INSTRUCTIONS TO CANDIDATES

1. The time allowed for this examination paper is **TWO hours**.

2. This examination paper contains a total of **FIVE** questions and comprises **FIVE** printed pages including this instruction sheet.

3. Answer **ALL** questions. You should aim to spend an average of 24 mins per question. Please be aware that the questions differ in level of difficulty; some questions may require more than 24 minutes (and others significantly less).

4. The total number of points in this examination is **50**. Each question carries **10** points.
1(a) (i) Write down the definition of the derivative. (1 pt)

(ii) Starting from the definition of the derivative, find the derivative of \( f(x) = \ln x \). (1 pt)

(iii) Show that \( \ln(1 + x) \approx x \), for small \( x \). (1 pt)

(iv) Let \( z = Ax^\alpha y^\beta \) where \( \alpha \) and \( \beta \) are positive constants. Show that if \( x \) and \( y \) both increase in value by 1 percent, then \( z \) increases in value by \( \alpha + \beta \) percent. (2 pts)

(b) Find the derivative of \( f(x) = x|x| \). Is \( f'(x) \) continuous at \( x = 0 \)? Find \( f''(x) \). Is \( f''(x) \) continuous at \( x = 0 \)? (5 pts)

2(a) The equation

\[
y^3 + 3x^2 y = 13
\]

implicitly defines a function \( y = f(x) \).

(i) Find \( f'(x) \). (1 pt)

(ii) Find the global maximum of this function. (2 pts)

(iii) Explain why this function cannot be concave everywhere. (2 pts)

(b) Let \( Q = F(K, L) \), and suppose

\[
\frac{dK}{dt} = \gamma Q, \quad Q = K^\alpha L, \quad \frac{dL}{dt} = \beta
\]

where \( \gamma \), \( \alpha \), and \( \beta \) are positive constants, \( \alpha < 1 \).

(i) Derive a differential equation that describes the evolution of \( K \) over time. (2 pts)

(ii) Show that a solution to this differential equation is

\[
K = \left[ (1-\alpha)\gamma \left( \frac{1}{2} \beta t^2 + L_0 t \right) + K_0^\alpha \right]^{\frac{1}{1-\alpha}}
\]

where \( L_0 = L(0) \) and \( K_0 = K(0) \). (1 pt)

(iii) Plot \( K(t) \) for \( t \geq 0 \), with \( K_0 > 0 \) and \( L_0 > 0 \). What if \( K_0 = 0 \) or \( L_0 = 0 \)? (2 pts)
3. Let
\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \]

(a) Write down two different Laplace expansions for \(|A|\), the determinant of \(A\), and show that both give the same result. \((2 \text{ pts})\)

(b) An important result for determinants is that \(|AB| = |A||B|\) for two square matrices of the same dimensions. Use this fact to explain why:

(i) multiplying a single row or column of \(A\) by a constant \(\alpha\) causes its determinant to be multiplied by \(\alpha\). \((1 \text{ pt})\)

(ii) interchanging two rows (or two columns) causes its determinant to change signs. \((1 \text{ pt})\)

(iii) adding a multiple of one row of \(A\) to another row does not change its determinant. \((1 \text{ pt})\)

(c) Find the determinant of the matrix
\[ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & 4 \\ 1 & 0 & 3 & -1 \\ -3 & -6 & -9 & -11 \end{bmatrix} \]
\((2 \text{ pts})\)

(d) Suppose \(A\) and \(B\) are two matrices such that \(AB\) is square and non-singular. Show that \((AB)^{-1} = B^{-1}A^{-1}\)? Under what conditions will this result not hold? \((3 \text{ pts})\)
4. Suppose that a firm’s problem is to

\[
\min_{K,L} \ rK + wL \ \text{subject to} \ F(K,L) = Q
\]

where \( r, w, \) and \( Q \) are taken as given, and where \( F(K,L) \) represents the firm’s production function. Assume that \( F'_1(K,L) > 0, \ F'_2(K,L) > 0 \).

(i) Write down the Lagrangian and first order conditions for this minimization problem. \( \text{(2 pts)} \)

Assume that the second-order condition for this minimization problem holds, and denote the solutions by \( K^* = K^*(r,w,Q) \) and \( L^* = L^*(r,w,Q) \). These are called the “factor demand functions”.

Define the firm’s cost function to be

\[
C^*(r,w,Q) = wK^* + rL^*
\]

In particular, define the partial derivative \( \partial C^* / \partial Q \) to be the firm’s “marginal cost”.

(ii) Show, using the appropriate Envelope Theorem, that \( \partial C^* / \partial Q = \lambda^* \). \( \text{(1 pt)} \)

(iii) Show that \( \partial K^* / \partial Q, \ \partial L^* / \partial Q, \) and \( \partial \lambda^* / \partial Q \) satisfies

\[
\begin{bmatrix}
-\lambda^* F''_{11} & -\lambda^* F''_{12} & -F'_1 \\
-\lambda^* F''_{12} & -\lambda^* F''_{22} & -F'_2 \\
-F'_1 & -F'_2 & 0
\end{bmatrix}
\begin{bmatrix}
\partial K^* / \partial Q \\
\partial L^* / \partial Q \\
\partial \lambda^* / \partial Q
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\]

What is the sign of the determinant of the \((3 \times 3)\) matrix on the left-hand-side? \( \text{(3 pts)} \)

(iv) Obtain an expression for \( \partial \lambda^* / \partial Q \) using Cramer’s rule. Under what condition will the firm’s marginal cost curve be upward sloping (with respect to \( Q \))? \( \text{(2 pts)} \)

(v) Suppose the firm’s production function is \( F(K,L) = AK^\alpha L^\beta \) where \( \alpha \) and \( \beta \) are positive constants. Is this production function homogenous, and if yes, to what degree? Show that the firm’s marginal cost curve is upward sloping if \( \alpha + \beta < 1 \). \( \text{(2 pts)} \)
5(a) Suppose, for given \( \alpha \), that the variables \( x \) and \( y \) satisfy the equations
\[
\begin{align*}
g(x, y, a) &= c_1 \\
h(x, y, a) &= c_2
\end{align*}
\]
simultaneously. Find expressions for \( \frac{dx}{da} \) and \( \frac{dy}{da} \). \hspace{1cm} (3 pts)

(b) (i) Taking \( a \) as fixed, the stationary points \( (x^*, y^*) \) of the function
\[
f(x, y, a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2 y^2.
\]
satisfy the first order conditions
\[
\begin{align*}
f'_1(x^*, y^*, a) &= x^* - 1 + ay^* = 0, \\
f'_2(x^*, y^*, a) &= ax^* - y^2 + 2a^2 y^* = 0.
\end{align*}
\]
Without first solving for \( x^* \) and \( y^* \), find expressions for \( \frac{dx^*}{da} \) and \( \frac{dy^*}{da} \). \hspace{1cm} (2 pts)

(ii) Let \( f^* = f(x^*, y^*, a) \), where \( (x^*, y^*) \) is a stationary point of \( f \), taking \( a \) as fixed. Prove that
\[
\frac{df^*}{da} = f'_1(x^*, y^*, a),
\]
and use this to obtain an expression for \( \frac{df^*}{da} \) in terms of \( x^* \), \( y^* \) and \( a \) for the function in (b.i). \hspace{1cm} (3 pts)

(iii) Where, in the \( x-y \) plane, is the function \( f(x, y, a) \) in (b.i) convex? \hspace{1cm} (2 pts)

~~~~~ Done! ~~~~