{5}{3} & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
3 & 1 & 2 \\
5 & 2 & 1 \\
0 & 2 & 4
\end{bmatrix}
= 
\begin{bmatrix}
3 & 1 & 2 \\
0 & \frac{1}{3} & -\frac{7}{3} \\
0 & 2 & 4
\end{bmatrix}
\]

To subtract \( 6 \times \text{row} \) 2 from row 3, premultiply by

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -6 & 1
\end{bmatrix}
\]

which is obtained by taking an identity matrix and subtracting \( 6 \times \text{row} \) 1 from row 2. Verify that

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -6 & 1
\end{bmatrix}
\begin{bmatrix}
3 & 1 & 2 \\
0 & \frac{1}{3} & -\frac{7}{3} \\
0 & 2 & 4
\end{bmatrix}
= 
\begin{bmatrix}
3 & 1 & 2 \\
0 & \frac{1}{3} & -\frac{7}{3} \\
0 & 0 & 18
\end{bmatrix}
\]

The pattern should be clear: to find the appropriate matrix for executing any particular row operation, take the identity matrix and apply that same operation to it.
To multiple row three by \( \frac{1}{18} \), premultiply by

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{18}
\end{bmatrix}
\]

We have

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{18}
\end{bmatrix}
\begin{bmatrix}
3 & 1 & 2 \\
0 & \frac{1}{3} & -\frac{7}{3} \\
0 & 0 & 18
\end{bmatrix}
= \begin{bmatrix}
3 & 1 & 2 \\
0 & \frac{1}{3} & -\frac{7}{3} \\
0 & 0 & 1
\end{bmatrix}.
\]

If the inverse of a matrix \( A \) exists, then there is a series of elementary row operators that reduces \( A \) to the identity matrix. Suppose the required elementary row operators are (in order) \( E_1, E_2, ..., E_n \), then

\[
E_n ... E_2 E_1 A = I
\]

which means that \( E_n ... E_2 E_1 = A^{-1} \). Furthermore, because \( E_n ... E_2 E_1 I = E_n ... E_2 E_1 \), we can use the following technique:

Write \( A \) and \( I \) side-by-side. Then apply the same row operators to both \( A \) and \( I \) until \( A \) is reduced to the identity matrix. At the same time, the identity matrix will be “reduced” to the inverse matrix.

Here is the fully worked out example for our example matrix:

\[
\begin{array}{ccc|ccc}
0 & 2 & 4 & 1 & 0 & 0 \\
3 & 1 & 2 & 0 & 1 & 0 \\
6 & 2 & 1 & 0 & 0 & 1 \\
\end{array}
\]

switch rows 1 and 2

\[
\begin{array}{ccc|ccc}
3 & 1 & 2 & 0 & 1 & 0 \\
0 & 2 & 4 & 1 & 0 & 0 \\
6 & 2 & 1 & 0 & 0 & 1 \\
\end{array}
\]

row 3 minus \( 2 \times \) row 1,

\[
\begin{array}{ccc|ccc}
3 & 1 & 2 & 0 & 1 & 0 \\
0 & 2 & 4 & 1 & 0 & 0 \\
0 & 0 & -3 & 0 & -2 & 1 \\
\end{array}
\]
row 2 minus $-\frac{4}{3} \times$ row 3
\[
\begin{bmatrix}
3 & 1 & 0 \\
0 & 2 & 0 \\
0 & 0 & -3
\end{bmatrix} - \frac{4}{3}
\begin{bmatrix}
0 & -\frac{1}{3} & \frac{2}{3} \\
1 & -\frac{8}{3} & \frac{4}{3} \\
0 & -2 & 1
\end{bmatrix}
\]
row 1 minus $-\frac{2}{3} \times$ row 3
\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -3
\end{bmatrix} - \frac{2}{3}
\begin{bmatrix}
0 & -\frac{1}{3} & \frac{2}{3} \\
1 & -\frac{8}{3} & \frac{4}{3} \\
0 & -2 & 1
\end{bmatrix}
\]
row 1 minus $\frac{1}{2} \times$ row 2
\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -3
\end{bmatrix} - \frac{1}{2}
\begin{bmatrix}
0 & 1 & 0 \\
1 & -\frac{4}{3} & \frac{2}{3} \\
0 & 3 & -\frac{1}{3}
\end{bmatrix}
\]
row 1 minus $\frac{1}{3} \times$ row 3
\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -3
\end{bmatrix} - \frac{1}{3}
\begin{bmatrix}
0 & 2 & 0 \\
3 & 1 & 2 \\
6 & 2 & 1
\end{bmatrix}
\]

The inverse of $A = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 2 \\ 6 & 2 & 1 \end{bmatrix}$ is therefore
\[
\begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 2 \\ 6 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{4}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}
\]

You should verify this by computing
\[
\begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{4}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 2 \\ 6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{4}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}
\]

and checking to see if you get the identity matrix.

If we are using this technique to solve the equation $Ax = b$, i.e. to compute $x = A^{-1}b$: write down $A$ and $b$ side-by-side and apply the row operations to reduce $A$ into the identity matrix. The rationale for this is the same as previously:

\[
\begin{bmatrix} A & b \\ E_1A & E_1b \\ E_2E_1A & E_2E_1b \\ \vdots \end{bmatrix}
\]

Reduced to $I$:

\[
\begin{bmatrix} E_n...E_2E_1A & E_n...E_2E_1b \end{bmatrix} = A^{-1}b
\]
Exercises

1. Find the inverse of the matrix \( A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{bmatrix} \).

2. The matrix \( A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 1 & 2 & -6 \end{bmatrix} \) has no inverse. Apply elementary row operations to \( A \) as though finding the inverse. What happens?

3. Write down the simultaneous equations

\[
3x_1 + 3x_2 + 2x_3 = 6 \\
2x_1 + x_2 + 3x_3 = 3 \\
x_1 + 5x_3 + 2x_3 = 4
\]

in the form \( Ax = b \). Solve this by

(i) finding the inverse of \( A \) and computing \( x = A^{-1}b \);

(ii) writing down \( A \) and \( b \) side-by-side:

\[
\begin{bmatrix}
3 & 3 & 2 & | & 6 \\
2 & 1 & 3 & | & 3 \\
1 & 5 & 2 & | & 4
\end{bmatrix}
\]

and applying the necessary row operations to reduce the left side of the matrix to the identity matrix.

4. Find the inverse of the matrix

\[
D = \begin{bmatrix}
2 & 2 & 3 & 4 \\
0 & -1 & 0 & 11 \\
1 & -1 & 0 & 3 \\
-2 & 0 & -1 & 3
\end{bmatrix}
\]
Section 8  An Introduction to Determinants and Cramer’s Rule

In this section, you are introduced to a formula for solving systems of simultaneous equations, called Cramer’s Rule. We begin with solutions to systems with two equations in two unknowns, and move our way up to the general \( n \) equations in \( n \) unknowns case. The formula in the general case is demonstrated here but not derived; the derivation will be given later. The key ingredient of the formula for solving systems of equations is the determinant of a matrix.

Determinants of \((2 \times 2)\) Matrices

A system of two simultaneous equations in two unknowns

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 &= b_1 \\
    a_{21}x_1 + a_{22}x_2 &= b_2
\end{align*}
\]

can be written as \( \mathbf{Ax} = \mathbf{b} \), where

\[
\mathbf{A} = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix}
    b_1 \\
    b_2
\end{bmatrix}.
\]

We have seen two (intimately related) ways to solve system of equations. One way is by elimination; another way is by computing the inverse of \( \mathbf{A}^{-1} \) and then finding \( \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \). The elimination method for computing the inverse of \( \mathbf{A} \) clearly shows the connection between the two methods. In this section, we consider yet another method, centered on the concept of a determinant (which as you will see later, is also closely connected to the inverse of \( \mathbf{A} \)).

By solving the system in the usual way, we can show that the general solution to the system is

\[
\begin{align*}
    x_1 &= \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}.
\end{align*}
\]

Some observations:

- The denominators of both \( x_1 \) and \( x_2 \) are the same, and are made up from only the elements of the coefficient matrix \( \mathbf{A} \). We call the expression in the denominator the ‘determinant’ of \( \mathbf{A} \), denoted by

\[
|\mathbf{A}| = a_{11}a_{22} - a_{21}a_{12}
\]

Determinants are also sometimes denoted as

\[
\det(\mathbf{A}) = \begin{vmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{vmatrix}.
\]
- If the denominator is zero, the system will not have a unique solution, since we cannot divide by zero (the system will have either no solution, or an infinite number of solutions). This expression therefore determines whether or not the system of equations has a unique solution; the system will have a unique solution only if $|A| \neq 0$.

- Observe that the numerator of $x_1$ can be expressed as the determinant of the matrix $\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$, which is just the matrix $A$ with its first column replaced by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Likewise, the numerator of $x_2$ is the determinant of the matrix $\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$, which is the matrix $A$ with its second column replaced by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

In other words, the solutions can be written in terms of determinants as

$$x_1 = \frac{b_1 a_{12} - b_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}} \quad \text{and} \quad x_2 = \frac{a_{11} b_1 - a_{21} b_2}{a_{11} a_{22} - a_{12} a_{21}}$$

Why is this important? Because this pattern extends to larger systems of equations (it is called ‘Cramer’s Rule’). Furthermore, by using the properties of ‘Determinants’, we can (i) find ways of helping us compute solutions of larger systems more easily, and (ii) we can often say a lot regarding the characteristics of solutions of systems of equations, without actually solving them. More on all this later. For now, memorize the formula for the determinant of a $(2 \times 2)$ matrix.

When will the determinant of the coefficient matrix be zero? There are the obvious cases where all the elements one or more row or one or more column are zero:

$$\begin{vmatrix} 0 & 0 \\ a_{21} & a_{22} \end{vmatrix} = 0, \quad \begin{vmatrix} a_{11} & 0 \\ a_{21} & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

It might be a good exercise for you to write out the systems of equations that correspond to these coefficient matrices.
A less obvious case is when one row is a multiple of the other (or is the same as the other):

\[
\begin{bmatrix}
  a_{11} & a_{12} \\
  ca_{11} & ca_{12}
\end{bmatrix} = ca_{11}a_{12} - ca_{11}a_{12} = 0;
\]

Take for example the simultaneous equations systems

\[
\begin{align*}
  2x_1 + 4x_2 &= 1 \\
  3x_1 + 6x_2 &= 1.5
\end{align*}
\]

\[
\begin{align*}
  2x_1 + 4x_2 &= 1 \\
  3x_1 + 6x_2 &= 2
\end{align*}
\]

The coefficients in the second equation in both systems of equations are 1.5 times of the first equation. You can quickly calculate the determinant of the coefficient matrix to be zero. For the system of equations on the left, the constant on the right hand side of the equality for the second equation is also 1.5 times that of the first. In this case, the two equations are the essentially the same: dividing the second equation throughout by 1.5 gives the first. There is effectively only one equation here, in two unknowns, so there are infinitely many solutions.

The system on the right, on the other hand, obviously has no solution.

One special case should be considered separately. Consider the system

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 &= 0 \\
  a_{21}x_1 + a_{22}x_2 &= 0
\end{align*}
\]

Such a system is called a homogenous system. If the determinant of the coefficient matrix

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\]

is non-zero, then there is a unique solution $x_1 = 0$ and $x_2 = 0$. This is sometimes referred to as the trivial solution (these are just two lines that intersect at the origin). The question then is: when will this system of equations have a non-trivial solution? It should be straightforward to see that non-trivial solutions will exist only if the determinant of the coefficient matrix is zero. (Cramer’s rule will not give you the solutions in this case, however – you get a “0/0” result).
Determinants of $(3 \times 3)$ Matrices

The general $3$ equation $3$ unknown system

\begin{align*}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{align*}

takes a bit more work to solve, but with a little bit of patience you can show that the solution is

\begin{align*}
x_1 &= \frac{b_1a_{23}a_{33} + a_{12}a_{23}b_3 + a_{13}b_2a_{33} - a_{13}a_{23}b_2 - b_1a_{23}a_{32} - a_{12}b_2a_{33}}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}} \\
x_2 &= \frac{a_{11}b_2a_{33} + b_1a_{23}a_{31} + a_{13}b_2a_{31} - a_{13}b_2a_{32} - b_1a_{23}a_{31} - a_{12}b_2a_{33}}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}} \\
x_3 &= \frac{a_{11}a_{22}b_3 + a_{12}b_2a_{31} + b_1a_{23}a_{32} - b_1a_{22}a_{32} - a_{11}b_2a_{33} - a_{12}b_2a_{33}}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}}
\end{align*}

Obviously you wouldn’t want to memorize this, at least not in this form. However, observe again that the denominators of all three are the same, and composed only of the coefficients (the $a$’s). Again, the system will have a unique solution only if the expression in the denominators does not equal zero; if it equals zero, then the system will either have infinitely many solutions, or none. The expression in the denominator again determines whether or not the system will have a unique solution, so we will collect the coefficients into the matrix

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

and define the determinant of this $(3 \times 3)$ matrix to be

\[
|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.
\]

This formula is also not easily memorized. There is a shortcut that is sometimes useful: extending the matrix to include the first two columns.
The formula is then easily remembered in terms of the “diagonals” where the terms under the solid arrows are added, whereas the terms under the dashed arrows are subtract.

Next, observe that the numerators of $x_1$, $x_2$, and $x_3$ can then be written as

$$
\begin{vmatrix}
  b_1 & a_{12} & a_{13} \\
  b_2 & a_{22} & a_{23} \\
  b_3 & a_{32} & a_{33}
\end{vmatrix},
\begin{vmatrix}
  a_{11} & b_1 & a_{13} \\
  a_{21} & b_2 & a_{23} \\
  a_{31} & b_3 & a_{33}
\end{vmatrix},
\begin{vmatrix}
  a_{11} & a_{12} & b_1 \\
  a_{21} & a_{22} & b_2 \\
  a_{31} & a_{32} & b_3
\end{vmatrix}
$$

respectively, i.e. we can write

$$
\begin{align*}
x_1 &= \frac{\begin{vmatrix}
  b_1 & a_{12} & a_{13} \\
  b_2 & a_{22} & a_{23} \\
  b_3 & a_{32} & a_{33}
\end{vmatrix}}{|A|}, & x_2 &= \frac{\begin{vmatrix}
  a_{11} & b_1 & a_{13} \\
  a_{21} & b_2 & a_{23} \\
  a_{31} & b_3 & a_{33}
\end{vmatrix}}{|A|}, & x_3 &= \frac{\begin{vmatrix}
  a_{11} & a_{12} & b_1 \\
  a_{21} & a_{22} & b_2 \\
  a_{31} & a_{32} & b_3
\end{vmatrix}}{|A|}.
\end{align*}
$$

We will save ourselves some tedium by using $A_i(b)$ to represent the matrix $A$ with the $i$th column replaced by $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, thus writing

$$
\begin{align*}
x_1 &= \frac{|A_1(b)|}{|A|}, & x_2 &= \frac{|A_2(b)|}{|A|}, & x_3 &= \frac{|A_3(b)|}{|A|}.
\end{align*}
$$

**Example**

Use Cramer’s Rule to solve

\[
\begin{align*}
2y - z &= -7 \\
x + y + 3z &= 2 \\
-3x + 3y + 2z &= 0
\end{align*}
\]

The coefficient matrix is

$$
A = \begin{bmatrix}
0 & 2 & -1 \\
1 & 1 & 3 \\
-3 & 3 & 2
\end{bmatrix}
$$

Using the ‘shortcut’ rule for computing determinants, we have
which gives $|A| = 0 + (-18) + (-3) - 3 - 0 - 4 = -28$.

Also, $A_1(b) = \begin{bmatrix} -7 & 2 & -1 \\ 2 & 1 & 3 \\ 0 & 3 & 2 \end{bmatrix}$, and the determinant of this matrix is

$$\begin{vmatrix} -7 & 2 & -1 \\ 0 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

which gives $|A_1(b)| = (-14) + 0 + (-6) - 0 - (-63) - 8 = 35$, so the solution for $x$ is

$$x = \frac{35}{28}$$

Similarly, the matrices $A_2(b)$ and $A_3(b)$ are

$$A_2(b) = \begin{bmatrix} 0 & -7 & -1 \\ 1 & 2 & 3 \\ -3 & 0 & 2 \end{bmatrix}$$

and

$$A_3(b) = \begin{bmatrix} 0 & 2 & -7 \\ 1 & 1 & 2 \\ -3 & 3 & 0 \end{bmatrix}$$

with determinants $|A_2(b)| = 71$ and $|A_3(b)| = -54$, so the full solution is

$$x = \frac{35}{28}, \quad y = \frac{71}{28}, \quad z = \frac{54}{28} = \frac{27}{14}.$$
Exercises

1. Find the determinants of the following matrices

(i) \[
\begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}
\]  
(ii) \[
\begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix}
\]  
(iii) \[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\]  
(iv) \[
\begin{bmatrix}
1 & 0 \\
2 & 0
\end{bmatrix}
\]  
(v) \[
\begin{bmatrix}
0 & 0 \\
2 & 1
\end{bmatrix}
\]  
(vi) \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\]  
(vii) \[
\begin{bmatrix}
2 & 2 \\
2 & 2
\end{bmatrix}
\]  
(viii) \[
\begin{bmatrix}
3 & 2 \\
2 & 1
\end{bmatrix}
\]  
(ix) \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]  
(x) \[
\begin{bmatrix}
7 & 1 \\
2 & 4
\end{bmatrix}
\]  

2. Solve the following systems of equations using Cramer’s Rule

(i) \[
\begin{align*}
2x_1 - x_2 &= 4 \\
x_1 + 2x_2 &= 2
\end{align*}
\]

(ii) \[
\begin{align*}
3x_1 + 5x_2 &= 6 \\
6x_1 + 10x_2 &= 12
\end{align*}
\]

(iii) \[
\begin{align*}
3x_1 + 5x_2 &= 6 \\
6x_1 + 10x_2 &= 10
\end{align*}
\]

(iv) \[
\begin{align*}
2y - x &= 4 + a \\
y + 2x &= 2
\end{align*}
\]

(v) \[
\begin{align*}
2x_1 - 3x_2 &= 0 \\
x_1 + 2x_2 &= 0
\end{align*}
\]

(vi) \[
\begin{align*}
3x_1 + 5x_2 &= 0 \\
6x_1 + 10x_2 &= 0
\end{align*}
\]

For the systems that do not have a unique solution, find out whether it has zero or infinitely many solutions.

3. Solve, using Cramer’s rule, the following system of equations for \( C \) and \( Y \) (take everything else as fixed)

\[
C = a + bY , \, 0 < b < 1
\]

\[
Y = C + G.
\]

4. Suppose we have the following demand and supply equation

\[
Q^d = \alpha_0 + \alpha_1 P , \, \alpha_1 < 0
\]

\[
Q^s = \beta_0 + \beta_1 P + \beta_2 R , \, \beta_1 > 0 , \, \beta_2 < 0
\]

where \( Q^d \) and \( Q^s \) are the quantities demanded (by consumers) and supplied (by firms). Suppose that in equilibrium, \( P \) is the market price of the good, and \( R \) is rainfall. In equilibrium, we have

\[
Q^d = Q^s \ (= Q) .
\]

Using Cramer’s Rule, solve this system of equations for the equilibrium price \( P \) and quantity \( Q \), treating everything else as fixed. What happens to equilibrium price and quantity when rainfall \( R \) increases? How would your answer compare with the case when demand is completely inelastic (i.e. \( \alpha_1 = 0 \)?)
5. Find the determinants of the following matrices

(i) \[
\begin{bmatrix}
4 & 0 & 1 \\
19 & 1 & -3 \\
7 & 1 & 0
\end{bmatrix}
\]

(ii) \[
\begin{bmatrix}
0 & 2 & 0 \\
3 & 0 & 4 \\
2 & 3 & 0
\end{bmatrix}
\]

6. Use Cramer’s Rule to solve

\[
\begin{align*}
4x + z &= 4 \\
19x + y - 3z &= 3 \\
7x + y &= 1
\end{align*}
\]

7. Let \( A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \) and \( B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \). Prove that \( |AB| = |A||B| \)

This is a very useful result, and holds for the general case, not only for \((2 \times 2)\) case: if \( A \) and \( B \) are \((n \times n)\) matrices, then \( |AB| = |A||B| \). Note that it is essential, however, for both \( A \) and \( B \) to be square matrices of the size dimension (why?)

8. Find the determinants of

(i) \[
\begin{bmatrix}
a & b \\
0 & d
\end{bmatrix}
\]

(ii) \[
\begin{bmatrix}
a & 0 \\
c & d
\end{bmatrix}
\]

(iii) \[
\begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}
\]

9. Find the determinants of

(i) \[
\begin{bmatrix}
a & b & c \\
0 & e & f \\
0 & 0 & i
\end{bmatrix}
\]

(ii) \[
\begin{bmatrix}
a & 0 & 0 \\
d & e & 0 \\
g & h & i
\end{bmatrix}
\]

10. Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \\ 0 & -3 & -1 \end{bmatrix} \).

(a) Show that the third equation can be written as a linear combination of the first two rows (i.e., you can find \( c_1 \) and \( c_2 \) such that

\[
c_1 [1 \ 2 \ 3] + c_2 [2 \ 1 \ 5] = [0 \ -3 \ -1].
\]

(b) Show that \( \det A = 0 \).
Section 9  The Laplace Expansion

In the last section, we defined the determinant of a \((3 \times 3)\) matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]
to be

\[
|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{11}a_{22}a_{32}.
\]

In this section, we introduce a general formula for computing determinants. Rewriting

\[
|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{11}a_{22}a_{32}.
\]

note that the terms outside the brackets are the terms along the first row of the matrix

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

The term in brackets associated with \(a_{11}\) is the determinant of the \((2 \times 2)\) matrix after deleting the 1\(^{st}\) row and 1\(^{st}\) column of \(A\):

\[
(a_{22}a_{33} - a_{23}a_{32}) = \begin{vmatrix}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{vmatrix}
\]

The term in brackets associated with \(a_{12}\) is the determinant of the \((2 \times 2)\) matrix after deleting the 1\(^{st}\) row and 2\(^{nd}\) column of \(A\):

\[
(a_{21}a_{33} - a_{23}a_{31}) = \begin{vmatrix}
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{vmatrix}
\]

The term in brackets associated with \(a_{13}\) is the determinant of the \((2 \times 2)\) matrix after deleting the 1\(^{st}\) row and 3\(^{rd}\) column of \(A\):

\[
(a_{21}a_{32} - a_{22}a_{31}) = \begin{vmatrix}
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{vmatrix}
\]

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There is the matter of the minus sign in front of $a_{12}$. This can be achieved by multiplying into each term $(-1)^{i+j}$. Therefore, we can write

$$
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
= a_{11} \begin{vmatrix}
  a_{22} & a_{23} \\
  a_{32} & a_{33}
\end{vmatrix} \cdot (-1)^{1+1} + a_{12} \begin{vmatrix}
  a_{21} & a_{23} \\
  a_{31} & a_{33}
\end{vmatrix} \cdot (-1)^{1+2} + a_{13} \begin{vmatrix}
  a_{21} & a_{22} \\
  a_{31} & a_{32}
\end{vmatrix} \cdot (-1)^{1+3}
$$

Some names have been introduced in the formula above: the determinant of the matrix after removing the $i$th row and $j$th column is called the $(i,j)$th “minor” of $A$; including the sign $(-1)^{i+j}$ gives us the $(i,j)$th “cofactor” of $A$.

**Example**

The determinant of the matrix

$$
A = \begin{bmatrix}
  0 & 2 & -1 \\
  1 & 1 & 3 \\
  -3 & 3 & 2
\end{bmatrix}
$$

is

$$
\det(A) = (0)(-1)^{1+1} \begin{vmatrix}
  2 & 3 \\
  3 & 2
\end{vmatrix} + (2)(-1)^{1+2} \begin{vmatrix}
  1 & 3 \\
  -3 & 2
\end{vmatrix} + (-1)(-1)^{1+3} \begin{vmatrix}
  1 & 1 \\
  -3 & 3
\end{vmatrix} = -28;
$$

The cofactor expansion for computing determinants is not unique. For instance, we could have written the original formula as

$$
\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{23}a_{32} - a_{11}a_{22}a_{33}
$$

which can be written as
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\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{vmatrix}
= 
\begin{vmatrix}
5 & 6 \\
8 & 9
\end{vmatrix}
+ \begin{vmatrix}
1 & 2 \\
7 & 9
\end{vmatrix}
+ \begin{vmatrix}
1 & 3 \\
4 & 8
\end{vmatrix}
\]

Alternatively, we could have expanded along a column:

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{vmatrix}
= 
\begin{vmatrix}
5 & 6 \\
8 & 9
\end{vmatrix}
+ \begin{vmatrix}
1 & 2 \\
7 & 9
\end{vmatrix}
+ \begin{vmatrix}
1 & 3 \\
4 & 8
\end{vmatrix}
\]

Note that we achieved the correct signs by multiplying the \((i,j)\)th minor with \((-1)^{i+j}\).

Expanding along any row or column would in fact give us the same expression; we can write

\[
\det(A) = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij} \quad \text{for any row } i
\]

or

\[
\det(A) = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} M_{ij} \quad \text{for any column } j.
\]

where \(M_{ij}\) is the \((i,j)\)th minor of A. This formula is known as the Laplace Expansion.
The fact that you can expand along any row or column can simplify computations substantially if there is a row or column with many zeros.

**Example** Compute the determinant of the matrix

\[
A = \begin{bmatrix}
1 & 0 & 11 \\
2 & 0 & 3 \\
5 & 4 & 6
\end{bmatrix}
\]

using the Laplace Expansion (i) expanding along the first row, (ii) expanding down the second column.

(i) \[
\begin{vmatrix}
1 & 0 & 11 \\
2 & 0 & 3 \\
5 & 4 & 6
\end{vmatrix} = (1)(-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 5 & 6 \end{vmatrix} + (0)(-1)^{1+2} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (11)(-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 5 & 4 \end{vmatrix} = -12 + 0 + 88 = 76
\]

(ii) \[
\begin{vmatrix}
1 & 0 & 11 \\
2 & 0 & 3 \\
5 & 4 & 6
\end{vmatrix} = (0)(-1)^{2+2} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (0)(-1)^{2+3} \begin{vmatrix} 1 & 11 \\ 5 & 6 \end{vmatrix} + (4)(-1)^{3+3} \begin{vmatrix} 1 & 11 \\ 2 & 3 \end{vmatrix} = 76
\]

\[(n \times n)\text{ Determinants}\]

Cramer’s Rule and the Laplace Expansion extend to larger systems. The determinant for a general \((n \times n)\) matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

is \[|A| = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij} \] for any row \(i\)

or \[|A| = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij} \] for any column \(j\)

where the \((i, j)\)th minor \(M_{ij}\) of an \(n \times n\) matrix \(A\) is the determinant of the \((n-1) \times (n-1)\) matrix that remains after removing the \(i\)th row and \(j\)th column of \(A\).
Example

The determinant of \( D = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 1 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{bmatrix} \) is

\[
|D| = 3(-1)^{1+3} \begin{vmatrix} 0 & -1 & 11 \\ 1 & -1 & 3 \\ -2 & 0 & 3 \end{vmatrix} + 0(-1)^{2+3} \begin{vmatrix} 2 & 2 & 4 \\ 1 & -1 & 3 \\ -2 & 0 & 3 \end{vmatrix} + 0(-1)^{3+3} \begin{vmatrix} 2 & 2 & 4 \\ 0 & -1 & 11 \\ -2 & 0 & 3 \end{vmatrix} + (-1)(-1)^{4+3} \begin{vmatrix} 2 & 2 & 4 \\ 0 & -1 & 11 \\ 1 & -1 & 3 \end{vmatrix} = 3
\]

where we have expanded down the third column.

The Laplace expansion includes the \((2 \times 2)\) case. Define the determinant of a single number as \(|a| = a\). Note that in this context the symbol \(|\cdot|\) does not refer to absolute values, e.g. \(|-2| = -2\), not 2. Then taking, say, a first row expansion, we have

\[
|A| = \sum_{j=1}^{2} a_{i j}(-1)^{i+j} M_{i j} = a_{i 1}(-1)^{1+1} M_{i 1} + a_{i 2}(-1)^{1+2} M_{i 2} = a_{i 1}a_{22} - a_{i 2}a_{21}.
\]

Cramer’s Rule also extends to general \((n \times n)\) systems of equations. The solution to

\[
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n
\]

is

\[
x_i = \frac{|A_i(b)|}{|A|}, \quad i = 1, \ldots, n,
\]

where

\[
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
\]

and \(A_i(b)\) is the matrix \(A\) with its \(i\)th column replaced by \(b\).
Exercises

1. Find the determinants of the following matrices using the Laplace expansion.

   \[
   \begin{bmatrix}
   4 & 0 & 1 \\
   19 & 1 & -3 \\
   7 & 1 & 0
   \end{bmatrix}
   \quad \begin{bmatrix}
   0 & 2 & 0 \\
   3 & 0 & 4 \\
   2 & 3 & 0
   \end{bmatrix}
   \quad \begin{bmatrix}
   a_{21} & a_{22} & a_{23} \\
   a_{31} & a_{32} & a_{33} \\
   a_{41} & a_{42} & a_{43}
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   a_{11} & 0 & 0 \\
   a_{21} & a_{22} & 0 \\
   a_{31} & a_{32} & a_{33}
   \end{bmatrix}
   \quad \begin{bmatrix}
   a_{11} & a_{12} & a_{13} \\
   0 & a_{22} & a_{23} \\
   0 & 0 & a_{33}
   \end{bmatrix}
   \quad \begin{bmatrix}
   a_{11} & 0 & 0 \\
   0 & a_{22} & 0 \\
   0 & 0 & a_{33}
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   0 & 0 & a_{23} \\
   0 & a_{32} & 0 \\
   a_{41} & 0 & 0
   \end{bmatrix}
   \quad \begin{bmatrix}
   4 & 3 & 2 \\
   6 & 4.5 & 3 \\
   7 & 1 & 0
   \end{bmatrix}
   \quad \begin{bmatrix}
   4 & 3 & 0 \\
   9 & 1 & 3 \\
   7 & 1 & 0
   \end{bmatrix}
   \]

2. Use Cramer’s Rule to solve

\[
\begin{align*}
4x + z &= 4 \\
19x + y - 3z &= 3 \\
7x + y &= 1
\end{align*}
\]

3. Solve the following system of equations

\[
\begin{align*}
2x_1 + 2x_2 + 3x_3 + 4x_4 &= 2 \\
-x_2 + 11x_4 &= 3 \\
x_1 - x_2 + 3x_4 &= 1 \\
-2x_1 - x_3 + 3x_4 &= 2
\end{align*}
\]

4. Find the determinants using the Laplace expansion

\[
\begin{bmatrix}
a_{11} & 0 & 0 & a_{41} \\
a_{21} & a_{22} & 0 & 0 \\
a_{31} & 0 & a_{33} & 0 \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\quad \begin{bmatrix}
a_{11} & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 \\
a_{31} & a_{32} & a_{33} & 0 \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\quad \begin{bmatrix}
a_{11} & a_{21} & a_{31} & a_{41} \\
0 & a_{22} & a_{32} & a_{42} \\
0 & 0 & a_{33} & a_{43} \\
0 & 0 & 0 & a_{44}
\end{bmatrix}
\]

\[
\begin{bmatrix}
a_{11} & 0 & 0 & 0 \\
0 & a_{22} & 0 & 0 \\
0 & 0 & a_{33} & 0 \\
0 & 0 & 0 & a_{44}
\end{bmatrix}
\quad \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\quad \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]
5. Let $A$ be an arbitrary $(n \times n)$ matrix. Show that if we multiply every element of a single row or column by $c$, then the determinant of the new matrix is $c |A|$. What is the determinant of $cA$?

6. Let $E$ be the matrix obtained by switching the 1st and last rows of an $(n \times n)$ identity matrix, i.e.

$$
E = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0 & 0
\end{bmatrix}
$$

(a) Show using the Laplace expansion that $\det E = -1$;

(b) Does your answer depend on whether $n$ is even or odd?

(c) Use your result to prove that the determinant of the matrix formed by switching any two rows of the identity matrix is $-1$.

7. (a) Let

$$
E = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
a & 0 & 1
\end{bmatrix}
$$

and suppose $A$ is some $(3 \times n)$ matrix. Describe the rows of the product $EA$ in terms of the rows of $A$; What is the determinant of the matrix $E$?

(b) Let $E$ be a matrix that carries out the elementary row operation of subtracting a multiple of one row from another row. What is the determinant of $E$?
Section 10 Properties of Determinants

A very important result, stated here without proof, is that

$$|AB| = |A||B|$$

if $A$ and $B$ are square matrices that can be multiplied together.

We can use this result to relate elementary row operations to the determinants of square matrices, generating several important properties of determinants. This in turn provides a way to simplify the computation of determinants (including determinants of larger matrices).

1. If two rows of $A$ are interchanged, the sign of the determinant changes.
2. If all the elements of a single row or column of $A$ are multiplied by $\alpha$, then the determinant of the new matrix is $\alpha |A|$.
3. If a multiple of a row (column) is added to another row (column), the determinant remains unchanged.

We illustrate these using examples. It should be easy for you to generalize from these examples. Throughout, we use

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix}$$

as an example. You can verify that the determinant of this matrix is $-24$.

Recall that switching two rows of the matrix:

$$\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 3 & 3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$

e.g. switching rows 1 and 2:

is equivalent to premultiplying $A$ with that matrix

$$E_{(1)\leftrightarrow(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

You can verify this:

$$E_{(1)\leftrightarrow(2)}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$
The matrix $E_{(1)\leftrightarrow(2)}$ is obtained by switching the 1st and 2nd rows of the $(3\times3)$ identity matrix. Using the Laplace expansion, you can easily show that

$$
\begin{vmatrix}
E_{(1)\leftrightarrow(2)}
\end{vmatrix} = \begin{vmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{vmatrix} = -1
$$

This is true in general: the determinant of the identity matrix is one; when two rows of the identity matrix are switched, the determinant of the new matrix becomes $-1$.

This means that switching two rows of a matrix results in a switch in the sign of its determinant.

$$
\begin{vmatrix}
E_{(1)\leftrightarrow(2)}A
\end{vmatrix} = \begin{vmatrix}
E_{(1)\leftrightarrow(2)}
\end{vmatrix} \begin{vmatrix}
A
\end{vmatrix} = -\begin{vmatrix}
A
\end{vmatrix}
$$

In our example:

$$
\begin{vmatrix}
2 & 1 & 3 \\
3 & 3 & 2 \\
1 & 5 & 2
\end{vmatrix} = \begin{vmatrix}
E_{(1)\leftrightarrow(2)}A
\end{vmatrix} = \begin{vmatrix}
E_{(1)\leftrightarrow(2)}
\end{vmatrix} \begin{vmatrix}
A
\end{vmatrix} = (-1)(-24) = 24
$$

**In general, switching two rows of a matrix switches the sign of its determinant.**

The second type of row operator is multiplying a row of a matrix by some number:

e.g. multiply row 1 of $A$ by 1/3:

$$
\begin{bmatrix}
3 & 3 & 2 \\
2 & 1 & 3 \\
1 & 5 & 2
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 1 & \frac{2}{3} \\
2 & 1 & 3 \\
1 & 5 & 2
\end{bmatrix}
$$

This is equivalent to premultiplying the matrix with

$$
E_{\frac{1}{3}(1)} = \begin{bmatrix}
\frac{1}{3} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

You can easily verify that the determinant of this matrix is $\frac{1}{3}$. Therefore

$$
\begin{vmatrix}
E_{\frac{1}{3}(1)}A
\end{vmatrix} = \begin{vmatrix}
E_{\frac{1}{3}(1)}
\end{vmatrix} \begin{vmatrix}
A
\end{vmatrix} = \left(\frac{1}{3}\right)(-24) = -8
$$

**Multiplying one row of a matrix by $\alpha$ multiplies the determinant by $\alpha$.**
Finally, the third type of elementary row operator is to add/subtract a constant times one row to another row:

e.g. subtract half of row 2 of $A$ from row 3:

\[
\begin{pmatrix}
3 & 3 & 2 \\
2 & 1 & 3 \\
1 & 5 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
3 & 3 & 2 \\
2 & 1 & 3 \\
0 & 4.5 & 0.5
\end{pmatrix}
\]

This is equivalent to premultiplying the matrix with

\[
E_{(3) \rightarrow \frac{1}{2}(2) \rightarrow (3)} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -\frac{1}{2} & 1
\end{bmatrix}
\]

obtained by subtracting half of the second row of the identity matrix from row 3 of the identity matrix. It is obvious that the determinant of this matrix is one. Therefore

\[
\begin{vmatrix}
3 & 3 & 2 \\
2 & 1 & 3 \\
0 & 4.5 & 0.5
\end{vmatrix} = \begin{vmatrix} E_{(3) \rightarrow \frac{1}{2}(2) \rightarrow (3)} \end{vmatrix} A = \begin{vmatrix} E_{(3) \rightarrow \frac{1}{2}(2) \rightarrow (3)} \end{vmatrix} \begin{vmatrix} A \end{vmatrix} = (1) \begin{vmatrix} A \end{vmatrix} = -24
\]

Adding/subtracting a constant times one row of a matrix to another row of the matrix does not change its determinant.

Since the determinant of a triangular matrix is simply the product of the elements in the diagonal, these results imply that to compute a determinant, we can reduce a matrix to a triangular matrix, compute the determinant of that, and reverse the effects of the row operations used.

<table>
<thead>
<tr>
<th>Row Operation</th>
<th>gives</th>
<th>Effect on det.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 3 2</td>
<td>1 5 2</td>
<td></td>
</tr>
<tr>
<td>2 1 3</td>
<td>2 1 3</td>
<td>$\times (-1)$</td>
</tr>
<tr>
<td>1 5 2</td>
<td>3 3 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 5 2</td>
<td></td>
</tr>
<tr>
<td>row 2 minus 2 times row 1, $\downarrow$</td>
<td>0 0 0</td>
<td>[no change]</td>
</tr>
<tr>
<td></td>
<td>3 3 2</td>
<td></td>
</tr>
<tr>
<td>row 3 minus 3 times row 1, $\downarrow$</td>
<td>0 0 0</td>
<td>[no change]</td>
</tr>
</tbody>
</table>
Row Operation gives Effect on det.

1 5 2
row 2 minus row 3, \( \perp \) 0 3 3 [no change]
0 \(-12\) \(-4\)

1 5 2
row 3 plus 4 times row 2, \( \perp \) 0 3 3 [no change]
0 0 8

Determinant of this last triangular matrix is \((1 \times 3 \times 8) = 24\).

Looking back at the row operations used, we find we switch the sign once. Therefore, the determinant of the original matrix is

\[
\begin{vmatrix}
3 & 3 & 2 \\
2 & 1 & 3 \\
1 & 5 & 2 \\
\end{vmatrix} = -24.
\]

**Exercises**

1. The following matrices are derived from the matrix

\[
\begin{bmatrix}
3 & 3 & 2 \\
2 & 1 & 3 \\
1 & 5 & 2 \\
\end{bmatrix}
\]

using elementary row operations. Find their determinants.

(i) \[
\begin{bmatrix}
1 & 5 & 2 \\
2 & 1 & 3 \\
3 & 3 & 2 \\
\end{bmatrix}
\]

(ii) \[
\begin{bmatrix}
3 & 3 & 2 \\
4 & 2 & 6 \\
1 & 5 & 2 \\
\end{bmatrix}
\]

(iii) \[
\begin{bmatrix}
5 & 13 & 6 \\
2 & 1 & 3 \\
1 & 5 & 2 \\
\end{bmatrix}
\]

2. It is true generally that \(|A| = |A'|\). Verify this for \((2 \times 2)\) and \((3 \times 3)\) matrices.

3. Find the determinant of the matrix

\[
D = \begin{bmatrix}
2 & 2 & 3 & 4 \\
0 & -1 & 1 & 11 \\
1 & -1 & 7 & 3 \\
-2 & 0 & -1 & 3 \\
\end{bmatrix}
\]

by first reducing to a triangular matrix.
Section 11  
Rank and Linear Dependence

In an early section, we learnt that the number of solutions in a system of linear equations depends on the number of “independent” equations in the system relative to the number of unknowns. From Cramer’s Rule, we learnt that a system of \( n \)-linear equations in \( n \)-unknowns will have a unique solution only when the determinant of the coefficient matrix is not zero. When will the determinant of a matrix be zero? This has a lot to do with the linear dependence in the coefficient matrix.

The presence of any zero row or column will produce a zero determinant (this is easy to verify: just use the Laplace expansion along that zero row or column).

In the \( (2 \times 2) \) case we obtained a zero determinant when one row was a multiple of the other (or is the same as the other), and this remains true of the \( (3 \times 3) \) case. A more subtle case is when one row is a linear combination of the other two (i.e., we can write one row as the sum of some multiple of a second row plus some multiple of the third). For instance, take the matrix

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 2 & 4 \\
3 & 3 & 5 \\
\end{bmatrix}
\]

which you can verify has a zero determinant. Here the third row is equal to the first row plus half of the second row:

\[
\begin{bmatrix}
3 & 3 & 5 \\
\end{bmatrix} = \begin{bmatrix}
1 & 2 & 3 \\
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
4 & 2 & 4 \\
\end{bmatrix}.
\]

(Alternatively, we can say that the first row is the third row minus half of the second

\[
\begin{bmatrix}
1 & 2 & 3 \\
\end{bmatrix} = \begin{bmatrix}
3 & 3 & 5 \\
\end{bmatrix} - \frac{1}{2} \begin{bmatrix}
4 & 2 & 4 \\
\end{bmatrix}.
\]

or that the second row is two times the third row minus two times the first.)

We say that the three row vectors that make up the matrix are “linearly dependent”. This “linear dependence” between the rows leads to a zero determinant because it means that row operations can be applied to ultimately create a zero row.

A \( (3 \times 3) \) matrix will have a non-zero determinant only if it has three linearly independent rows (or columns). More generally, think of a \( (3 \times 3) \) square matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
\end{bmatrix}
\]

as a stack of three row vectors.
We say that three (column or row) vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly dependent if there are constants $c_1$, $c_2$, and $c_3$, not all zero, such that

$$c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + c_3 \mathbf{a}_3 = \mathbf{0}.$$ 

If the only case where this is true is $c_1 = c_2 = c_3 = 0$, then the vectors are said to be linearly independent. For now, define the **rank** of the matrix is the number of linearly independent row vectors that it contains.

**Examples**

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 1 & 3 \end{bmatrix}$. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

Then these vectors are linearly dependent, since $2\mathbf{a}_1 - \mathbf{a}_2 + 0\mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The second row is just twice that of the first row (or the first is half of the second). Removing one of the dependent vectors, say $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, the remaining vectors $\mathbf{a}_1$ and $\mathbf{a}_3$ are linearly independent: the only constants $c_1$ and $c_2$ such that $c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ are $c_1 = c_2 = 0$. The matrix has two linear independent vectors, and is said to be of rank 2.

If we imagine these three vectors in the usual representation as ‘arrows’ in three-dimensional co-ordinate space, we will see that the vector $\mathbf{a}_2$ is just an extension of $\mathbf{a}_1$. Combinations of the three vectors are effectively combinations of only two vectors.

$$c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + c_3 \mathbf{a}_3 = c_1 \mathbf{a}_1 + c_2 2\mathbf{a}_1 + c_3 \mathbf{a}_3 = (c_1 + 2c_2)\mathbf{a}_1 + c_3 \mathbf{a}_3$$

Different combinations of the three vectors will therefore result in new vectors that all lie on a plane (a two-dimensional space), and cannot ‘span’ or cover the entire 3-d space.

2. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \\ 2 & 1 & 3 \end{bmatrix}$. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

Then these vectors are linearly dependent, since $1\mathbf{a}_1 - 1\mathbf{a}_2 + 1\mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The second row “depends” on the other two in that it is the sum of $\mathbf{a}_1$ and $\mathbf{a}_3$. 

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ a \\ a \\ a \\ a \\ a \\ a \end{bmatrix}$$
(Equivalently, \( \mathbf{a}_3 \) “depends on” \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \) because it is \( \mathbf{a}_2 - \mathbf{a}_1 \)). Removing one of these ‘dependent’ vectors (any one of them) leaves two independent vectors. For instance, removing \( \mathbf{a}_3 \), then the only constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 = c_1 [1 \ 2 \ 3] + c_2 [3 \ 3 \ 6] = [0 \ 0 \ 0]
\]

are \( c_1 = c_2 = 0 \). The matrix has two linear independent vectors, and is said to be of rank 2.

If we imagine these three vectors geometrically in three-dimensional co-ordinate space, we will see that although no one vector is an extension of another, all three vectors nonetheless lie in a single plane, so again combinations of the three vectors will ‘create’ new vectors that also lie on that plane. The three vectors cannot ‘span’, or cover, the entire 3-d space.

3. Let \( \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \). Let \( \mathbf{a}_1 = [1 \ 2 \ 3] \), \( \mathbf{a}_2 = [2 \ 4 \ 6] \), and \( \mathbf{a}_3 = [3 \ 6 \ 9] \).

Then these vectors are linearly dependent, since \( \mathbf{A} \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = [0 \ 0 \ 0] \). In this case, there is only one linearly independent vector. If we remove \( \mathbf{a}_3 \), we find that the remaining two are still linearly dependent: \( \mathbf{a}_2 \) is twice that of \( \mathbf{a}_1 \). The rank of this matrix is one: \( \text{rank}(\mathbf{A}) = 1 \).

Geometrically, all three vectors lie on a single line. The vector \( \mathbf{a}_2 \) is twice that of \( \mathbf{a}_1 \), and \( \mathbf{a}_3 \) is three times that of \( \mathbf{a}_1 \). Combinations of these vectors therefore will only create new vectors that are also extensions of \( \mathbf{a}_1 \).

\[
c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + c_3 \mathbf{a}_3 = c_1 \mathbf{a}_1 + c_2 2\mathbf{a}_1 + c_3 3\mathbf{a}_1 = (c_1 + 2c_2 + 3c_3)\mathbf{a}_1.
\]

4. Let \( \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 6 \\ 2 & 1 & 3 \end{bmatrix} \). Let \( \mathbf{a}_1 = [0 \ 0 \ 0] \), \( \mathbf{a}_2 = [2 \ 4 \ 6] \), and \( \mathbf{a}_3 = [2 \ 1 \ 3] \).

Then these vectors are linearly dependent, since \( \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = [0 \ 0 \ 0] \) for any \( c_1 \). Removing vector \( \mathbf{a}_1 \) leaves two linearly independent vectors. The rank of this matrix is two.
5. Let \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & 1 \end{bmatrix} \). Let \( a_1 = [1 \ 0 \ 0] \), \( a_2 = [0 \ 2 \ 4] \), and \( a_3 = [0 \ 2 \ 1] \).

Then all three row vectors are linearly independent; it is impossible to find \( c_1, c_2 \) and \( c_3 \), not all zero, such that \( c_1a_1 + c_2a_2 + c_3a_3 = [0 \ 0 \ 0] \). This matrix is of “full rank”.

A square matrix will have a non-zero determinant if and only if (‘iff’) it has full rank.

There are many ways to determine the rank of a matrix. One way is to do ‘Gaussian Elimination’ (the row operations) and see how many non-zero pivots you get (see Section 2 for ‘pivots’). Other ways will be discussed in later sections, or in more advanced classes. For the time being we will proceed by observation.

We conclude this section with several remarks regarding rank. Earlier, we viewed a matrix as a stack of row vectors. The rank concept that we discussed could be called ‘row rank’. But we can also view a matrix as the concatenation (joining together) of three column vectors

\[
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\]

1. Everything that we said here follows for column vectors as for row vectors, and in fact a matrix will have the same number of linearly independent column vectors as there are linearly independent row vectors: if \( A \) has two linearly independent row vectors, it will also have two linearly independent column vectors; if it has only one linearly independent row vector, then it will one have one linearly independent column vector, and so on. A matrix’s row rank is the same as its “column rank”.

2. We can also speak of the rank of non-square matrices. Take for example

\[
A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 3 \end{bmatrix}
\]

This is in fact the last two rows of an earlier example we were looking at. Looking at the rows, we see that these two rows are linearly independent (verify!) The ‘row-rank’ is two. What if we view this matrix as a concatenation of three two-dimensional vectors? Note that three two-dimensional vectors can only span a two-dimensional
space. Draw three two dimensional vectors on a “x-y” co-ordinate space on a piece of paper. Can you generate a three-dimensional space from this? (The answer is no.)

If we observe the three vectors

\[
\begin{bmatrix}
2 \\
2 \\
4 \\
1 \\
3
\end{bmatrix}
\]

we observe that these do span the entire two-dimensional plane. So the column rank of this matrix is also 2. The principle that a matrix’s row and column ranks are the same continues to hold, and holds generally.

As another illustration, consider

\[
A = \begin{bmatrix}
2 & 4 & 6 \\
1 & 2 & 3
\end{bmatrix}
\]

It is easy to see that the row rank of this matrix is one, and the column rank of this matrix is also one.

Because the row and column ranks are always the same, we can always speak unambiguously of “the rank of a matrix”. Furthermore,

\[
rank(A) \leq \min(\#\text{rows}, \#\text{columns})
\]

3. Finally, we often need to find the rank of a product. Here are two very useful results:

3a If \(A\) is \((m \times n)\) and \(B\) is \((n \times p)\), then

\[
rank(AB) \leq \min(rank(A), rank(B))
\]

**Proof** Let \(C = AB\). Every column of \(C\) is some linear combination of the columns of \(A\), therefore the columns in \(C\) cannot span a space of dimension greater than the column rank of \(A\). Similarly, every row of \(C\) is some linear combination of the rows of \(B\), therefore the rows in \(C\) cannot span a space of dimension greater than the row rank of \(B\). Since row rank and column rank are the same, we have our result.

3b If \(A\) is \((m \times n)\) and \(B\) is \((n \times n)\) and full rank, then

\[
rank(AB) = rank(A)
\]
Proof If \( x \leq \min(a, b) \), then obviously \( x \leq a \) and \( x \leq b \) are both true. Since 
\[
\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)).
\]
we have 
\[
\text{rank}(A) = \text{rank}(ABB^{-1}) \quad [B^{-1}\text{exists since } B \text{ is full rank}]
\]
\[
\leq \min(\text{rank}(AB), B^{-1})
\]
\[
\leq \text{rank}(AB)
\]
\[
\leq \min(\text{rank}(A), \text{rank}(B))
\]
\[
\leq \text{rank}(A)
\]
This extends to the product \( CAB \) where both \( C \) and \( B \) are full rank: 
\[
\text{rank}(CAB) = \text{rank}(A).
\]

Exercises

1. Find the determinants of the following matrices using the Laplace expansion. If you find the determinant of a matrix to be zero, find out how many linear independent rows the matrix has.

(i) \[
\begin{bmatrix}
4 & 0 & 1 \\
19 & 1 & -3 \\
7 & 1 & 0
\end{bmatrix}
\]
(ii) \[
\begin{bmatrix}
4 & 3 & 0 \\
19 & 1 & 0 \\
7 & 1 & 0
\end{bmatrix}
\]
(iii) \[
\begin{bmatrix}
4 & 3 & 0 \\
1 & 1 & 2 \\
7 & 6 & 6
\end{bmatrix}
\]
(vi) \[
\begin{bmatrix}
1 & 3 & 1 \\
-2 & -6 & -2 \\
4 & 12 & 4
\end{bmatrix}
\]

2. Show using the formula
\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}
\]

that the determinant of a \((3 \times 3)\) matrix will be zero if

(i) one row is a multiple of another;

(ii) one row is a linear combination of the other two.

3. Show that \( \text{rank}(A^T A) = \text{rank}(A) \).
Section 12  A Formula for the Inverse

In this chapter, we develop a formula for the inverse of an \((n \times n)\) matrix, based on cofactors. We will not be using this formula for computing inverses – for that the elementary row operations approach is the most efficient. The objective in studying the formula for the inverse is, for us, to understand where Cramer’s Rule comes from.

Recall that the \((i, j)\)th “minor” of an \((n \times n)\) matrix \(A\) is the determinant of the \((n-1) \times (n-1)\) matrix after removing the \(i\)th row and \(j\)th column of \(A\). We will sometimes refer to this as the minor associated with the \((i, j)\)th element of \(A\). The \((i, j)\)th times \((-1)^{i+j}\) gives us the \((i, j)\)th “cofactor” of \(A\), or the cofactor associated with the \((i, j)\)th element of \(A\).

For example, let \(A = \begin{bmatrix} 3 & 5 & 6 & 7 \\ 5 & 4 & 7 & 3 \\ 1 & 2 & 9 & 10 \\ 2 & 8 & 2 & 1 \end{bmatrix}\).

\[
A = \begin{bmatrix} x & 5 & 6 & 7 \\ x & x & x & x \\ 2 & 9 & 10 \\ x & 8 & 2 & 1 \end{bmatrix}
\]

The \((2,1)\)th minor is \(M_{21} = \begin{vmatrix} 5 & 6 & 7 \\ 2 & 9 & 10 \\ 8 & 2 & 1 \end{vmatrix} \).

The \((2,1)\)th cofactor is \(C_{21} = (-1)^{2+1} \begin{vmatrix} 5 & 6 & 7 \\ 2 & 9 & 10 \\ 8 & 2 & 1 \end{vmatrix} \).

\[
A = \begin{bmatrix} 3 & 5 & x & 7 \\ 5 & 4 & x & 3 \\ x & x & x & x \\ 2 & 8 & x & 1 \end{bmatrix}
\]

The \((3,3)\)th minor is \(M_{33} = \begin{vmatrix} 3 & 5 & 7 \\ 5 & 4 & 3 \\ 2 & 8 & 1 \end{vmatrix} \).

The \((3,3)\)th cofactor is \(C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 5 & 7 \\ 5 & 4 & 3 \\ 2 & 8 & 1 \end{vmatrix} \).

\[
A = \begin{bmatrix} x & x & x \boxed{x} \\ 5 & 4 & 7 \boxed{x} \\ 1 & 2 & 9 \boxed{x} \\ 2 & 8 & 2 \boxed{x} \end{bmatrix}
\]

The \((1,4)\)th minor is \(M_{14} = \begin{vmatrix} 5 & 4 & 7 \\ 1 & 2 & 9 \\ 2 & 8 & 2 \end{vmatrix} \).

The \((1,4)\)th cofactor is \(C_{14} = (-1)^{1+4} \begin{vmatrix} 5 & 4 & 7 \\ 1 & 2 & 9 \\ 2 & 8 & 2 \end{vmatrix} \).
We can collect all the cofactors of a matrix -- one cofactor for each element of the matrix -- into a single cofactor matrix. For example, the cofactor matrix of \( \mathbf{A} \), denoted \( C(\mathbf{A}) \), is the \((4 \times 4)\) matrix

\[
C(\mathbf{A}) = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{21} & C_{22} & C_{23} & C_{24} \\
C_{31} & C_{32} & C_{33} & C_{34} \\
C_{41} & C_{42} & C_{43} & C_{44}
\end{bmatrix} = \begin{bmatrix}
M_{11} & -M_{12} & M_{13} & -M_{14} \\
-M_{21} & M_{22} & -M_{23} & M_{24} \\
M_{31} & -M_{32} & M_{33} & -M_{34} \\
-M_{41} & M_{42} & -M_{43} & M_{44}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
4 & 7 & 3 & | & 5 & 7 & 3 & | & 5 & 4 & 3 & | & 5 & 4 & 7 \\
2 & 9 & 10 & | & -1 & 9 & 10 & | & 1 & 2 & 10 & | & -1 & 2 & 9 \\
8 & 2 & 1 & | & 2 & 2 & 1 & | & 2 & 8 & 1 & | & 2 & 8 & 2 \\
| & 5 & 6 & 7 & | & 3 & 6 & 7 & | & 3 & 5 & 7 & | & 3 & 5 & 6 \\
-2 & 9 & 10 & | & 1 & 9 & 10 & | & -1 & 2 & 10 & | & 1 & 2 & 9 \\
8 & 2 & 1 & | & 2 & 2 & 1 & | & 2 & 8 & 1 & | & 2 & 8 & 2 \\
| & 5 & 6 & 7 & | & 3 & 6 & 7 & | & 3 & 5 & 7 & | & 3 & 5 & 6 \\
4 & 7 & 3 & | & -5 & 7 & 3 & | & 5 & 4 & 3 & | & -5 & 4 & 7 \\
8 & 2 & 1 & | & 2 & 2 & 1 & | & 2 & 8 & 1 & | & 2 & 8 & 2 \\
| & 5 & 6 & 7 & | & 3 & 6 & 7 & | & 3 & 5 & 7 & | & 3 & 5 & 6 \\
-4 & 7 & 3 & | & 5 & 7 & 3 & | & -5 & 4 & 3 & | & 5 & 4 & 7 \\
2 & 9 & 10 & | & 1 & 9 & 10 & | & 1 & 2 & 10 & | & 1 & 2 & 9
\end{bmatrix}
\]

Obviously, computing the cofactor matrix of a large matrix by hand is a tedious affair. Fortunately we won’t have to compute these matrices explicitly, but it is nonetheless useful in helping us to derive useful results.

**Exercise**

1. Compute the cofactor matrix of the matrix \( \mathbf{A} = \begin{bmatrix}
3 & 1 & 2 \\
5 & 2 & 0 \\
4 & 6 & 9
\end{bmatrix} \).

2. Compute the cofactor matrix of the matrix

\[
\mathbf{A} = \begin{bmatrix}
3 & 1 & 2 & 4 & 2 \\
5 & 2 & 0 & 9 & 5 \\
4 & 6 & 9 & 0 & 8 \\
10 & -3 & 2 & -7 & 1 \\
0 & 2 & 4 & 1 & 11
\end{bmatrix}
\]

leaving each element as a determinant of a \((4 \times 4)\) matrix -- don’t compute them!
A Property of Cofactors

Recall the Laplace expansion for the determinant of a matrix:

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij}$$ for any column $j$

where we have given only the expansion along a column (all of the following statements works if we replace ‘column’ by ‘row’). That is, the determinant can be computed as the sum of the product of the cofactors of a column with the corresponding elements of that column. What happens if we were to take the sum of the product of the cofactors of a column with the corresponding elements of a different column? To take a specific example, consider a $(4 \times 4)$ matrix $A$ and its corresponding cofactor matrix $C(A)$, and take an expansion along the second column.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad C(A) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

The expansion $a_{12} c_{12} + a_{22} c_{22} + a_{32} c_{32} + a_{42} c_{42}$ gives the determinant of $A$. What happens if we take, for example, the sum of the column 2 cofactors multiplied by the column 1 elements:

$$a_{11} c_{12} + a_{21} c_{22} + a_{31} c_{32} + a_{41} c_{42}$$

To answer this question, take the following matrix $\tilde{A}$ and the corresponding cofactor matrix

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{11} & a_{13} & a_{14} \\ a_{21} & a_{21} & a_{23} & a_{24} \\ a_{31} & a_{31} & a_{33} & a_{34} \\ a_{41} & a_{41} & a_{43} & a_{44} \end{bmatrix} \quad \text{and} \quad C(\tilde{A}) = \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13} & \tilde{c}_{14} \\ \tilde{c}_{21} & \tilde{c}_{22} & \tilde{c}_{23} & \tilde{c}_{24} \\ \tilde{c}_{31} & \tilde{c}_{32} & \tilde{c}_{33} & \tilde{c}_{34} \\ \tilde{c}_{41} & \tilde{c}_{42} & \tilde{c}_{43} & \tilde{c}_{44} \end{bmatrix}$$

and make the following observations:

(i) the determinant of $\tilde{A}$ can be computed as

$$|\tilde{A}| = a_{11} \tilde{c}_{12} + a_{21} \tilde{c}_{22} + a_{31} \tilde{c}_{32} + a_{41} \tilde{c}_{42}$$

where we have expanded along the second column;

(ii) the cofactors associated with the second column of $\tilde{A}$ are identical to the cofactors associated with the second column of $A$: $c_{12} = \tilde{c}_{12}$, $c_{22} = \tilde{c}_{22}$, $c_{32} = \tilde{c}_{32}$, $c_{42} = \tilde{c}_{42}$, since the second column is removed when computing these cofactors. Therefore $|\tilde{A}| = a_{11} c_{12} + a_{21} c_{22} + a_{31} c_{32} + a_{41} c_{42}$.
(iii) The determinant of $\tilde{A}$ is zero, because it has two identical rows. Therefore

$$a_{11}C_{12} + a_{21}C_{22} + a_{31}C_{32} + a_{41}C_{42} = 0.$$ 

In general

*The sum of the product of the cofactors of one column and the elements of another column is zero.*

**Exercises**

1. (a) Write down the three cofactors $C_{13}$, $C_{23}$, and $C_{33}$ corresponding to the three elements in the third column of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 2 & 0 \\ 4 & 6 & 9 \end{bmatrix}.$$ 

and compute the determinant of $A$ by expanding down the third column.

(b) Now find the sum of the products of the same three cofactors and the corresponding elements of a different column, e.g compute

$$3C_{13} + 5C_{23} + 4C_{33}$$

and

$$1C_{13} + 2C_{23} + 6C_{33}$$

2. For the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

write down the cofactors $C_{21}$ and $C_{22}$ corresponding to the elements of the second row. Find $cC_{21} + dC_{22}$ and $aC_{21} + bC_{22}$. 

A Formula for the Inverse, and Cramer’s Rule

The results from the preceding section can be used to develop a formula for the inverse. Suppose we premultiply $A$ by the transpose of the cofactor matrix of $A$. What we get is

$$C^T(A)A = \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 & 0 \\ 0 & |A| & 0 & 0 \\ 0 & 0 & |A| & 0 \\ 0 & 0 & 0 & |A| \end{bmatrix}$$

Each diagonal element in the right-most matrix is the sum of the product of the cofactors of a column of $A$ and the elements of that column, and is therefore equal to the determinant. One example is marked out. (To reiterate: the cofactor matrix here has been transposed.) The other elements are the sum of the product of the cofactors of a column of $A$ and the elements of a different column of $A$, therefore is zero. Multiplying both sides by the reciprocal of the determinant, we get

$$\frac{1}{|A|} C^T(A)A = I$$

and therefore:

$$A^{-1} = \frac{1}{|A|} C^T(A) = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix}.$$

The transpose of the cofactor matrix is called the adjoint of $A$, $adj(A)$, so the formula for the inverse is often written as

$$A^{-1} = \frac{1}{|A|} adj(A),$$

which is called the adjoint formula for the inverse.

Now we prove Cramer’s Rule. Take the general $n$ equations in $n$ unknowns system of simultaneous equations.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$...$$
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$
and write it as 

$$Ax = b$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$ 

The solution is, using the inverse $x = A^{-1}b$. Writing this using the adjoint formula:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}b = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$ 

Take as a specific example the solution for $x_2$. We have

$$x_2 = \frac{b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2}}{|A|}.$$ 

Finally, observe that the numerator of this expression is the determinant of the matrix

$$A_2(b) = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}.$$ 

(recall again that in computing the cofactors of the column, that column is deleted.)

This same argument holds for any $x_i$, therefore we get Cramer’s Rule:

$$x_i = \frac{|A_i(b)|}{|A|}, \quad i = 1, \ldots, n.$$ 

In other words, solving a system by Cramer’s Rule is exactly equivalent to solving the system by using the inverse. Cramer’s Rule is simply a shortcut for implementing the inverse matrix approach.
Exercises

1. Under what condition will a system of \( n \) equations in \( n \) unknowns have a unique solution?

2. Earlier you computed the cofactor matrix and determinant of the matrix

\[
A = \begin{bmatrix}
3 & 1 & 2 \\
5 & 2 & 0 \\
4 & 6 & 9
\end{bmatrix}.
\]

Use the adjoint formula to compute the inverse of \( A \). Verify your answer by computing the product \( A^{-1}A \).

3. Use the adjoint formula to compute the inverse of the matrix

\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]

4. Find the inverse of matrix

\[
A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 6 & 0 \\
0 & 0 & 0 & 5
\end{bmatrix}.
\]

Can you generalize to arbitrary diagonal matrices of dimension \( (n \times n) \)?
Properties of the Inverse

We summarize here a few properties of matrix inverses that will be useful for us. You will notice from the proofs that these properties arise from the nature of what an inverse is, and that we do not have to appeal to the adjoint formula to obtain the results. That is, all the proofs arise from the fact that by definition, the inverse (where it exists) of a square matrix $A$ is the square matrix $A^{-1}$ of similar size, such that

$$A^{-1}A = I.$$

(1) $(AB)^{-1} = B^{-1}A^{-1}$ when the inverses exist.

Premultiplying $B^{-1}A^{-1}$ by $AB$ gives the identity matrix:

$$ABB^{-1}A^{-1} = AB\underbrace{B^{-1}A^{-1}}_{=I} = AA^{-1} = I.$$

But $AB\left(B^{-1}A^{-1}\right) = I$ says that $B^{-1}A^{-1}$ is the inverse of $AB$.

(2) $(A^{-1})^T = (A^T)^{-1}$

This says that the inverse of a transpose is the transpose of the inverse. It doesn’t matter whether you transpose before computing the inverse, or transpose the inverse. We can take $A^{-1}A = I$ as the starting point. Taking transpose, and noting that the transpose of the identity matrix is itself, we have

$$\left(A^{-1}A\right)^T = I^T \iff A^T(A^{-1})^T = I.$$

Now premultiply both sides by $(A^T)^{-1}$, we get

$$\underbrace{(A^T)^{-1}}_{=I}A^T(A^{-1})^T = (A^T)^{-1}I$$

which gives us $(A^{-1})^T = (A^T)^{-1}$. A corollary is that the inverse of a symmetric matrix is symmetric: if $A^T = A$, then $(A^{-1})^T = (A^T)^{-1} = A^{-1}$.

(3) $|A^{-1}| = \frac{1}{|A|}$.

This arises from the fact that the determinant of a product is the product of the determinants: $|A^{-1}| |A| = |I|$.
Exercises

1(a) Let \( A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 5 & 2 \\ 5 & -1 \end{bmatrix} \).

Find \( (AB)^{-1} \) and verify that \( (AB)^{-1} = B^{-1}A^{-1} \).

(b) Let \( A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 \end{bmatrix} \) and \( B = \begin{bmatrix} 5 & 2 \\ 5 & -1 \\ -2 & 1 \end{bmatrix} \).

Find \( (AB)^{-1} \). Why would it inappropriate to write \( (AB)^{-1} = B^{-1}A^{-1} \)?

Note that an alternative notation for transpose is the ‘prime’, i.e., \( X^T = X' \). It is important to be comfortable with both notations. In the following questions, we will use the ‘prime’ notation. Thereafter, we will switch between the two, depending on which looks better.

2. Let \( X \) be a matrix of dimension \( (n \times k) \) such \( (XX')^{-1} \) exists. We want to show that \( ((XX')^{-1})' = (X'X)^{-1} \). Which of the following arguments are correct?

A. \( XX \) is symmetric \( (XX')' = XX'' = XX \). We know that the inverse of a symmetric matrix is symmetric, therefore \( ((XX')^{-1})' = (X'X)^{-1} \).

B. \( ((XX')^{-1})' = (X^{-1}(X')^{-1})' = ((X')^{-1})' (X^{-1})' = (X')^{-1} = (X'X)^{-1} \)

C. \( ((XX')^{-1})' = (XX')^{-1} = (XX'')^{-1} = (XX)^{-1} \)

(a) A only  (b) B and C only  (c) All three  (d) A and C only.

Explain your choice.

3. Let \( A = \begin{bmatrix} 2 & 4 & -1 & 2 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \). Find \( |A^{-1}| \).
Section 13  Eigenvalues and Eigenvectors

Eigenvalues are numbers associated with matrices that are useful in many applications, including dynamic problems involving differential or difference equations. Eigenvalues (and eigenvectors) are also intimately connected to other matrix concepts such as the determinant, the rank, and definiteness.

13.1 Definitions

For any \( n \times n \) matrix \( A \), a scalar \( \lambda \) is an eigenvalue of \( A \) if there is a nonzero vector \( x \) such that

\[
Ax = \lambda x \tag{1.1}
\]

The vector \( x \) is said to be an eigenvector of \( A \) associated with \( \lambda \).

Example

The scalar \( \lambda = 3 \) is an eigenvalue of

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}
\]

and \( x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is an eigenvector associated with this eigenvalue, because \( \lambda \) and \( x \) satisfies equation (1.1):

\[
\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

The scalar \( \lambda = -2 \) is also an eigenvalue of

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}
\]

with associated eigenvector \( x = \begin{bmatrix} -2 \\ \frac{3}{2} \\ 1 \end{bmatrix} \), because:

\[
\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ \frac{3}{2} \\ 1 \end{bmatrix} = -2 \begin{bmatrix} -2 \\ \frac{3}{2} \\ 1 \end{bmatrix}.
\]

There are no other eigenvalues for this matrix.

Remark  Geometrically, pre-multiplying a vector \( x = [x_1 \ x_2]^T \) by a matrix \( A \) changes the direction of the vector. An eigenvector of \( A \) is a vector whose direction is not changed when pre-multiplied by \( A \); doing so merely scales the vector by a factor of \( \lambda \).
To find the eigenvectors and eigenvalues of a $n \times n$ matrix $A$:

We use the fact that

$$Ax = \lambda x \iff Ax - \lambda x = 0 \iff (A - \lambda I)x = 0.$$ 

This system of equations has a non-trivial solution $x \neq 0$ only if

$$\det(A - \lambda I) = 0$$

(this is called the characteristic equation or characteristic polynomial of the matrix $A$). Therefore, we can find the eigenvalues of $A$ by finding all $\lambda$ that satisfy $\det(A - \lambda I) = 0$. Then, for each $\lambda$, we find the associated eigenvector $x$ from the equation $(A - \lambda I)x = 0$.

**Example** Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$.

The eigenvalues satisfy $\det(A - \lambda I) = 0$. Since

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 2 \\ 3 & -\lambda \end{bmatrix}$$

$$= -(1 - \lambda)(-\lambda) - 6$$

$$= \lambda^2 - \lambda - 6$$

$$= (\lambda - 3)(\lambda + 2),$$

the eigenvalues are $\lambda = 3$ and $\lambda = -2$.

For $\lambda = 3$, the system $(A - \lambda I)x = 0$ is

$$\begin{bmatrix} 1 - \lambda & 2 \\ 3 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives $x_1 = x_2$, i.e. any vector of the form $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ associated with $\lambda = 3$.  

Similarly, for $\lambda = -2$, the system $(A - \lambda I)x = 0$ is

$$
\begin{bmatrix}
  1 - \lambda & 2 \\
  3 & -\lambda
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= 
\begin{bmatrix}
  3 & 2 \\
  3 & 2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}
$$

which gives $x_1 = -\frac{2}{3}x_2$, i.e., any vector of the form $x = s \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$ is an eigenvector of $A$ associated with $\lambda = -2$.

**Remark** Note that if $x = [x_1 \ x_2]^T$ is an eigenvector associated with an eigenvalue $\lambda$, then the vector $sx = [sx_1 \ sx_2]^T$ for any $s$ is also an eigenvector of $A$ associated with $\lambda$, since

$$Ax = \lambda x \iff A(sx) = \lambda(sx).$$

That is, for every eigenvalue, we have an entire ‘line’ of eigenvectors. For example, for the matrix

$$A = \begin{bmatrix}
  1 & 2 \\
  3 & 0
\end{bmatrix},$$

the vectors $x = [1 \ 1]^T$, $x = [2 \ 2]^T$, $x = [3 \ 3]^T$, etc. are all eigenvectors associated with the eigenvalue $\lambda = 3$.

---

**Exercises**

1. Find the eigenvalues and associated eigenvectors of the matrices

   (a) $\begin{bmatrix}
       5 & -2 \\
       4 & -1
\end{bmatrix}$;

   Ans: Eigenvalues are $\lambda = 3$ and $\lambda = 1$ with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ resp.

   (b) $\begin{bmatrix}
       1 & 1 \\
       4 & 1
\end{bmatrix}$

   (c) $\begin{bmatrix}
       0.8 & 0.3 \\
       0.2 & 0.7
\end{bmatrix}$

   (d) $\begin{bmatrix}
       1 & 2 \\
       2 & 4
\end{bmatrix}$

   (e) $\begin{bmatrix}
       2 & 1 \\
       -1 & 4
\end{bmatrix}$

   (f) $\begin{bmatrix}
       1 & 0 & 0 \\
       2 & 1 & 0 \\
       5 & 3 & 2
\end{bmatrix}$
Exercise 1(d) shows that eigenvalues may take value zero. There is nothing unusual about this, although it does say something important regarding the matrix (more on that later).

In general, an \((n \times n)\) matrix will have \(n\) eigenvalues, but exercises 1(e) and (f) shows that these eigenvalues need not be distinct. The matrix in exercise 1(e) has eigenvalue \(\lambda = 3\), occurring twice. In exercise 1(f), the eigenvalues are \(\lambda = 1, 1, \text{ and } 2\). We say that the eigenvalue \(\lambda = 1\) occurs with multiplicity 2.

There will usually be one ‘line’ of eigenvectors associated with each distinct eigenvalue. In 1(f), there is only one line of eigenvectors

\[
\mathbf{x} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

associated with the eigenvalue \(\lambda = 3\). In 1(e), the eigenvectors are

\[
\mathbf{x} = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ (corresponding to } \lambda = 2 \text{) and } \mathbf{x} = s \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \text{ corresponding to } \lambda = 1.
\]

However, there are some unusual cases, where multiple lines of eigenvectors are associated with a single eigenvalue:

2. Find the eigenvalues of the matrix

\[
\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.
\]

Show that any vector \(\mathbf{x} \neq \mathbf{0}\) is an eigenvector associated with the eigenvalues. Does this make any geometric sense?

Note also that eigenvalues can take complex values!

3. Find the eigenvalues and associated eigenvectors of the matrix \(\mathbf{A} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}\).

You should find \(\lambda = 4 \pm 2i\). The eigenvectors can be expressed in many ways. One possibility is \(s \begin{bmatrix} 1-i \\ 1 \end{bmatrix}\) and \(t \begin{bmatrix} 1 \\ 1-i \\ 2 \end{bmatrix}\) respectively.
13.2 A detailed look at the (2×2) case

We study the (2×2) case in detail. A few of the results obtained here apply only to the (2×2) case, but many generalize to the \((n \times n)\) case (albeit with more complicated proofs). Studying the (2×2) case will help us develop intuition for these results with a minimum of algebra.

Let \(A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\). The characteristic polynomial is

\[
\rho(\lambda) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})
\]

i.e.,

\[
\rho(\lambda) = \lambda^2 - tr(A)\lambda + \det(A)
\]

Therefore the eigenvalues are

\[
\lambda_{1,2} = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4\det(A)}}{2}
\]

Useful results

1. The two eigenvalues of \(A\) are

- real identical complex

if \(\begin{cases} tr(A)^2 \geq 4\det(A) \\ tr(A)^2 = 4\det(A) \\ tr(A)^2 < 4\det(A) \end{cases}\).

The eigenvalues of a (2×2) matrix can be identical only if they are real, since complex roots of a polynomial always appear in conjugate pairs. However, matrices that are (4×4) or larger can have repeated pairs of complex roots.

It should be clear from the characteristic polynomial that if \(A\) is triangular (or diagonal), i.e. if \(a_{12} = 0\) or \(a_{21} = 0\), then the eigenvalues of \(A\) are simply its diagonal elements:

\[
\lambda_1 = a_{11} \quad \text{and} \quad \lambda_2 = a_{22}.
\]

Naturally, in this case the eigenvalues are real (we are only considering matrices of real numbers).
2. Another important case where the eigenvalues are guaranteed to be real is when the matrix is symmetric: the eigenvalues of a \((2 \times 2)\) symmetric matrix are always real. If \(a_{12} = a_{21}\), then

\[
tr(A)^2 - 4\det(A) = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}^2) = (a_{11} - a_{22})^2 + 4a_{12}^2 \geq 0
\]

Note that the eigenvalues will also be distinct, unless \(a_{11} = a_{22}\) and \(a_{12} = a_{21}\).

3. The product of the eigenvalues of a \((2 \times 2)\) matrix gives its determinant. The sum of the eigenvalues gives the trace:

\[
\det(A) = \lambda_1 \lambda_2 \quad \text{and} \quad tr(A) = \lambda_1 + \lambda_2
\]

In detail:

\[
\lambda_1 \lambda_2 = \frac{tr(A) + \sqrt{tr(A)^2 - 4\det(A)}}{2} \cdot \frac{tr(A) - \sqrt{tr(A)^2 - 4\det(A)}}{2}
\]

\[
= \frac{1}{4}(tr(A)^2 - tr(A)^2 + 4\det(A))
\]

\[
= \det(A)
\]

\[
\lambda_1 + \lambda_2 = \frac{2tr(A)}{2} = tr(A).
\]

We will see that similar results hold for larger matrices. For the \((2 \times 2)\) case, one consequence of this result is that (when the eigenvalues are real):

- both eigenvalues are positive \(\iff\) \(\det(A) > 0\) and \(tr(A) > 0\)
- both eigenvalues are negative \(\iff\) \(\det(A) > 0\) and \(tr(A) < 0\)
- the two eigenvalues have opposite signs \(\iff\) \(\det(A) < 0\)

4. If \(A\) is singular, i.e., \(\det(A) = 0\), then at least one of the eigenvalues are zero:

- if \(\det(A) = 0\), then \(\lambda_1 = tr(A), \lambda_2 = 0\);
- if \(tr(A) = 0\) and \(\det(A) = 0\), then \(\lambda_1 = \lambda_2 = 0\).
5. If \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of \( A \), then the eigenvalues of \( A^2 = AA \) are \( \lambda_1^2 \) and \( \lambda_2^2 \). The associated eigenvectors remain the same.

We can see this without referring to the formula for the eigenvalues. An eigenvalue \( \lambda_i \) and the associated eigenvector \( x_i \) satisfies

\[
Ax_i = \lambda_i x_i
\]

Pre-multiplying by \( A \) on both sides, we have

\[
A^2x_i = \lambda_i Ax_i = \lambda_i (\lambda_i x_i) = \lambda_i^2 x_i.
\]

This says that \( \lambda_i^2 \) is an eigenvalue of \( A^2 \), with associated eigenvector \( x_i \). This result obviously generalizes to arbitrary powers of \( A \).

Note a very important special case when the matrix \( A \) is idempotent (that is, when \( AA = A \)). In such cases, the eigenvalues can only take values 1 and 0. Since \( AA = A \), both \( AA \) and \( A \) have identical eigenvalues, i.e., \( \lambda_i^2 = \lambda_i \), so 1 and 0 are the only possible values.

6. If \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of \( A \), and \( A \) is invertible, then the eigenvalues of \( A^{-1} \) are \( 1/\lambda_1 \) and \( 1/\lambda_2 \), with the same associated eigenvectors.

Starting with \( Ax_i = \lambda_i x_i \) and pre-multiplying by \( A^{-1} \), we have

\[
A^{-1}Ax_i = \lambda_i A^{-1}x_i
\]

But the LHS is \( x_i \), therefore \( \lambda_i A^{-1}x_i = x_i \), from which we get

\[
A^{-1}x_i = (1/\lambda_i)x_i.
\]

7. The eigenvalues of \( A^T \) are the same as those of \( A \).

That the eigenvalues of the transpose are the same as those of the original matrix is easy to see – a matrix and its transpose share the same characteristic polynomial.

**Diagonalization**

The following important results come under the general topic of diagonalization of matrices:

8. Suppose that the eigenvalues of \( A \) are real and distinct, i.e. \( \lambda_1 \neq \lambda_2 \). Then the eigenvectors \( x_1 \) and \( x_2 \) are linearly independent, i.e.

\[
c_1x_1 + c_2x_2 = 0 \Rightarrow c_1 = c_2 = 0.
\]
**Proof** Starting with \( c_1 x_1 + c_2 x_2 = 0 \), premultiply by \( A \) to get

\[
c_1Ax_1 + c_2Ax_2 = A0 = 0.
\]

Substituting \( Ax_1 = \lambda_1 x_1 \) and \( Ax_2 = \lambda_2 x_2 \) gives

\[
c_1\lambda_1 x_1 + c_2\lambda_2 x_2 = 0.
\]

Multiplying \( c_1 x_1 + c_2 x_2 = 0 \) throughout by \( \lambda_1 \) gives

\[
c_1\lambda_1 x_1 + c_2\lambda_1 x_2 = 0.
\]

Substituting \( c_1\lambda_1 x_1 = -c_2\lambda_2 x_2 \) into \( c_1\lambda_1 x_1 + c_2\lambda_2 x_2 = 0 \) gives

\[
-c_2\lambda_2 x_2 + c_2\lambda_1 x_2 = 0,
\]

or

\[
c_2(\lambda_2 - \lambda_1)x_2 = 0.
\]

Since \( x_2 \neq 0 \) and \( \lambda_1 \neq \lambda_2 \), we have \( c_2 = 0 \). A similar argument shows that \( c_1 = 0 \).

9. Suppose that the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( A \) are real and distinct, with eigenvectors \( x_1 \) and \( x_2 \). Let

\[
P = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.
\]

Then

\[
P^{-1}AP = \Lambda, \quad \text{or equivalently} \quad A = P\Lambda P^{-1}.
\]

We say that \( A \) is diagonalizable.

We note first that \( P^{-1} \) exists, since \( x_1 \) and \( x_2 \) are linearly independent. The equations \( Ax_1 = \lambda_1 x_1 \) and \( Ax_2 = \lambda_2 x_2 \) can be put together into a single equation \( AP = PA \). The result follows.

Result (9) is extremely helpful when large powers of matrices have to be computed, because

\[
A^n = P\Lambda P^{-1}P\Lambda P^{-1}...P\Lambda P^{-1} = P\Lambda\Lambda...\Lambda P^{-1} = P\Lambda^n P^{-1}
\]

Furthermore, \( \Lambda^n \) is computationally cheap because it is diagonal, so computing \( PA^nP^{-1} \) tends to produce more accurate results than multiplying \( A^n \) directly, as relatively fewer computational steps are required. This result is also helpful for deriving theoretical results, e.g. for which matrices will \( A^n \to 0 \) as \( n \to \infty \)? Ans: matrices whose eigenvalues are all less than one in absolute value.
10. Suppose $A$ is symmetric (so we know its eigenvalues are real) with distinct eigenvalues $\lambda_1 \neq \lambda_2$. Then the eigenvectors $x_1$ and $x_2$ are orthogonal, i.e. $x_1^T x_2 = 0$.

**Proof**

We have $Ax_1 = \lambda_1 x_1$. Pre-multiplying the first by $x_2^T$ gives $x_2^T Ax_1 = \lambda_1 x_2^T x_1$. Taking transpose gives $(x_1^T Ax_2)^T = x_1^T A^T x_2 = x_1^T Ax_2 = \lambda_2 x_1^T x_2$, where we have used the fact that $A$ is symmetric. We also have $Ax_2 = \lambda_2 x_2$. Pre-multiplying by $x_1^T$ gives $x_1^T Ax_2 = \lambda_2 x_1^T x_2$. Therefore

$$\lambda_1 x_1^T x_2 - \lambda_2 x_1^T x_2 = (\lambda_1 - \lambda_2) x_1^T x_2 = 0.$$ 

Since $\lambda_1 \neq \lambda_2$, we have $x_1^T x_2 = 0$. If the eigenvectors were chosen to have unit length, then the eigenvectors are orthonormal: 

$$x_1^T x_2 = 0 \text{ and } x_1^T x_1 = 1.$$ 

If we pick the unit eigenvectors when constructing the matrix $P$ in result (9), we would have $P^TP = I$. In other words, we have $P^{-1} = P^T$. We say that the matrix $P$ is orthonormal.

What about the case where the two eigenvalues are not distinct? For a $(2 \times 2)$ symmetric matrix $A$, $\lambda_1 = \lambda_2 (= \lambda)$ iff

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

which is already diagonal. (Or put differently, every vector $x$ is an eigenvector, and we are at liberty to pick a pair of orthonormal eigenvectors to construct the matrix $P$, so we pick $x_1 = [1 \ 0]^T$ and $x_2 = [0 \ 1]^T$, i.e., we pick $P$ to be the identity matrix.)

We can therefore re-state result (10) as the next result:

11. Suppose $A$ is symmetric (in which case we know its eigenvalues are real, though perhaps not distinct). Then there exists an orthonormal matrix $P$ such that

$$P^{-1}AP = \Lambda,$$

or equivalently $A = P\Lambda P^{-1}$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ and } P = [x_1 \ x_2],$$

and where $x_1$ and $x_2$ are eigenvectors with unit length associated with the eigenvalues $\lambda_1$ and $\lambda_2$ respectively. This is the “Spectral Theorem for Symmetric Matrices”, stated and proved here for $(2 \times 2)$ matrices. (The result applies to general symmetric matrices).
Exercises

1. The (non-symmetric) matrix \( \mathbf{A} = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \) has (real, distinct) eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = -1 \).

(a) Verify that \( tr(\mathbf{A}) = \lambda_1 + \lambda_2 \) and \( \det(\mathbf{A}) = \lambda_1 \lambda_2 \);

(b) Verify that the eigenvalues of \( \mathbf{A}^2 \) are \( \lambda_1^2 \) and \( \lambda_2^2 \);

(c) Verify that the eigenvalues of \( \mathbf{A}^{-1} \) are \( 1/\lambda_1 \) and \( 1/\lambda_2 \);

(d) Verify that the eigenvalues of \( \mathbf{A}' \) are the same as those of \( \mathbf{A} \). What are the associated eigenvectors?

(e) Verify that the eigenvectors of \( \mathbf{A} \) are linearly independent;

(f) Verify the diagonalization formula in result (10);

(g) Show that the eigenvectors are not orthogonal.

2. Find the eigenvalues and associated eigenvectors of \( \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \). Verify result (11).

3. Eigenvectors are often normalized to length one. For instance, the eigenvector \( \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) of the matrix \( \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \) can be normalized to \( \mathbf{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \) so that

\[
|\mathbf{x}| = \left[ (1/\sqrt{2})^2 + (1/\sqrt{2})^2 \right]^{1/2} = 1.
\]

Normalize the eigenvectors in Qn.1.
13.3 The general case

We take a brief look at the $(3 \times 3)$ case before moving to the general case. Let

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

Its characteristic equation is

\[
\det(A - \lambda I) = (a_{11} - \lambda) \det\begin{pmatrix} a_{22} - \lambda & a_{23} \\ a_{32} & a_{33} - \lambda \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} - \lambda \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} - \lambda \\ a_{31} & a_{32} \end{pmatrix}
\]

(3.1)

where we have expanded the determinant along the first row. The eigenvalues of $A$ are the roots of this equation. As before, the roots may be real or complex, they may all be distinct or there may be repeated roots. If a root is repeated once, we say it has multiplicity 2. In the special case where $A$ is (upper or lower) triangular or diagonal, the characteristic equation is particular easy to compute, since the determinant of such matrices is simply the product of its diagonal. In such cases, the eigenvalues are simply the diagonal elements of the matrix.

Without expanding the expansion any further, we note that the characteristic equation is an order 3 polynomial, which we can write as

\[
\rho(\lambda) = b_3 \lambda^3 + b_2 \lambda^2 + b_1 \lambda + b_0
\]

(3.2)

The third power of $\lambda$ appears only in the first term of the determinant expansion, and has coefficient $(-1)^3$. An order three polynomial has three roots (the eigenvalues), so we can write this equation as

\[
\rho(\lambda) = (-1)^3(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)
\]

(3.3)

From (3.1), the determinant of $A$ can be found by setting $\lambda = 0$. Doing so in equation (3.3) show that

\[
\det(A) = \lambda_1 \lambda_2 \lambda_3
\]

As in the $(2 \times 2)$ case, the $\det(A) = 0$ iff one or more of the eigenvalues are zero.
Observe that the second power of \( \lambda \) also appears only in the first term in the expansion (3.1). Expanding the first term in (3.1) we have

\[
(a_{11} - \lambda)[(a_{22} - \lambda)(a_{33} - \lambda) - a_{23}a_{32}]
= (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) - (a_{11} - \lambda)a_{23}a_{32}
\]

(3.4)

so the second power of \( \lambda \) in fact only appears in \((a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)\). Expanding this expression further, you can easily verify that the coefficient on \( \lambda^2 \) is \( a_{11} + a_{22} + a_{33} \), which is \( \text{tr}(A) \). Expanding (3.3), we see that the coefficient on \( \lambda^2 \) there is \( \lambda_1 + \lambda_2 + \lambda_3 \). Matching coefficients, we see that

\[
\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3.
\]

All these results extend to the general \((n \times n)\) case: for an \((n \times n)\) matrix \( A \) we have

\[
\det(A) = \lambda_1\lambda_2\ldots\lambda_n \quad \text{and} \quad \text{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n.
\]

The arguments for results (5)-(10) in Section 2 also carry over to the \((n \times n)\) case with little or no amendment: if \( \lambda \) is an eigenvalue of \( A \), then \( \lambda \) is also an eigenvalue of \( A^T \), \( \lambda_i^* \) is an eigenvalue of \( A^* \), and if \( A \) is invertible, then \( 1/\lambda \) is an eigenvalue of \( A^{-1} \); Eigenvectors associated with distinct eigenvalues are linearly independent, and \( A \) has \( n \) distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \), with associated eigenvectors \( x_1, x_2, \ldots, x_n \), then it is diagonalizable: letting

\[
P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.
\]

Then

\[
P^{-1}AP = \Lambda, \quad \text{or equivalently} \quad A = P\Lambda P^{-1}.
\]

The Spectral Theorem for Symmetric Matrices also continues to hold: if the \((n \times n)\) matrix \( A \) is symmetric, then

(i) all its eigenvalues are real;

(ii) the eigenvectors the correspond to different eigenvalues are orthogonal,

(iii) there exists an orthonormal matrix \( P \) (i.e. \( P^{-1} = P^T \)) comprising the unit eigenvectors such that

\[
P^{-1}AP = \Lambda, \quad \text{or equivalently} \quad A = P\Lambda P^{-1}.
\]
The proof in the case when the eigenvalues are distinct is a simply extrapolation of the argument following result (11). For the proof when there are repeated roots, see Strang (2009) Section 6.4.

Exercises

1. Suppose $\Sigma$ is symmetric, with non-zero eigenvalues. Use the spectral theorem to define the matrix $\Sigma^{1/2}$ such that $\Sigma = \Sigma^{1/2} \Sigma^{1/2}$. Similarly, find $\Sigma^{-1/2}$ such that $\Sigma^{-1} = \Sigma^{-1/2} \Sigma^{-1/2}$. 
Section 14  Applications of Eigenvalues

Eigenvalues and the Rank of a Matrix  We often have to determine the rank of a matrix (the number of linear independent rows or columns contained in the matrix). The Spectral Theorem for symmetric matrices makes it very easy to do so. We use the following result:

If \( A \) is \((m \times n)\) and \( B \) is \((n \times n)\) and full rank, then \( \text{rank}(AB) = \text{rank}(A) \)

which extends to the product \( CAB \) where both \( C \) and \( B \) are full rank:

\[
\text{rank}(CAB) = \text{rank}(A) .
\]

This means that if \( A \) is a square diagonalizable matrix, then

\[
\text{rank}(A) = \text{rank}(P\Lambda P^{-1}) = \text{rank}(\Lambda)
\]

The rank of \( \Lambda \) is simply the number of non-zero terms on the diagonal, i.e. the number of non-zero eigenvalues of \( A \).

Furthermore, if \( A \) is symmetric and idempotent, then the eigenvalues of \( A \) take on values 1 or 0 only, i.e., the diagonal of \( A \) comprises only 1’s and 0’s. In this case,

\[
\text{rank}(A) = \text{trace}(A)
\]

(remember that the trace of a square matrix is just the sum of the elements on its diagonal.) Furthermore, because

\[
A = P^{-1}AP
\]

we have

\[
\text{trace}(A) = \text{trace}(P^{-1}AP) = \text{trace}(PP^{-1}A) = \text{trace}(A)
\]

where we have used the fact that \( \text{trace}(AB) = \text{trace}(BA) \) when both products exist. It is therefore very easy to find the rank of a symmetric idempotent matrix. Simply add up the elements in its diagonal – you don’t even have to find the eigenvalues!

The result

\[
\text{rank}(A) = \text{rank}(P^{-1}AP) = \text{rank}(A)
\]

always works for symmetric matrices, since symmetric matrices are always diagonalizable. But this approach can also work with non-diagonalizable matrices, and even non-square matrices. Recall that for any matrix \( A \),

\[
\text{rank}(A^TA) = \text{rank}(A)
\]

Since \( A^TA \) is symmetric, we can compute its rank, and therefore the rank of \( A \), by computing and counting the number of non-zero eigenvalues possessed by \( A^TA \).
Exercises (incomplete)

1. Let $X$ be an arbitrary $(n \times k)$ matrix such that $(X'X)^{-1}$ exists. Because $X'X$ is symmetric (why?), we know that $(X'X)^{-1}$ is also symmetric (why?). Use this fact to show that the matrix

$$I - X(X'X)X'$$

is symmetric and idempotent. Find its rank.
Positive Definiteness of Quadratic Forms

A quadratic form is an expression of the form

\[ Q = x^T Ax \]

When \( A \) is \((2 \times 2)\), this expression is

\[
Q = x^T Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2
\]

For a general \((n \times n)\) matrix, we have

\[
Q = x^T Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_i x_j
\]

The following are examples of quadratic forms:

(i) \[ q_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 2x_1x_2 + x_2^2 \]

(ii) \[ q_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + x_2^2 \]

(iii) \[ q_3 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 4x_1x_2 + 3x_2^2 \]

(iv) \[ q_4 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 6x_1x_2 + 3x_2^2 \]

The quadratic forms in (i), (ii) and (iii) involve symmetric matrices, whereas (iv) does not. Note, however, that for any quadratic form involving a non-symmetric matrix, there is an equivalent one using a symmetric matrix. In the \((2 \times 2)\) case, replace both \(a_{12}\) and \(a_{21}\) by \((a_{12} + a_{21})/2\). For example, corresponding to (iv) we have

\[
q_4 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 6x_1x_2 + 3x_2^2.
\]

In other words, we can limit ourselves to studying quadratic forms involving symmetric matrices.

For larger \((n \times n)\) matrices, replace \(A\) with \(\frac{A + A^T}{2}\).

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Matrix Algebra Notes
Anthony Tay
14-3
In applications, we often need to determine the sign of a quadratic form for arbitrary values of \( \mathbf{x} \neq \mathbf{0} \). For instance, the quadratic form \( q_1 \) can be written as

\[
q_1 = x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2
\]

so we know that \( q_1 \geq 0 \) no matter what values of \( x_1 \) and \( x_2 \) are chosen. We call such quadratic forms positive semi-definite. The quadratic form \( q_2 \) has a similar (but stronger) behavior. This quadratic form can be written as

\[
q_2 = 2x_1^2 + x_2^2 > 0
\]

for all values of \( x_1 \) and \( x_2 \), not both equal to zero at the same time. This quadratic form is said to be positive definite. The quadratic form \( q_3 \), however, is “indefinite”: writing

\[
q_3(x_1, x_2) = x_1^2 + 4x_1x_2 + 3x_2^2
\]

we have \( q_3(-2,1) = 4 - 8 + 3 = -1 \) whereas \( q_3(2,1) = 4 + 8 + 3 = 15 \). That is, the quadratic form is negative for some values of \( x_1 \) and \( x_2 \), and positive for others.

A quadratic form \( \mathbf{Q} = \mathbf{x}^T \mathbf{A} \mathbf{x} \) is said to be

- positive definite if \( \mathbf{x}^T \mathbf{A} \mathbf{x} > 0 \) for all \( \mathbf{x} \neq \mathbf{0} \);
- positive semi-definite if \( \mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0 \) for all \( \mathbf{x} \neq \mathbf{0} \);
- negative definite if \( \mathbf{x}^T \mathbf{A} \mathbf{x} < 0 \) for all \( \mathbf{x} \neq \mathbf{0} \);
- negative semi-definite if \( \mathbf{x}^T \mathbf{A} \mathbf{x} \leq 0 \) for all \( \mathbf{x} \neq \mathbf{0} \);

Although definiteness pertain to the quadratic form \( \mathbf{Q} = \mathbf{x}^T \mathbf{A} \mathbf{x} \), we often apply the term to the matrix \( \mathbf{A} \). One application of these concepts is in optimization theory where the second order condition often involves determining the definiteness of the Hessian matrix. Another application is in ‘comparing matrices’.

One interesting fact about definite symmetric matrices is they can be factorized into the product of a triangle matrix and its transpose. We state and explain this for positive definite symmetric matrices:

**Triangular Factorization**

Any positive definite symmetric \((n \times n)\) matrix \( \mathbf{A} \) has a unique representation of the form

\[
\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T
\]

where \( \mathbf{L} \) is lower triangular with ones down the diagonal, and \( \mathbf{D} \) is diagonal with positive diagonal elements.
We demonstrate this for a positive definite symmetric \((3 \times 3)\) matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}.
\]

Because \(A\) is positive definite, we have \(a_{11} > 0\) (pick \(x^T = [1 \ 0 \ 0]\)). We can construct the elimination matrix

\[
E_1 = \begin{bmatrix}
1 & 0 & 0 \\
-\frac{a_{22}}{a_{11}} & 1 & 0 \\
-\frac{a_{33}}{a_{11}} & 0 & 1
\end{bmatrix}
\]

You can easily verify that

\[
E_1^T A E_1^T = \begin{bmatrix}
a_{11} & 0 & 0 \\
0 & b_{22} & b_{23} \\
0 & b_{32} & b_{33}
\end{bmatrix}
\]

where \(b_{ij} = a_{ij} - a_{1j}a_{1i} / a_{11}\).

Note that if \(A\) is positive definite, then \(E_1^T A E_1^T\) must also be positive definite: define \(y\) such that \(x = E_1^T y\) which then allows us to write \(y^T E_1^T A E_1^T y = x^T A x\). Because \(E_1\) is non-singular, \(x = 0 \iff y = 0\). Therefore \(x^T A x > 0\) for all \(x \neq 0\) implies \(y^T E_1^T A E_1^T y > 0\) for all \(y \neq 0\).

Because \(E_1^T A E_1^T\) is positive definite, \(b_{22} > 0\) (pick \(y^T = [0 \ 1 \ 0]\)), so we can define

\[
E_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -\frac{b_{32}}{b_{22}} & 1
\end{bmatrix}
\]

You can verify that

\[
E_2 E_1^T A E_1^T E_2^T = \begin{bmatrix}
a_{11} & 0 & 0 \\
0 & b_{22} & 0 \\
0 & 0 & c_{33}
\end{bmatrix}
\]

where \(c_{33} = b_{33} - \frac{b_{32}^2}{b_{22}}\).
Call the resulting diagonal matrix $D$. Because $E_1$ and $E_2$ are lower triangular with ones down the diagonal, $E_2 E_1$ has the same structure. Furthermore, $L = (E_2 E_1)^{-1}$ exists, and has the same structure. Therefore

$$E_2 E_1 A E_1^T E_2^T = D \iff A = LDL^T.$$  

We can go one step further. Defining

$$D = \begin{bmatrix}
\sqrt{a_{11}} & 0 & 0 \\
0 & \sqrt{b_{22}} & 0 \\
0 & 0 & \sqrt{c_{33}}
\end{bmatrix},$$

we have $A = LDL^T = LD^{1/2}D^{1/2}L^T = CC^T$ where $C = LD^{1/2}$ is also lower triangular. This is the Cholesky factorization (or Cholesky decomposition) of $A$. These decompositions are related to, but different from, the eigenvalue-eigenvector decomposition discussed earlier.

These arguments readily extend to the general case.

Proving the indefiniteness of a matrix merely requires providing counter examples, as we did for $q_3$ above. To prove the definiteness of a matrix is much harder, because we have to show that the requisite sign holds for all $x \neq 0$. The objective here is to develop criteria for evaluating the definiteness of a matrix. We provide two sets of results, one using principal minors, and the other using eigenvalues. It is worthwhile previewing the results with $(2 \times 2)$ matrices, before results for the general case are given.

For a given $(2 \times 2)$ symmetric matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$, let

$$Q(x_1, x_2) = x^T A x = a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2,$$

for arbitrary $x \neq 0$. Then

(i) $Q(x_1, x_2)$ is positive semi-definite $\iff a_{11} \geq 0$, $a_{22} \geq 0$, and $a_{11} a_{22} - a_{12}^2 \geq 0$;
(ii) $Q(x_1, x_2)$ is positive definite $\iff a_{11} > 0$ and $a_{11} a_{22} - a_{12}^2 > 0$;
(iii) $Q(x_1, x_2)$ is negative semi-definite $\iff a_{11} \leq 0$, $a_{22} \leq 0$, and $a_{11} a_{22} - a_{12}^2 \geq 0$;
(iv) $Q(x_1, x_2)$ is negative definite $\iff a_{11} < 0$ and $a_{11} a_{22} - a_{12}^2 > 0$.
Proof:

(i) Suppose $a_{11} \geq 0$, $a_{22} \geq 0$, and $a_{11}a_{22} - a_{12}^2 \geq 0$.

Case 1: $a_{11} = 0$. Then $a_{11}a_{22} - a_{12}^2 \geq 0$ implies $a_{12} = 0$, so the quadratic form is $Q(x_1, x_2) = a_{22}x_2^2 \geq 0$.

Case 2: $a_{11} > 0$. Then

$$Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

$$= a_{11}(x_1^2 + 2\frac{a_{12}}{a_{11}}x_1x_2 + \frac{a_{22}}{a_{11}}x_2^2)$$

$$= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2\right)^2 + \frac{a_{22}}{a_{11}}x_2^2 - \left(\frac{a_{12}}{a_{11}}\right)^2 x_2^2$$

$$= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2\right)^2 + \frac{a_{11}a_{22} - a_{12}^2}{a_{11}^2}x_2^2 \quad (*)$$

With $a_{11} > 0$ and $a_{11}a_{22} - a_{12}^2 \geq 0$, clearly $Q(x_1, x_2) \geq 0$ for all $x_1, x_2$ not both equal to zero.

We now show the converse: Suppose $Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \geq 0$ for all $x_1, x_2$ not both equal to zero. Then, in particular, we have

$$Q(1, 0) = a_{11} \geq 0 \quad \text{and} \quad Q(0, 1) = a_{22} \geq 0.$$ 

If $a_{11} = 0$, then $Q(x_1, 1) = 2a_{12}x_1 + a_{22} \geq 0$, which implies $a_{12} = 0$, since if $a_{12} > 0$, we can make $Q(x_1, 1) < 0$ by choosing $x_1$ a large enough negative number, and if $a_{12} < 0$, we can make $Q(x_1, 1) < 0$ by choosing $x_1$ to be a large enough positive number. Then $a_{11}a_{22} - a_{12}^2 = 0$. If $a_{11} > 0$, then we must have $a_{11}a_{22} - a_{12}^2 \geq 0$, otherwise we can make $Q(x_1, x_2) < 0$ by choosing $x_1$ and $x_2$ to make

$$x_1 + \frac{a_{12}}{a_{11}}x_2 = 0.$$ 

Result (iii) is proved in a similar fashion.

(ii) Suppose $a_{11} > 0$ and $a_{11}a_{22} - a_{12}^2 > 0$. Then from $(*)$, $Q(x_1, x_2) > 0$.

Suppose $Q(x_1, x_2) > 0$ for all $x_1, x_2$ not both equal to zero. Then $Q(1, 0) = a_{11} > 0$. Because $a_{11} > 0$, we can write $(*)$. Then $Q(-a_{12}/a_{11}, 1) = (a_{11}a_{22} - a_{12}^2)/a_{11} > 0$, which implies $a_{11}a_{22} - a_{12}^2 > 0$.

Result (iv) is proved in similar fashion.
We re-state this result for general \((n \times n)\) symmetric matrices (without proof):

**Theorem**  Consider the \((n \times n)\) symmetric matrix \(A\) and the associated quadratic form

\[
Q(x) = x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j ,
\]

where \(x\) is an arbitrary non-zero \(n\)-dimensional vector. Let \(D_k\) be the \(k\)-th leading principal minor of \(A\), and \(\Delta_k\) denote an arbitrary principal minors of order \(k\). Then \(Q(x)\) is

\[
(a) \text{ positive definite } \iff D_k > 0 \text{ for } k = 1, 2, \ldots, n
\]
\[
(b) \text{ positive semi-definite } \iff \Delta_k \geq 0 \text{ for } k = 1, 2, \ldots, n .
\]
\[
(c) \text{ negative definite } \iff (-1)^k D_k > 0 \text{ for } k = 1, 2, \ldots, n .
\]
\[
(d) \text{ negative semi-definite } \iff (-1)^k \Delta_k \geq 0 \text{ for } k = 1, 2, \ldots, n .
\]

It should be clear that the results stated and proved for the \((2 \times 2)\) case is a special case of this theorem.

*(Note: to add definition of principal minors and leading principal minors. For the moment, look it up.)*

**An Eigenvalue Approach**  Eigenvalues provide a more convenient way to determine definiteness of quadratic forms. Because we are dealing with symmetric matrices, which are diagonalizable, we can rewrite our quadratic form as

\[
Q(x) = x^T A x = x^T P \Lambda P^T x = y^T \Lambda y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_{n-1} y_{n-1}^2 + \lambda_n y_n^2
\]

where \(y = P^T x\). It should be clear that \(Q(x) > 0\) for all \(x \neq 0\) is equivalent to \(Q(x) > 0\) for all \(y \neq 0\), which is in turn equivalent to \(\lambda_r > 0\), \(r = 1, \ldots, n\). Similarly, \(Q(x) \geq 0\) for all \(x \neq 0\) is equivalent to \(Q(y) \geq 0\) for all \(y \neq 0\), which is in turn equivalent to \(\lambda_r \geq 0\), \(r = 1, \ldots, n\). That is,

For a given symmetric \((n \times n)\) matrix \(A\), let \(Q(x) = x^T A x\), for arbitrary \(x \neq 0\). Then

\[
\begin{align*}
(i) & \quad Q(x) \text{ is positive semi-definite } \iff \lambda_r \geq 0, \ r = 1, \ldots, n \\
(ii) & \quad Q(x) \text{ is positive definite } \iff \lambda_r > 0, \ r = 1, \ldots, n \\
(iii) & \quad Q(x) \text{ is negative semi-definite } \iff \lambda_r \leq 0, \ r = 1, \ldots, n \\
(iv) & \quad Q(x) \text{ is negative definite } \iff \lambda_r < 0, \ r = 1, \ldots, n
\end{align*}
\]
Exercises (not available yet)
Section 15   Vectors of Random Variables

When working with several random variables $X_1, X_2, \ldots, X_n$, it is often convenient to arrange them in vector form

$$\mathbf{x} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

We can then make use of matrix algebra to help us organize and manipulate large numbers of random variables simultaneously. We define the expectation of a random vector as element-by-element expectation:

$$E[\mathbf{x}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}.$$ 

If $\mathbf{X}$ is an $(m \times n)$ matrix of random variables, then $E[\mathbf{X}]$ is the $(m \times n)$ matrix where the $(i, j)$th element is the mean of the $(i, j)$th element of $\mathbf{X}$, i.e.,

$$E[\mathbf{X}] = \begin{bmatrix} E[X_{11}] & E[X_{12}] & \cdots & E[X_{1n}] \\ E[X_{21}] & E[X_{22}] & \cdots & E[X_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ E[X_{m1}] & E[X_{m2}] & \cdots & E[X_{mn}] \end{bmatrix}.$$ 

These definitions provide a neat way for computing the variances and covariances of the variables in $\mathbf{X}$ “all at once”:

$$\text{var}[\mathbf{x}] = E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^\prime]$$

$$= E \begin{bmatrix} (X_1 - E[X_1])(X_1 - E[X_1]) & (X_1 - E[X_1])(X_2 - E[X_2]) & \cdots & (X_1 - E[X_1])(X_n - E[X_n]) \\ (X_2 - E[X_2])(X_1 - E[X_1]) & (X_2 - E[X_2])(X_2 - E[X_2]) & \cdots & (X_2 - E[X_2])(X_n - E[X_n]) \\ \vdots & \vdots & \ddots & \vdots \\ (X_n - E[X_n])(X_1 - E[X_1]) & (X_n - E[X_n])(X_2 - E[X_2]) & \cdots & (X_n - E[X_n])(X_n - E[X_n]) \end{bmatrix}$$

$$= \begin{bmatrix} \text{var}[X_1] & \text{cov}[X_1, X_2] & \cdots & \text{cov}[X_1, X_n] \\ \text{cov}[X_2, X_1] & \text{var}[X_2] & \cdots & \text{cov}[X_2, X_n] \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}[X_n, X_1] & \text{cov}[X_n, X_2] & \cdots & \text{var}[X_n] \end{bmatrix}.$$ 

We call $\text{var}[\mathbf{x}]$ the variance-covariance matrix of $\mathbf{x}$. 

Matrix Algebra Notes  15-1
Anthony Tay
The formula \( \text{var}[\mathbf{x}] = E(\mathbf{x} - E[\mathbf{x}])' (\mathbf{x} - E[\mathbf{x}]) \) can be viewed as the matrix version of the variance formula \( \text{var}[X] = E((X - E[X])^2) \) for a single variable.

Sometimes we want to compute a ‘covariance matrix’ between two vectors of random variables \( \mathbf{x} \) and \( \mathbf{y} \). We can compute

\[
\text{cov}[\mathbf{x}, \mathbf{y}] = E(\mathbf{x} - E[\mathbf{x}])(\mathbf{y} - E[\mathbf{y}])'
\]

\[
= E \begin{bmatrix}
(X_1 - E[X_1])(Y_1 - E[Y_1]) & (X_1 - E[X_1])(Y_2 - E[Y_2]) & \cdots & (X_1 - E[X_1])(Y_n - E[Y_n]) \\
(X_2 - E[X_2])(Y_1 - E[Y_1]) & (X_2 - E[X_2])(Y_2 - E[Y_2]) & \cdots & (X_2 - E[X_2])(Y_n - E[Y_n]) \\
\vdots & \vdots & \ddots & \vdots \\
(X_n - E[X_n])(Y_1 - E[Y_1]) & (X_n - E[X_n])(Y_2 - E[Y_2]) & \cdots & (X_n - E[X_n])(Y_n - E[Y_n])
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\text{cov}[X_1, Y_1] & \text{cov}[X_1, Y_2] & \cdots & \text{cov}[X_1, Y_n] \\
\text{cov}[X_2, Y_1] & \text{cov}[X_2, Y_2] & \cdots & \text{cov}[X_2, Y_n] \\
\vdots & \vdots & \ddots & \vdots \\
\text{cov}[X_n, Y_1] & \text{cov}[X_n, Y_2] & \cdots & \text{cov}[X_n, Y_n]
\end{bmatrix}
\]

Rules for dealing with the mean vector and the variance-covariance matrix

If \( \mathbf{x} \) is an \((n \times 1)\) vector of random variables, \( \mathbf{X} \) is an \((m \times n)\) matrix of random variables, \( \mathbf{b} \) is an \((m \times 1)\) vector of constants, and \( \mathbf{A} \) is an \((m \times n)\) matrix of constants, then

1. \( E[\mathbf{A}\mathbf{x} + \mathbf{b}] = \mathbf{A}E[\mathbf{x}] + \mathbf{b} \)
2. \( \text{var-cov}[\mathbf{A}\mathbf{x} + \mathbf{b}] = \mathbf{A} \text{var}[\mathbf{x}]
\]

In particular,

\[
\text{var}[c_1X_1 + c_2X_2 + \ldots + c_nX_n] = \text{var}[\mathbf{c'}\mathbf{x}] = \mathbf{c'} \text{var}[\mathbf{x}]\mathbf{c} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \text{cov}[X_i, X_j]
\]

3. A useful result is

\[
E[\text{tr}[\mathbf{X}]] = E[X_{11} + X_{22} + \ldots + X_{nn}]
\]

\[
= E[X_{11}] + E[X_{22}] + \ldots + E[X_{nn}].
\]

\[
= \text{tr}[E[\mathbf{X}]].
\]

The first of these is straightforward to show by simply writing out the expression \( \mathbf{A}\mathbf{x} + \mathbf{b} \) in full and taking expectations. This formula is the matrix version of the usual single variable result

\[
E[aX + b] = aE[X] + b
\]
To show (2), plug $\mathbf{Ax + b}$ into the variance formula:

\[
\text{var}[\mathbf{Ax + b}] = E[(\mathbf{Ax + b} - E[\mathbf{Ax + b}])(\mathbf{Ax + b} - E[\mathbf{Ax + b}])']
\]
\[
= E[(\mathbf{Ax + b} - \mathbf{AE}[\mathbf{x}] - \mathbf{b})(\mathbf{Ax + b} - \mathbf{AE}[\mathbf{x}] - \mathbf{b})']
\]
\[
= E[(\mathbf{Ax - AE}[\mathbf{x}])((\mathbf{Ax - AE}[\mathbf{x}])']
\]
\[
= E[\mathbf{A}(\mathbf{x} - E[\mathbf{x}])((\mathbf{x} - E[\mathbf{x}])')]
\]
\[
= E[\mathbf{A}(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])']\mathbf{A}'
\]
\[
= \mathbf{A} \text{var[\mathbf{x}]}\mathbf{A}'
\]

This is the matrix version of the single variable result

\[
\text{var}[a\mathbf{x} + \mathbf{b}] = a^2 \text{var}[\mathbf{x}].
\]

Note that a variance-covariance matrix must be positive-definite --

\[
\text{var}[c_1X_1 + c_2X_2 + \ldots + c_nX_n] = \text{var}[c'\mathbf{x}] = c' \text{var}[\mathbf{x}]c
\]

has to be positive for all $\mathbf{c} \neq \mathbf{0}$, since variances must be positive.

**Exercise**

Let $\mathbf{x} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a vector of random variables, and

\[
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\]

and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be constants. Write out $\mathbf{Ax + b}$ in full, and take expectations to show that

\[
E[\mathbf{Ax + b}] = \mathbf{A}E[\mathbf{x}] + \mathbf{b}
\]

**The Multivariate Normal Distribution**

The random vector $\mathbf{x}$ follows the **multivariate normal distribution** with

\[
\text{mean } E[\mathbf{x}] = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}
\]

and variance-covariance matrix $\Sigma = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \ldots & \sigma_{1n}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \ldots & \sigma_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1}^2 & \sigma_{n2}^2 & \ldots & \sigma_{nn}^2 \end{bmatrix}$

if its distribution has the form

\[
f(\mathbf{x}) = (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp\{-1/2(\mathbf{x} - \mu)'\Sigma^{-1}(\mathbf{x} - \mu)\}.
\]
We denote this by \( \mathbf{x} \sim N(\mu, \Sigma) \). This is analogous to the univariate normal pdf:

\[
f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2}\right)
\]

**Exercise** Write the joint pdf out without matrix notation for the bivariate case

\[
\mathbf{x} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}
\]

Then show that \( f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) \) when \( \sigma_{12} = \sigma_{21} = 0 \). What does this say?

**Exercise** Use computer software (say matlab) to plot the distribution of the bivariate normal for various parameter values.

The following are important properties of the multivariate normal:

4. The conditional distributions are also normal. In particular, if

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left( \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right)
\]

where \( x_1 \) and \( x_2 \) are (\( n_1 \times 1 \)) , \( x_1 \) and \( x_2 \) are (\( n_2 \times 1 \)) , \( \Sigma_{11} \) is (\( n_1 \times n_1 \)) , \( \Sigma_{12} \) is (\( n_1 \times n_2 \)) , \( \Sigma_{21} \) is (\( n_2 \times n_1 \)) , and (\( n_2 \times n_2 \)) , then

a. the marginal distribution for \( x_1 \) is \( N(\mu_1, \Sigma_{11}) \)

b. the conditional distribution for \( x_1 \) given \( x_2 \) is \( N(\mu_{1|2}, \Sigma_{1|2}) \) where

\[
\mu_{1|2} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1}(x_2 - \mu_2) \quad \text{and} \quad \Sigma_{1|2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}
\]

**Exercise** In the bivariate case,

\[
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left( \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} \\ \sigma_{21} & \sigma_{22}^2 \end{bmatrix} \right)
\]

(4a) says that \( X_1 \sim N(\mu_1, \sigma_{11}^2) \) and \( X_2 \sim N(\mu_2, \sigma_{22}^2) \). Write out the expressions in (4b). Note in particular that the conditional mean of \( X_1 \) given \( X_2 \) is a linear function of \( X_2 \).
Exercise Using the expressions for the conditional mean and conditional variance of \( X_2 \) given \( X_1 \), show that

\[
f_{X_1,x_2}(x_1,x_2) = f_{X_1|X_2}(x_2 \mid x_1) f_{X_1}(x_1).
\]

(You can make a similar argument for the general \( x_1 \) and \( x_2 \) case.)

5. If \( x \sim N(\mu, \Sigma) \), then \( Ax + b \sim N(A \mu + b, A \Sigma A') \); The expression \( Ax + b \) is normal because linear combinations of normal random variables remain normal. The formulae for the mean and variance-covariance matrix are the usual ones.

6. If \( x \sim N(\mu, \Sigma) \) and \( \Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) \), then the random variables in \( x \) are independent.

The following make use of the fact that

The square of a standard normal variable has a \( \chi_1^2 \) distribution;

The sum of \( n \) independent \( \chi_1^2 \) is a \( \chi_n^2 \);

If \( X \sim N(0,1), Y \sim \chi_n^2 \) and \( X \) and \( Y \) are independent, then \( \frac{X}{\sqrt{Y/n}} \sim t_n \)

If \( Y_1 \sim \chi_n^2, Y_2 \sim \chi_n^2 \), and \( Y_1 \) and \( Y_2 \) are independent, then \( \frac{Y_1/n}{Y_2/m} \sim F_{(n,m)} \)

We have

7. If \( x \sim N(0, I) \) and \( A \) is symmetric, and idempotent with rank \( J \), then the scalar random variable \( x'A x \sim \chi_J^2 \). In particular, \( x'A x \sim \chi_n^2 \).

Proof

Because \( A \) is symmetric, we can write \( A = C \Lambda C' \), with \( C'C = I \). Note that \( C'x \sim N(0, I) \) because \( \text{var}(C'x) = C'IC = C'C = I \). That is, \( C'x \) is a vector of independently distributed standard normal variables.

Write

\[
x'A x = x'A C \Lambda C' x = y'\Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2
\]

where \( y = C'x \). Each \( y_i^2 \) is an independent chi-sq degree one, since the \( y_i \)'s are independent standard normal variables. Because \( A \) is idempotent, there are \( J \) \( \lambda_i \)'s equal to one, and \( (n - J) \) \( \lambda_i \)'s that are zero. Relabeling the \( y_i \)'s so that the first \( \lambda_i \)'s are equal to one, we have

\[
x'A x = \sum_{i=1}^{J} y_i^2
\]

which is a sum of \( J \) independent \( \chi_J^2 \). Therefore \( x'A x \sim \chi_J^2 \).
8. If \( \mathbf{x} \sim N(\mu, \Sigma) \), then \( (\mathbf{x} - \mu)\Sigma^{-1} (\mathbf{x} - \mu) \sim \chi^2_n \).

**Proof**

\( \Sigma \) is positive definite, symmetric, and full rank. Therefore we can write \( \Sigma^{-1} = \Sigma^{-1/2} \Sigma^{-1/2} \). Note that \( \mathbf{z} = \Sigma^{-1/2} (\mathbf{x} - \mu) \sim N(\mathbf{0}, \mathbf{I}) \), therefore

\[
(\mathbf{x} - \mu)' \Sigma^{-1} (\mathbf{x} - \mu) = (\Sigma^{-1/2} (\mathbf{x} - \mu))' \Sigma^{-1/2} (\mathbf{x} - \mu) = \mathbf{z}' \mathbf{z} \sim \chi^2_n.
\]

9. If \( \mathbf{x} \sim N(\mathbf{0}, \mathbf{I}) \), and \( \mathbf{A} \) and \( \mathbf{B} \) are symmetric and idempotent, then \( \mathbf{x}'\mathbf{A}\mathbf{x} \) and \( \mathbf{x}'\mathbf{B}\mathbf{x} \) are independent if \( \mathbf{AB} = \mathbf{0} \).

**Proof**

Because \( \mathbf{A} \) and \( \mathbf{B} \) are symmetric and idempotent, we have \( \mathbf{A}'\mathbf{A} = \mathbf{A} \) and \( \mathbf{B}'\mathbf{B} = \mathbf{B} \). Therefore we can write the quadratic forms as \( \mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})'(\mathbf{A}\mathbf{x}) \). Because \( \mathbf{x} \) is normal with mean \( \mathbf{0} \), \( \mathbf{A}\mathbf{x} \) also normal with mean \( \mathbf{0} \). For vectors of zero mean random variables, \( \text{cov}[\mathbf{x}, \mathbf{y}] = \mathbb{E}[\mathbf{x}\mathbf{y}'] \) (why?). We have

\[
\text{cov} \left[ \mathbf{A}\mathbf{x}, \mathbf{B}\mathbf{x} \right] = \mathbb{E} \left[ \mathbf{A}\mathbf{x}\mathbf{B}' \right] = \mathbf{A}\mathbb{E}[\mathbf{x}\mathbf{x}']\mathbf{B}' = \mathbf{A}\mathbf{B}' = \mathbf{AB}.
\]

Therefore, \( \mathbf{AB} = \mathbf{0} \) implies that \( \mathbf{A}\mathbf{x} \) and \( \mathbf{B}\mathbf{x} \) are normally distributed, with covariance \( \mathbf{0} \). This implies that \( \mathbf{A}\mathbf{x} \) and \( \mathbf{B}\mathbf{x} \) are independent (why?), and therefore the quadratic forms \( \mathbf{x}'\mathbf{A}\mathbf{x} \) and \( \mathbf{x}'\mathbf{B}\mathbf{x} \) are also independent.

It follows from (7) that \( \frac{[\mathbf{x}'\mathbf{A}\mathbf{x} / \text{rank}(\mathbf{A})]}{[\mathbf{x}'\mathbf{B}\mathbf{x} / \text{rank}(\mathbf{B})]} \) is distributed \( F_{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})} \).

10. If \( \mathbf{x} \sim N(\mathbf{0}, \mathbf{I}) \), and \( \mathbf{A} \) is symmetric and idempotent, then \( \mathbf{L}\mathbf{x} \) and \( \mathbf{x}'\mathbf{A}\mathbf{x} \) are independent if \( \mathbf{LA} = \mathbf{0} \).

**Proof**

Same idea as in (9).

We use these results to prove some standard results in statistics. Suppose \( X_1, X_2, \ldots, X_n \) are \( n \) independent draws from a \( N(\mu, \sigma^2) \) distribution, i.e.

\[
\mathbf{x} \sim N(\mu, \sigma^2\mathbf{I}).
\]

\[
\begin{bmatrix}
\mu \\
\mu \\
\vdots \\
\mu
\end{bmatrix} = 
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix} = \mu \mathbf{i}.
\]
We know that the sample mean
\[ \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \]
is normally distributed (because a linear combination of normal variables is normal) with mean
\[ E[\bar{X}] = E\left[ \frac{1}{n} \sum_{i=1}^{n} X_i \right] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \frac{1}{n} n \mu = \mu \]
and variance
\[ \text{var}[\bar{X}] = \text{var}\left[ \frac{1}{n} \sum_{i=1}^{n} X_i \right] = \frac{1}{n^2} \sum_{i=1}^{n} \text{var}[X_i] = \frac{1}{n^2} n \sigma^2 = \frac{\sigma^2}{n} \]

Furthermore, \( \bar{X} \sim N\left( \mu, \frac{\sigma^2}{n} \right) \) implies that
\[ \frac{\bar{X} - \mu}{\sqrt{\sigma^2 / n}} \sim N(0,1) \]

Unfortunately, this result is not very helpful if, for instance, you want to test a hypothesis on \( \mu \), since \( \sigma^2 \) is general unknown, and must be estimated. An unbiased estimator is
\[ \hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \]

To show this, note that \( \sum_{i=1}^{n} (X_i - \bar{X})^2 = \left( \sum_{i=1}^{n} X_i^2 \right) - n \bar{X}^2 \). Each \( X_i \) has variance
\[ \text{var}[X_i] = E[X_i^2] - \mu^2 = \sigma^2, \]
so
\[ E\left[ \sum_{i=1}^{n} X_i^2 \right] = \sum_{i=1}^{n} E[X_i^2] = \sum_{i=1}^{n} (\sigma^2 + \mu^2) = n \sigma^2 + n \mu^2. \]

Also, \( E[\bar{X}^2] = \text{var}[\bar{X}] + E[\bar{X}]^2 = \frac{\sigma^2}{n} + \mu^2 \). Putting all this together, we have:
One idea, then, is to substitute $\sigma^2$ with $\hat{\sigma}^2$ in $\frac{\bar{X} - \mu}{\sigma^2/n}$ to get the ‘t-statistic’:

$$ t = \frac{\bar{X} - \mu}{\sqrt{\hat{\sigma}^2/n}}. $$

Unfortunately, the $t$-statistic does not have a standard normal distribution.

We use the results discussed earlier to derive the distribution of the $t$-statistic. We begin by deriving a matrix expression for $\hat{\sigma}^2$. Observe first that

$$ E[\sigma^2] = \frac{1}{n-1} E \left[ \sum_{i=1}^{n} (X_i - \bar{X})^2 \right] $$

$$ = \frac{1}{n-1} E \left[ \sum_{i=1}^{n} X_i^2 \right] - n\bar{X}^2 $$

$$ = \frac{1}{n-1} \left( n\sigma^2 + n\mu^2 - n\left( \frac{\sigma^2}{n} + \mu^2 \right) \right) $$

$$ = \frac{1}{n-1} (n-1)\sigma^2 $$

$$ = \sigma^2. $$
An interesting property of the matrix $M = I - i(i'\bar{i})^{-1}i'$ is that it is symmetric and idempotent, with rank $n - 1$.

**Symmetric:**

$M' = (I - i(i')^{-1}i')' = I' - i'\left((i')^{-1}\right)'i = I - i(i')^{-1}i' = M$

**Idempotent:**

$$MM = \left(I - i(i')^{-1}i'\right)\left(I - i(i')^{-1}i'\right)$$

$$= \begin{bmatrix} I - i(i')^{-1}i & -i(i')^{-1}i' \\ -i(i')^{-1}i' & I - i(i')^{-1}i' \end{bmatrix}$$

Since $i(i')^{-1}i' = i$ cancels out,

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Because $M$ is symmetric and idempotent, $\text{rank}(M) = \text{trace}(M)$, and

$$\text{trace}\left(I - i(i')^{-1}i'\right) = \text{trace}(I) - \text{trace}(i(i')^{-1}i') = n - \text{trace}(i(i')^{-1}i') = n - 1$$

Therefore

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$= \frac{1}{n-1} \begin{bmatrix} X_1 - \bar{X} & X_2 - \bar{X} & \ldots & X_n - \bar{X} \\ X_2 - \bar{X} & \ldots & \ldots & \ldots \\ \vdots & \ldots & \ddots & \ldots \\ X_n - \bar{X} & \ldots & \ldots & X_n - \bar{X} \end{bmatrix}$$

$$= \frac{1}{n-1} x'M'Mx$$

$$= \frac{1}{n-1} x'Mx$$

Furthermore, note that

$$\frac{1}{\sigma}(x - \mu) \sim N(0, I)$$

$$Mx = M(x - \mu) \text{ since } M = 0 \text{ (why?)}$$

Together with the fact that $M$ is symmetric and idempotent with rank $n - 1$, result (7) implies that

$$\frac{1}{\sigma^2} x'Mx = \frac{1}{\sigma^2}(x - \mu)'M(x - \mu) \sim \chi^2_{n-1}.$$
This result is consistent with the fact that \( E[\sigma^2] = \sigma^2 \). The mean of a \( \chi^2_{n-1} \) is \( n-1 \), i.e.

\[
\frac{1}{\sigma^2}E[x'Mx] = n-1.
\]

Since \( \sigma^2 = \frac{1}{n-1}x'Mx \), the result follows. The fact that

\[
\frac{(n-1)\hat{\sigma}^2}{\sigma^2} = \frac{1}{\sigma^2}x'Mx \sim \chi^2_{n-1},
\]

which we have just shown, is of course a much stronger one. The result that \( E[\sigma^2] = \sigma^2 \) does not depend on the normality of \( x \). If we have normality of \( x \), then we have the \( \chi^2 \) result.

Finally, note that

\[
\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu) = \frac{1}{\sigma}(i'i)^{-1}i'(x - \mu) \sim N(0,1),
\]

and that

\[
(i'i)^{-1}i'M = \left[(i'i)^{-1}i'ight] \left[I - i(i'i)^{-1}i'\right] = (i'i)^{-1}i' - (i'i)^{-1}i(i'i)^{-1}i' = 0
\]

cancels out

which says that

\[
\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu) = \frac{1}{\sigma}(i'i)^{-1}i'(x - \mu) \sim N(0,1) \quad \text{and} \quad \frac{(n-1)\hat{\sigma}^2}{\sigma^2} = \frac{1}{\sigma^2}x'Mx \sim \chi^2_{n-1}
\]

are independent. Therefore,

\[
t = \frac{\bar{X} - \mu}{\sqrt{\sigma^2}/n} = \frac{\sqrt{n}}{\sigma}(\bar{X} - \mu) \quad \frac{(n-1)\hat{\sigma}^2}{\sigma^2} \left/ (n-1) \right. \right|^{\chi^2_{n-1}} \sim t_{n-1}.
\]

Of course, if \( X_i \)'s are not random draws from \( N(\mu, \sigma^2) \), then all of these results do not hold (except \( E[\bar{X}] = \mu \), \( \text{var}[\bar{X}] = \sigma^2 / n \) and \( E[\sigma^2] = \sigma^2 \), which does not require the normality assumption). Under reasonable conditions, the \( t \)-statistic will converge to the normal as the sample size grows.
Section 16  Differentiation of Matrix Forms

There are no new ‘calculus’ results here. We merely have a few definitions so that we can take derivatives of functions expressed in matrix form.

Definitions

Given \( y = f(x_1, x_2, \ldots, x_n) \), let \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \), then define \( \frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix} \).

Example  If \( y = x_1^2 x_2^2 x_3 \), then \( \frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} 2x_1 x_2^2 x_3 \\ 2x_1 x_1 x_2 x_3 \\ x_1^2 x_2^2 \end{bmatrix} \).

Example  If \( y = x_1 x_2 + x_3 x_4 \), then \( \frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{bmatrix} \).

Example  Let \( y = \det \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = x_1 x_4 - x_2 x_3 \), then \( \frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} x_4 \\ -x_3 \\ -x_2 \\ x_1 \end{bmatrix} \).

The case of linear functions and quadratic forms are particularly important. We have:

Example  If \( y = \mathbf{c}' \mathbf{x} \) where \( \mathbf{c} \) and \( \mathbf{x} \) are \((n \times 1)\), then \( \frac{\partial y}{\partial \mathbf{x}} = \mathbf{c} \).

To see this, simply expand and differentiate: \( y = \mathbf{c}' \mathbf{x} = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \), so

\[
\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{c}
\]
Example If \( y = x' Ax \) where \( x \) is \((n \times 1)\) and \( A \) is \((n \times n)\), then
\[
\frac{\partial y}{\partial x} = (A' + A)x
\]
It is easiest to see this in the \( n = 2 \) case: let
\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\]
Then
\[
y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2
\]
Then
\[
\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2a_{11}x_1 + (a_{12} + a_{21})x_2 \\ (a_{12} + a_{21})x_1 + 2a_{22}x_2 \end{bmatrix} = \begin{bmatrix} 2a_{11} & a_{12} + a_{21} \\ a_{12} + a_{21} & 2a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]
and it is easily seen that \[
\begin{bmatrix} 2a_{11} & a_{12} + a_{21} \\ a_{12} + a_{21} & 2a_{22} \end{bmatrix} = A + A'.
\]
If \( A \) is symmetric, then \( A = A' \), so
\[
\frac{\partial y}{\partial x} = 2Ax,
\]
which bears considerable resemblance to the univariate case where \( dy/dx = 2ax \) when \( y = ax^2 \). Remember that \( x'Ax \) is the matrix version of the quadratic function.

**********

The general principle is that the shape of the derivative matrix follows that of the denominator of the derivative, so that for \( y = f(x_1, x_2, \ldots, x_n) \),
\[
\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}
\]
since \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \).

Example If \( y = c'x \) where \( c \) and \( x \) are \((n \times 1)\), then \( \frac{\partial y}{\partial \mathbf{x}} = c' \).
Example  If
\[
y = \det \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = x_1x_4 - x_2x_3,
\]
we have \[
\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_4} \end{bmatrix} = \begin{bmatrix} x_4 & -x_3 & -x_2 & x_1 \end{bmatrix}.
\]

This “row” form of the vector derivative is most often applied to situations where we are differentiating a vector of functions, i.e. when
\[
y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix},
\]
in which case \[
\frac{\partial y}{\partial \mathbf{x}'} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} \frac{\partial y_1}{\partial x_2} \ldots \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} \frac{\partial y_2}{\partial x_2} \ldots \frac{\partial y_2}{\partial x_n} \\ \vdots \\ \frac{\partial y_m}{\partial x_1} \frac{\partial y_m}{\partial x_2} \ldots \frac{\partial y_m}{\partial x_n} \end{bmatrix}.
\]

Example  If \( \mathbf{y} = \mathbf{A} \mathbf{x} \) where \( \mathbf{A} \) is \((m \times n)\) and \( \mathbf{x} \) is \((n \times 1)\), then \[
\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \mathbf{A}.
\]

Then
\[
\frac{\partial \mathbf{y}}{\partial \mathbf{x}'} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} \frac{\partial y_1}{\partial x_2} \ldots \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} \frac{\partial y_2}{\partial x_2} \ldots \frac{\partial y_2}{\partial x_n} \\ \vdots \\ \frac{\partial y_m}{\partial x_1} \frac{\partial y_m}{\partial x_2} \ldots \frac{\partial y_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \mathbf{A}.
\]
Another application is to get a matrix of second derivatives: if \( y = f(x_1, x_2, \ldots, x_n) \), we have

\[
\frac{\partial^2 y}{\partial x \partial x'} = \frac{\partial}{\partial x'} \left( \frac{\partial y}{\partial x} \right) = \begin{bmatrix}
\frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_1 \partial x_n} \\
\frac{\partial^2 y}{\partial x_1 \partial x_2} & \frac{\partial^2 y}{\partial x_2^2} & \cdots & \frac{\partial^2 y}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 y}{\partial x_n \partial x_1} & \frac{\partial^2 y}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_n^2}
\end{bmatrix}.
\]

This is called the Hessian of \( f(x) \) and written \( f''(x) \). Young’s Theorem says that this will be a symmetric matrix.

**Example** If \( y = x'Ax \) where \( x \) is \((n \times 1)\) and \( A \) is \((n \times n)\), then \( \frac{\partial y}{\partial x} = (A' + A)x \), and therefore

\[
\frac{\partial^2 y}{\partial x \partial x'} = (A' + A), \quad = 2A \text{ if } A \text{ is symmetric.}
\]

This should remind you of the fact that \( d^2 y / dx^2 = 2a \) when \( y = ax^2 \).

Finally, we can use the principle that the shape of the derivative matrix follows that of the denominator of the derivative, to get derivatives with respect to matrices (as opposed to with respect to vectors):

**Example** If \( X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \), then

\[
\frac{\partial \det(X)}{\partial X} = \begin{bmatrix} \frac{\partial \det(X)}{\partial x_1} & \frac{\partial \det(X)}{\partial x_2} \\ \frac{\partial \det(X)}{\partial x_3} & \frac{\partial \det(X)}{\partial x_4} \end{bmatrix} = \begin{bmatrix} x_4 & -x_3 \\ -x_2 & x_1 \end{bmatrix}.
\]

Taking this further, consider the derivative \( \frac{\partial \ln(\det(X))}{\partial X} \). Applying the chain rule:

\[
\frac{\partial \ln(\det(X))}{\partial X} = \frac{1}{\det X} \begin{bmatrix} \frac{\partial \det(X)}{\partial x_1} & \frac{\partial \det(X)}{\partial x_2} \\ \frac{\partial \det(X)}{\partial x_3} & \frac{\partial \det(X)}{\partial x_4} \end{bmatrix} = \frac{1}{\det X} \begin{bmatrix} x_4 & -x_3 \\ -x_2 & x_1 \end{bmatrix} = (X^{-1})'
\]

since \( X^{-1} = \frac{1}{\det X} \begin{bmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{bmatrix} \).
This is true also for general \((n \times n)\) matrices. Note: to write down \(\ln(\det X)\) we ought to have \(\det X > 0\), which is true if \(X\) is positive-definite. Furthermore, if \(A\) is symmetric, then the inverse is symmetric, and we do not require the transposition of the inverse.

As a final example, consider the quadratic form \(y = x'Ax\), and consider differentiating it with respect to \(A\) (earlier we differentiated with respect to \(x\)). To simplify the exposition, we again focus on the \((2 \times 2)\) case, where

\[
y = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2
\]

We have

\[
\frac{\partial x'Ax}{\partial A} = \begin{bmatrix} \frac{\partial y}{\partial a_{11}} & \frac{\partial y}{\partial a_{12}} \\ \frac{\partial y}{\partial a_{21}} & \frac{\partial y}{\partial a_{22}} \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix} = \begin{bmatrix} x_1 \\\n \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = xx'
\]

There are other important forms. These will do for now.

**Summary**

\[
\frac{\partial c'x}{\partial x} = c \; ;
\]

\[
\frac{\partial x'Ax}{\partial x} = (A' + A)x \; , \; = 2Ax \; \text{if} \; A \; \text{is symmetric};
\]

\[
\frac{\partial^2 y}{\partial x \partial x'} = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_1 \partial x_n} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} & \cdots & \frac{\partial^2 y}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 y}{\partial x_n \partial x_1} & \frac{\partial^2 y}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_n^2} \end{bmatrix}, \; \frac{\partial x'Ax}{\partial x \partial x'} = (A' + A) \; , \; = 2A \; \text{if} \; A \; \text{is symm.}
\]

\[
\frac{\partial A}{\partial x'} = A \; , \; \frac{\partial \ln(\det X)}{\partial X} = (X^{-1})' \; , \; \frac{\partial x'Ax}{\partial A} = xx'
\]