# Tests for Spatial Dependence under Distributional Misspecifications[1]

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**Abstract**

The most of the existing LM tests for spatial dependence are derived under the assumption that errors of the model are normally distributed and hence are sensitive to the departure from normality. In this paper we present score-based tests for identifying the existence of various forms of spatial dependence in cross-sectional models as well as in panel models. The proposed tests typically modify the existing ones based on the techniques of Kelejian and Prucha (2001) or Bera, Bilias and Yoon (2008) by allowing the error distributions to be non-normal; hence they are robust against distributional misspecifications. Monte Carlo simulation shows that the new tests compare favorably against the existing popular ones such as Moran-flavored tests (see, e.g., Florax and de Graaff, 2004), LM tests for error components in cross-sectional models (Anselin and Moreno, 2003), and LM tests for spatial dependence in panel models (Baltagi, et al., 2003, 2007).

**Key Words:** Distributional misspecification; Group interaction; Moran's $I$ Test; Spatial error components; Spatial panel models.

**JEL Classification:** C23, C5

## 1 Introduction.

The most of the existing tests for spatial dependence are derived under the assumption that innovations of the model are normally distributed. The earliest of this kind may be the popular Moran's $I$ test for spacial dependence based on a random sample (Moran, 1950). Subsequently, Cliff and Ord (1972, 1973, 1981) generalized Moran's $I$ test to linear models; Burridge (1980) derived an LM test for this model which turns out to be asymptotically equivalent to Cliff-Ord's test; Anselin (2001) derived an LM test for spatial error components model proposed by Kelejian and Robinson (1995); Baltagi, Song and Koh (2003) derived an LM test for spatial dependence in panel models with random effets; Baltagi, et al. (2007) derived an LM test for spatial dependence in panel models with random effects and serial correlation, etc. Anselin and Bera (1998) and Florax and de Graaff (2004) provide excellent reviews on tests of spatial dependence in linear models. While Anselin and Moreno (2003) recognized that the LM test for spatial error components of Anselin (2001) is not robust against distributional misspecifications and the spatial layouts, it is generally not clear how

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those normal-theory based LM tests perform under alternative distributions for the errors and under different spatial layouts.

Kelejian and Prucha (2001) considered the asymptotic distribution of the Moran's I type of tests under a fairly general set-up allowing the innovations to be non-normal and heteroscedastic. Their central limit theorem for linear-quadratic forms provides us a useful tool for the development of the robust tests in this paper.

In this paper we present score-based tests for identifying the existence of various forms of spatial dependence in cross-sectional models as well as in panel models. The proposed tests typically modify the existing ones based on the techniques of Kelejian and Prucha (2001) or Bera, Bilias and Yoon (2008) by allowing the error distributions to be non-normal; hence they are robust against distributional misspecifications. Monte Carlo simulation shows that the new tests compare favorably against the existing popular ones such as Moran-flavored tests (see, e.g., Florax and de Graaff, 2004), LM tests for spatial error components in cross-sectional models (Anselin and Moreno, 2003), and LM tests for spatial dependence in panel models (Baltagi, Song and Koh, 2003). The most contrasting results are found in LM tests for spatial error components where the original LM test can perform very poorly with empirical frequencies of rejection being as high as 20% for a 5% test, but the newly proposed test drastically improves the original LM test with the empirical sizes being generally quite close to their nominal level. One important point revealed in this study is that the spatial layout can have a significant impact on the performance of the classical LM tests.

Section 2 introduces a modified LM test for spatial error in a linear regression model. Section 3 introduces a robustified version of the LM test for spatial error components in a linear regression model. Section 4 introduces two modified versions of the LM test for spatial error in a panel regression model. Section 5 presents Monte Carlo results. Section 6 concludes the paper.

## 2 LM Tests for Spatial Error

The original form of Moran's $I$ test (Moran, 1950) is based a sample of observations $Y_1, Y_2, \cdots, Y_N$ on a variable of interest $Y$, which takes the form

$$I = \frac{N}{T_0} \frac{\sum_i \sum_j w_{ij}(Y_i - \bar{Y})(Y_j - \bar{Y})}{\sum_i (Y_i - \bar{Y})^2} \tag{1}$$

where $T_0 = \sum_i \sum_j w_{ij}$, $w_{ij}$'s are the elements of an $N \times N$ spatial weight matrix $W$, and $\bar{Y}$ is the average of the $Y_i$'s. If the observations are normal, then the null distribution of Moran's $I$ test statistic is shown to be asymptotic normal with mean $E(I) = -1/(N-1)$ and a finite variance (see the general expression below).

Cliff and Ord (1972) extended Moran's $I$ test to the case of a spatial linear model:

$$Y = X\beta + u \tag{2}$$

where $Y$ is an $N \times 1$ vector of observations on the response variable, $X$ is an $N \times k$ matrix containing the values of explanatory (exogenous) variables, and $u$ is an $N \times 1$ vector of disturbances, which follow either a spatial autoregressive (SAR) process or a spatial moving average (SMA) process. That is, $u = \lambda W u + \varepsilon$ or $u = \lambda W \varepsilon + \varepsilon$, where $W$ is defined similarly as above, $\lambda$ is the spatial parameter, and $\varepsilon$ is a vector of independent and identically distributed (iid) innovations. For this model, the hypothesis of no spatial error correlation corresponds to $H_0 : \lambda = 0$ vs $H_a : \lambda \neq 0$ or one sided alternative. For a row-standardized $W$ matrix (in this case $T_0 = N$), the Moran's $I$ test takes the form

$$I = \frac{\tilde{u}'W\tilde{u}}{\tilde{u}'\tilde{u}} \tag{3}$$

where $\tilde{u}$ is a vector of OLS residuals when regressing $Y$ on $X$. Under the assumptions that the elements of $\varepsilon$ are iid normal, the null distribution of $I$ is asymptotic normal with mean $\mathrm{E}(I) = \frac{1}{N-p}\mathrm{tr}(MW)$, and variance

$$\mathrm{Var}(I) = \frac{\mathrm{tr}(MWMW') + \mathrm{tr}((MW)^2) + [\mathrm{tr}(MW)]^2}{(N-k)(N-k+2)},$$

where $M = I_N - X(X'X)^{-1}X'$. Denoting $I^0 = (I - \mathrm{E}I)/\mathrm{Var}^{\frac{1}{2}}(I)$, then $I^0$ is asymptotically $N(0,1)$.

Burridge (1980) derived an LM test for $H_0$:

$$\mathrm{LM}_B = \frac{N}{\sqrt{T_1}} \frac{\tilde{u}'W\tilde{u}}{\tilde{u}'\tilde{u}} \tag{4}$$

where $T_1 = \mathrm{tr}(\mathrm{W}'\mathrm{W} + \mathrm{W}^2)$. Under the null hypothesis of no spatial error correlation, $LM \xrightarrow{D} N(0,1)$. Clearly, the $\mathrm{LM}_B$ statistic is proportional to Moran's $I$ statistic, and thus is expected to perform similarly to the standardized Moran's $I$ test statistic. However, our Monte Carlo simulation shows that it is not the case under certain spatial layouts. Some heuristic arguments for this are given after Theorem 1.

Both test statistics are derived under the assumption that errors are normally distributed. It is not clear how these tests perform when the error distributions are not normal. In Theorem 1 given below we show that they both behave well asymptotically under non-normality. The question remained is how they behave under finite samples. We now present a modified version of these tests allowing the error distributions to be non-normal. The following basic regularity conditions are necessary for studying the asymptotic behavior of the test statistics.

**Assumption A1:** *The innovations $\{\varepsilon_i\}$ are iid with mean zero, variance $\sigma_\varepsilon^2$, and excess kurtosis $\kappa_\varepsilon$. Also, the moment $E|\varepsilon_i|^{4+\eta}$ exists for some $\eta > 0$.*

**Assumption A2:** *The elements of $W$ are at most of order $h_N^{-1}$ uniformly for all i. j, with the rate sequence $\{h_n\}$, bounded or divergent, satisfying $h_N/N \to 0$ as $N$ goes to infinity. As normalizations, the diagonal elements $w_{ii} = 0$, and $\sum_j W_{ij} = 1$ for all $i$.*

**Assumption A3:** $\lim_{N\to\infty} \frac{1}{N}X'X$ *exists and is nonsingular.*

The Assumption A1 corresponds to one assumption of Kelejian and Prucha (2001) for their central limit theorem of linear-quadratic forms. Assumption A2 corresponds to one assumption in Lee (2004a) which identifies the different types of spatial dependence. Typically, one type of spatial dependence corresponds to the case where each unit has fixed number of neighbors and in this case $h_N$ is bounded, the other type of spatial dependence corresponds to the case where the number of neighbors of each spatial unit grows as $N$ goes to infinity, and in this case $h_N$ is divergent. To limit the spatial dependence to a manageable degree, it is thus required that $h_N/N \to 0$ as $N \to \infty$.

**Theorem 1:** *Under Assumptions A1-A3, the modified LM test for testing $H_0 : \lambda = 0$ vs $H_a : \lambda \neq 0$ (or $\lambda < 0$, or $\lambda > 0$) takes the form*

$$\text{LM}_B^* = \frac{\tilde{u}'W\tilde{u}/\tilde{u}'\tilde{u} - S_1}{(\tilde{\kappa}_\varepsilon S_2 + S_3)^{\frac{1}{2}}/N} \tag{5}$$

*where $S_1 = \frac{1}{N-k}\text{tr}(WM), S_2 = \sum_{i=1}^N a_{ii}^2$, and $S_3 = \text{tr}(AA' + A^2)$, $A = MWM - S_1M$, $a_{ii}$ are the diagonal elements of $A$, and $\tilde{\kappa}_\varepsilon$ is the excess sample kurtosis of $\tilde{u}$. Under $H_0$, we have (i) $LM_B^* \xrightarrow{D} N(0,1)$; and (ii) The three test statistics, $I^0$, $\text{LM}_B$ and $\text{LM}_B^*$ are asymptotically equivalent.*

**Proof:** We have $\tilde{u}'W\tilde{u} - S_1\tilde{u}'\tilde{u} = \tilde{u}'(W - S_1I_N)\tilde{u} = u'M(W - S_1I_N)Mu = u'Au$. By Lemma 1, $\text{E}u'Au = \sigma_\varepsilon^2\text{tr}A = 0$ and $\text{Var}(u'Au) = \sigma_\varepsilon^4\kappa_\varepsilon\sum_{i=1}^n a_{ii}^2 + \sigma_\varepsilon^4[\text{tr}(AA') + \text{tr}(A^2)]$. It is easy to show that the column sums of the matrix $A$ is uniformly bounded. By the central limit theorem of linear-quadratic forms of Kelejian and Prucha (2001), $u'Au$ is asymptotically normal, and hence,

$$\frac{u'Au}{\sigma_\varepsilon^2(\kappa_\varepsilon S_2 + S_3)^{\frac{1}{2}}} = \frac{\tilde{u}'W\tilde{u} - S_1\tilde{u}'\tilde{u}}{\sigma_\varepsilon^2(\kappa_\varepsilon S_2 + S_3)^{\frac{1}{2}}} \xrightarrow{D} N(0,1).$$

Now, $\tilde{\sigma}_\varepsilon^2 = \tilde{u}'\tilde{u}/N \xrightarrow{P} \sigma_\varepsilon^2$ and $\tilde{\kappa}_\varepsilon = \frac{1}{N\tilde{\sigma}_\varepsilon^4}\sum_{i=1}^n \tilde{u}_i^4 - 3 \xrightarrow{P} \kappa_\varepsilon$, where $\tilde{\kappa}_\varepsilon$ is the sample excess kurtosis of the OLS residuals. The result of the theorem thus follows from Slutsky's theorem by replacing $\sigma_\varepsilon$ by $\tilde{\sigma}_\varepsilon$ and $\kappa_\varepsilon$ by $\tilde{\kappa}_\varepsilon$.

To prove the asymptotic equivalence of the three test statistics, we note, from Lemma 2 (i) in Appendix that $\text{tr}(WM) = O(1)$, and thus $S_1 = O(N^{-1})$. The elements of $W^* = W - S_1I_N$ are uniformly $O(h_N^{-1})$. Lemma 2 (vi) gives $S_2 = \sum_{i=1}^N a_{ii}^2 = \sum_{i=1}^N (w_{ii}^*)^2 + O(h_N^{-1}) = O(h_N^{-1})$. Lemma 2 (ii) and (iii) lead to $S_3 - T_1 = o(1)$. Since the elements of $W$ are uniformly $O(h_N^{-1})$ and the row sums of $W$ are uniformly bounded, it follows that the elements of $WW'$ and $W^2$ are uniformly $O(h_N^{-1})$. Hence, $S_3$ and $T_1$ are both $O(N/h_N)$. Furthermore, $\tilde{\kappa}_\varepsilon = O_p(1)$. These give $S_1/(\tilde{\kappa}_\varepsilon S_2 + S_3)^{\frac{1}{2}} = O_p(\sqrt{h_N/N}) = o_p(1)$, which leads to $\text{LM}_B \sim \text{LM}_B^*$. Similarly, one shows $\text{Var}(I) \sim T_1$, and hence $\text{LM}_B \sim I^o$. *Q.E.D.*

Although both Moran's $I$ and the $\text{LM}_B$ test statistics are derived under the assumption that the innovations are normally distributed, Theorem 1 shows that they are asymptotically equivalent to the modified LM test derived under relaxed conditions on error distributions. This means that all the three tests are robust against the distributional misspecification when sample sizes are large. One question that still remains is that whether the three tests behave similarly under finite samples. Following heuristic arguments show that their finite sample performance may be different.

As it can be seen that the major difference between $\text{LM}_B$ and $\text{LM}_B^*$ is that the latter puts a correction on the mean of the key quantity $\tilde{u}'W\tilde{u}/\tilde{u}'\tilde{u}$. This correction may quickly become negligible as sample size goes large under certain spatial layouts, but not under the others. From the proof of the theorem, we see that the mean correction factor is

$$\frac{S_1}{(\tilde{\kappa}_\varepsilon S_2 + S_3)^{\frac{1}{2}}} = O_p((h_N/N)^{\frac{1}{2}}),$$

which shows that the magnitude of mean correction depends on the ratio $(h_N/N)^{\frac{1}{2}}$. For example, when $h_N = N^{0.8}$, $(h_N/N)^{\frac{1}{2}} = N^{-0.1}$. Thus, if $N = 20, 100$, and 1000, $N^{-0.1} = 0.74, 0.63$, and 0.50. This shows that the mean of $\text{LM}_B$ can differ from the means of $\text{LM}_B^*$ and $I^o$ by 0.74 when $N = 20$, and 0.50 when $N = 1000$. Note that the situations leading to $h_N = N^{0.8}$ or similar may be the spatial layouts constructed under large group interactions where the group sizes are large and the number of groups is small.[2] Our theory shows that in this situation, the regular LM test may be misleading. Monte Carlo simulations presented in Section 5 confirm the above discussions.

## 3 Robust LM Test for Spatial Error Components

While the bulk of the spatial econometrics literature is devoted to models where the spatial dependence is expressed in the form of a SAR or SMA process, an alternative, called spatial error components (SEC) model, was proposed by Kelejian and Robinson (1995).

$$Y = X\beta + u \quad \text{with } u = W\nu + \varepsilon \tag{6}$$

where $\nu$ is an $N \times 1$ vector of errors that together with $W$ incorporate the spatial dependence, and all the other quantities have the same meaning as in Model (1). The two vectors of error components $\nu$ and $\varepsilon$ are assumed to be independent, each with iid elements of mean zero and variances $\sigma_\nu^2$ and $\sigma_\varepsilon^2$, respectively. So, in this model the null hypothesis of no spatial effect can be either $H_0 : \sigma_\nu^2 = 0$, or $\theta = \sigma_\nu^2/\sigma_\varepsilon^2 = 0$. The alternative hypothesis can only be one-sided, i.e., $H_a : \sigma_\nu^2 > 0$ as $\sigma_\nu^2$ is non-negative. Anselin (2001) derived an LM test based on the assumptions that both error components are normally distributed. The test is of the form

$$\text{LM}_{\text{SEC}} = \frac{\tilde{u}'WW'\tilde{u}/\tilde{\sigma}_\varepsilon^2 - T_1}{(2T_2 - \frac{2}{N}T_1^2)^{\frac{1}{2}}} \tag{7}$$

where $\tilde{\sigma}_\varepsilon^2 = \tilde{u}'\tilde{u}/N$, $\tilde{u}$ is the vector of OLS residuals, $T_1 = \text{tr}(WW')$ and $T_2 = \text{tr}(WW'WW')$. Under $H_0$, the positive part of $\text{LM}_{\text{SEC}}$ converges to that of $N(0, 1)$. This means that the above one sided test can be carried out as per normal. Alternatively, if the squared version $LM_{\text{SEC}}^2$ is used, the reference null distribution of the test statistic for testing this one sided test is a chi-square mixture. See Verbeke and Molenberghs (2003) for a detailed discussion on tests where the parameter value under the null hypothesis falls on the boundary of parameter space. Anselin and Moreno (2003) provide Monte Carlo evidence for the finite sample performance of $\text{LM}_{\text{sec}}$ and find that it is not robust against distributional misspecifications. We now present a robustified version of the above LM test statistic.

[2] See, e.g., Lee (2007) for detailed discussions on spatial models with group interactions.

**Theorem 2:** *If $W$, $\{\varepsilon_i\}$ and $X$ of Model (2) satisfy the Assumptions A1-A3, then a robust LM test statistic for testing $H_0 : \sigma_\nu^2 = 0$ vs $H_a : \sigma_\nu^2 > 0$ takes the form*

$$\mathrm{LM}^*_{\mathrm{SEC}} = \frac{\tilde{u}'WW'\tilde{u}/\tilde{\sigma}^2_\varepsilon - S_1}{(\tilde{\kappa}_\varepsilon S_2 + S_3)^{\frac{1}{2}}} \tag{8}$$

*where $S_1 = \frac{N}{N-k}\mathrm{tr}(WW'M), S_2 = \sum_i a_{ii}^2$ with $a_{ii}$ being the diagonal elements of $A = M'WW'M - \frac{1}{N}S_1 M$, $S_3 = \mathrm{tr}(AA' + A^2)$, and $\tilde{\kappa}_\varepsilon$ is the excess sample kurtosis of $\tilde{u}$. Under $H_0$, (i) the positive part of $LM^*_{\mathrm{SEC}}$ converges to that of $N(0,1)$, and (ii) $\mathrm{LM}^*_{\mathrm{SEC}}$ is asymptotically equivalent to $\mathrm{LM}_{\mathrm{SEC}}$ when errors are normal.*

**Proof:** We have $\tilde{u}'WW'\tilde{u} - \tilde{\sigma}^2_\varepsilon S_1 = \tilde{u}'(WW' - \frac{1}{N}S_1 I_N)\tilde{u} = u'M(WW' - \frac{1}{N}S_1 I_N)Mu$. It follows from the central limit theorem of linear-quadratic forms of Kelejian and Prucha (2001) and Lemma 1 of Appendix that this quantity is asymptotic normal with mean zero and variance $\sigma^4_\varepsilon(\kappa_\varepsilon S_2 + S_3)$. The rest of the proof is similar to that of Theorem 1.

*Q.E.D.*

To have a feeling for the finite sample difference between $\mathrm{LM}^*_{\mathrm{SEC}}$ and $\mathrm{LM}_{\mathrm{SEC}}$, we note that $S_1 \sim T_1$. Lemma 2 (iv) of Appendix shows that $S_1 = O(N/h_N^2)$, and Lemma 2 (ii)-(iii) shows that $S_3 = O(N/h_N)$. Take for example, $h_N = N^{0.2}$. Then, we have $S_1 = O(N^{0.6})$ and $S_3 = O(N^{0.8})$. With $\tilde{\kappa}_\varepsilon$ being $O_p(1)$, it follows that the excess kurtosis may have significant impact on the variance and hence on the test statistic. The Monte Carlo simulation results given in Section 5 indeed confirm this observation.

# 4 Modified LM Test for Spatial Error in Panel Models

Baltagi, Song and Koh (2003, BSK here after) introduced an LM test for SAR error in the presence of random individual effects in panel models. The model they considered is of the form:

$$Y_t = X_t\beta + u_t, \;\text{ with }\; u_t = \mu + \varepsilon_t \;\text{ and }\; \varepsilon_t = \lambda W \varepsilon_t + v_t, t = 1, \cdots, T, \tag{9}$$

where $Y_t, u_t$ and $v_t$ are all $N \times 1$ ($t$-dependent) random vectors, $\mu$ is an $N \times 1$ random vector representing individual specific effects, $X_t$ is an $N \times k$ matrix containing the values of explanatory variables at time period $t$, $\beta$ is a $k \times 1$ vector of regression coefficients, $W$ is an $N \times N$ spatial weight matrix, and $\lambda$ is the spatial parameter. The vectors $u$ and $v$ are assumed to be independent, each with iid elements of means zero and variances $\sigma^2_\mu$ and $\sigma^2_v$, respectively. Stacking the vectors $(Y_t, u_t, v_t)$ and the matrix $X_t$, the model is written in matrix form:

$$Y = X\beta + u, \;\; u = (\iota_T \otimes I_N)\mu + (I_T \otimes B^{-1})v \tag{10}$$

where $\iota_m$ represents an $m \times 1$ vector of ones, $I_m$ represents an $m$ dimensional identity matrix and $B = I_N - \lambda W$. The Gaussian log-likelihood is

$$\ell(\beta, \sigma^2_v, \sigma^2_\mu, \lambda) = -\frac{NT}{2}\log(2\pi\sigma^2_v) - \frac{1}{2}\log|\Sigma| - \frac{1}{2\sigma^2_v}u'\Sigma^{-1}u \tag{11}$$

where $\Sigma = \phi(J_T \otimes I_N) + I_T \otimes (B'B)^{-1}$, $\Sigma^{-1} = \bar{J}_T \otimes (T\phi I_N + (B'B)^{-1})^{-1} + E_T \otimes (B'B)^{-1}$, $\phi = \sigma^2_\mu/\sigma^2_v, J_T = \iota_T\iota_T', \bar{J}_T = \frac{1}{T}J_T$, and $E_T = I_T - \bar{J}_T$. Note that $\Sigma = \frac{1}{\sigma^2_v}\mathrm{E}(uu')$.

Maximizing (11) gives the maximum likelihood estimators (MLE) of the model parameters if the error components are normally distributed, otherwise quasi-maximum likelihood estimators (QMLE). The BSK's LM test of the hypothesis $H_0 : \lambda = 0$, under the assumption that error components are normally distributed, takes the following form:

$$\mathrm{LM}_{\mathrm{BSK}} = \frac{\tilde{u}'[\tilde{\rho}^2(\bar{J}_T \otimes W^o) + E_T \otimes W^o]\tilde{u}}{\tilde{\sigma}_v^2[(T-1+\tilde{\rho}^2)b]^{\frac{1}{2}}} \tag{12}$$

where $W^o = \frac{1}{2}(W + W')$, $\rho = \sigma_v^2/(T\sigma_\mu^2 + \sigma_v^2)$, $b = \mathrm{tr}(W^2 + W'W)$, $\tilde{\rho}$ and $\tilde{\sigma_v^2}$ are the constrained MLEs under $H_0$ of $\rho$ and $\sigma_v^2$, respectively, and $\tilde{u}$ is the vector of constrained MLE residuals.

A nice feature of the LM test is that it requires only the estimates of the model under $H_0$. However, even under $H_0$, the constrained QMLE of $\rho$ (or $\phi$) does not posses an explicit expression, meaning that $\tilde{\rho}$ has to be obtained through a numerical optimization. The detail is as follows. Under the $H_0$, the partially maximized (with respect to $\beta$ and $\sigma_v^2$) log-likelihood is

$$\ell_{\max}(\rho) = \text{constant} - \frac{NT}{2}\log\tilde{\sigma}_v^2(\rho) + \frac{N}{2}\log\rho \tag{13}$$

where $\tilde{\sigma}_v^2(\rho) = \frac{1}{NT}\tilde{u}'(\rho)\Sigma^{-1}\tilde{u}(\rho)$, $\tilde{u}(\rho) = Y - X\tilde{\beta}(\rho)$, $\tilde{\beta}(\rho) = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y$, and $\Sigma^{-1} = \rho\bar{J}_T \otimes I_N + E_T \otimes I_N$. Maximizing (13) gives the QMLE $\tilde{\rho}$ of $\rho$, which in turn gives the QMLEs $\tilde{\beta} = \tilde{\beta}(\tilde{\rho})$, $\tilde{\sigma}_v^2 = \tilde{\sigma}_v^2(\tilde{\rho})$, $\tilde{\Sigma}^{-1} = \tilde{\rho}\bar{J}_T \otimes I_N + E_T \otimes I_N$, and $\tilde{u} = \tilde{u}(\tilde{\rho})$.

We now present modified versions of $\mathrm{LM}_{\mathrm{BSK}}$ test, aiming for a better finite sample performance. The idea is that the numerator of $\mathrm{LM}_{\mathrm{BSK}}$ does not have a zero mean at finite samples, thus causing biasedness in the test. Finite sample variance of it may also need to be corrected. Lemma 3 given in Appendix is essential in deriving the modified test statistics. Some basic regularity conditions are listed below.

**Assumption B1:** *The random effects $\{\mu_i\}$ are iid with mean zero, variance $\sigma_\mu^2$, and excess kurtosis $\kappa_\mu$. The idiosyncratic errors $\{v_{it}\}$ are iid with mean zero, variance $\sigma_v^2$, and excess kurtosis $\kappa_v$. Also, the moments $E|\mu_i|^{4+\eta_1}$ and $E|v_{it}|^{4+\eta_2}$ exist for some $\eta_1, \eta_2 > 0$.*

**Assumption B2:** *The elements of $W$ are at most of order $h_N^{-1}$ uniformly for all i. j, with the rate sequence $\{h_n\}$, bounded or divergent, satisfying $h_N/N \to 0$ as $N$ goes to infinity. As normalizations, the diagonal elements $w_{ii} = 0$, and $\sum_j W_{ij} = 1$ for all i.*

**Assumption B3:** *Both $\lim_{N\to\infty} \frac{1}{N}X'X$ and $\lim_{N\to\infty} \frac{1}{N}X'(\bar{J}_T \otimes I_N)X$ exist and are nonsingular.*

Now, define $A(\rho) = \rho^2(\bar{J}_T \otimes W^o) + E_T \otimes W^o$ and $M(\tilde{\rho}) = I_{NT} - X(X'\tilde{\Sigma}^{-1}X)^{-1}X'\tilde{\Sigma}^{-1}$. Putting $C(\rho) = M'(\rho)A(\rho)M(\rho)$, we have the following theorem.

**Theorem 3:** *Under Assumptions B1-B3, for testing $H_0; \lambda = 0$, the mean-corrected LM test for spatial SAR in panel random effects model takes the form:*

$$\mathrm{LM}_{\mathrm{BSK}}^* = \frac{\tilde{u}'\tilde{A}\tilde{u}/\tilde{\sigma}_v^2 - \mathrm{tr}(\tilde{\Sigma}\tilde{C})}{[(T-1+\tilde{\rho}^2)b]^{\frac{1}{2}}}, \tag{14}$$

*where $\tilde{A} = A(\tilde{\rho})$ and $\tilde{C} = C(\tilde{\rho})$. Under $H_0$, $\text{LM}^*_{\text{BSK}} \xrightarrow{D} N(0,1)$. The modified LM test which corrects both the mean and variance takes the form:*

$$\text{LM}^{**}_{\text{BSK}} = \frac{\tilde{u}'\tilde{A}\tilde{u}/\tilde{\sigma}_v^2 - \text{tr}(\tilde{\Sigma}\tilde{C})}{[\tilde{\phi}^2\tilde{\kappa}_\mu \tilde{a}_1'\tilde{a}_1 + \tilde{\kappa}_v\tilde{a}_2'\tilde{a}_2 + 2\text{tr}(\tilde{\Sigma}\tilde{C}\tilde{\Sigma}\tilde{C})]^{\frac{1}{2}}}, \tag{15}$$

*where $\tilde{\kappa}_\mu$ is the sample excess kurtosis of $\tilde{\mu} = (\bar{J}_T \otimes I_N)\tilde{u}$, $\tilde{\kappa}_v$ is the sample excess kurtosis of $\tilde{v} = \tilde{u} - (\iota \otimes I_N)\tilde{\mu}$, $\tilde{a}_1 = \text{diagv}[(\iota_T' \otimes I_N)\tilde{A}(\iota_T \otimes I_N)]$, and $\tilde{a}_2 = \text{diagv}(\tilde{A})$ with 'diagv' being an operator which forms a column vector from the diagonal elements of the corresponding matrix. Under $H_0$, $\text{LM}^{**}_{\text{BSK}} \xrightarrow{D} N(0,1)$. Finally, the three LM tests (12), (14) and (15) are asymptotically equivalent.*

**Proof:** We have $\tilde{u} = Y - X\tilde{\beta} = Y - X(X'\tilde{\Sigma}^{-1}X)^{-1}X'\tilde{\Sigma}^{-1}Y \equiv M(\tilde{\rho})Y$. The numerator of $\text{LM}_{\text{BSK}}$ becomes $\tilde{u}'A(\tilde{\rho})\tilde{u} = Y'M'(\tilde{\rho})A(\tilde{\rho})M(\tilde{\rho})Y = u'M'(\tilde{\rho})A(\tilde{\rho})M(\tilde{\rho})u$. It is easy to see that the last term above is asymptotically equivalent to $u'M'(\rho)A(\rho)M(\rho)u$, which can be decomposed to the following three terms

$$\mu'(\iota_T' \otimes I_N)M(\rho)A(\rho)M(\rho)(\iota_T \otimes I_N)\mu + v'M(\rho)A(\rho)M(\rho)v + \mu'(\iota_T' \otimes I_N)M(\rho)A(\rho)M(\rho)v,$$

which are either independent or asymptotically independent. Thus, the asymptotic normality of $u'M'(\rho)A(\rho)M(\rho)u$ follows from the central limit theorem for linear-quadratic forms of Kelejian and Prucha (2001). The mean and variance of it can be easily be obtained from Lemma 3 of Appendix, which are $\text{E}(u'C(\rho)u) = \sigma_v^2\text{tr}(\Sigma C(\rho))$, giving the mean-corrected LM test, and

$$\text{Var}(u'C(\rho)u) = \sigma_v^2[\phi^2\kappa_\mu a_1'a_1 + \kappa_v a_2'a_2 + 2\text{tr}(\Sigma C(\rho)\Sigma C(\rho))],$$

which together with the mean correction gives the modified LM test which corrects both the mean and variance. The rest of the proof follows closely from the proof of Theorem 1. *Q.E.D.*

We note that the results of Theorem 3 are quite similar to the results of Theorem 1. Thus, it is expected that the three statistics will contrast themselves in a similar way. The Monte Carlo results given in Section 5 indeed reveal this, though the three statistics contrast themselves in a lesser degree. Nevertheless, the main message is that it is sometimes very necessary to do finite sample corrections on the supposingly robust LM test.

We also note that the asymptotic equivalence of the three test statistics can also be seen using the technique of adjusting Rao's score test for distributional misspecification, outlined in Bera, Bilias and Yoon (2008).

# 5 Monte Carlo Results

The finite sample performance of the test statistics introduced in this paper is evaluated based on a series of Monte Carlo experiments under a number of different error distributions and a number of different spatial layouts. Also comparison are made between the newly introduced tests and the standard ones to see the improvement of the new tests in the situations where there is a distributional misspecification.

Two general spatial layouts are considered in the Monte Carlo experiments and they are applied to all the test statistics involved in the experiments. One is based on the Rook contiguity and the other is based on the notion of group or social interactions with the number of groups $G = N^\delta$ where $0 < \delta < 1$. In the former case, the number of neighbors for each spatial unit stays at 2 to 4 and does not change when sample size $N$ increases, whereas in the later case, the number of neighbors for each spatial unit increases with the increase of sample size but with different rates.

The detail for generating the $W$ matrix under Rook contiguity is as follows: (i) index the $N$ spatial units by $\{1, 2, \cdots, N\}$, randomly permute these indices and then allocate them into a lattice of $r \times m(\geq N)$ squares, (ii) let $W_{ij} = 1$ if the index $j$ is in a square which is on immediate left, or right, or above, or below the square which contains the index $i$, otherwise $W_{ij} = 0$, and (iii) divide each element of $W$ by its row sum.

To generate the $W$ matrix according to the group interaction scheme, (i) calculate the number of groups according to $G = \text{Round}(N^\delta)$, and the approximate average group size $m = N/G$, (ii) generate the group sizes $(n_1, n_2, \cdots, n_G)$ according to a discrete uniform distribution from $m/2$ to $3m/2$, (iii) adjust the group sizes so that $\sum_{i=1}^{G} n_i = N$, and (iv) define $W = \text{diag}\{W_i/(n_i - 1), i = 1, \cdots, G\}$, a matrix formed by placing the submatrices $W_i$ along the diagonal direction, where $W_i$ is an $n_i \times n_i$ matrix with ones on the off-diagonal positions and zeros on the diagonal positions. In our Monte Carlo experiments, we choose $\delta = 0.2, 0.5$, and $0.8$, representing respectively the situations where (i) there are few groups and many spatial units in a group, (ii) the number of groups and the sizes of the groups are of the same magnitude, and (iii) there are many groups of few elements in each. Clearly, under Rook contiguity, $h_N$ defined in the theorems is bounded, whereas under group interaction $h_N$ is divergent with rates $N^{1-\delta}$.

For the error distributions, the reported Monte Carlo results correspond to the following three: (i) standard normal, (ii) mixture normal standardized to have mean zero and variance 1, and (iii) log-normal also standardized to have mean zero and variance one. The standardized normal-mixture variates are generated according to

$$u_i = ((1 - \xi_i)Z_i + \xi_i \sigma Z_i)/(1 - p + p * \sigma^2)^{0.5},$$

where $\xi$ is a Bernoulli random variable with probability of success $p$ and $Z_i$ is standard normal independent of $\xi$. The parameter $p$ in this case also represents the proportion of mixing the two normal populations. In our experiments, we choose $p = 0.05$, meaning that 95% of the random variates are from standard normal and the remaining 5% are from another normal population with standard deviation $\sigma$. We choose $\sigma = 10$ to simulate the situation where there are gross errors in the data. The standardized lognormal random variates are generated according to

$$u_i = [\exp(Z_i) - \exp(0.5)]/[\exp(2) - \exp(1)]^{0.5}.$$

This gives an error distribution that is both skewed and leptokurtic. The normal mixture gives an error distribution that is still symmetric like normal but leptokurtic. Other non-normal distributions, such as normal-gamma mixture and chi-squared, are also considered and the results are available from the author upon request. All the Monte Carlo experiments are based on 10,000 Monte Carlo samples.

## 5.1 Moran's I, LM and modified LM tests

The performance of the modified LM test statistic ($\text{LM}_B^*$) introduced in Section 2 is compared with the standardized Moran's $I^0$ and the LM statistics of Buridge (1980) ($\text{LM}_B$). The Monte Carlo experiments are carried out based on the following data generating process:

$$Y_i = \beta_0 + X_{1i}\beta_1 + X_{2i}\beta_2 + u_i$$

where $X_{1i}$'s are drawn from $10U(0,1)$ and $X_{2i}$'s are drawn from $5N(0,1)+5$. Both are treated as fixed in the experiments. The parameters $\beta = \{5, 1, 0.5\}'$. Six different sample sizes are considered, i.e., $N = 20, 50, 100, 200, 500$, and 1000.

**Size of the tests.** Empirical sizes of the three tests are summarized in Table 1((a)-(d)). From the results we see that the $\text{LM}_B^*$ and $I^0$ perform similarly in almost all the situations. There are situations where the $\text{LM}_B$ performs poorly.

In the situation when the spatial layout is Rook, the results show that the three tests perform similarly, though $\text{LM}_B$ can perform less satisfactorily when the sample sizes are not large. The most contrasting results are obtained from the cases with group interaction where there are few groups and each group contains many spatial units, e.g., $G = N^{0.2}$. In this case, the empirical means, SDs, and the rejection frequencies of $\text{LM}_B$ test can all be far below their nominal values (0, 1, and 0.05). For samples sizes ranging from 20 to 1000, the empirical mean of $\text{LM}_B$ ranges from $-0.7$ to $-0.35$, empirical SD changes from 0.45 to 0.88, and the empirical rejection rates ranges from 0.0004 to 0.0264. In contrast, the Moran's $I^0$ and the modified LM test $\text{LM}_B^*$ still perform equally well, although with a slight edge toward the latter. When $G$ is changed to $N^{0.5}$ and $N^{0.8}$, the performance of $\text{LM}_B$ improved. The Monte Carlo results are consistent with the theoretical findings given in and after Theorem 1, i.e., the mean bias of the LM test cannot be ignored when each spatial unit has many neighbors.

**Power of the tests.** Empirical frequencies of rejection of the three tests are plotted in Figures 1-3 against the values of $\lambda$ (horizontal line). Results show that $\text{LM}_B^*$ almost alway has higher power than the other two. The difference diminishes when sample size is increased. Figure 3 reveals an interesting phenomenon: in the situation where $G = N^{0.5}$, the power of the tests can be very low when $\lambda$ is begative and the sample size is not large. It requires a very large sample (e.g., 1000) for the tests to be able to detect a negative spatial dependence. However, the tests behave similarly to the other spatial two structures.

**Table 1a.** Empirical Means, SDs and Rejection Frequencies at 5% Level: Rook

| | | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | Test | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| 20 | 1 | 0.0031 | 1.0141 | 0.0526 | 0.0021 | 0.9013 | 0.0352 | 0.0288 | 0.9196 | 0.0329 |
| | 2 | -0.3526 | 0.9138 | 0.0441 | -0.3535 | 0.8122 | 0.0267 | -0.3294 | 0.8287 | 0.0207 |
| | 3 | 0.0034 | 1.1306 | 0.0822 | 0.0030 | 0.9924 | 0.0555 | 0.0318 | 1.0118 | 0.0522 |
| 50 | 1 | 0.0044 | 1.0086 | 0.0516 | -0.0042 | 0.8379 | 0.0292 | 0.0200 | 0.9070 | 0.0347 |
| | 2 | -0.1112 | 0.9756 | 0.0452 | -0.1195 | 0.8105 | 0.0262 | -0.0961 | 0.8774 | 0.0272 |
| | 3 | 0.0045 | 1.0510 | 0.0611 | -0.0044 | 0.8673 | 0.0337 | 0.0207 | 0.9407 | 0.0399 |
| 100 | 1 | -0.0168 | 0.9979 | 0.0501 | 0.0011 | 0.8737 | 0.0368 | -0.0053 | 0.9289 | 0.0381 |
| | 2 | -0.1454 | 0.9755 | 0.0471 | -0.1279 | 0.8541 | 0.0355 | -0.1342 | 0.9080 | 0.0322 |
| | 3 | -0.0172 | 1.0183 | 0.0537 | 0.0011 | 0.8865 | 0.0388 | -0.0054 | 0.9447 | 0.0408 |
| 200 | 1 | -0.0133 | 0.9956 | 0.0491 | -0.0097 | 0.9111 | 0.0453 | 0.0082 | 0.9477 | 0.0414 |
| | 2 | -0.0937 | 0.9867 | 0.0479 | -0.0901 | 0.9030 | 0.0445 | -0.0723 | 0.9392 | 0.0374 |
| | 3 | -0.0135 | 1.0057 | 0.0508 | -0.0098 | 0.9192 | 0.0465 | 0.0083 | 0.9564 | 0.0424 |
| 500 | 1 | 0.0035 | 1.0032 | 0.0504 | 0.0032 | 0.9695 | 0.0505 | 0.0068 | 0.9577 | 0.0416 |
| | 2 | -0.0573 | 0.9998 | 0.0506 | -0.0576 | 0.9662 | 0.0507 | -0.0540 | 0.9545 | 0.0405 |
| | 3 | 0.0035 | 1.0072 | 0.0516 | 0.0032 | 0.9731 | 0.0514 | 0.0068 | 0.9613 | 0.0427 |
| 1000 | 1 | -0.0267 | 0.9926 | 0.0502 | 0.0067 | 0.9841 | 0.0498 | 0.0049 | 0.9874 | 0.0445 |
| | 2 | -0.0665 | 0.9909 | 0.0502 | -0.0331 | 0.9824 | 0.0505 | -0.0349 | 0.9857 | 0.0446 |
| | 3 | -0.0268 | 0.9946 | 0.0505 | 0.0067 | 0.9860 | 0.0501 | 0.0049 | 0.9893 | 0.0452 |

Note: Test 1 = $I^0$, Test 2 = $\text{LM}_B$, Test 3 = $\text{LM}_B^*$.

**Table 1b.** Empirical Means, SDs and Rejection Frequencies at 5% Level: Group, $G = N^{0.2}$

| | | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | Test | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| 20 | 1 | 0.0125 | 1.0214 | 0.0608 | -0.0041 | 0.9727 | 0.0517 | -0.0084 | 0.9666 | 0.0519 |
| | 2 | -0.6912 | 0.4727 | 0.0009 | -0.6989 | 0.4501 | 0.0009 | -0.7009 | 0.4473 | 0.0004 |
| | 3 | 0.0138 | 1.1439 | 0.0728 | -0.0076 | 1.0365 | 0.0562 | -0.0116 | 1.0327 | 0.0560 |
| 50 | 1 | 0.0065 | 1.0102 | 0.0546 | 0.0180 | 0.8844 | 0.0419 | 0.0027 | 0.8999 | 0.0463 |
| | 2 | -0.5091 | 0.7018 | 0.0132 | -0.5011 | 0.6144 | 0.0066 | -0.5117 | 0.6252 | 0.0085 |
| | 3 | 0.0067 | 1.0527 | 0.0593 | 0.0187 | 0.9091 | 0.0439 | 0.0028 | 0.9288 | 0.0479 |
| 100 | 1 | 0.0056 | 1.0081 | 0.0557 | 0.0114 | 0.8817 | 0.0380 | 0.0185 | 0.9270 | 0.0438 |
| | 2 | -0.3902 | 0.8379 | 0.0235 | -0.3855 | 0.7328 | 0.0099 | -0.3795 | 0.7705 | 0.0142 |
| | 3 | 0.0057 | 1.0287 | 0.0579 | 0.0116 | 0.8986 | 0.0398 | 0.0189 | 0.9453 | 0.0449 |
| 200 | 1 | -0.0077 | 1.0017 | 0.0519 | 0.0024 | 0.9126 | 0.0414 | 0.0017 | 0.9323 | 0.0445 |
| | 2 | -0.4204 | 0.8130 | 0.0189 | -0.4123 | 0.7408 | 0.0119 | -0.4128 | 0.7568 | 0.0140 |
| | 3 | -0.0077 | 1.0118 | 0.0526 | 0.0024 | 0.9193 | 0.0420 | 0.0017 | 0.9400 | 0.0454 |
| 500 | 1 | -0.0088 | 0.9848 | 0.0495 | -0.0060 | 0.9556 | 0.0489 | -0.0037 | 0.9813 | 0.0467 |
| | 2 | -0.4140 | 0.8059 | 0.0166 | -0.4117 | 0.7820 | 0.0169 | -0.4098 | 0.8030 | 0.0175 |
| | 3 | -0.0088 | 0.9888 | 0.0497 | -0.0061 | 0.9593 | 0.0489 | -0.0037 | 0.9851 | 0.0470 |
| 1000 | 1 | 0.0085 | 1.0197 | 0.0519 | 0.0098 | 0.9809 | 0.0478 | -0.0008 | 0.9838 | 0.0473 |
| | 2 | -0.3461 | 0.8833 | 0.0264 | -0.3450 | 0.8497 | 0.0212 | -0.3542 | 0.8522 | 0.0206 |
| | 3 | 0.0085 | 1.0217 | 0.0520 | 0.0098 | 0.9828 | 0.0481 | -0.0008 | 0.9857 | 0.0475 |

Note: Test 1 = $I^0$, Test 2 = $\text{LM}_B$, Test 3 = $\text{LM}_B^*$.

**Table 1c.** Empirical Means, SDs and Rejection Frequencies at 5% Level: Group, $G = N^{0.5}$

| | | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | Test | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| 20 | 1 | 0.0128 | 1.0145 | 0.0491 | -0.0090 | 0.9148 | 0.0394 | 0.0029 | 0.9272 | 0.0422 |
| | 2 | -0.3477 | 0.8818 | 0.0195 | -0.3667 | 0.7952 | 0.0144 | -0.3563 | 0.8060 | 0.0157 |
| | 3 | 0.0143 | 1.1320 | 0.0668 | -0.0090 | 1.0036 | 0.0507 | 0.0042 | 1.0169 | 0.0535 |
| 50 | 1 | 0.0180 | 1.0074 | 0.0448 | -0.0112 | 0.8461 | 0.0252 | -0.0095 | 0.9058 | 0.0338 |
| | 2 | -0.2286 | 0.9431 | 0.0259 | -0.2559 | 0.7921 | 0.0141 | -0.2544 | 0.8480 | 0.0195 |
| | 3 | 0.0187 | 1.0497 | 0.0509 | -0.0116 | 0.8747 | 0.0273 | -0.0099 | 0.9389 | 0.0387 |
| 100 | 1 | 0.0100 | 1.0089 | 0.0448 | 0.0008 | 0.8830 | 0.0308 | 0.0007 | 0.9138 | 0.0354 |
| | 2 | -0.1961 | 0.9653 | 0.0338 | -0.2049 | 0.8449 | 0.0235 | -0.2049 | 0.8744 | 0.0237 |
| | 3 | 0.0102 | 1.0295 | 0.0478 | 0.0008 | 0.8980 | 0.0326 | 0.0008 | 0.9305 | 0.0377 |
| 200 | 1 | -0.0037 | 1.0132 | 0.0470 | -0.0064 | 0.9104 | 0.0350 | -0.0065 | 0.9390 | 0.0345 |
| | 2 | -0.1869 | 0.9787 | 0.0381 | -0.1894 | 0.8794 | 0.0294 | -0.1896 | 0.9070 | 0.0259 |
| | 3 | -0.0038 | 1.0235 | 0.0491 | -0.0065 | 0.9182 | 0.0359 | -0.0066 | 0.9475 | 0.0354 |
| 500 | 1 | 0.0118 | 1.0014 | 0.0486 | -0.0096 | 0.9664 | 0.0417 | -0.0077 | 0.9616 | 0.0403 |
| | 2 | -0.1357 | 0.9795 | 0.0427 | -0.1567 | 0.9453 | 0.0377 | -0.1548 | 0.9406 | 0.0335 |
| | 3 | 0.0118 | 1.0054 | 0.0498 | -0.0097 | 0.9700 | 0.0427 | -0.0078 | 0.9652 | 0.0409 |
| 1000 | 1 | -0.0066 | 1.0057 | 0.0493 | -0.0106 | 0.9723 | 0.0409 | 0.0084 | 0.9812 | 0.0440 |
| | 2 | -0.1308 | 0.9901 | 0.0434 | -0.1347 | 0.9572 | 0.0374 | -0.1160 | 0.9660 | 0.0387 |
| | 3 | -0.0067 | 1.0077 | 0.0495 | -0.0106 | 0.9741 | 0.0413 | 0.0084 | 0.9830 | 0.0443 |

Note: Test 1 = $I^0$, Test 2 = $\text{LM}_B$, Test 3 = $\text{LM}_B^*$.

**Table 1d.** Empirical Means, SDs and Rejection Frequencies at 5% Level: Group, $G = N^{0.8}$

| | | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | Test | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| 20 | 1 | -0.0138 | 0.9883 | 0.0462 | -0.0009 | 0.8930 | 0.0329 | 0.0054 | 0.9031 | 0.0353 |
| | 2 | -0.2379 | 0.9243 | 0.0368 | -0.2258 | 0.8352 | 0.0263 | -0.2199 | 0.8447 | 0.0238 |
| | 3 | -0.0154 | 1.1014 | 0.0727 | -0.0007 | 0.9872 | 0.0534 | 0.0061 | 0.9976 | 0.0497 |
| 50 | 1 | 0.0061 | 0.9968 | 0.0482 | -0.0061 | 0.8498 | 0.0314 | 0.0023 | 0.9071 | 0.0353 |
| | 2 | -0.1150 | 0.9803 | 0.0466 | -0.1270 | 0.8358 | 0.0295 | -0.1187 | 0.8922 | 0.0328 |
| | 3 | 0.0064 | 1.0386 | 0.0594 | -0.0063 | 0.8784 | 0.0347 | 0.0024 | 0.9402 | 0.0416 |
| 100 | 1 | 0.0040 | 1.0096 | 0.0528 | 0.0134 | 0.8858 | 0.0368 | -0.0001 | 0.9307 | 0.0374 |
| | 2 | -0.1066 | 1.0010 | 0.0511 | -0.0973 | 0.8782 | 0.0360 | -0.1107 | 0.9228 | 0.0326 |
| | 3 | 0.0041 | 1.0303 | 0.0586 | 0.0136 | 0.9011 | 0.0388 | -0.0001 | 0.9479 | 0.0394 |
| 200 | 1 | 0.0031 | 0.9978 | 0.0483 | -0.0011 | 0.9188 | 0.0419 | 0.0056 | 0.9471 | 0.0384 |
| | 2 | -0.0512 | 0.9940 | 0.0474 | -0.0554 | 0.9153 | 0.0419 | -0.0488 | 0.9436 | 0.0370 |
| | 3 | 0.0032 | 1.0079 | 0.0508 | -0.0011 | 0.9270 | 0.0428 | 0.0056 | 0.9560 | 0.0400 |
| 500 | 1 | -0.0017 | 0.9902 | 0.0500 | -0.0097 | 0.9822 | 0.0503 | 0.0033 | 0.9581 | 0.0415 |
| | 2 | -0.0501 | 0.9879 | 0.0497 | -0.0581 | 0.9798 | 0.0501 | -0.0451 | 0.9558 | 0.0411 |
| | 3 | -0.0017 | 0.9942 | 0.0509 | -0.0098 | 0.9858 | 0.0506 | 0.0033 | 0.9617 | 0.0422 |
| 1000 | 1 | 0.0110 | 0.9993 | 0.0480 | -0.0015 | 0.9883 | 0.0511 | 0.0170 | 0.9823 | 0.0420 |
| | 2 | -0.0257 | 0.9979 | 0.0471 | -0.0382 | 0.9869 | 0.0510 | -0.0197 | 0.9809 | 0.0412 |
| | 3 | 0.0110 | 1.0013 | 0.0486 | -0.0015 | 0.9902 | 0.0514 | 0.0170 | 0.9841 | 0.0423 |

Note: Test 1 = $I^0$, Test 2 = $\text{LM}_B$, Test 3 = $\text{LM}_B^*$.

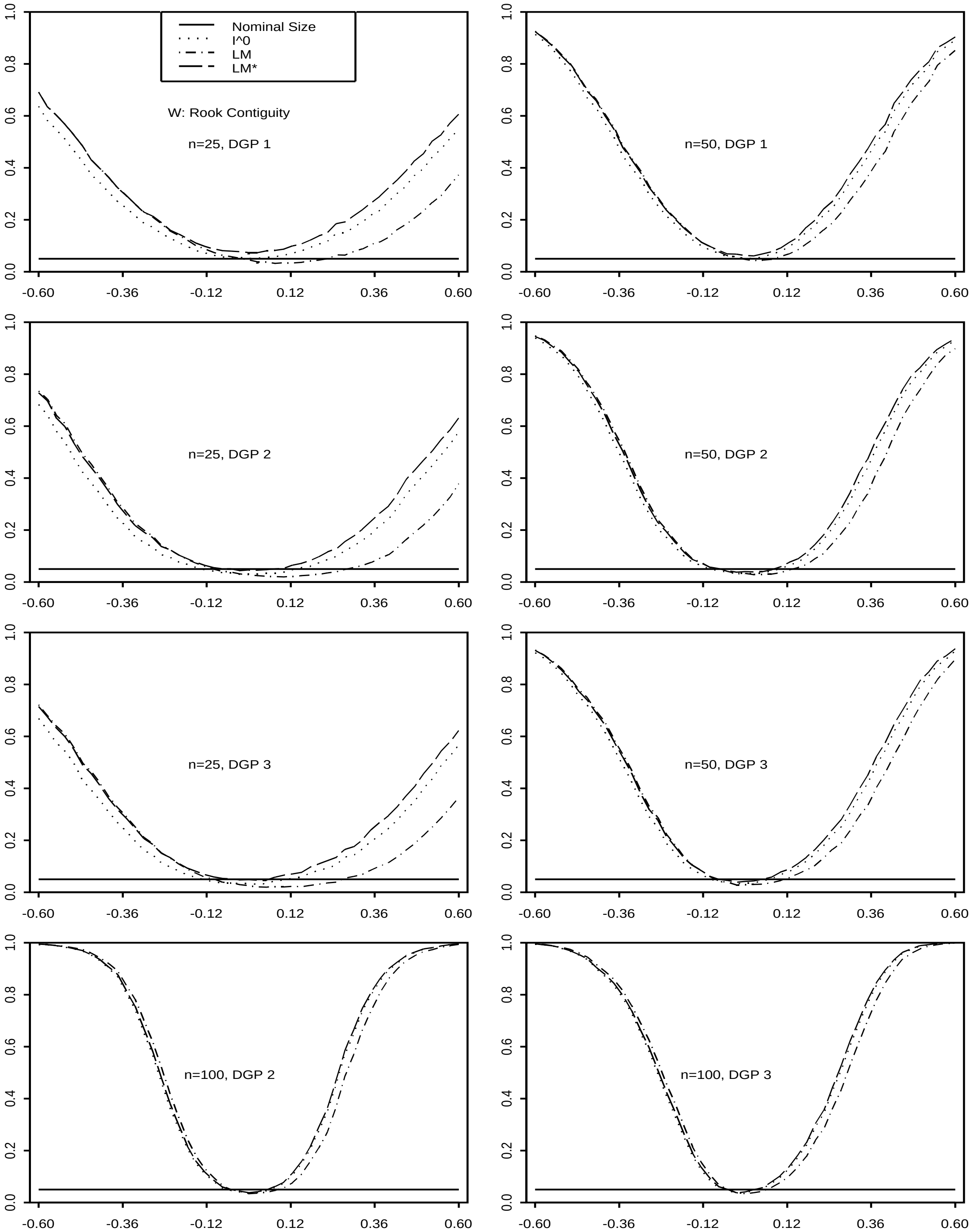


**Figure 1.** Empirical Powers of the Tests: Rook Contiguity

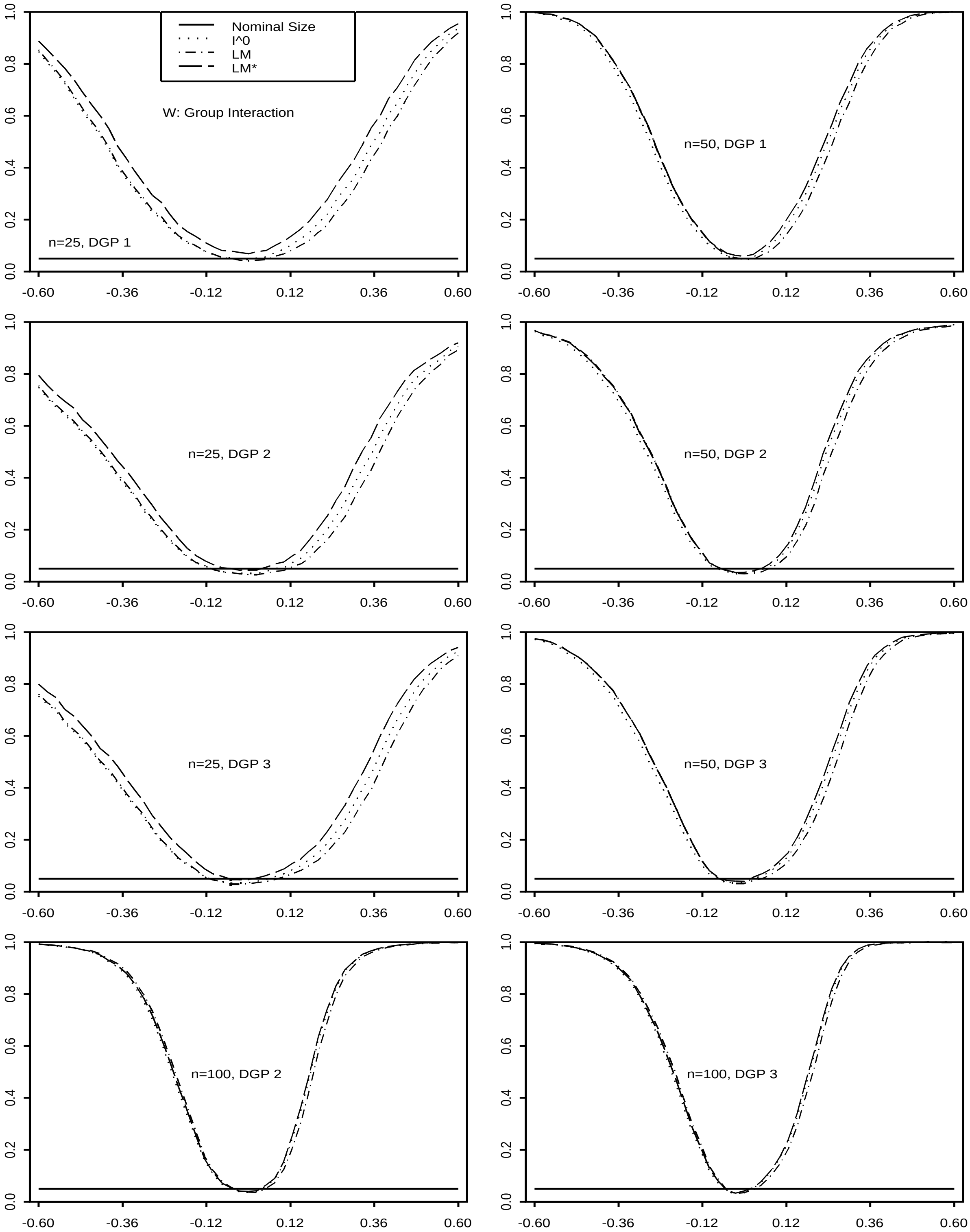


**Figure 2.** Empirical Powers of the Tests: Group Interaction, $G = N^{0.8}$

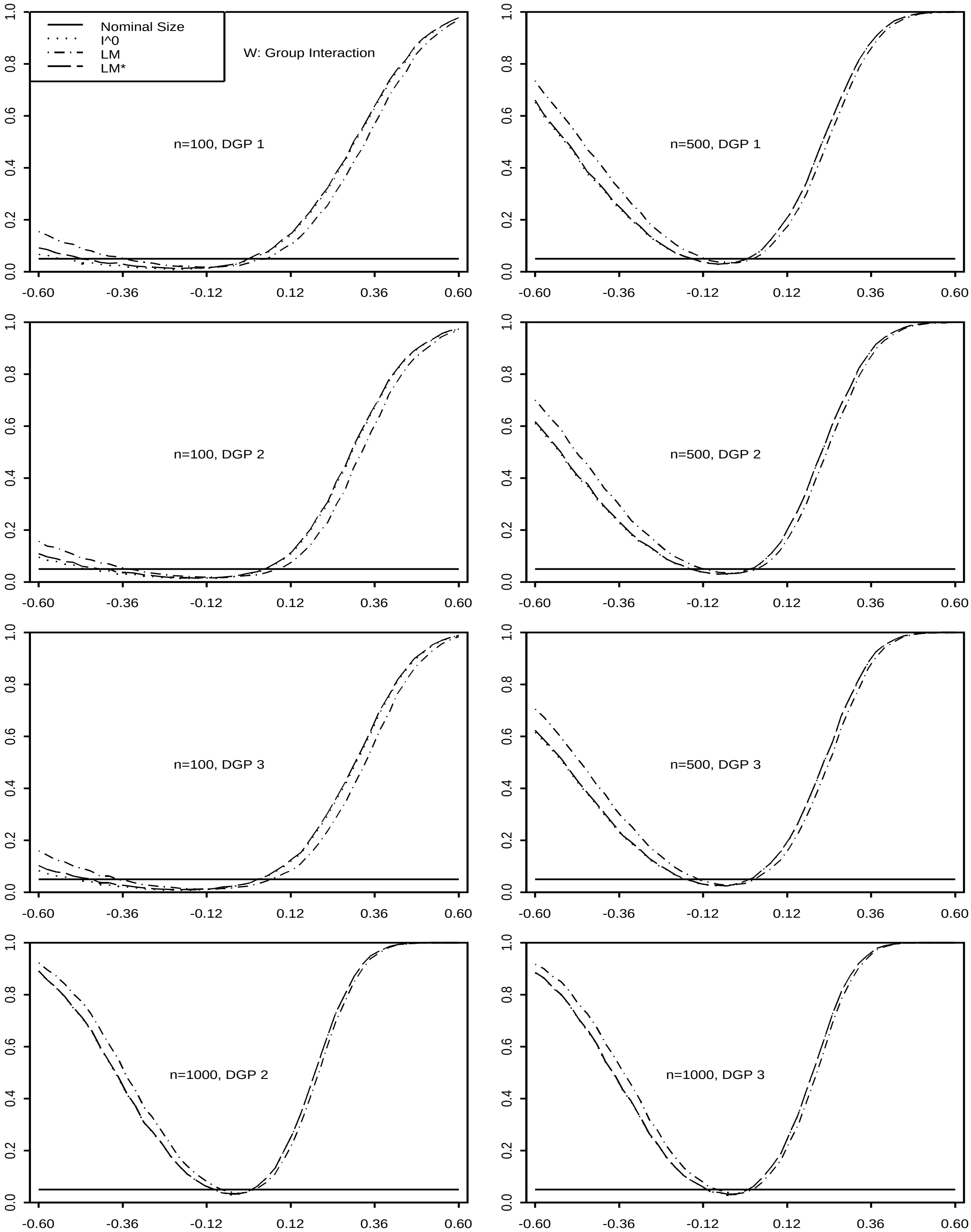


**Figure 3.** Empirical Powers of the Tests: Group Interaction, $G = N^{0.5}$

### 5.2 Tests of spatial error components

In this subsection, the performance of the robust LM test of spatial error components ($\text{LM}^*_{\text{SEC}}$) introduced in Section 3 is evaluated and is compared with the LM test ($\text{LM}_{\text{SEC}}$) introduced by Anselin (2001). We adopt the same designs as in Section 5.1 for the Monte Carlo experiments. Note that these designs are also quite similar to those considered in Anselin and Moreno (2003) when they compared the $\text{LM}_{\text{SEC}}$ with several other tests. They concluded from their Monte Carlo results that essentially none of the tests considered is satisfactory in all situations. In particular, their Monte Carlo results show that the $\text{LM}_{\text{SEC}}$ can perform badly in the situation that the spatial layout is irregular. Our Monte Carlo results are summarized in Table 2. Like in Anselin and Moreno (2003), our results (though under different spatial layout) also show that the $\text{LM}_{\text{SEC}}$ can perform quite badly in situations where the number of neighbors grow with the increase of sample size. However, the modified LM test introduced in this paper perform very well in all the situations considered including those not reported.

The details are as follows. Under Rook contiguity, the two tests seem to perform similarly in most of the situations. However, a closer examination for the cases of non-normal error distributions and large samples, we see that there is sign that the size of the $\text{LM}_{\text{SEC}}$ starts to increase with the sample size. This is a worrisome sign as it indicates the test is not consistent when the error distribution is not normal. To confirm this point, we repeated the experiment using an even larger sample size 1500 and it indeed shows this point.

Under group interaction, the performance of the modified test is very robust against the error distributions as well the spatial layouts. In contrast, the performance of the LM test depends very much on these two factors. When $G = N^{0.2}$, i.e, there are few groups and each group contains many spatial units, the $\text{LM}_{\text{SEC}}$ test tends to under-reject the null hypothesis, where as when $G = N^{0.8}$, i.e., there are many groups and each group contains a few spatial units, the $\text{LM}_{\text{SEC}}$ test tends to over-rejects the null hypothesis severely, where the empirical frequencies of rejection are often around 20% in the situations where the error distributions are non-normal. For those reasons, we feel that it may not be necessary to compare the power of the two tests.

**Table 2.** Empirical Means, SDs and Rejection Frequencies for One-sided 5% Tests

| | | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | Test | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| | | | | | Rook | | | | | |
| 20 | 1 | -0.3745 | 0.8720 | 0.0210 | -0.3586 | 0.8825 | 0.0160 | -0.3653 | 0.8670 | 0.0194 |
| | 2 | -0.0084 | 1.1180 | 0.0804 | 0.0139 | 1.0674 | 0.0700 | 0.0058 | 1.0514 | 0.0703 |
| 50 | 1 | -0.2180 | 0.9411 | 0.0334 | -0.2019 | 0.9949 | 0.0483 | -0.2003 | 0.9763 | 0.0431 |
| | 2 | -0.0174 | 1.0359 | 0.0643 | -0.0006 | 0.9660 | 0.0603 | 0.0018 | 0.9843 | 0.0605 |
| 100 | 1 | -0.1484 | 0.9744 | 0.0411 | -0.1521 | 1.0651 | 0.0507 | -0.1436 | 1.0387 | 0.0537 |
| | 2 | 0.0013 | 1.0185 | 0.0601 | -0.0012 | 0.9580 | 0.0509 | 0.0052 | 0.9812 | 0.0584 |
| 200 | 1 | -0.1165 | 0.9886 | 0.0450 | -0.1003 | 1.1164 | 0.0609 | -0.1112 | 1.0754 | 0.0592 |
| | 2 | -0.0054 | 1.0118 | 0.0589 | 0.0096 | 0.9637 | 0.0497 | 0.0019 | 0.9706 | 0.0547 |
| 500 | 1 | -0.0886 | 0.9970 | 0.0459 | -0.0817 | 1.1578 | 0.0668 | -0.0718 | 1.1437 | 0.0702 |
| | 2 | -0.0128 | 1.0069 | 0.0550 | -0.0047 | 0.9928 | 0.0507 | 0.0031 | 0.9872 | 0.0544 |
| 1000 | 1 | -0.0601 | 0.9910 | 0.0453 | -0.0604 | 1.1541 | 0.0708 | -0.0509 | 1.1808 | 0.0763 |
| | 2 | -0.0090 | 0.9957 | 0.0509 | -0.0083 | 0.9896 | 0.0497 | 0.0010 | 0.9885 | 0.0521 |
| 1500 | 1 | -0.0335 | 0.9956 | 0.0479 | -0.0438 | 1.1470 | 0.0699 | -0.0446 | 1.1971 | 0.0806 |
| | 2 | 0.0092 | 0.9988 | 0.0522 | -0.0007 | 0.9839 | 0.0486 | -0.0023 | 0.9823 | 0.0485 |
| | | | | | Group: $G = N^{0.2}$ | | | | | |
| 20 | 1 | -0.4628 | 0.7543 | 0.0244 | -0.4702 | 0.6964 | 0.0169 | -0.4719 | 0.7002 | 0.0188 |
| | 2 | 0.0045 | 1.1095 | 0.0898 | -0.0073 | 1.0017 | 0.0710 | -0.0088 | 1.0079 | 0.0736 |
| 50 | 1 | -0.5056 | 0.7119 | 0.0223 | -0.5103 | 0.6286 | 0.0128 | -0.5105 | 0.6606 | 0.0165 |
| | 2 | 0.0057 | 1.0590 | 0.0779 | -0.0008 | 0.9091 | 0.0595 | -0.0011 | 0.9638 | 0.0686 |
| 100 | 1 | -0.4245 | 0.8067 | 0.0283 | -0.4348 | 0.7164 | 0.0163 | -0.4227 | 0.7555 | 0.0216 |
| | 2 | 0.0116 | 1.0376 | 0.0774 | -0.0017 | 0.8953 | 0.0610 | 0.0136 | 0.9552 | 0.0659 |
| 200 | 1 | -0.4088 | 0.8194 | 0.0299 | -0.4184 | 0.7426 | 0.0207 | -0.4074 | 0.7927 | 0.0276 |
| | 2 | -0.0025 | 1.0106 | 0.0721 | -0.0141 | 0.9057 | 0.0615 | -0.0008 | 0.9704 | 0.0675 |
| 500 | 1 | -0.4051 | 0.8099 | 0.0305 | -0.4040 | 0.7957 | 0.0281 | -0.4116 | 0.7814 | 0.0276 |
| | 2 | -0.0003 | 0.9913 | 0.0719 | 0.0010 | 0.9733 | 0.0707 | -0.0083 | 0.9558 | 0.0655 |
| 1000 | 1 | -0.3417 | 0.8900 | 0.0384 | -0.3488 | 0.8472 | 0.0318 | -0.3356 | 0.8579 | 0.0318 |
| | 2 | 0.0125 | 1.0286 | 0.0735 | 0.0043 | 0.9789 | 0.0687 | 0.0195 | 0.9912 | 0.0685 |
| | | | | | Group: $G = N^{0.8}$ | | | | | |
| 20 | 1 | -0.0573 | 0.9520 | 0.0036 | -0.0456 | 1.5576 | 0.0327 | -0.0778 | 1.5179 | 0.0179 |
| | 2 | 0.0073 | 1.1491 | 0.0465 | 0.0187 | 1.2307 | 0.0229 | -0.0005 | 1.2560 | 0.0185 |
| 50 | 1 | -0.0423 | 0.9979 | 0.0410 | -0.0357 | 2.3038 | 0.2748 | -0.0467 | 1.9405 | 0.2149 |
| | 2 | 0.0058 | 1.0685 | 0.0583 | 0.0044 | 1.0989 | 0.0311 | -0.0019 | 1.0919 | 0.0420 |
| 100 | 1 | -0.0501 | 1.0030 | 0.0424 | -0.0211 | 2.5926 | 0.2811 | -0.0355 | 2.1466 | 0.2250 |
| | 2 | 0.0008 | 1.0356 | 0.0542 | 0.0107 | 1.0518 | 0.0576 | 0.0045 | 1.0515 | 0.0547 |
| 200 | 1 | -0.0679 | 1.0096 | 0.0475 | -0.1192 | 2.4816 | 0.2176 | -0.0723 | 2.2211 | 0.1853 |
| | 2 | -0.0083 | 1.0281 | 0.0568 | -0.0256 | 1.0122 | 0.0776 | -0.0015 | 1.0273 | 0.0857 |
| 500 | 1 | -0.0216 | 0.9957 | 0.0484 | -0.0674 | 2.5125 | 0.2182 | -0.0660 | 2.4502 | 0.1843 |
| | 2 | 0.0240 | 1.0022 | 0.0549 | -0.0093 | 1.0119 | 0.0716 | -0.0042 | 1.0010 | 0.0758 |
| 1000 | 1 | -0.0130 | 0.9989 | 0.0499 | -0.0395 | 2.2213 | 0.2045 | -0.0231 | 2.3749 | 0.1791 |
| | 2 | 0.0221 | 1.0024 | 0.0547 | -0.0019 | 1.0179 | 0.0669 | 0.0067 | 1.0101 | 0.0650 |

Note: Test 1 = $\mathrm{LM}_{\mathrm{SEC}}$, Test 2 = $\mathrm{LM}^{*}_{\mathrm{SEC}}$

## 5.3 Tests of spatial error in panel models

Consider the following DGP

$$Y_t = \beta_0 + X_{1t}\beta_1 + X_{2t}\beta_2 + u_t, \text{ with } u_t = \mu + \varepsilon_t, \ \ t = 1, \cdots, T,$$

where the error components $\mu$ and $\varepsilon_t$ can be drawn from any of the three distributions used in the previous two subsections, or the combination of any two distributions. For example, $\mu$ and $\varepsilon_t$ can both be drawn from the normal mixture, or $\mu$ from the normal mixture but $\varepsilon_t$ from the normal or log-normal distribution. The beta parameters are set at the same values as before, $\sigma_v^2 = 1.0$ and $\sigma_\mu^2 = 1.5$. For sample sizes, $T = 3$, and $N = 20, 50, 100, 200$, and 500. The same spatial layouts as above are used.

**Size of the tests.** The results presented in Table 3 correspond to the cases where both $\mu$ and $v_t$ are normal, both are normal mixture, and both are log-normal. Some observations are as follows. The results confirm the theory that the three tests are asymptotically equivalent and that they are robust against excess skewness and excess kurtosis. However, the results also show there that there is indeed a need for finite sample corrections. The two modified statistics correct the mean or both the mean and variance. The results show a certain improvement, though it still may not seem to be enough in many situations.

Some more results are shown in Table 4 where the standard deviations of the error components are changed to $\sigma_v^2 = 5$, and $\sigma_\mu^2 = 0.5$. The results contrast further between the LM and the Modified LM tests. In particular, when sample sizes are not large, LM test severely under-rejects the null hypothesis, and as the sample size increases its empirical size of the test converges to the nominal level very slowly. This is true even when the error components are both normally distributed. A detailed examination shows that the cause of this size distortion is that the mean of the LM test statistic is downward shifted.

**Power of the tests.** Similar to the case of Section 5.1, the sizes of the tests are similar in many situations, it is necessary to compare the power. Empirical frequencies of rejection of the three tests are plotted in Figures 4-6 against the values of $\lambda$ (horizontal line). From the plots we see that the $\text{LM}_{\text{BKS}}^{**}$ performs noticeably better than the other two. Again, the tests show weakness in detecting a negative spatial dependence when $G = N^{0.5}$ and sample size is not large.

**Table 3a.** Empirical Means, SDs and Rejection Frequencies at 5% Level: Rook

| | | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | Test | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| 20 | 1 | -0.0024 | 0.9472 | 0.0379 | -0.0244 | 0.8474 | 0.0270 | -0.0207 | 0.8809 | 0.0296 |
| | 2 | -0.0038 | 0.9471 | 0.0379 | -0.0030 | 0.8461 | 0.0272 | -0.0077 | 0.8801 | 0.0296 |
| | 3 | -0.0041 | 1.0063 | 0.0532 | -0.0031 | 0.8967 | 0.0332 | -0.0082 | 0.9333 | 0.0387 |
| 50 | 1 | -0.0341 | 0.9936 | 0.0502 | -0.0602 | 0.8989 | 0.0385 | -0.0507 | 0.9179 | 0.0356 |
| | 2 | -0.0003 | 0.9936 | 0.0514 | -0.0184 | 0.8988 | 0.0394 | -0.0118 | 0.9177 | 0.0354 |
| | 3 | -0.0003 | 1.0122 | 0.0544 | -0.0187 | 0.9150 | 0.0418 | -0.0120 | 0.9344 | 0.0381 |
| 100 | 1 | -0.0123 | 0.9900 | 0.0492 | -0.0350 | 0.9409 | 0.0448 | -0.0261 | 0.9573 | 0.0417 |
| | 2 | 0.0053 | 0.9900 | 0.0493 | -0.0141 | 0.9408 | 0.0449 | -0.0059 | 0.9572 | 0.0411 |
| | 3 | 0.0053 | 1.0002 | 0.0515 | -0.0143 | 0.9502 | 0.0464 | -0.0060 | 0.9669 | 0.0435 |
| 200 | 1 | -0.0115 | 1.0014 | 0.0483 | -0.0054 | 0.9645 | 0.0472 | -0.0160 | 0.9628 | 0.0432 |
| | 2 | -0.0062 | 1.0014 | 0.0481 | 0.0012 | 0.9644 | 0.0471 | -0.0095 | 0.9628 | 0.0428 |
| | 3 | -0.0062 | 1.0061 | 0.0498 | 0.0012 | 0.9689 | 0.0478 | -0.0095 | 0.9673 | 0.0441 |
| 500 | 1 | 0.0093 | 1.0072 | 0.0532 | -0.0053 | 0.9906 | 0.0515 | -0.0029 | 0.9804 | 0.0453 |
| | 2 | 0.0156 | 1.0072 | 0.0533 | 0.0013 | 0.9906 | 0.0514 | 0.0038 | 0.9804 | 0.0454 |
| | 3 | 0.0156 | 1.0091 | 0.0539 | 0.0013 | 0.9925 | 0.0521 | 0.0038 | 0.9822 | 0.0457 |

Note: Test 1 = $\mathrm{LM_{BSK}}$, Test 2 = $\mathrm{LM^{*}_{BSK}}$, Test 3 = $\mathrm{LM^{**}_{BSK}}$.

**Table 3b.** Empirical Means, SDs and Rejection Frequencies at 5% Level: Group, $G = N^{0.2}$

| | | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | Test | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| 20 | 1 | -0.0630 | 0.9706 | 0.0415 | -0.0917 | 0.8334 | 0.0224 | -0.0890 | 0.8833 | 0.0281 |
| | 2 | -0.0119 | 0.9705 | 0.0454 | -0.0034 | 0.8310 | 0.0238 | -0.0136 | 0.8810 | 0.0313 |
| | 3 | -0.0126 | 1.0221 | 0.0510 | -0.0035 | 0.8797 | 0.0301 | -0.0144 | 0.9303 | 0.0362 |
| 50 | 1 | -0.0736 | 0.9773 | 0.0431 | -0.0624 | 0.8928 | 0.0283 | -0.0677 | 0.9237 | 0.0372 |
| | 2 | -0.0326 | 0.9770 | 0.0460 | 0.0025 | 0.8911 | 0.0315 | -0.0105 | 0.9224 | 0.0404 |
| | 3 | -0.0332 | 0.9948 | 0.0473 | 0.0025 | 0.9112 | 0.0337 | -0.0108 | 0.9409 | 0.0431 |
| 100 | 1 | -0.0576 | 0.9785 | 0.0413 | -0.0388 | 0.9485 | 0.0380 | -0.0526 | 0.9407 | 0.0336 |
| | 2 | -0.0138 | 0.9785 | 0.0431 | 0.0156 | 0.9483 | 0.0402 | -0.0002 | 0.9404 | 0.0365 |
| | 3 | -0.0140 | 0.9929 | 0.0447 | 0.0159 | 0.9633 | 0.0423 | -0.0002 | 0.9549 | 0.0383 |
| 200 | 1 | -0.0330 | 1.0093 | 0.0451 | -0.0645 | 0.9519 | 0.0360 | -0.0447 | 0.9590 | 0.0381 |
| | 2 | 0.0036 | 1.0092 | 0.0470 | -0.0223 | 0.9519 | 0.0382 | -0.0025 | 0.9588 | 0.0406 |
| | 3 | 0.0036 | 1.0150 | 0.0475 | -0.0225 | 0.9580 | 0.0387 | -0.0025 | 0.9648 | 0.0416 |
| 500 | 1 | -0.0330 | 0.9971 | 0.0464 | -0.0205 | 0.9873 | 0.0416 | -0.0536 | 0.9743 | 0.0389 |
| | 2 | 0.0049 | 0.9970 | 0.0488 | 0.0195 | 0.9870 | 0.0440 | -0.0128 | 0.9741 | 0.0413 |
| | 3 | 0.0049 | 1.0007 | 0.0491 | 0.0196 | 0.9908 | 0.0445 | -0.0129 | 0.9780 | 0.0414 |

Note: Test 1 = $\mathrm{LM_{BSK}}$, Test 2 = $\mathrm{LM^{*}_{BSK}}$, Test 3 = $\mathrm{LM^{**}_{BSK}}$.

**Table 3c.** Empirical Means, SDs and Rejection Frequencies at 5% Level: Group, $G = N^{0.5}$

| $N$ | Test | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| 20 | 1 | -0.0693 | 0.9631 | 0.0326 | -0.1020 | 0.8448 | 0.0225 | -0.1015 | 0.8900 | 0.0287 |
| | 2 | 0.0033 | 0.9632 | 0.0353 | -0.0047 | 0.8422 | 0.0236 | -0.0144 | 0.8892 | 0.0309 |
| | 3 | 0.0035 | 1.0241 | 0.0435 | -0.0050 | 0.8942 | 0.0308 | -0.0154 | 0.9444 | 0.0380 |
| 50 | 1 | -0.0178 | 0.9924 | 0.0434 | -0.0240 | 0.9115 | 0.0336 | -0.0316 | 0.9242 | 0.0351 |
| | 2 | 0.0071 | 0.9924 | 0.0440 | 0.0128 | 0.9105 | 0.0335 | 0.0008 | 0.9240 | 0.0352 |
| | 3 | 0.0072 | 1.0123 | 0.0484 | 0.0130 | 0.9289 | 0.0371 | 0.0009 | 0.9424 | 0.0380 |
| 100 | 1 | -0.0232 | 0.9960 | 0.0456 | -0.0158 | 0.9412 | 0.0378 | -0.0172 | 0.9611 | 0.0400 |
| | 2 | -0.0026 | 0.9960 | 0.0457 | 0.0104 | 0.9410 | 0.0384 | 0.0077 | 0.9609 | 0.0411 |
| | 3 | -0.0026 | 1.0061 | 0.0481 | 0.0105 | 0.9506 | 0.0395 | 0.0078 | 0.9707 | 0.0427 |
| 200 | 1 | -0.0190 | 1.0013 | 0.0472 | -0.0084 | 0.9741 | 0.0426 | -0.0077 | 0.9808 | 0.0433 |
| | 2 | -0.0074 | 1.0013 | 0.0470 | 0.0058 | 0.9740 | 0.0429 | 0.0064 | 0.9808 | 0.0437 |
| | 3 | -0.0074 | 1.0051 | 0.0480 | 0.0058 | 0.9778 | 0.0431 | 0.0065 | 0.9845 | 0.0445 |
| 500 | 1 | -0.0208 | 1.0011 | 0.0476 | -0.0348 | 0.9863 | 0.0422 | -0.0077 | 0.9763 | 0.0426 |
| | 2 | -0.0068 | 1.0011 | 0.0479 | -0.0201 | 0.9863 | 0.0423 | 0.0073 | 0.9764 | 0.0433 |
| | 3 | -0.0068 | 1.0032 | 0.0483 | -0.0201 | 0.9884 | 0.0425 | 0.0073 | 0.9785 | 0.0438 |

Note: Test 1 = $\text{LM}_{\text{BSK}}$, Test 2 = $\text{LM}^*_{\text{BSK}}$, Test 3 = $\text{LM}^{**}_{\text{BSK}}$.

**Table 3d.** Empirical Means, SDs and Rejection Frequencies at 5% Level: Group, $G = N^{0.8}$

| $N$ | Test | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| 20 | 1 | 0.0393 | 0.9865 | 0.0456 | -0.0009 | 0.8627 | 0.0333 | 0.0203 | 0.9110 | 0.0399 |
| | 2 | 0.0122 | 0.9864 | 0.0451 | -0.0084 | 0.8618 | 0.0329 | 0.0054 | 0.9104 | 0.0401 |
| | 3 | 0.0127 | 1.0270 | 0.0554 | -0.0087 | 0.8952 | 0.0377 | 0.0057 | 0.9462 | 0.0465 |
| 50 | 1 | -0.0083 | 0.9786 | 0.0460 | -0.0277 | 0.9030 | 0.0400 | -0.0209 | 0.9250 | 0.0393 |
| | 2 | 0.0043 | 0.9786 | 0.0453 | -0.0102 | 0.9029 | 0.0399 | -0.0049 | 0.9247 | 0.0392 |
| | 3 | 0.0044 | 0.9974 | 0.0494 | -0.0104 | 0.9194 | 0.0418 | -0.0049 | 0.9419 | 0.0425 |
| 100 | 1 | 0.0010 | 1.0013 | 0.0512 | 0.0069 | 0.9431 | 0.0435 | 0.0035 | 0.9479 | 0.0437 |
| | 2 | 0.0021 | 1.0013 | 0.0512 | 0.0106 | 0.9431 | 0.0437 | 0.0067 | 0.9479 | 0.0435 |
| | 3 | 0.0022 | 1.0108 | 0.0531 | 0.0107 | 0.9517 | 0.0453 | 0.0067 | 0.9567 | 0.0453 |
| 200 | 1 | 0.0078 | 0.9838 | 0.0456 | -0.0008 | 0.9443 | 0.0420 | -0.0069 | 0.9718 | 0.0461 |
| | 2 | 0.0138 | 0.9838 | 0.0454 | 0.0060 | 0.9443 | 0.0421 | 0.0000 | 0.9719 | 0.0463 |
| | 3 | 0.0138 | 0.9883 | 0.0469 | 0.0061 | 0.9485 | 0.0426 | 0.0000 | 0.9762 | 0.0469 |
| 500 | 1 | 0.0059 | 0.9982 | 0.0497 | -0.0003 | 0.9978 | 0.0522 | -0.0036 | 0.9816 | 0.0447 |
| | 2 | 0.0110 | 0.9982 | 0.0499 | 0.0051 | 0.9978 | 0.0522 | 0.0019 | 0.9815 | 0.0453 |
| | 3 | 0.0110 | 1.0002 | 0.0504 | 0.0051 | 0.9998 | 0.0525 | 0.0019 | 0.9835 | 0.0458 |

Note: Test 1 = $\text{LM}_{\text{BSK}}$, Test 2 = $\text{LM}^*_{\text{BSK}}$, Test 3 = $\text{LM}^{**}_{\text{BSK}}$.

**Table 4.** More Monte Carlo Results: $\sigma_\mu = 0.5, \sigma_v = 5.0$

| | | Normal | | | Normal Mixture | | | Log-normal | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | Test | Mean | SD | Prob | Mean | SD | Prob | Mean | SD | Prob |
| | | | | | Group: $G = N^{0.2}$ | | | | | |
| 20 | 1 | -0.2084 | 0.9524 | 0.0297 | -0.2027 | 0.8085 | 0.0191 | -0.2136 | 0.8629 | 0.0215 |
| | 2 | 0.0127 | 0.9528 | 0.0418 | 0.0211 | 0.8088 | 0.0237 | 0.0097 | 0.8627 | 0.0305 |
| | 3 | 0.0146 | 1.0431 | 0.0513 | 0.0234 | 0.8799 | 0.0340 | 0.0109 | 0.9412 | 0.0389 |
| 50 | 1 | -0.2383 | 0.9327 | 0.0276 | -0.2618 | 0.8287 | 0.0175 | -0.2404 | 0.8771 | 0.0224 |
| | 2 | 0.0113 | 0.9331 | 0.0396 | -0.0102 | 0.8287 | 0.0253 | 0.0099 | 0.8771 | 0.0328 |
| | 3 | 0.0127 | 1.0125 | 0.0488 | -0.0108 | 0.8940 | 0.0331 | 0.0109 | 0.9478 | 0.0417 |
| 100 | 1 | -0.2067 | 0.9496 | 0.0295 | -0.2095 | 0.9118 | 0.0286 | -0.2085 | 0.9209 | 0.0259 |
| | 2 | 0.0100 | 0.9493 | 0.0394 | 0.0083 | 0.9121 | 0.0349 | 0.0089 | 0.9207 | 0.0339 |
| | 3 | 0.0105 | 1.0041 | 0.0465 | 0.0089 | 0.9624 | 0.0402 | 0.0094 | 0.9719 | 0.0389 |
| 200 | 1 | -0.2084 | 0.9707 | 0.0318 | -0.2342 | 0.9262 | 0.0265 | -0.2085 | 0.9191 | 0.0264 |
| | 2 | 0.0160 | 0.9707 | 0.0427 | -0.0091 | 0.9262 | 0.0346 | 0.0162 | 0.9190 | 0.0360 |
| | 3 | 0.0170 | 1.0275 | 0.0515 | -0.0096 | 0.9789 | 0.0395 | 0.0171 | 0.9710 | 0.0419 |
| | | | | | Group: $G = N^{0.5}$ | | | | | |
| 20 | 1 | -0.0922 | 0.9894 | 0.0472 | -0.1024 | 0.8541 | 0.0295 | -0.0926 | 0.9006 | 0.0346 |
| | 2 | 0.0048 | 0.9892 | 0.0462 | -0.0046 | 0.8541 | 0.0294 | 0.0056 | 0.9004 | 0.0360 |
| | 3 | 0.0050 | 1.0348 | 0.0577 | -0.0048 | 0.8912 | 0.0346 | 0.0058 | 0.9407 | 0.0429 |
| 50 | 1 | -0.0702 | 1.0049 | 0.0509 | -0.0599 | 0.9107 | 0.0376 | -0.0501 | 0.9453 | 0.0372 |
| | 2 | -0.0077 | 1.0048 | 0.0501 | 0.0029 | 0.9107 | 0.0380 | 0.0124 | 0.9452 | 0.0385 |
| | 3 | -0.0079 | 1.0230 | 0.0548 | 0.0030 | 0.9264 | 0.0397 | 0.0126 | 0.9618 | 0.0411 |
| 100 | 1 | -0.0584 | 0.9925 | 0.0492 | -0.0531 | 0.9247 | 0.0416 | -0.0426 | 0.9573 | 0.0391 |
| | 2 | -0.0104 | 0.9924 | 0.0481 | -0.0049 | 0.9247 | 0.0424 | 0.0055 | 0.9573 | 0.0402 |
| | 3 | -0.0105 | 1.0014 | 0.0501 | -0.0049 | 0.9328 | 0.0427 | 0.0055 | 0.9657 | 0.0413 |
| 200 | 1 | -0.0493 | 0.9970 | 0.0492 | -0.0373 | 0.9792 | 0.0500 | -0.0338 | 0.9786 | 0.0406 |
| | 2 | -0.0085 | 0.9970 | 0.0491 | 0.0035 | 0.9792 | 0.0503 | 0.0070 | 0.9786 | 0.0415 |
| | 3 | -0.0085 | 1.0020 | 0.0502 | 0.0036 | 0.9840 | 0.0511 | 0.0070 | 0.9833 | 0.0422 |
| | | | | | Group: $G = N^{0.8}$ | | | | | |
| 20 | 1 | -0.0948 | 0.9828 | 0.0465 | -0.0868 | 0.8530 | 0.0293 | -0.0936 | 0.8953 | 0.0330 |
| | 2 | -0.0042 | 0.9827 | 0.0442 | 0.0044 | 0.8531 | 0.0292 | -0.0024 | 0.8953 | 0.0344 |
| | 3 | -0.0044 | 1.0289 | 0.0570 | 0.0046 | 0.8905 | 0.0331 | -0.0025 | 0.9358 | 0.0395 |
| 50 | 1 | -0.0491 | 0.9978 | 0.0497 | -0.0556 | 0.8764 | 0.0351 | -0.0650 | 0.9340 | 0.0365 |
| | 2 | 0.0147 | 0.9979 | 0.0491 | 0.0086 | 0.8764 | 0.0351 | -0.0011 | 0.9340 | 0.0383 |
| | 3 | 0.0149 | 1.0159 | 0.0526 | 0.0087 | 0.8912 | 0.0367 | -0.0011 | 0.9503 | 0.0408 |
| 100 | 1 | -0.0451 | 0.9851 | 0.0441 | -0.0578 | 0.9314 | 0.0451 | -0.0783 | 0.9332 | 0.0340 |
| | 2 | 0.0124 | 0.9851 | 0.0435 | -0.0001 | 0.9314 | 0.0442 | -0.0207 | 0.9331 | 0.0351 |
| | 3 | 0.0126 | 0.9947 | 0.0452 | -0.0001 | 0.9402 | 0.0454 | -0.0209 | 0.9420 | 0.0359 |
| 200 | 1 | -0.0372 | 1.0004 | 0.0487 | -0.0289 | 0.9765 | 0.0469 | -0.0402 | 0.9675 | 0.0427 |
| | 2 | 0.0026 | 1.0003 | 0.0488 | 0.0110 | 0.9764 | 0.0474 | -0.0003 | 0.9675 | 0.0428 |
| | 3 | 0.0026 | 1.0051 | 0.0497 | 0.0110 | 0.9810 | 0.0481 | -0.0003 | 0.9720 | 0.0433 |

Note: Test 1 = $\text{LM}_{\text{BSK}}$, Test 2 = $\text{LM}^*_{\text{BSK}}$, Test 3 = $\text{LM}^{**}_{\text{BSK}}$.

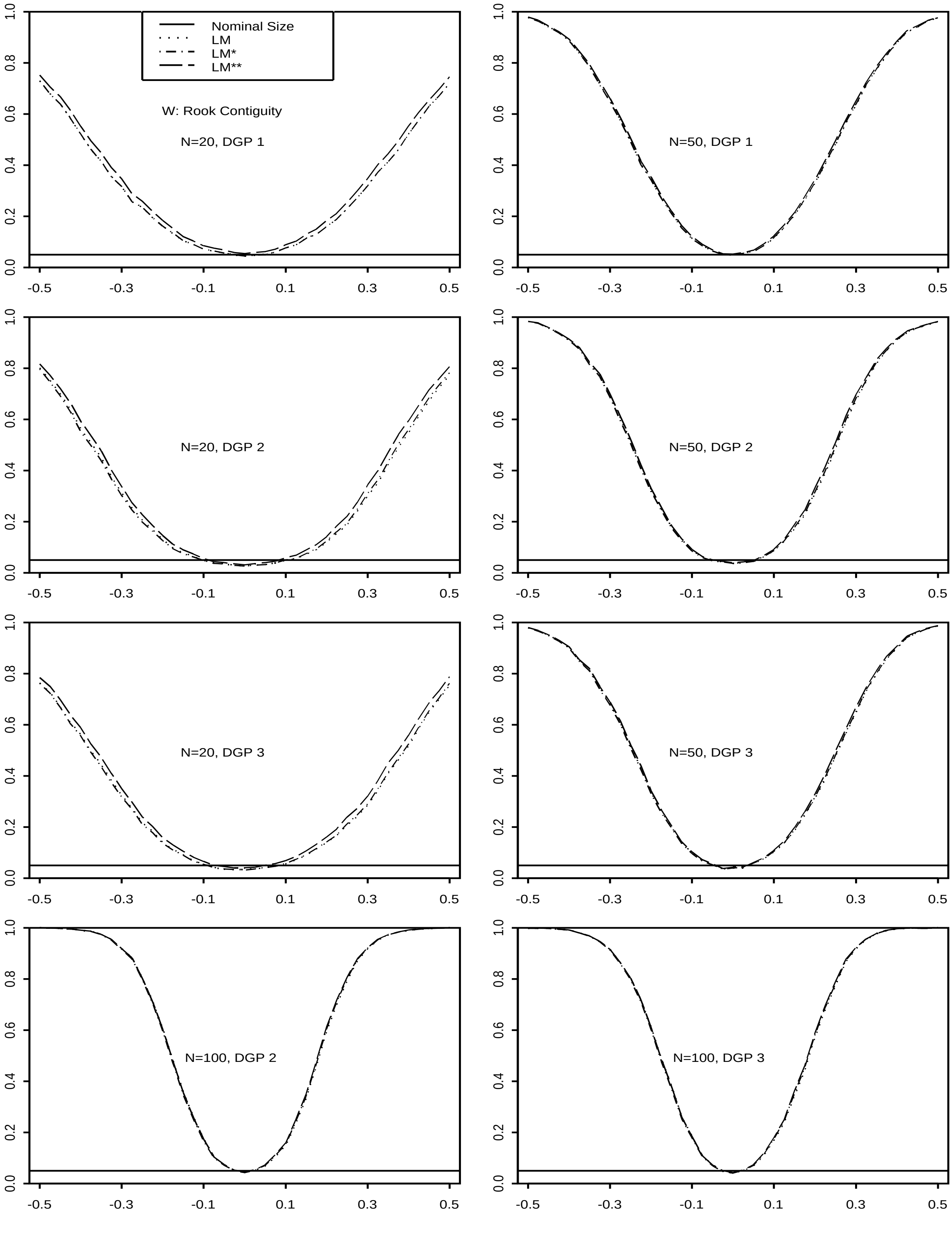


**Figure 4.** Empirical Powers of Panel LM Tests: Rook Contiguity

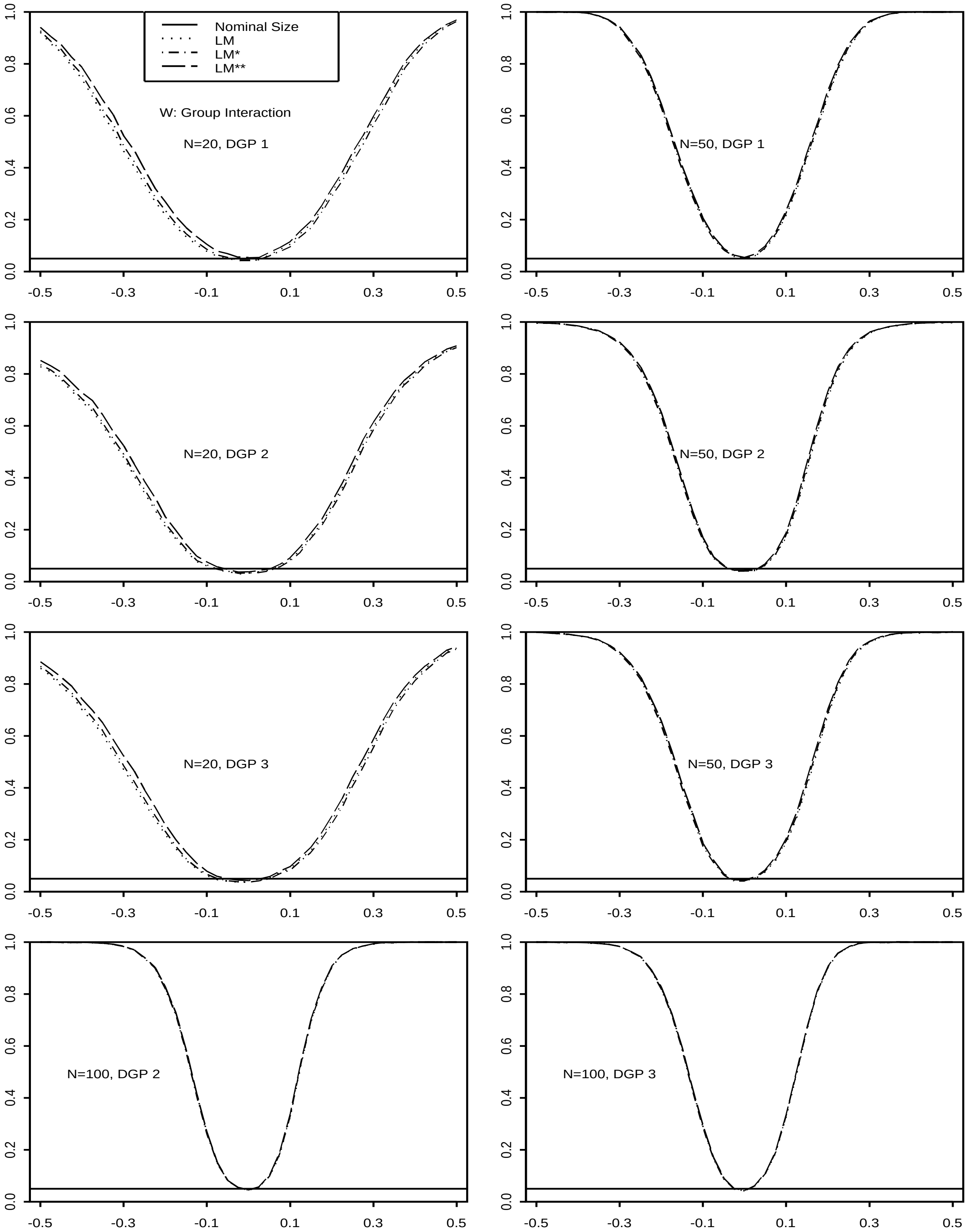


**Figure 5.** Empirical Powers of Panel LM Tests: Group Interaction, $G = N^{0.8}$

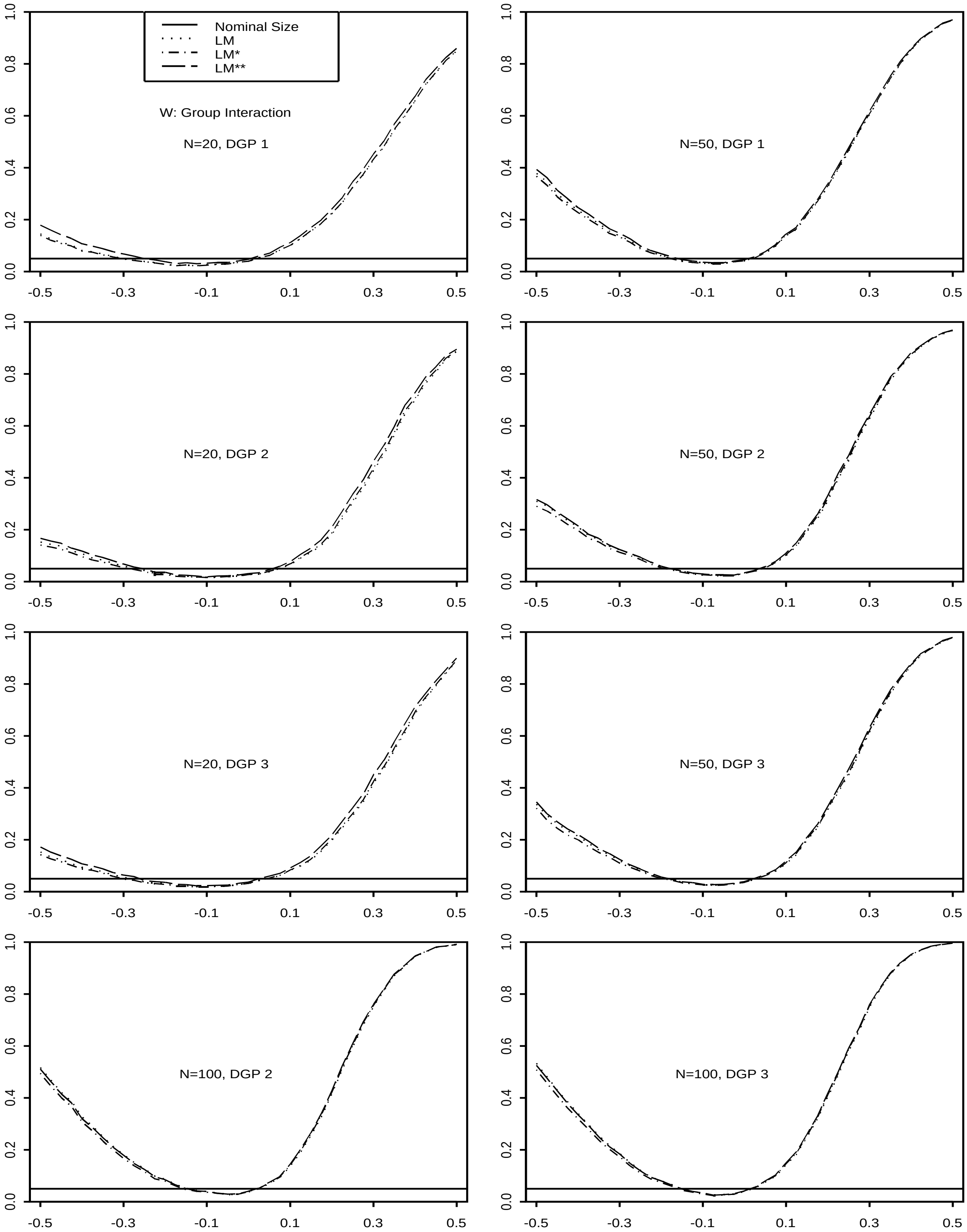


**Figure 6.** Empirical Powers of Panel LM Tests: Group Interaction, $G = N^{0.5}$

# 6 Conclusions

We obtained three sets of modified tests for spatial dependence in the errors of each of the three type of models: (i) linear regression with possible SAR or SMA effect in the errors, (ii) linear regression with possible spatial error components, and (iii) linear panel regression with random effects and possible SAR effect in the pure error term. For case (i), standardized Moran's $I$ and modified LM test perform well in general. The LM test can be seriously affected by the degree of spatial dependence. A large degree of spatial dependence (two many neighbors for each spatial unit) may reduce the mean, variance and the tail probability of the LM test statistic greatly when sample sizes are not large, and noticeably when the sample size is as large as 1000. For case (ii), the LM test is sensitive to both the error distributions and to the spatial layout. In contrast, the proposed test performs well in general. Case (iii) is somehow similar to case (i), and thus the proposed tests outperform the LM test in a similar manner, though in a lesser degree.

There are other LM tests for other spatial models that are derived under normal assumptions such as Baltagi, et al. (2007), and the LM test for spatial lag, which can be studied in a similar manner. It is seen that the new tests presented in this paper not only offer improvements in robustness over the standard LM tests, but also preserve simplicity of the original LM tests so that they can be easily adopted by the applied researchers.

# Appendix: Some Useful Lemmas

**Lemma 1** (Lee, 2004a): *Let $v$ be a random vector of iid elements with mean zero, variance $\sigma^2$, and finite excess kurtosis $\kappa$. Let $A$ be an $N$ dimensional square matrix. Then* $\mathrm{E}(v'Av) = \sigma^2 \mathrm{tr}(A)$ *and* $\mathrm{Var}(v'Av) = \sigma^4 \kappa \sum_{i=1}^N a_{ii}^2 + \sigma^4 \mathrm{tr}(AA' + A^2)$.

**Lemma 2** (Lemma A.9, Lee, 2004b): *Suppose that $A$ represents a sequence of $N \times N$ matrices that are uniformly bounded in both row and column sums. Elements of the $N \times k$ matrix $X$ are uniformly bounded; and $\lim_{n\to\infty} \frac{1}{N} X'X$ exists and is nonsingular. Let $M = I_N - X(X'X)^{-1}X'$. Then*

*(i)* $\mathrm{tr}(MA) = \mathrm{tr}(A) + O(1)$
*(ii)* $\mathrm{tr}(A'MA) = \mathrm{tr}(A'A) + O(1)$
*(iii)* $\mathrm{tr}[(MA)^2] = \mathrm{tr}(A^2) + O(1)$*, and*
*(iv)* $\mathrm{tr}[(A'MA)^2] = \mathrm{tr}[(MA'A)^2] = \mathrm{tr}[A'A)^2] + O(1)$

*Furthermore, if $A_{ij} = O(h_N^{-1})$ for all $i$ and $j$, then*

*(vi)* $\mathrm{tr}^2(MA) = \mathrm{tr}^2(A) + O(\frac{n}{h_N})$*, and*
*(vii)* $\sum_{i=1}^n [(MA)_{ii}]^2 = \sum_{i=1}^n (a_{ii})^2 + O(h_N^{-1})$,

*where $(MA)_{ii}$ are the diagonal elements of $MA$, and $a_{ii}$ are the diagonal elements of $A$.*

**Lemma 3:** *Let $u = G_1\mu + G_2 v$, where $u$ and $v$ are independent vectors, each containing iid elements of means zero, variances $\sigma_\mu^2$ and $\sigma_v^2$, skewness $\alpha_\mu$ and $\alpha_v$, and excess kurtosis $\kappa_\mu$ and $\kappa_v$; $G_1$ and $G_2$ are two conformable non-stochastic matrices. Let $\phi = \sigma_\mu^2/\sigma_v^2$. We have for a symmetric matrix $A$,*

*(i)*$\mathrm{E}(u'Au) = \sigma_v^2 \mathrm{tr}(\Sigma A)$,
*(ii)*$\mathrm{Var}(u'Au) = \sigma_\mu^4 \kappa_\mu a_1' a_1 + \sigma_v^4 \kappa_v a_2' a_2 + 2\sigma_v^4 \mathrm{tr}(\Sigma A \Sigma A)$,

*where* $\Sigma = \sigma_v^{-2} \mathrm{E}(uu') = \phi G_1 G_1' + G_2 G_2'$, $a_1 = \mathrm{diagv}(G_1' A G_1)$*, and* $a_2 = \mathrm{diagv}(G_2' A G_2)$.

**Proof:** Proof of (i) is simple. For ii), we have $u'Au = \mu' G_1' A G_1 \mu + v' G_2' A G_2 v + 2\mu' G_1' A G_2 v$. It is easy to see that the three terms are uncorrelated. Thus,

$$\mathrm{Var}(u'Au) = \mathrm{Var}(\mu' G_1' A G_1 \mu) + \mathrm{Var}(v' G_2' A G_2 v) + 4\mathrm{Var}(\mu' G_1' A G_2 v).$$

From Lemma 1, we obtain

$$\mathrm{Var}(\mu' G_1' A G_1 \mu) = \sigma_\mu^4 \kappa_\mu a_1' a_1 + 2\sigma_\mu^4 \mathrm{tr}(G_1' A G_1 G_1' A G_1), \text{ and}$$

$$\mathrm{Var}(v' G_2' A G_2 v) = \sigma_v^4 \kappa_v a_2' a_2 + 2\sigma_v^4 \mathrm{tr}(G_2' A G_2 G_2' A G_2).$$

Finally, it is easy to show that $\mathrm{Var}(\mu' G_1' A G_2 v) = \sigma_\mu^2 \sigma_v^2 G_1 G_1' A G_2 G_2' A$. Putting these three expressions together gives (ii). *Q.E.D.*

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