Journal of Statistical Computation and Simulation
Vol. 00, No. 00, February 2013, 1–14

Manuscript

Bias-Correction for Weibull Common Shape Estimation

Yan Shen[a*] and Zhenlin Yang[b]

[a] *Department of Statistics, School of Economics, Xiamen University, P.R.China*; [b] *School of Economics, Singapore Management University, Singapore*

Email: sheny@xmu.edu.cn; zlyang@smu.edu.sg

(*v0.0 released February 2013*)

A general method for correcting the bias of the maximum likelihood estimator (MLE) of the common shape parameter of Weibull populations, allowing a general right censorship, is proposed in this paper. Extensive simulation results show that the new method is very effective and robust in correcting the bias of the MLE, regardless of censoring mechanism, sample size, censoring proportion and number of populations involved. The method can be extended to more complicated Weibull models.

Keywords: Bias correction; Bootstrap; Right censoring; Stochastic expansion; Weibull models

AMS Subject Classification: 62NXX

1. Introduction

Weibull distribution is a parametric model popular in reliability and biostatistics. It plays a particularly important role for the analysis of failure time data. Suppose the failure time of an item, denoted by T, follows a Weibull distribution $\text{WB}(\alpha, \beta)$. Then the probability density function (pdf) of T has the form: $f(t) = \alpha^{-\beta}\beta t^{\beta-1}\exp\{-(t/\alpha)^{\beta}\}$, $t \geq 0$, where $\alpha > 0$ is the *scale parameter* and $\beta > 0$ is the *shape parameter*. The flexibility (e.g., pdf has many different shapes) and simplicity (e.g., cumulative distribution function has a closed form) are perhaps the main reasons for the popularity of Weibull distribution. As we know, the shape of Weibull distribution is determined primarily by its shape parameter β. Therefore, how to estimate β accurately has been one of the most important research focuses since the Weibull literature begun in 1951 (see, e.g., [1–10]). A more interesting and general problem may be the estimation of the common shape parameter of several Weibull populations. As indicated in [11], equality of Weibull shape parameters across different groups of individuals is an important and simplifying assumption in many applications. In Weibull regression models, such an assumption is analogous to the constant variance assumption in normal regression models.

The most common method to estimate the Weibull shape parameter is the maximum likelihood method. However, it is widely recognized that the maximum likelihood estimator (MLE) can be quite biased, in particular when sample size is small, data are heavily censored, or many Weibull populations are involved. To deal with this problem in the case of small complete sample, Hirose [3] proposed a bias-correction method for a single

*Corresponding author.

Weibull population by expanding the bias as a nonlinear function. In [5], a modified MLE (MMLE) is proposed for the shape parameter of a Weibull population through modifying the profile likelihood. This study was further generalized to the common shape parameter of several Weibull populations [6]. While these methods work well as shown by the Monte Carlo results, they can only be applied to complete data or Type I, Type II censored data. Furthermore, under Type I censored data, it is seen that their methods have rooms for further improvements, in particular when sample size is small and censorship is heavy.

In this paper, we propose a general method of bias-correction for the MLE of Weibull common shape parameter that allows a general right censoring mechanism, including Type I censoring, Type II censoring, random censoring, progressive Type II censoring, adaptive Type II progressive censoring, etc [12, 13]. The method is based on a third-order stochastic expansion for the MLE of the shape parameter [14] and a simple bootstrap procedure for estimating various expectations involved in the expansion [15]. Besides its simplicity, the method is also quite general as it is essentially applicable to any situation where a smooth estimating equation (not necessarily the concentrated score function) for the parameter of interest (common shape in this case) is available. Extensive simulation experiments are designed and carried out to assess the performance of the new method under different types of data. The results show that the new method is generally very effective and robust in correcting the bias of the MLE of β, regardless of censoring mechanism, sample size, censoring proportion and number of groups. Compared with the methods of [6], we see that the proposed method performs equally well under Type II censored data, but better under Type I censored data. Furthermore, the proposed method performs excellently under the random censoring mechanism; in contrast, the methods of [6] may not perform satisfactorily when sample size is small and censorship is heavy. This is because they are developed particularly under either Type I or Type II censoring mechanisms. With the new method, the bias-correction can be easily made up to third-order, and more complicated lifetime distribution models can be handled in a similar fashion.

The paper is organized as follows. Section 2 describes the general methodology. Section 3 presents the bias-correction method for the MLE of the common shape parameter of several Weibull populations. Section 4 presents Monte Carlo results. Section 5 concludes the paper.

2. The Method

In studying the finite sample properties of the parameter estimator, $\hat{\theta}_n$ say, defined as $\hat{\theta}_n = \arg\{\psi_n(\theta) = 0\}$ based on a sample of data of size n, Rilstone et al. [14] and Bao and Ullah [16] developed a stochastic expansion from which bias-correction on $\hat{\theta}_n$ can be made. Often, the vector of parameters θ contains a set of *linear parameters*, α say, and one *nonlinear parameter*, β say, in the sense that given β, the constrained estimator $\hat{\alpha}_n(\beta)$ of the vector α possesses an explicit expression and the estimation of β has to be done through numerical optimization. In this case, Yang [15] argued that it is more effective to work with the *concentrated estimating equation*: $\tilde{\psi}_n(\beta) = \psi_n(\hat{\alpha}_n(\beta), \beta)$, and to perform stochastic expansion and hence bias correction only on the nonlinear estimator defined by

$$\hat{\beta}_n = \arg\{\tilde{\psi}_n(\beta) = 0\}. \qquad (1)$$

Doing so, a multi-dimensional problem is reduced to a one-dimensional problem, and the additional variability from the estimation of the 'nuisance' parameters α is taken into account in bias-correcting the estimation of the nonlinear parameter β. Let $H_{rn}(\beta) = \frac{d^r}{d\beta^r}\tilde{\psi}_n(\beta), r = 1, 2, 3$. Under some general smoothness conditions on $\tilde{\psi}_n(\beta)$, [15] presented a third-order stochastic expansion for $\hat{\beta}_n$ at the true parameter value β_0,

$$\hat{\beta}_n - \beta_0 = a_{-1/2} + a_{-1} + a_{-3/2} + O_p(n^{-2}), \tag{2}$$

where $a_{-s/2}$ represents terms of order $O_p(n^{-s/2})$ for $s = 1, 2, 3$, and they are

$$a_{-1/2} = \Omega_n\tilde{\psi}_n,$$

$$a_{-1} = \Omega_n H^\circ_{1n} a_{-1/2} + \frac{1}{2}\Omega_n \mathrm{E}(H_{2n})(a^2_{-1/2}), \text{ and}$$

$$a_{-3/2} = \Omega_n H^\circ_{1n} a_{-1} + \frac{1}{2}\Omega_n H^\circ_{2n}(a^2_{-1/2}) + \Omega_n \mathrm{E}(H_{2n})(a_{-1/2}a_{-1}) + \frac{1}{6}\Omega_n \mathrm{E}(H_{3n})(a^3_{-1/2}),$$

where $\tilde{\psi}_n \equiv \tilde{\psi}_n(\beta_0)$, $H_{rn} \equiv H_{rn}(\beta_0), r = 1, 2, 3$, $H^\circ_{rn} = H_{rn} - E(H_{rn})$, and $\Omega_n = -E^{-1}(H_{1n})$.

The above stochastic expansion leads immediately to a second-order bias $B_2(\hat{\beta}_n) = E(a_{-1/2} + a_{-1})$, and a third-order bias $B_3(\hat{\beta}_n) = E(a_{-1/2} + a_{-1} + a_{-3/2})$, which may be used for performing bias corrections on $\hat{\beta}_n$, provided that analytical expressions for the various expected quantities in the expansion can be derived so that they can be estimated through a plug-in method. Several applications of this plug-in method to some simple models have appeared in the literature: [16] for a pure spatial autoregressive process, [17] for time-series models, [18] for a Poisson regression model, and [19] for an exponential regression.

However, for slightly more complicated models such as the Weibull model considered in this paper, $B_2(\hat{\beta}_n)$ and $B_3(\hat{\beta}_n)$ typically do not possess analytical expressions and the plug-in method cannot be applied. To overcome this major difficulty, a general non-parametric bootstrap method was proposed in [15] to estimate those expectations, which sheds light on the parametric bootstrap procedure designed in this work. Kundhi and Rilstone [18] considered standard bootstrap correction: bootstrapping $\hat{\beta}_n$ directly for bias-reduction. However, their Monte Carlos results showed that this method does not work as well compared with the analytical method they proposed.

It was argued that in many situations there is a sole nonlinear parameter (like the shape parameter in Weibull models) that is the main source of bias in model estimation, and that given this parameter the estimation of other parameters incurs much less biasness and usually can be done analytically as well [15]. Thus, for the purpose of bias-correction, it may only be necessary to focus on the estimation of this parameter.

3. Bias-Correction for Weibull Common Shape Estimation

Consider the case of estimating the common shape parameter of several Weibull populations based on the maximum likelihood estimation (MLE) method. For the ith Weibull population WB(α_i, β), $i = 1, 2, \ldots, k$, let t_{ij} $(j = 1, 2, \ldots, n_i)$ be the observed failure times or censoring times of n_i independent and identically distributed (iid) items in the sample, $\delta_{ij}(j = 1, 2, \ldots, n_i)$ be the failure indicators with $\delta_{ij} = 1$ for the actual failure time and $\delta_{ij} = 0$ for the censored lifetime, $r_i = \sum_{j=1}^{n_i}\delta_{ij}$ be the number of observed failure times. Also let $m = \sum_{i=1}^{k} r_i$ be the total number of observed failure times in all k samples and $n = \sum_{i=1}^{k} n_i$ be the total number of items. Let $\theta = (\alpha_1, \ldots, \alpha_k, \beta)'$. The log-likelihood function can be written as

$$\ell_n(\theta) = m\log\ \beta + (\beta-1)\sum_{i=1}^{k}\sum_{j=1}^{n_i}\delta_{ij}\log t_{ij} - \beta\sum_{i=1}^{k} r_i\log\alpha_i - \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left\{\frac{t_{ij}}{\alpha_i}\right\}^{\beta} \quad (3)$$

Maximizing $\ell_n(\theta)$ with respect to α_i $(i = 1, 2, \ldots, k)$ gives the constrained MLEs:

$$\hat{\alpha}_{n,i}(\beta) = \left\{\frac{1}{r_i}\sum_{j=1}^{n_i} t_{ij}^{\beta}\right\}^{1/\beta}, i = 1, \ldots, k. \quad (4)$$

Substituting the $\hat{\alpha}_{n,i}(\beta)$ back into (3) yields the concentrated log-likelihood function

$$\ell_n^c(\beta) = \sum_{i=1}^{k} r_i\log r_i - m + m\log\beta + (\beta-1)\sum_{i=1}^{k}\sum_{j=1}^{n_i}\delta_{ij}\log t_{ij} - \sum_{i=1}^{k} r_i\log\sum_{j=1}^{n_i} t_{ij}^{\beta}. \quad (5)$$

Maximizing $\ell_n^c(\beta)$, or equivalently solving $\tilde{\psi}_n(\beta) \equiv n^{-1}\frac{d}{d\beta}\ell_n^c(\beta) = 0$, where

$$\tilde{\psi}_n(\beta) \equiv \frac{m}{n\beta} + \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\delta_{ij}\log t_{ij} - \frac{1}{n}\sum_{i=1}^{k} r_i\left\{\frac{\sum_{j=1}^{n_i} t_{ij}^{\beta}\log t_{ij}}{\sum_{j=1}^{n_i} t_{ij}^{\beta}}\right\}, \quad (6)$$

gives the unconstrained MLE $\hat{\beta}_n$ of β, and hence the unconstrained MLEs of α_i as $\hat{\alpha}_{n,i} \equiv \hat{\alpha}_{n,i}(\hat{\beta}_n)$, $i = 1, 2, \ldots, k$. See [11, 12] for details on likelihood estimation of censored failure time models, including Weibull models.

It is well known that the MLE $\hat{\beta}_n$ can be significantly biased for small sample sizes, heavy censorship, or complicated Weibull models. Such a bias would make the subsequent statistical inferences inaccurate. Various attempts have been made to reduce the bias of the Weibull shape estimation. The approach adopted by [5] and [6] is to modify the concentrated log-likelihood defined in (5). Alternative methods can be found in, e.g., [2, 7, 9].

As discussed in the introduction, the approach of [5] and [6] applies only to Type I and Type II right censored data. Apparently, the log-likelihood function (3) is not restricted to these types of censoring. Random censoring and progressively Type II censoring, etc, are also included [12]. It is thus of a great interest to develop a general method that works for any type of censoring mechanism. In theory, the method outlined in Section 2 indeed works for any type of censoring mechanism as long as log-likelihood function possesses explicit derivatives up to fourth-order. In this paper, we focus on the general right censoring mechanism for estimating the common shape parameter of Weibull populations, where the concentrated estimating function for the common shape parameter is defined in (6).

3.1 *The 2nd and 3rd-order bias corrections*

Note that the function $\tilde{\psi}_n(\beta)$ defined in (6) is of order $O_p(n^{-1/2})$ for regular MLE problems. Let $H_{rn}(\beta) = \frac{d^r}{d\beta^r}\tilde{\psi}_n(\beta), r = 1, 2, 3$. We have after some algebra,

$$H_{1n}(\beta) = \sum_{i=1}^{k} \frac{r_i}{n}\left(-\frac{1}{\beta^2} - \frac{\Lambda_{2i}}{T_i} + \frac{\Lambda_{1i}^2}{T_i^2}\right),$$

$$H_{2n}(\beta) = \sum_{i=1}^{k} \frac{r_i}{n}\left(\frac{2}{\beta^3} - \frac{\Lambda_{3i}}{T_i} + \frac{3\Lambda_{1i}\Lambda_{2i}}{T_i^2} - \frac{2\Lambda_{1i}^3}{T_i^3}\right),$$

$$H_{3n}(\beta) = \sum_{i=1}^{k} \frac{r_i}{n}\left(-\frac{6}{\beta^4} - \frac{\Lambda_{4i}}{T_i} + \frac{4\Lambda_{1i}\Lambda_{3i}}{T_i^2} + \frac{3\Lambda_{2i}^2}{T_i^2} - \frac{12\Lambda_{1i}^2\Lambda_{2i}}{T_i^3} + \frac{6\Lambda_{1i}^4}{T_i^4}\right),$$

where $T_i \equiv T_i(\beta) = \sum_{j=1}^{n_i} t_{ij}^{\beta}$, and $\Lambda_{si} \equiv \Lambda_{si}(\beta) = \sum_{j=1}^{n_i} t_{ij}^{\beta}(\log t_{ij})^s$, $s = 1, 2, 3, 4$, $i = 1, \ldots, k$. The validity of the asymptotic expansion (2) depends crucially on the $\sqrt{n}$-consistency of $\hat{\beta}_n$, and the proper stochastic behavior of the various ratios of the quantities T_i and Λ_{si}. Along the lines of the general results of [15], the following set of simplified regularity conditions is sufficient.

Assumption 1. *The true β_0 is an interior point of an open subset of the real line.*

Assumption 2. $\frac{1}{n}\ell_n^c(\beta)$ *converges in probability to a nonstochastic function $\ell(\beta)$ uniformly in β in an open neighborhood of β_0, and $\ell(\beta)$ attains the global maximum at β_0.*

Assumption 3. *(i)* $\lim_{n\to\infty} \mathrm{Var}\left(\sqrt{n}\tilde{\psi}_n(\beta_0)\right)$ *exists and (ii)* $H_{1n}(\tilde{\beta}_n) \xrightarrow{p} c(\beta_0)$, $-\infty < c(\beta_0) < 0$, *for any sequence $\tilde{\beta}_n$ such that $\tilde{\beta}_n \xrightarrow{p} \beta_0$.*

Assumptions 1-3 are sufficient conditions for the $\sqrt{n}$-consistency of $\hat{\beta}_n$ (see [20]). It is easy to see that the concentrated log-likelihood function $\ell_n^c(\beta)$ is globally concave, hence these assumptions are not restrictive. The assumptions given below ensure the proper behavior of the higher order terms.

Assumption 4. *For each i, (i)* $\mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{1i}^5(\beta_0)}{T_i^5(\beta_0)}]$, $\mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{2i}^2(\beta_0)}{T_i^2(\beta_0)}]$, $\mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{3i}^2(\beta_0)}{T_i^2(\beta_0)}]$, *and* $\mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{4i}(\beta_0)}{T_i(\beta_0)}]$ *exist; (ii)* $\frac{r_i}{n}\frac{\Lambda_{si}(\beta_0)}{T_i(\beta_0)} = \mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{si}(\beta_0)}{T_i(\beta_0)}] + O_p(n^{-\frac{1}{2}}), s = 1, \ldots, 4$*; and (iii)* $\frac{r_i}{n}\left|\frac{\Lambda_{si}(\beta)}{T_i(\beta)} - \frac{\Lambda_{si}(\beta_0)}{T_i(\beta_0)}\right| = |\beta - \beta_0|X_{n,s}$, $\mathrm{E}|X_{n,s}| < c_s < \infty$, $s = 1, \ldots, 4$, *for β in a neighborhood of β_0.*

Theorem 3.1. *Under Assumptions 1-4, we have, respectively, the 2nd-order ($O(n^{-1})$) bias and the 3rd-order ($O(n^{-3/2})$) bias for the MLE $\hat{\beta}_n$ of the shape parameter β:*

$$B_2(\hat{\beta}_n) = 2\Omega_n E(\tilde{\psi}_n) + \Omega_n^2 E(H_{1n}\tilde{\psi}_n) + \frac{1}{2}\Omega_n^3 E(H_{2n})E(\tilde{\psi}_n^2), \tag{7}$$

$$\begin{aligned} B_3(\hat{\beta}_n) = {} & \Omega_n E(\tilde{\psi}_n) + 2\Omega_n^2 E(H_{1n}\tilde{\psi}_n) + \Omega_n^3 E(H_{2n})E(\tilde{\psi}_n^2) + \Omega_n^3 E(H_{1n}^2\tilde{\psi}_n), \\ & + \frac{1}{2}\Omega_n^3 E(H_{2n}\tilde{\psi}_n^2) + \frac{3}{2}\Omega_n^4 E(H_{2n})E(H_{1n}\tilde{\psi}_n^2) + \frac{1}{2}\Omega_n^5 (EH_{2n})^2 E(\tilde{\psi}_n^3) \\ & + \frac{1}{6}\Omega_n^4 E(H_{3n})E(\tilde{\psi}_n), \end{aligned} \tag{8}$$

where $\tilde{\psi}_n \equiv \tilde{\psi}_n(\beta_0)$, $H_{rn} \equiv H_{rn}(\beta_0)$, $r = 1, 2, 3$, *and* $\Omega_n = -E^{-1}(H_{1n})$.

The proof of Theorem 1 is given in the Appendix A. As noted in [15], $\tilde{\psi}_n(\beta)$ represents the concentrated estimating equation, which incorporates the extra variabilites resulted from the estimation of the nuisance parameters α_i's, hence, $E[\tilde{\psi}_n(\beta_0)] \neq 0$ at the true value β_0 of β. We show in the proof of Theorem 1 in Appendix A that $E[\tilde{\psi}_n(\beta_0)] = O(n^{-1})$. This is in contrast to the case of using the joint estimating equation, $\psi_n(\theta) = 0$, introduced at the beginning of Section 2, for which we have $E[\psi_n(\theta_0)] = 0$.

The above equations (7) and (8) lead immediately to the second- or third-order bias-corrected MLEs of β as

$$\hat{\beta}_n^{bc2} = \hat{\beta}_n - \hat{B}_2(\hat{\beta}_n) \quad \textit{and} \quad \hat{\beta}_n^{bc3} = \hat{\beta}_n - \hat{B}_2(\hat{\beta}_n) - \hat{B}_3(\hat{\beta}_n). \tag{9}$$

provided that the estimates of the bias terms, $\hat{B}_2(\hat{\beta}_n)$ and $\hat{B}_3(\hat{\beta}_n)$, are readily available, and that they are valid in the sense that the estimation of the biases does not introduce extra variability that is higher than the remainder. Obviously, the analytical expressions of $B_2(\hat{\beta}_n)$ and $B_3(\hat{\beta}_n)$ are not available, and hence the usual 'plug-in' method does not work. We introduce a parametric bootstrap method to overcome this difficulty and give formal justifications on its validity in next section.

Substituting (9) back into (4) gives the corresponding estimators of scale parameters as $\hat{\alpha}_n^{bc2} \equiv \hat{\alpha}_n(\hat{\beta}_n^{bc2})$ and $\hat{\alpha}_n^{bc3} \equiv \hat{\alpha}_n(\hat{\beta}_n^{bc3})$. If there is only one complete sample composed of n failure times, denoted by $t_j, j = 1, \ldots, n$. The associated concentrated estimating function $\tilde{\psi}_n$ reduces to

$$\tilde{\psi}_n(\beta) = \frac{1}{\beta} + \frac{1}{n}\sum_{j=1}^{n} \log\, t_j - (\sum_{j=1}^{n} t_j^{\beta})^{-1} \sum_{j=1}^{n} t_j^{\beta} \log t_j.$$

Similarly, the quantities $H_{1n}(\beta)$,$H_{2n}(\beta)$ and $H_{3n}(\beta)$ reduce to

$$H_{1n}(\beta) = -\frac{1}{\beta^2} - \frac{\Lambda_2}{T} + \frac{\Lambda_1^2}{T^2},$$

$$H_{2n}(\beta) = \frac{2}{\beta^3} - \frac{\Lambda_3}{T} + \frac{3\Lambda_1\Lambda_2}{T^2} - \frac{2\Lambda_{1n}^3}{T_n^3},$$

$$H_{3n}(\beta) = -\frac{6}{\beta^4} - \frac{\Lambda_4}{T} + \frac{4\Lambda_1\Lambda_3}{T^2} + \frac{3\Lambda_2^2}{T^2} - \frac{12\Lambda_1^2\Lambda_2}{T^3} + \frac{6\Lambda_1^4}{T^4},$$

respectively, where $T \equiv T(\beta) = \sum_{j=1}^{n} t_j^{\beta}$, and $\Lambda_s \equiv \Lambda_s(\beta) = \sum_{j=1}^{n} t_j^{\beta}(\log t_j)^s$, $s = 1, 2, 3, 4$.

3.2 *The bootstrap method for practical implementations*

A typical way of obtaining the estimate of the bias term is to find its analytical expression, and then plug-in the estimates for the parameters [14, 16, 17]. However, this approach often runs into difficulty if more complicated models are considered, simply because this analytical expression is either unavailable, or difficult to obtain, or too tedious to be practically tractable. Apparently, the problem we are considering falls into the first category. This indicates that the usefulness of stochastic expansion in conducting bias correction is rather limited if one does not have a general method for estimating the expectations of various quantities involved in the expansions. Thus, alternative methods are desired. In working with the bias-correction problem for a general spatial autoregressive model, [15] proposed a simple but rather general nonparametric bootstrap method, leading to bias-corrected estimators of the spatial parameter that are nearly unbiased.

The situation we are facing now is on one hand simpler than that of [15] in that the distribution of the model is completely specified, but on the other hand more complicated in that various censoring mechanisms are allowed. Following the general idea of [15] and taking advantage of a known distribution, we propose a parametric bootstrap method for estimating the expectations involved in (7) and (8):

(1) Compute the MLEs $\hat{\beta}_n$ and $\hat{\alpha}_{n,i}, i = 1, \ldots, k$, based on the original data;
(2) For each Weibull population $\text{WB}(\hat{\alpha}_{n,i}, \hat{\beta}_n), i = 1, \ldots, k$, generate n_i random data, censor them according to the original censoring mechanism if necessary, and denote the generated (bootstrapped) data as $\{t_{ij}^b, j = 1, \ldots, n_i, i = 1, \ldots, k\}$;
(3) Compute $\tilde{\psi}_{n,b}(\hat{\beta}_n)$, $H_{1n,b}(\hat{\beta}_n)$, $H_{2n,b}(\hat{\beta}_n)$, and $H_{3n,b}(\hat{\beta}_n)$ based on the bootstrapped data $\{t_{ij}^b, j = 1, \ldots, n_i, i = 1, \ldots, k\}$;
(4) Repeat the steps (2)-(3) B times ($b = 1, \ldots$,B) to get sequences of bootstrapped values for $\tilde{\psi}_n(\hat{\beta}_n)$, $H_{1n}(\hat{\beta}_n)$, $H_{2n}(\hat{\beta}_n)$, and $H_{3n}(\hat{\beta}_n)$.

The bootstrap estimates of various expectations in (7) and (8) thus follow. For example, the bootstrap estimates for $E(\tilde{\psi}_n^2)$ and $E(H_{1n}\tilde{\psi}_n)$ are, respectively,

$$\hat{E}(\tilde{\psi}_n^2) = \tfrac{1}{B}\textstyle\sum_{b=1}^{B}[\tilde{\psi}_{n,b}(\hat{\beta}_n)]^2 \quad \text{and} \quad \hat{E}(H_{1n}\tilde{\psi}_n) = \tfrac{1}{B}\textstyle\sum_{b=1}^{B} H_{1n,b}(\hat{\beta}_n)\tilde{\psi}_{n,b}(\hat{\beta}_n).$$

The estimates of other expectations can be obtained in a similar fashion, and hence the the bootstrap estimates of the second- and third-order biases, denoted as $\hat{B}_2(\hat{\beta}_n)$ and $\hat{B}_3(\hat{\beta}_n)$.

Corollary 3.2. *Under the assumptions of Theorem 1, if further (i)* $\frac{\partial^k}{\partial\theta_0^k}B_i(\hat{\beta}_n) \sim B_i(\hat{\beta}_n)$, *for* $k = 1, 2, i = 2, 3$, *and (ii) a quantity bounded in probability has a finite expectation, then the bootstrap estimates of the 2nd- and 3rd-order biases for the MLE* $\hat{\beta}_n$ *are such that:*

$$\hat{B}_2(\hat{\beta}_n) = B_2(\hat{\beta}_n) + O_p(n^{-2}) \quad \textit{and} \quad \hat{B}_3(\hat{\beta}_n) = B_3(\hat{\beta}_n) + O_p(n^{-2}),$$

where $\sim$ *indicates that the two quantities are of the same order of magnitude.*

The results of Corollary 1 say that estimating the bias terms using the bootstrap method only (possibly) introduces additional bias of order $O_p(n^{-2})$ or higher. This makes the third-order bootstrap bias correction valid. Thus, the validity of the second-order bootstrap bias correction follows. The proof of Corollary 1 is put in the Appendix B.

4. Monte Carlo Simulation

To investigate the finite sample performance of the proposed method of bias-correcting the MLE of Weibull common shape parameter, extensive Monte Carlo simulations are performed. Tables 1-12 summarize the empirical mean, root-mean-squared-error (rmse) and standard error (se) of the original and bias-corrected MLEs under various combinations of models, censoring schemes, and the values of n, α, β and p, where p denotes the non-censoring proportion. We consider four scenarios: (i) complete samples, (ii) Type I censored samples, (iii) Type II censored samples, and (iv) randomly censored samples. Under each scenario, the numbers of groups considered are $k = 1, 2$ and 8, respectively. Furthermore, we also compare the proposed method with the modified MLE (MMLE) discussed in [5] and [6].

In the entire simulation study, the parametric bootstrapping procedure is adopted, which (i) fits original data to Weibull model, (ii) draws random samples from this fitted

Table 1. Empirical mean [rmse](se) of MLE-type estimators of β, complete data, $k = 1$

n	β	$\hat{\beta}_{\mathrm{MLE}}$	$\hat{\beta}^{bc2}_{\mathrm{MLE}}$	$\hat{\beta}^{bc3}_{\mathrm{MLE}}$	$\hat{\beta}_{\mathrm{MMLE}}$
10	0.5	0.584 [.198](.179)	0.500 [.154](.154)	0.500 [.154](.154)	0.508 [.155](.155)
	0.8	0.934 [.311](.281)	0.800 [.241](.241)	0.800 [.241](.241)	0.812 [.244](.244)
	1.0	1.164 [.384](.347)	0.997 [.298](.298)	0.997 [.298](.298)	1.012 [.302](.301)
	2.0	2.344 [.795](.717)	2.008 [.615](.615)	2.008 [.615](.615)	2.037 [.622](.621)
	5.0	5.859 [1.95](1.75)	5.020 [1.50](1.50)	5.020 [1.50](1.50)	5.094 [1.52](1.52)
20	0.5	0.539 [.111](.104)	0.502 [.097](.097)	0.502 [.097](.097)	0.505 [.098](.097)
	0.8	0.863 [.177](.166)	0.803 [.154](.154)	0.804 [.154](.154)	0.809 [.155](.155)
	1.0	1.074 [.215](.201)	0.999 [.188](.188)	1.000 [.188](.188)	1.006 [.189](.189)
	2.0	2.152 [.435](.407)	2.001 [.379](.379)	2.004 [.379](.379)	2.016 [.382](.381)
	5.0	5.375 [1.10](1.03)	4.998 [.959](.959)	5.004 [.960](.960)	5.035 [.966](.965)
50	0.5	0.514 [.060](.058)	0.499 [.057](.057)	0.500 [.057](.057)	0.501 [.057](.057)
	0.8	0.823 [.097](.095)	0.800 [.092](.092)	0.800 [.092](.092)	0.802 [.092](.092)
	1.0	1.028 [.122](.118)	1.000 [.115](.115)	1.000 [.115](.115)	1.003 [.115](.115)
	2.0	2.060 [.244](.236)	2.003 [.230](.230)	2.004 [.230](.230)	2.009 [.230](.230)
	5.0	5.141 [.595](.579)	4.998 [.563](.563)	5.000 [.563](.563)	5.014 [.564](.564)
100	0.5	0.507 [.041](.040)	0.500 [.040](.040)	0.500 [.040](.040)	0.501 [.040](.040)
	0.8	0.812 [.066](.065)	0.800 [.064](.064)	0.801 [.064](.064)	0.802 [.064](.064)
	1.0	1.015 [.082](.080)	1.001 [.079](.079)	1.001 [.079](.079)	1.002 [.079](.079)
	2.0	2.030 [.163](.160)	2.002 [.158](.158)	2.002 [.158](.158)	2.005 [.158](.158)
	5.0	5.070 [.410](.404)	5.000 [.398](.398)	5.000 [.398](.398)	5.008 [.399](.399)

Table 2. Empirical mean [rmse](se) of MLE-type estimators of β, complete data, $k = 2$

n	β	$\hat{\beta}_{\mathrm{MLE}}$	$\hat{\beta}^{bc2}_{\mathrm{MLE}}$	$\hat{\beta}^{bc3}_{\mathrm{MLE}}$	$\hat{\beta}_{\mathrm{MMLE}}$
10	0.5	0.556 [.123](.110)	0.497 [.099](.098)	0.495 [.098](.098)	0.502 [.099](.099)
	0.8	0.895 [.204](.180)	0.800 [.161](.161)	0.797 [.161](.161)	0.808 [.163](.163)
	1.0	1.117 [.250](.221)	0.999 [.198](.198)	0.995 [.197](.197)	1.008 [.199](.199)
	2.0	2.230 [.492](.435)	1.994 [.390](.390)	1.986 [.388](.388)	2.013 [.392](.392)
	5.0	5.588 [1.26](1.11)	4.995 [.995](.995)	4.975 [.992](.991)	5.043 [1.00](1.00)
20	0.5	0.528 [.075](.070)	0.501 [.066](.066)	0.501 [.066](.066)	0.503 [.066](.066)
	0.8	0.845 [.119](.110)	0.802 [.104](.104)	0.801 [.104](.104)	0.805 [.105](.105)
	1.0	1.053 [.149](.140)	0.999 [.133](.133)	0.999 [.132](.132)	1.003 [.133](.133)
	2.0	2.110 [.297](.276)	2.002 [.262](.262)	2.001 [.262](.262)	2.011 [.263](.263)
	5.0	5.257 [.720](.673)	4.988 [.639](.639)	4.985 [.639](.638)	5.009 [.641](.641)
50	0.5	0.511 [.043](.041)	0.501 [.040](.040)	0.501 [.040](.040)	0.502 [.040](.040)
	0.8	0.817 [.067](.065)	0.801 [.064](.064)	0.801 [.064](.064)	0.802 [.064](.064)
	1.0	1.021 [.085](.082)	1.000 [.081](.081)	1.000 [.081](.081)	1.002 [.081](.081)
	2.0	2.040 [.169](.164)	1.999 [.161](.161)	1.999 [.161](.161)	2.003 [.161](.161)
	5.0	5.105 [.414](.400)	5.003 [.393](.393)	5.002 [.393](.393)	5.011 [.393](.393)
100	0.5	0.505 [.028](.028)	0.500 [.028](.028)	0.500 [.028](.028)	0.500 [.028](.028)
	0.8	0.808 [.046](.045)	0.800 [.045](.045)	0.800 [.045](.045)	0.801 [.045](.045)
	1.0	1.010 [.057](.056)	1.000 [.055](.055)	1.000 [.055](.055)	1.001 [.055](.055)
	2.0	2.019 [.114](.113)	1.999 [.112](.112)	1.999 [.112](.112)	2.001 [.112](.112)
	5.0	5.051 [.287](.282)	5.000 [.279](.280)	5.000 [.280](.280)	5.004 [.279](.279)

distribution with the size being the same as the original sample size, and then (*iii*) censors the data in the identical way as the original data. For all the experiments, 10,000 replications are run in each simulation and the number of bootstrap is set to be 699. Also, for convenience, the values of p are set as 0.3, 0.5 and 0.7, respectively, so that np are integers for $n = 10$, 20, 50, and 100.

4.1 *Complete samples*

Tables 1-3 present the results corresponding to the cases of complete samples with $k = 1, 2, 8$. From the tables, we see that the second-order and third-order bias-corrected MLEs, $\hat{\beta}_{bc2}$ and β_{bc3}, are generally nearly unbiased and are much superior to the original MLE $\hat{\beta}$ regardless of the values of n and k. Some details are: (*i*) $\hat{\beta}$ always over-estimates the shape parameter, (*ii*) $\hat{\beta}_{bc2}$ and $\hat{\beta}_{bc3}$ have smaller rmses and ses compared with these of $\hat{\beta}$, (*iii*) the second-order bias-correction seems sufficient and a higher order bias correction may not be necessary, at least for the cases considered in this work, and (*iv*) $\hat{\beta}_{bc2}$ and $\hat{\beta}_{bc3}$ are generally better than the MMLE of [6], except $n = 10$, $k = 8$.

Table 3. Empirical mean [rmse](se) of MLE-type estimators of β, complete data, $k = 8$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
10	0.5	0.540 [.065](.051)	0.498 [.047](.047)	0.497 [.047](.047)	0.501 [.047](.047)
	0.8	0.865 [.104](.082)	0.797 [.075](.075)	0.795 [.075](.075)	0.802 [.076](.076)
	1.0	1.080 [.129](.102)	0.995 [.094](.094)	0.993 [.094](.094)	1.001 [.095](.095)
	2.0	2.162 [.263](.207)	1.993 [.191](.191)	1.989 [.191](.191)	2.004 [.192](.192)
	5.0	5.395 [.642](.506)	4.973 [.468](.467)	4.964 [.468](.467)	5.002 [.469](.469)
20	0.5	0.519 [.038](.034)	0.499 [.032](.032)	0.499 [.032](.032)	0.500 [.032](.032)
	0.8	0.830 [.061](.053)	0.799 [.051](.051)	0.799 [.051](.051)	0.801 [.051](.051)
	1.0	1.038 [.077](.067)	1.000 [.065](.065)	0.999 [.065](.065)	1.001 [.065](.065)
	2.0	2.076 [.153](.133)	1.998 [.128](.128)	1.998 [.128](.128)	2.002 [.128](.128)
	5.0	5.188 [.382](.333)	4.995 [.321](.321)	4.993 [.321](.321)	5.004 [.321](.321)
50	0.5	0.507 [.021](.020)	0.500 [.020](.020)	0.500 [.020](.020)	0.500 [.020](.020)
	0.8	0.812 [.034](.032)	0.800 [.032](.032)	0.800 [.032](.032)	0.800 [.032](.032)
	1.0	1.014 [.043](.040)	1.000 [.040](.040)	1.000 [.040](.040)	1.000 [.040](.040)
	2.0	2.030 [.087](.081)	2.001 [.080](.080)	2.000 [.080](.080)	2.002 [.080](.080)
	5.0	5.074 [.216](.203)	5.001 [.200](.200)	5.001 [.200](.200)	5.004 [.200](.200)
100	0.5	0.504 [.014](.014)	0.500 [.014](.014)	0.500 [.014](.014)	0.500 [.014](.014)
	0.8	0.806 [.023](.023)	0.800 [.023](.023)	0.800 [.023](.023)	0.800 [.022](.022)
	1.0	1.007 [.029](.028)	1.000 [.028](.028)	1.000 [.028](.028)	1.000 [.028](.028)
	2.0	2.015 [.058](.056)	2.000 [.056](.056)	2.000 [.056](.056)	2.001 [.056](.056)
	5.0	5.035 [.145](.141)	4.999 [.140](.104)	4.999 [.140](.140)	5.000 [.140](.140)

4.2 *Type I censoring*

The mechanism for generating the Type I censored data is simply described as follows. For a given Weibull population and a given non-censoring proportion p, set the censoring time C as the pth quantile of Weibull distribution and then generate a random sample $\{T_1, \ldots, T_n\}$. The observed lifetimes are $\min(T_j, C)$ and the failure indicators are $\delta_j = 1$ if $T_j < C$ and 0 otherwise. To generate bootstrap samples from the ith estimated Weibull population obtained from a given Monte Carlo replication, the censoring time is set to be the (r_i/n_i)th quantile of the estimated distribution rather than the given censoring time in the real sample. This is because, for the bootstrap procedure to work, the bootstrap world must be set up so that it mimics the real world. Following this fundamental principle, the same censoring mechanism should be followed for generating the bootstrap sample.

Tables 4-6 reports Monte Carlo results for comparing the four estimators under Type I censoring, where p denotes the proportion that the data is observed in a sample (non-censoring proportion). From Tables 4-6, we observe that the two bias-corrected estimators, $\hat{\beta}_{bc2}$ and $\hat{\beta}_{bc3}$, have a much better finite sample performance in terms of the empirical mean, rmse and se, than the original MLE $\hat{\beta}$ in almost all combinations simulated, except the combination of $k = 8$, $p = 0.3$, $\beta = 2.0, 5.0$ with larger bias. Similar to the previous scenario, the second-order bias-corrected MLE $\hat{\beta}_{bc2}$ still shows the superiority over the third-order one; it dramatically reduces the bias rooted in the original MLE $\hat{\beta}$ and thus provides much more accurate estimation. Generally, the sample size, the non-censoring proportion and the number of groups do not affect the performance of $\hat{\beta}_{bc2}$ and $\hat{\beta}_{bc3}$ much. A notable phenomena worth mentioning is that $\hat{\beta}_{bc3}$ seems to perform best for the case of $k = 1$, $p = 0.3$, $n = 10$. Furthermore, the two bias-corrected estimators have smaller variabilities compared with $\hat{\beta}$.

As pointed out in [6], with several Type I censored samples, the MMLE does not perform so well as with complete and Type II censored data. Here we see again that compared to true parameter value, the MMLE has obvious positive bias for $k = 1, 2$ and negative bias for $k = 8$. In contrast, the proposed bias-corrected MLEs, especially $\hat{\beta}_{bc2}$, show very good performances for Type I censoring and are robust against the number of groups k and the sample sizes n_i. Thus, to deal with Type I censored data, the bias-corrected MLEs is strongly recommended.

4.3 *Type II censoring*

To generate Type II censored data, a random sample $\{T_1, \ldots, T_n\}$ is drawn from each of k Weibull distributions. As the non-censoring proportion p now represents the proportion of the observed lifetimes, i.e. the ratio of the number of observed lifetimes to the sample size n, the censoring time C should be chosen to be T_{np}, the (np)th smallest exact Weibull failure time. Thus, the observed Type II censored lifetimes are $\{\min(T_j, C), j = 1 \ldots, n\}$ and the failure indicators are $\delta_j, j = 1, \ldots, n$, where $\delta_j = 1$ if $T_j < C$ and 0 otherwise. In an identical manner as the original data being generated, Type II censored bootstrap samples are generated from the estimated Weibull populations with the bootstrap censoring time C^* being the (np)th smallest order statistic in the bootstrap sample.

The Monte Carlo results reported in Tables 7-9 show that the accuracy of estimation is in general greatly improved after bias correction, as the bias-corrected estimators $\hat{\beta}_{bc2}$ and $\hat{\beta}_{bc3}$ have the means much closer to the true β than the MLE $\hat{\beta}$, as well as the smaller rmses and standard errors. Again, the simulation results indicate that the use of the second-order bias-corrected MLE $\hat{\beta}_{bc2}$ seems sufficient for this situation as well. Although the third-order estimator $\hat{\beta}_{bc3}$ has the smallest errors, its bias is relatively large and can be rather substantial for heavy censoring.

The two bias-corrected estimators are superior to the original MLE, but they do not outperform the MMLE in regard of bias, in particular when $k = 2, 8$, $n = 10, 20$. This imply that the MMLE is preferred in the case of Type II censoring. Occasionally, the MMLE would have slightly larger errors or less efficiency. But overall, the MMLE is a better choice than others in Type II censoring.

4.4 *Random censoring*

We also consider the case of samples with random censoring. To simplify our descriptions, we assume $k = 1$. For each Monte Carlo replication, two sets of observations $T = \{T_1, \ldots, T_n\}$ and $C = \{C_1, \ldots, C_n\}$ are generated, with T_i from a Weibull distribution and C_i from Uniform$(0.5q_p, 1.5q_p)$, where q_p is the pth quantile of the Weibull population involved. The observed lifetimes $Y = \{\min(T_i, C_i), i = 1 \ldots, n\}$ and the failure indicators $\{\delta_i\}$ are recorded. To generate a bootstrap sample, $Y_j^*, j = 1, \ldots, n$, we follow a procedure described in [21]:

1) generate $T_1^*, \ldots, T_n^*$ independently from $WB(\hat{\alpha}, \hat{\beta})$ where $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs;
2) sort the observed lifetimes Y as $(Y_{(1)}, ..., Y_{(n)})$ and denote the corresponding failure indicators by $(\delta_{(1)}, ..., \delta_{(n)})$;
3) if $\delta_{(i)} = 0$, then the bootstrap censoring time $C_i^* = Y_{(i)}$; otherwise C_i^* is a value randomly chosen from $(Y_{(i)}, ..., Y_{(n)})$;
4) set $Y_j^* = \min(T_j^*, C_j^*)$, for $j = 1, \ldots, n$.

Monte Carlo results are summarized in Tables 10-12. From the results we see that the two bias-corrected MLEs $\hat{\beta}_{bc2}$ and $\hat{\beta}_{bc3}$ can reduce greatly the bias as well as the variability of $\hat{\beta}$ in all combinations under the random censoring mechanism. Different from the previous censoring scenarios, the third-order bias-corrected estimator in this case shows a greater accuracy and robustness as compared with the second-order bias-corrected estimator in almost all combinations for $k = 1$ and with large β values for $k = 2, 8$. In other parameter settings, the two bias-corrected MLEs performs almost equally well.

Since there is no specific MMLE designed for the randomly censored data, the MMLEs designed for Type I and Type II censoring schemes are included in the Monte Carlo experiments for comparison purpose. The two estimators are denoted, respectively, by MMLE-I and MMLE-II. Indeed, from the results we see that under a 'wrong censoring scheme', the MMLEs do not seem to be able to provide a reliable estimation of β. In general, MMLE-I can significantly over-estimate β when it is large, whereas MMLE-II can significantly under-estimate β when it is small. Based on these observations, we may conclude that the proposed method is more robust to the censoring schemes.

Remark: Theoretically, the third-order bias-corrected MLE should perform better than the second-order bias-corrected MLE. This is clearly observed from the Monte Carlo results corresponding to random censoring mechanism. However, in cases of Type I and II censored data with a very small sample and a very heavy censorship, the performance of the third-order bias-corrected MLEs may deteriorate slightly due to the numerical instability in the third-order correction term. In these, it is recommended to use the second-order bias-corrected MLEs when the sample size is very small and the censorship is very heavy.

5. Discussion and Conclusion

In this paper, we proposed a general method for correcting the bias of the MLE of the common Weibull shape parameter that allows a general censoring mechanism. The method is based a third-order stochastic expansion for the MLE of the common shape parameter, and a simple bootstrap procedure that allows easy estimation of various expected quantities involved in the expansions for bias. Extensive Monte Carlo simulation experiments are conducted and the results show that the proposed method performs excellently in general. Even for very small or heavy censored data, the proposed bias-corrected MLEs of the Weibull shape parameter, especially the second-order one, perform rather satisfactorily. When compared with the MMLE of [5, 6], the results show that the proposed estimators are more robust against different types of data.

Although four censoring mechanisms are examined in our simulation studies, it seems reasonable to believe that our asymptotic-expansion and bootstrap-based method should work well for the estimation of the shape parameter with most types of data. Also, we may infer that the proposed method should be a simple and good method that can be applied to other parametric distributions, as long as the concentrated estimating equations for the parameter(s) of interest, and their derivatives (up to third order) can be expressed in analytical forms.

In literature, there are various other likelihood-related approaches available for the type of estimation problems discussed in this paper, such as modified profile likelihood method [22], profile-kernel likelihood inference [23], marginal likelihood approach [**?**] and penalized maximum likelihood approach [25], etc. Thus it would be interesting for a possible future work to compare the existing approaches with our asymptotic-expansion and bootstrap-based method.

Moreover, as known, another commonly used method for the Weibull shape parameter is the least-square estimation (LSE), which is also biased. So it would be interesting to consider the stochastic expansion of the LSE, and then develop a LSE-based bias-corrected method and compare it to some existing approaches, such as [7].

Acknowledgement

This work is supported by research grant from MOE (Ministry of Education) Key Laboratory of Econometrics and Fujian Key Laboratory of Statistical Sciences, China.

Appendix A. Proof of Theorem 1

The proof of $\sqrt{n}$-consistency of $\hat{\beta}_n$ amounts to check the conditions of Theorem 4.1.2 and Theorem 4.1.3 of [20]. The differentiability and measureability of $\ell_n^c(\beta)$ are obvious. It is easy to see that $\ell_n^c(\beta)$ is globally concave, and thus $\ell_n^c(\beta)$ attains a unique global maximum at $\hat{\beta}_n$ which is the unique solution of $\tilde{\psi}_n(\beta) = 0$. These and Assumptions 1-3 lead to the $\sqrt{n}$-consistency of $\hat{\beta}_n$.

Now, the differentiability of $\tilde{\psi}_n(\beta)$ leads to the Taylor series expansion:

$$0 = \tilde{\psi}_n(\hat{\beta}_n) = \tilde{\psi}_n + H_{1n}(\hat{\beta}_n - \beta_0) + \frac{1}{2}H_{2n}(\hat{\beta}_n - \beta_0)^2 + \frac{1}{6}H_{3n}(\hat{\beta}_n - \beta_0)^3$$
$$+\frac{1}{6}[H_{3n}(\bar{\beta}_n) - H_{3n})(\hat{\beta}_n - \beta_0)^3.$$

where $\bar{\beta}_n$ lies between $\hat{\beta}_n$ and β_0. As $\hat{\beta}_n = \beta_0 + O_p(n^{-\frac{1}{2}})$, we have $\bar{\beta}_n = \beta_0 + O_p(n^{-\frac{1}{2}})$. Assumptions 3-4 lead to the following:

1) $\tilde{\psi}_n = O_p(n^{-1/2})$ and $E(\tilde{\psi}_n) = O(n^{-1})$;
2) $E(H_{rn}) = O(1)$ and $H_{rn}^{\circ} = O_p(n^{-\frac{1}{2}})$, $r = 1,2,3$;
3) $E(H_{1n})^{-1} = O(1)$ and $H_{1n}^{-1} = O_p(1)$;
4) $|H_{rn}(\beta) - H_{rn}| \leq |\beta - \beta_0| X_{n,r}$, $r = 1, 2, 3$, for some β in a neighborhood of β_0;
5) $H_{3n}(\bar{\beta}) - H_{3n} = O_p(n^{-\frac{1}{2}})$.

The proofs of these results are straightforward. The details parallel those of [15] and are available from the authors. These make the stochastic expansion (2) valid. Finally, Assumptions 4(i) and 4(iii) guarantee the transition from the stochastic expansion (2) to the results of Theorem 1.

Appendix B. Proof of Corollary 1

Write $B_2(\hat{\beta}_n) = f_2(\theta_0)$ and $B_3(\hat{\beta}_n) = f_3(\theta_0)$, which are, respectively, $O(n^{-1})$ and $O(n^{-3/2})$. If explicit expressions of $f_2(\theta)$ and $f_3(\theta)$ exist, then the "plug-in" estimates of $B_2(\hat{\beta}_n)$ and $B_3(\hat{\beta}_n)$ would be, respectively, $f_2(\hat{\theta}_n)$ and $f_3(\hat{\theta}_n)$, where $\hat{\theta}_n$ is the MLE of θ defined at the beginning of Section 3, which is a $\sqrt{n}$-consistent estimator of θ_0. We have under the additional assumptions in the corollary,

$$f_2(\hat{\theta}_n) = f_2(\theta_0) + \frac{\partial}{\partial\theta_0}f_2(\theta_0)(\hat{\theta}_n - \theta_0) + O_p(n^{-2}),$$

and $\mathrm{E}[f_2(\hat{\theta}_n)] = f_2(\theta_0) + \frac{\partial}{\partial\theta_0}f_2(\theta_0)\mathrm{E}(\hat{\theta}_n - \theta_0) + \mathrm{E}[O_p(n^{-2})] = f_2(\theta_0) + O(n^{-2})$. Similarly, $\mathrm{E}[f_3(\hat{\theta}_n)] = f_3(\theta_0) + O(n^{-2})$. These show that replacing θ_0 by $\hat{\theta}_n$ only (possibly) imposes additional bias of order $O_p(n^{-2})$ or higher.

Clearly, our bootstrap estimate has two step approximations, one is that described above, and the other is the bootstrap approximations to the various expectations in (7) and (8), given $\hat{\theta}_n$. For example,

$$\hat{E}(H_{1n}\tilde{\psi}_n) = \frac{1}{B}\sum_{b=1}^{B} H_{1n,b}(\hat{\beta}_n)\tilde{\psi}_{n,b}(\hat{\beta}_n).$$

However, these approximations can be made arbitrarily accurate, for a given $\hat{\theta}_n$, by choosing an arbitrarily large B. The results of Corollary 1 thus follow.

References

[1] Keats JB, Lawrence FP, Wang FK. Weibull maximum likelihood parameter estimates with censored data. Journal of Quality Technology. 1997;29(1):105-110.
[2] Montanari GC, Mazzanti G, Caccian M, Fothergill JC. In search of convenient techniques for reducing bias in the estimation of Weibull parameters for uncensored tests. IEEE Transactions on Dielectrics and Electrical Insulation. 1997;4(3):306-313.
[3] Hirose H. Bias correction for the maximum likelihood estimates in the two-parameter Weibull distribution. IEEE Transactions on Dielectrics and Electrical Insulation. 1999;6(1):66-69.
[4] Singh U, Gupta PK, Upadhyay SK. Estimation of parameters for exponentiated-Weibull family under type-II censoring scheme. Computational Statistics and Data Analysis. 2005;48(3):509-523.
[5] Yang ZL, Xie M. Efficient estimation of the Weibull shape parameter based on modified profile likelihood. Journal of Statistical Computation and Simulation. 2003;73:115-123.
[6] Yang ZL, Lin Dennis KJ. Improved maximum-likelihood estimation for the common shape parameter of several Weibull populations. Applied Stochastic Models in Business and Industry. 2007;23:373-383.
[7] Zhang LF, Xie M, Tang LC. Bias correction for the least squares estimator of Weibull shape parameter with complete and censored data. Reliability Engineering and System Safety. 2006;91:930-939.
[8] Balakrishnan N, Kateri M. On the maximum likelihood estimation of parameters of Weibull distribution based on. 2008;78(17):2971-2975.
[9] Ng HKT, Wang Z. Statistical estimation for the parameters of Weibull distribution based on progressively type-I interval censored sample. Journal of Statistical Computation and Simulation. 2009;79(2):145-159.
[10] Kim YJ. A comparative study of nonparametric estimation in Weibull regression: A penalized likelihood approach. Computational Statistics and Data Analysis. 2011;55(4):1884-1896.
[11] Lawless JF. Statistical Models and Methods for Lifetime Data. John Wiley and Sons, Inc; 1982.
[12] Lawless JF. Statistical Models and Methods for Lifetime Data (2nd edn). John Wiley and Sons, Inc; 2003.
[13] Ye ZS, Chan PS, Xie M, Ng HKT. Statistical inference for the extreme value distribution under adaptive Type-II progressive censoring schemes. Journal of Statistical Computation and Simulation. 2012; to appear.
[14] Rilstone P, Srivastava VK, Ullah A. The second-order bias and mean squared error of nonlinear estimators. Journal of Econometrics. 1996;75:369-395.
[15] Yang ZL. A general method for third-order bias and variance corrections on a nonlinear estimator. 2012. Working Paper, Singapore Management University. Located at http://www.mysmu.edu/faculty/zlyang/SubPages/research.htm.
[16] Bao Y, Ullah A. Finite sample properties of maximum likelihood estimator in spatial models. Journal of Econometrics. 2007a;150:396-413.
[17] Bao Y, Ullah A. The second-order bias and mean squared error of estimators in time-series models. Journal of Econometrics. 2007b;150:650-669.
[18] Chen QA, Giles, E. Finite-sample properties of the maximum likelihood estimation for the Poisson regression model with random covariates. Communications in Statistics - Theory and Methods. 2011;40(6):1000-1014.
[19] Kundhi G, Rilstone P. The third-order bias of nonlinear estimators. Communications in Statistics - Theory and Methods. 2008;37(16):2617-2633.
[20] Amemeiya T. Advanced Econometrics. Cambridge, Massachusetts: Harvard University Press; 1985.
[21] Davison AC, Hinkley DV. Bootstrap Methods and Their Applications. Cambridge University Press: London, 1997.

[22] Sartori N, Bellio R, Salvan A. The direct modified profile likelihood in models with nuisance parameters. Biometrika. 1999;86(3):735-742.
[23] Lam C, Fan JQ. Profile-kernel likelihood inference with diverging number of parameters. Annals of Statistics. 2008;36(5):2232-2260.
[24] Bell BM. Approximating the marginal likelihood estimate for models with random parameters. Applied Mathematics and Computation. 2001;119(1):57-75.
[25] Tran MN. Penalized maximum likelihood principle for choosing ridge parameter. Communications in Statistics - Simulation and Computation. 2009;38(8):1610-1624.

Table 4. Empirical mean [rmse](se) of MLE-type estimators of β, Type-I data, $k = 1$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.579 [.246](.233)	0.496 [.199](.199)	0.509 [.206](.205)	0.533 [.216](.214)
	0.8	0.937 [.417](.394)	0.803 [.337](.337)	0.823 [.344](.343)	0.863 [.367](.361)
	1.0	1.160 [.508](.482)	0.994 [.413](.413)	1.020 [.425](.425)	1.067 [.447](.442)
	2.0	2.322 [.974](.919)	1.990 [.783](.783)	2.041 [.800](.799)	2.136 [.854](.843)
	5.0	5.823 [2.43](2.28)	4.992 [1.96](1.96)	5.120 [2.01](2.01)	5.357 [2.12](2.09)
20	0.5	0.535 [.136](.131)	0.499 [.124](.124)	0.502 [.124](.124)	0.515 [.127](.126)
	0.8	0.858 [.223](.216)	0.800 [.202](.202)	0.804 [.203](.203)	0.825 [.209](.207)
	1.0	1.065 [.275](.267)	0.992 [.251](.251)	0.997 [.252](.252)	1.023 [.259](.257)
	2.0	2.148 [.563](.543)	2.002 [.510](.510)	2.012 [.511](.511)	2.065 [.527](.523)
	5.0	5.372 [1.41](1.37)	5.005 [1.28](1.28)	5.031 [1.29](1.29)	5.162 [1.32](1.31)
50	0.5	0.515 [.080](.078)	0.502 [.076](.076)	0.502 [.076](.076)	0.507 [.077](.077)
	0.8	0.822 [.130](.128)	0.800 [.125](.125)	0.800 [.125](.125)	0.809 [.126](.126)
	1.0	1.028 [.160](.157)	1.001 [.154](.154)	1.002 [.154](.154)	1.012 [.155](.155)
	2.0	2.062 [.325](.319)	2.007 [.312](.312)	2.009 [.312](.312)	2.030 [.316](.314)
	5.0	5.133 [.796](.785)	4.995 [.768](.768)	4.999 [.768](.768)	5.053 [.775](.774)
100	0.5	0.507 [.055](.054)	0.500 [.054](.054)	0.501 [.054](.054)	0.503 [.054](.054)
	0.8	0.809 [.087](.086)	0.799 [.085](.085)	0.799 [.085](.085)	0.803 [.086](.086)
	1.0	1.015 [.111](.110)	1.001 [.109](.109)	1.002 [.109](.109)	1.007 [.109](.109)
	2.0	2.026 [.215](.214)	1.999 [.211](.211)	2.000 [.211](.211)	2.011 [.212](.212)
	5.0	5.072 [.543](.538)	5.004 [.532](.532)	5.005 [.532](.532)	5.032 [.535](.534)
		$p = 0.5$			
10	0.5	0.649 [.510](.487)	0.506 [.306](.304)	0.549 [.340](.335)	0.595 [.463](.452)
	0.8	1.030 [.769](.733)	0.803 [.477](.475)	0.871 [.539](.534)	0.943 [.699](.683)
	1.0	1.287 [1.02](.973)	1.004 [.626](.625)	1.087 [.702](.696)	1.179 [.928](.910)
	2.0	2.557 [1.94](1.86)	1.996 [1.20](1.20)	2.166 [1.36](1.35)	2.343 [1.77](1.74)
	5.0	6.368 [4.47](4.26)	4.982 [2.80](2.80)	5.403 [3.11](3.09)	5.829 [4.01](3.92)
20	0.5	0.555 [.197](.190)	0.501 [.171](.171)	0.513 [.176](.176)	0.531 [.184](.181)
	0.8	0.887 [.322](.310)	0.801 [.274](.274)	0.819 [.283](.283)	0.849 [.301](.297)
	1.0	1.105 [.379](.365)	0.998 [.331](.331)	1.021 [.338](.337)	1.058 [.354](.349)
	2.0	2.212 [.778](.748)	1.998 [.671](.671)	2.044 [.691](.690)	2.118 [.726](.716)
	5.0	5.538 [1.94](1.86)	5.003 [1.68](1.68)	5.119 [1.72](1.72)	5.302 [1.81](1.78)
50	0.5	0.520 [.102](.100)	0.500 [.097](.097)	0.502 [.097](.097)	0.511 [.099](.099)
	0.8	0.830 [.163](.161)	0.798 [.155](.155)	0.801 [.155](.155)	0.815 [.159](.158)
	1.0	1.039 [.205](.202)	1.000 [.195](.195)	1.003 [.195](.195)	1.021 [.199](.198)
	2.0	2.077 [.408](.401)	1.999 [.388](.388)	2.005 [.388](.388)	2.041 [.396](.394)
	5.0	5.196 [1.03](1.02)	5.000 [.980](.980)	5.015 [.982](.982)	5.107 [1.00](.997)
100	0.5	0.510 [.069](.069)	0.500 [.068](.068)	0.501 [.068](.068)	0.505 [.068](.068)
	0.8	0.816 [.111](.110)	0.801 [.108](.108)	0.802 [.108](.108)	0.809 [.109](.109)
	1.0	1.020 [.135](.134)	1.001 [.132](.132)	1.001 [.132](.132)	1.011 [.133](.133)
	2.0	2.040 [.275](.272)	2.002 [.268](.268)	2.003 [.268](.268)	2.022 [.271](.270)
	5.0	5.097 [.690](.683)	5.001 [.672](.672)	5.005 [.672](.672)	5.053 [.679](.677)
		$p = 0.3$			
10	0.5	0.858 [1.17](1.10)	0.485 [.491](.469)	0.543 [.543](.522)	0.789 [1.10](1.05)
	0.8	1.352 [1.81](1.72)	0.763 [.778](.758)	0.859 [.873](.854)	1.240 [1.69](1.62)
	1.0	1.695 [2.32](2.20)	0.937 [.930](.911)	1.055 [1.06](1.05)	1.553 [2.17](2.09)
	2.0	3.334 [4.51](4.30)	1.850 [1.84](1.84)	2.092 [2.09](2.09)	3.054 [4.22](4.08)
	5.0	8.266 [11.2](10.7)	4.604 [4.61](4.63)	5.182 [5.20](5.23)	7.568 [10.4](10.1)
20	0.5	0.619 [.423](.406)	0.498 [.266](.265)	0.540 [.293](.289)	0.590 [.401](.390)
	0.8	1.003 [.739](.710)	0.806 [.447](.446)	0.876 [.499](.493)	0.958 [.704](.686)
	1.0	1.239 [.856](.822)	1.000 [.557](.556)	1.088 [.622](.615)	1.183 [.811](.790)
	2.0	2.516 [1.99](1.92)	2.008 [1.12](1.12)	2.184 [1.23](1.22)	2.402 [1.90](1.86)
	5.0	6.180 [4.30](4.13)	4.969 [2.61](2.61)	5.395 [2.92](2.89)	5.899 [4.08](3.98)
50	0.5	0.536 [.151](.146)	0.501 [.136](.136)	0.508 [.139](.138)	0.526 [.146](.143)
	0.8	0.858 [.241](.234)	0.801 [.218](.218)	0.813 [.222](.222)	0.842 [.234](.230)
	1.0	1.077 [.310](.301)	1.006 [.280](.279)	1.020 [.285](.285)	1.057 [.301](.295)
	2.0	2.143 [.601](.584)	2.001 [.544](.544)	2.030 [.554](.553)	2.104 [.582](.573)
	5.0	5.375 [1.54](1.49)	5.018 [1.39](1.39)	5.091 [1.41](1.41)	5.276 [1.49](1.46)
100	0.5	0.516 [.095](.093)	0.499 [.090](.090)	0.501 [.091](.091)	0.511 [.093](.092)
	0.8	0.826 [.153](.151)	0.800 [.146](.146)	0.802 [.147](.147)	0.819 [.151](.149)
	1.0	1.038 [.196](.192)	1.005 [.186](.186)	1.008 [.186](.186)	1.029 [.192](.190)
	2.0	2.068 [.387](.381)	2.001 [.369](.369)	2.007 [.371](.371)	2.049 [.381](.378)
	5.0	5.170 [.953](.938)	5.004 [.910](.910)	5.020 [.913](.912)	5.123 [.937](.929)

Table 5. Empirical mean [rmse](se) of MLE-type estimators of β, Type-I data, $k = 2$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.548 [.149](.141)	0.498 [.128](.128)	0.499 [.128](.128)	0.506 [.130](.130)
	0.8	0.876 [.235](.222)	0.795 [.202](.202)	0.796 [.202](.202)	0.808 [.206](.206)
	1.0	1.095 [.296](.281)	0.995 [.256](.255)	0.996 [.255](.255)	1.010 [.260](.260)
	2.0	2.201 [.594](.559)	1.999 [.508](.508)	2.001 [.507](.507)	2.031 [.519](.518)
	5.0	5.467 [1.47](1.39)	4.964 [1.27](1.27)	4.969 [1.26](1.26)	5.043 [1.29](1.29)
20	0.5	0.523 [.094](.091)	0.500 [.087](.087)	0.500 [.087](.087)	0.503 [.088](.088)
	0.8	0.834 [.150](.146)	0.798 [.141](.141)	0.798 [.140](.140)	0.801 [.141](.141)
	1.0	1.048 [.187](.181)	1.002 [.174](.174)	1.002 [.174](.174)	1.007 [.175](.174)
	2.0	2.087 [.372](.362)	1.996 [.347](.347)	1.997 [.347](.347)	2.006 [.348](.348)
	5.0	5.223 [.938](.911)	4.996 [.874](.874)	4.997 [.874](.874)	5.019 [.878](.877)
50	0.5	0.509 [.056](.055)	0.500 [.054](.054)	0.500 [.054](.054)	0.501 [.054](.054)
	0.8	0.814 [.088](.087)	0.800 [.086](.086)	0.800 [.086](.086)	0.801 [.086](.086)
	1.0	1.017 [.109](.108)	1.000 [.106](.106)	1.000 [.106](.106)	1.001 [.106](.106)
	2.0	2.034 [.218](.216)	2.000 [.213](.213)	2.000 [.213](.213)	2.002 [.212](.212)
	5.0	5.081 [.548](.542)	4.995 [.534](.534)	4.995 [.534](.534)	5.002 [.534](.534)
100	0.5	0.505 [.038](.038)	0.500 [.037](.037)	0.500 [.037](.037)	0.501 [.037](.037)
	0.8	0.807 [.061](.061)	0.800 [.060](.060)	0.800 [.060](.060)	0.801 [.060](.060)
	1.0	1.009 [.077](.076)	1.000 [.076](.076)	1.000 [.076](.076)	1.001 [.076](.076)
	2.0	2.019 [.153](.152)	2.002 [.151](.151)	2.002 [.151](.151)	2.003 [.151](.151)
	5.0	5.045 [.384](.382)	5.002 [.379](.379)	5.002 [.379](.379)	5.006 [.379](.379)
		$p = 0.5$			
10	0.5	0.569 [.208](.192)	0.506 [.175](.171)	0.514 [.179](.174)	0.522 [.182](.177)
	0.8	0.901 [.320](.301)	0.801 [.270](.268)	0.813 [.275](.272)	0.826 [.279](.276)
	1.0	1.125 [.403](.381)	0.998 [.338](.336)	1.014 [.347](.344)	1.030 [.352](.349)
	2.0	2.248 [.779](.738)	1.996 [.654](.654)	2.027 [.666](.665)	2.058 [.679](.676)
	5.0	5.616 [1.99](1.90)	4.985 [1.68](1.69)	5.064 [1.71](1.71)	5.142 [1.74](1.74)
20	0.5	0.530 [.119](.115)	0.501 [.109](.109)	0.502 [.109](.109)	0.507 [.110](.110)
	0.8	0.844 [.188](.183)	0.797 [.173](.173)	0.800 [.174](.174)	0.808 [.175](.175)
	1.0	1.060 [.239](.232)	1.001 [.220](.220)	1.005 [.220](.220)	1.014 [.222](.222)
	2.0	2.122 [.478](.462)	2.006 [.439](.439)	2.013 [.440](.440)	2.032 [.444](.443)
	5.0	5.305 [1.22](1.18)	5.014 [1.12](1.12)	5.031 [1.12](1.12)	5.080 [1.13](1.13)
50	0.5	0.510 [.070](.069)	0.499 [.068](.068)	0.500 [.068](.068)	0.502 [.068](.068)
	0.8	0.818 [.111](.109)	0.801 [.107](.107)	0.801 [.107](.107)	0.804 [.107](.107)
	1.0	1.022 [.138](.136)	0.999 [.134](.134)	1.000 [.134](.134)	1.004 [.134](.134)
	2.0	2.047 [.278](.274)	2.003 [.269](.269)	2.004 [.269](.269)	2.012 [.270](.269)
	5.0	5.107 [.698](.690)	4.995 [.676](.676)	4.998 [.677](.677)	5.019 [.678](.678)
100	0.5	0.505 [.048](.047)	0.499 [.047](.047)	0.499 [.047](.047)	0.501 [.047](.047)
	0.8	0.807 [.075](.074)	0.798 [.074](.074)	0.799 [.074](.074)	0.800 [.074](.074)
	1.0	1.012 [.095](.094)	1.001 [.093](.093)	1.001 [.093](.093)	1.003 [.094](.094)
	2.0	2.020 [.189](.188)	1.998 [.186](.186)	1.998 [.186](.186)	2.003 [.186](.186)
	5.0	5.056 [.477](.474)	5.001 [.470](.470)	5.002 [.470](.470)	5.013 [.470](.470)
		$p = 0.3$			
10	0.5	0.640 [.426](.344)	0.534 [.351](.280)	0.573 [.381](.310)	0.587 [.390](.318)
	0.8	0.989 [.587](.506)	0.822 [.462](.401)	0.882 [.507](.445)	0.905 [.525](.460)
	1.0	1.226 [.801](.733)	1.014 [.578](.531)	1.089 [.639](.590)	1.121 [.719](.671)
	2.0	2.399 [1.48](1.43)	1.974 [1.08](1.08)	2.124 [1.19](1.19)	2.188 [1.31](1.29)
	5.0	5.873 [3.43](3.44)	4.833 [2.55](2.71)	5.206 [2.85](2.99)	5.349 [3.00](3.12)
20	0.5	0.548 [.188](.178)	0.499 [.163](.160)	0.510 [.170](.167)	0.523 [.175](.170)
	0.8	0.882 [.295](.280)	0.804 [.258](.254)	0.820 [.265](.261)	0.842 [.274](.267)
	1.0	1.102 [.367](.352)	1.004 [.320](.319)	1.025 [.329](.327)	1.052 [.341](.336)
	2.0	2.206 [.729](.699)	2.010 [.634](.634)	2.052 [.652](.650)	2.105 [.675](.667)
	5.0	5.486 [1.82](1.77)	4.999 [1.59](1.60)	5.104 [1.64](1.64)	5.236 [1.69](1.69)
50	0.5	0.520 [.096](.094)	0.502 [.091](.091)	0.503 [.092](.092)	0.510 [.093](.093)
	0.8	0.831 [.156](.153)	0.802 [.147](.147)	0.804 [.148](.148)	0.816 [.151](.150)
	1.0	1.034 [.194](.191)	0.998 [.185](.185)	1.001 [.185](.185)	1.015 [.188](.187)
	2.0	2.078 [.384](.376)	2.006 [.363](.363)	2.011 [.364](.364)	2.040 [.371](.369)
	5.0	5.179 [.956](.939)	4.999 [.908](.908)	5.013 [.911](.911)	5.084 [.926](.922)
100	0.5	0.510 [.064](.064)	0.501 [.063](.063)	0.501 [.063](.063)	0.505 [.063](.063)
	0.8	0.815 [.104](.103)	0.801 [.101](.101)	0.801 [.101](.101)	0.808 [.102](.102)
	1.0	1.018 [.131](.130)	1.000 [.128](.128)	1.001 [.128](.128)	1.009 [.129](.128)
	2.0	2.037 [.261](.259)	2.002 [.255](.255)	2.003 [.255](.255)	2.018 [.257](.256)
	5.0	5.094 [.654](.647)	5.006 [.637](.637)	5.010 [.638](.638)	5.048 [.643](.641)

Table 6. Empirical mean [rmse](se) of MLE-type estimators of β, Type-I data, $k = 8$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.526 [.070](.065)	0.497 [.061](.061)	0.496 [.061](.061)	0.486 [.062](.060)
	0.8	0.845 [.113](.104)	0.798 [.097](.097)	0.797 [.097](.097)	0.779 [.098](.096)
	1.0	1.054 [.141](.130)	0.995 [.122](.122)	0.994 [.122](.122)	0.972 [.124](.121)
	2.0	2.109 [.279](.257)	1.991 [.242](.242)	1.989 [.241](.241)	1.945 [.245](.238)
	5.0	5.275 [.701](.645)	4.980 [.607](.606)	4.974 [.606](.605)	4.865 [.613](.598)
20	0.5	0.513 [.046](.044)	0.500 [.043](.043)	0.500 [.043](.043)	0.493 [.043](.042)
	0.8	0.820 [.072](.069)	0.799 [.067](.067)	0.799 [.067](.067)	0.788 [.068](.067)
	1.0	1.026 [.090](.086)	1.000 [.084](.084)	1.000 [.084](.084)	0.986 [.084](.083)
	2.0	2.049 [.183](.176)	1.996 [.172](.171)	1.996 [.171](.171)	1.969 [.172](.169)
	5.0	5.119 [.448](.432)	4.988 [.421](.421)	4.987 [.421](.421)	4.920 [.424](.416)
50	0.5	0.505 [.027](.027)	0.500 [.027](.027)	0.500 [.027](.027)	0.497 [.027](.026)
	0.8	0.808 [.044](.044)	0.800 [.043](.043)	0.800 [.043](.043)	0.796 [.043](.043)
	1.0	1.011 [.055](.054)	1.001 [.054](.054)	1.001 [.054](.054)	0.995 [.054](.054)
	2.0	2.017 [.110](.108)	1.997 [.107](.107)	1.997 [.107](.107)	1.985 [.108](.107)
	5.0	5.048 [.272](.268)	4.999 [.265](.265)	4.998 [.265](.265)	4.969 [.265](.264)
100	0.5	0.502 [.019](.019)	0.500 [.019](.019)	0.500 [.019](.019)	0.499 [.019](.019)
	0.8	0.804 [.031](.031)	0.800 [.031](.031)	0.800 [.031](.031)	0.797 [.031](.030)
	1.0	1.005 [.038](.038)	1.000 [.038](.038)	1.000 [.038](.038)	0.997 [.038](.037)
	2.0	2.010 [.077](.076)	2.001 [.076](.076)	2.001 [.076](.076)	1.995 [.075](.075)
	5.0	5.024 [.191](.190)	4.999 [.189](.189)	4.999 [.189](.189)	4.984 [.189](.188)
		$p = 0.5$			
10	0.5	0.530 [.123](.090)	0.502 [.117](.087)	0.501 [.117](.087)	0.486 [.117](.086)
	0.8	0.845 [.158](.128)	0.800 [.146](.122)	0.799 [.146](.121)	0.775 [.146](.118)
	1.0	1.054 [.194](.159)	0.998 [.179](.151)	0.996 [.179](.151)	0.966 [.179](.146)
	2.0	2.102 [.348](.333)	1.988 [.314](.313)	1.986 [.313](.313)	1.926 [.312](.303)
	5.0	5.231 [.842](.878)	4.946 [.757](.829)	4.941 [.757](.828)	4.791 [.757](.803)
20	0.5	0.513 [.056](.054)	0.499 [.053](.053)	0.499 [.053](.053)	0.491 [.053](.052)
	0.8	0.822 [.090](.087)	0.801 [.085](.085)	0.801 [.085](.085)	0.788 [.084](.083)
	1.0	1.025 [.112](.109)	0.998 [.106](.106)	0.998 [.106](.106)	0.981 [.106](.104)
	2.0	2.055 [.223](.216)	2.002 [.211](.211)	2.001 [.211](.211)	1.968 [.210](.207)
	5.0	5.134 [.560](.543)	5.000 [.530](.530)	4.999 [.530](.530)	4.916 [.527](.521)
50	0.5	0.505 [.034](.034)	0.500 [.034](.034)	0.500 [.034](.034)	0.497 [.033](.033)
	0.8	0.809 [.055](.054)	0.800 [.054](.054)	0.800 [.054](.054)	0.795 [.053](.053)
	1.0	1.011 [.067](.067)	1.000 [.066](.066)	1.000 [.066](.066)	0.993 [.066](.066)
	2.0	2.020 [.135](.133)	1.999 [.132](.132)	1.999 [.132](.132)	1.985 [.132](.131)
	5.0	5.053 [.336](.332)	5.001 [.329](.329)	5.001 [.329](.329)	4.966 [.328](.326)
100	0.5	0.503 [.023](.023)	0.500 [.023](.023)	0.500 [.023](.023)	0.498 [.023](.023)
	0.8	0.804 [.038](.038)	0.800 [.038](.038)	0.800 [.038](.038)	0.797 [.038](.037)
	1.0	1.005 [.047](.047)	1.000 [.047](.047)	1.000 [.047](.047)	0.996 [.047](.047)
	2.0	2.011 [.094](.093)	2.001 [.093](.093)	2.001 [.093](.093)	1.994 [.093](.092)
	5.0	5.025 [.237](.235)	5.000 [.234](.234)	5.000 [.234](.234)	4.982 [.234](.233)
		$p = 0.3$			
10	0.5	0.633 [.471](.213)	0.606 [.469](.223)	0.606 [.469](.223)	0.595 [.468](.227)
	0.8	0.887 [.480](.166)	0.842 [.474](.167)	0.843 [.474](.167)	0.825 [.473](.167)
	1.0	1.054 [.500](.196)	0.999 [.492](.182)	1.000 [.492](.182)	0.978 [.491](.177)
	2.0	1.906 [.610](.603)	1.797 [.581](.545)	1.798 [.582](.545)	1.753 [.577](.522)
	5.0	4.495 [1.13](2.00)	4.219 [1.03](1.85)	4.220 [1.03](1.85)	4.109 [1.01](1.79)
20	0.5	0.519 [.109](.082)	0.503 [.107](.081)	0.503 [.107](.081)	0.496 [.106](.080)
	0.8	0.827 [.143](.119)	0.801 [.137](.115)	0.801 [.137](.115)	0.790 [.136](.114)
	1.0	1.031 [.166](.147)	0.998 [.159](.143)	0.998 [.159](.143)	0.985 [.158](.141)
	2.0	2.063 [.315](.308)	1.997 [.298](.298)	1.997 [.298](.298)	1.970 [.295](.293)
	5.0	5.144 [.762](.808)	4.978 [.721](.782)	4.979 [.721](.783)	4.910 [.713](.770)
50	0.5	0.507 [.045](.045)	0.500 [.044](.044)	0.500 [.044](.044)	0.497 [.044](.044)
	0.8	0.810 [.072](.072)	0.800 [.071](.071)	0.800 [.071](.071)	0.795 [.070](.070)
	1.0	1.013 [.091](.090)	1.000 [.089](.089)	1.000 [.089](.089)	0.995 [.089](.088)
	2.0	2.025 [.181](.179)	1.999 [.177](.177)	1.999 [.177](.177)	1.988 [.176](.176)
	5.0	5.070 [.453](.447)	5.004 [.443](.443)	5.005 [.443](.443)	4.977 [.440](.439)
100	0.5	0.503 [.031](.031)	0.500 [.031](.031)	0.500 [.031](.031)	0.498 [.031](.031)
	0.8	0.805 [.051](.051)	0.800 [.050](.050)	0.800 [.050](.050)	0.798 [.050](.050)
	1.0	1.006 [.063](.063)	1.000 [.063](.063)	1.000 [.063](.063)	0.997 [.063](.062)
	2.0	2.011 [.124](.124)	1.998 [.123](.123)	1.998 [.123](.123)	1.993 [.123](.123)
	5.0	5.034 [.317](.316)	5.001 [.314](.314)	5.001 [.314](.314)	4.988 [.313](.313)

Table 7. Empirical mean [rmse](se) of MLE-type estimators of β, Type-II data, $k = 1$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.658 [.334](.294)	0.501 [.224](.224)	0.491 [.220](.220)	0.507 [.226](.226)
	0.8	1.058 [.528](.460)	0.804 [.350](.350)	0.789 [.343](.343)	0.815 [.355](.355)
	1.0	1.312 [.659](.581)	0.998 [.442](.442)	0.979 [.435](.434)	1.011 [.447](.447)
	2.0	2.650 [1.33](1.16)	2.015 [.883](.883)	1.977 [.868](.868)	2.040 [.893](.892)
	5.0	6.610 [3.31](2.89)	5.023 [2.20](2.20)	4.928 [2.16](2.16)	5.087 [2.22](2.22)
20	0.5	0.570 [.170](.155)	0.502 [.136](.136)	0.501 [.136](.136)	0.505 [.137](.137)
	0.8	0.907 [.261](.238)	0.800 [.210](.210)	0.797 [.209](.209)	0.804 [.211](.211)
	1.0	1.139 [.338](.308)	1.004 [.272](.272)	1.000 [.271](.271)	1.009 [.273](.273)
	2.0	2.275 [.672](.613)	2.006 [.541](.541)	1.999 [.539](.539)	2.017 [.543](.543)
	5.0	5.666 [1.66](1.52)	4.995 [1.34](1.34)	4.977 [1.33](1.33)	5.022 [1.34](1.34)
50	0.5	0.524 [.086](.083)	0.499 [.079](.079)	0.499 [.079](.079)	0.500 [.079](.079)
	0.8	0.840 [.137](.131)	0.800 [.125](.125)	0.800 [.125](.125)	0.802 [.125](.125)
	1.0	1.049 [.171](.163)	1.000 [.156](.156)	1.000 [.155](.155)	1.002 [.156](.156)
	2.0	2.101 [.343](.328)	2.002 [.313](.313)	2.001 [.313](.313)	2.006 [.314](.314)
	5.0	5.250 [.857](.820)	5.003 [.782](.782)	5.000 [.781](.781)	5.013 [.783](.783)
100	0.5	0.513 [.057](.056)	0.501 [.054](.054)	0.501 [.054](.054)	0.502 [.055](.055)
	0.8	0.820 [.091](.089)	0.801 [.087](.087)	0.801 [.087](.087)	0.802 [.087](.087)
	1.0	1.026 [.115](.112)	1.002 [.110](.110)	1.001 [.110](.110)	1.003 [.110](.110)
	2.0	2.049 [.225](.219)	2.001 [.214](.214)	2.001 [.214](.214)	2.003 [.214](.214)
	5.0	5.114 [.567](.555)	4.994 [.542](.542)	4.993 [.542](.542)	4.999 [.543](.543)
		$p = 0.5$			
10	0.5	0.777 [.556](.482)	0.497 [.309](.309)	0.471 [.295](.294)	0.505 [.318](.318)
	0.8	1.262 [.946](.826)	0.807 [.529](.529)	0.765 [.503](.502)	0.820 [.542](.542)
	1.0	1.572 [1.16](1.00)	1.006 [.643](.643)	0.953 [.612](.610)	1.022 [.652](.652)
	2.0	3.132 [2.33](2.03)	2.005 [1.31](1.31)	1.900 [1.24](1.24)	2.036 [1.32](1.32)
	5.0	7.858 [5.86](5.12)	5.029 [3.28](3.28)	4.762 [3.11](3.10)	5.108 [3.41](3.41)
20	0.5	0.611 [.242](.215)	0.503 [.177](.177)	0.498 [.175](.175)	0.505 [.178](.177)
	0.8	0.967 [.370](.330)	0.795 [.272](.272)	0.788 [.269](.269)	0.800 [.273](.273)
	1.0	1.214 [.470](.418)	0.998 [.344](.344)	0.988 [.341](.341)	1.004 [.346](.346)
	2.0	2.440 [.964](.858)	2.007 [.705](.705)	1.987 [.699](.699)	2.018 [.710](.709)
	5.0	6.075 [2.39](2.14)	4.994 [1.76](1.76)	4.945 [1.74](1.74)	5.022 [1.77](1.77)
50	0.5	0.537 [.111](.105)	0.499 [.098](.098)	0.499 [.098](.098)	0.500 [.098](.098)
	0.8	0.859 [.179](.169)	0.799 [.158](.158)	0.797 [.157](.157)	0.800 [.158](.158)
	1.0	1.078 [.228](.214)	1.002 [.199](.199)	1.001 [.199](.199)	1.004 [.199](.199)
	2.0	2.159 [.456](.427)	2.007 [.398](.398)	2.004 [.397](.397)	2.010 [.398](.398)
	5.0	5.371 [1.11](1.05)	4.991 [.976](.976)	4.984 [.974](.974)	5.001 [.977](.977)
100	0.5	0.517 [.072](.070)	0.498 [.068](.068)	0.498 [.068](.068)	0.499 [.068](.068)
	0.8	0.832 [.116](.112)	0.803 [.108](.108)	0.802 [.108](.108)	0.803 [.108](.108)
	1.0	1.037 [.146](.141)	1.001 [.136](.136)	1.000 [.136](.136)	1.002 [.136](.136)
	2.0	2.074 [.287](.278)	2.001 [.268](.268)	2.000 [.268](.268)	2.002 [.268](.268)
	5.0	5.172 [.719](.699)	4.990 [.674](.674)	4.989 [.674](.674)	4.995 [.674](.675)
		$p = 0.3$			
10	0.5	1.302 [1.64](1.43)	0.475 [.524](.524)	0.341 [.414](.382)	0.506 [.719](.719)
	0.8	2.140 [2.71](2.35)	0.782 [.861](.861)	0.561 [.672](.628)	0.829 [1.14](1.14)
	1.0	2.672 [3.41](2.97)	0.977 [1.09](1.09)	0.700 [.847](.793)	1.040 [1.48](1.48)
	2.0	5.250 [6.65](5.80)	1.917 [2.13](2.13)	1.374 [1.68](1.55)	2.053 [3.01](3.01)
	5.0	13.205[16.7](14.5)	4.819 [5.34](5.33)	3.447 [4.17](3.87)	5.123 [7.17](7.17)
20	0.5	0.731 [.481](.422)	0.501 [.289](.289)	0.483 [.279](.278)	0.505 [.292](.292)
	0.8	1.172 [.745](.646)	0.804 [.444](.444)	0.775 [.428](.428)	0.810 [.446](.446)
	1.0	1.450 [.911](.793)	0.995 [.545](.545)	0.958 [.528](.526)	1.002 [.548](.548)
	2.0	2.891 [1.80](1.56)	1.984 [1.07](1.07)	1.911 [1.04](1.03)	1.997 [1.08](1.08)
	5.0	7.248 [4.65](4.07)	4.975 [2.79](2.79)	4.793 [2.70](2.69)	5.009 [2.84](2.84)
50	0.5	0.570 [.171](.157)	0.499 [.137](.137)	0.496 [.137](.137)	0.500 [.137](.137)
	0.8	0.915 [.274](.249)	0.801 [.218](.218)	0.797 [.217](.217)	0.802 [.218](.218)
	1.0	1.142 [.349](.318)	1.000 [.279](.279)	0.996 [.278](.278)	1.002 [.279](.279)
	2.0	2.285 [.693](.632)	2.001 [.553](.553)	1.991 [.551](.551)	2.004 [.554](.554)
	5.0	5.750 [1.78](1.61)	5.033 [1.41](1.41)	5.010 [1.41](1.41)	5.043 [1.42](1.41)
100	0.5	0.535 [.105](.099)	0.502 [.093](.093)	0.502 [.093](.093)	0.503 [.093](.093)
	0.8	0.855 [.168](.159)	0.802 [.149](.149)	0.801 [.149](.149)	0.803 [.149](.149)
	1.0	1.068 [.209](.197)	1.002 [.185](.185)	1.001 [.185](.185)	1.003 [.185](.185)
	2.0	2.135 [.417](.395)	2.003 [.371](.371)	2.000 [.370](.370)	2.004 [.371](.371)
	5.0	5.316 [1.02](.969)	4.986 [.909](.909)	4.980 [.909](.909)	4.989 [.909](.909)

Table 8. Empirical mean [rmse](se) of MLE-type estimators of β, Type-II data, $k = 2$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.607 [.201](.170)	0.495 [.139](.139)	0.486 [.138](.137)	0.502 [.141](.141)
	0.8	0.971 [.321](.271)	0.792 [.222](.221)	0.779 [.219](.218)	0.803 [.224](.224)
	1.0	1.224 [.407](.340)	0.998 [.278](.278)	0.981 [.274](.273)	1.013 [.282](.281)
	2.0	2.429 [.807](.684)	1.981 [.558](.558)	1.947 [.551](.549)	2.009 [.565](.565)
	5.0	6.066 [1.99](1.68)	4.946 [1.37](1.37)	4.862 [1.36](1.35)	5.018 [1.39](1.39)
20	0.5	0.549 [.111](.100)	0.500 [.091](.091)	0.498 [.090](.090)	0.502 [.091](.091)
	0.8	0.874 [.172](.155)	0.796 [.142](.142)	0.794 [.141](.141)	0.800 [.142](.142)
	1.0	1.099 [.222](.199)	1.001 [.181](.181)	0.997 [.180](.180)	1.005 [.182](.182)
	2.0	2.198 [.447](.401)	2.002 [.365](.365)	1.995 [.364](.364)	2.011 [.366](.366)
	5.0	5.491 [1.11](.990)	5.001 [.903](.903)	4.983 [.900](.900)	5.023 [.906](.906)
50	0.5	0.517 [.059](.057)	0.499 [.055](.055)	0.499 [.055](.055)	0.500 [.055](.055)
	0.8	0.828 [.094](.090)	0.799 [.087](.087)	0.799 [.087](.087)	0.800 [.087](.087)
	1.0	1.036 [.119](.113)	1.000 [.109](.109)	0.999 [.109](.109)	1.001 [.109](.109)
	2.0	2.070 [.238](.227)	1.998 [.219](.219)	1.997 [.219](.219)	2.000 [.220](.220)
	5.0	5.173 [.591](.566)	4.992 [.546](.546)	4.989 [.546](.546)	4.998 [.546](.546)
100	0.5	0.509 [.040](.039)	0.500 [.038](.038)	0.500 [.038](.038)	0.501 [.038](.038)
	0.8	0.815 [.064](.062)	0.801 [.061](.061)	0.801 [.061](.061)	0.801 [.061](.061)
	1.0	1.018 [.079](.077)	1.001 [.076](.076)	1.001 [.076](.076)	1.001 [.076](.076)
	2.0	2.035 [.160](.156)	1.999 [.153](.153)	1.999 [.153](.153)	2.001 [.153](.153)
	5.0	5.082 [.392](.384)	4.993 [.377](.377)	4.992 [.377](.377)	4.996 [.377](.377)
		$p = 0.5$			
10	0.5	0.685 [.319](.260)	0.491 [.186](.186)	0.469 [.180](.178)	0.505 [.191](.191)
	0.8	1.101 [.513](.416)	0.789 [.299](.298)	0.754 [.289](.285)	0.811 [.306](.306)
	1.0	1.370 [.632](.513)	0.981 [.369](.368)	0.937 [.358](.352)	1.009 [.378](.378)
	2.0	2.725 [1.24](1.01)	1.952 [.724](.722)	1.865 [.704](.691)	2.008 [.742](.742)
	5.0	6.877 [3.21](2.60)	4.927 [1.87](1.86)	4.708 [1.81](1.78)	5.067 [1.92](1.92)
20	0.5	0.575 [.154](.134)	0.497 [.116](.116)	0.493 [.115](.115)	0.501 [.116](.116)
	0.8	0.917 [.240](.209)	0.793 [.181](.181)	0.785 [.180](.179)	0.798 [.182](.182)
	1.0	1.155 [.308](.267)	0.998 [.231](.231)	0.989 [.229](.229)	1.005 [.232](.232)
	2.0	2.302 [.616](.537)	1.989 [.465](.464)	1.971 [.461](.460)	2.002 [.467](.467)
	5.0	5.764 [1.53](1.33)	4.979 [1.15](1.15)	4.935 [1.14](1.14)	5.014 [1.15](1.15)
50	0.5	0.529 [.078](.073)	0.501 [.069](.069)	0.500 [.069](.069)	0.502 [.069](.069)
	0.8	0.845 [.123](.114)	0.801 [.109](.109)	0.800 [.108](.108)	0.802 [.109](.109)
	1.0	1.054 [.153](.143)	0.998 [.136](.136)	0.997 [.136](.136)	0.999 [.136](.136)
	2.0	2.111 [.310](.290)	1.999 [.274](.274)	1.996 [.274](.274)	2.002 [.275](.275)
	5.0	5.292 [.774](.717)	5.011 [.680](.679)	5.005 [.679](.679)	5.018 [.680](.680)
100	0.5	0.513 [.050](.048)	0.500 [.047](.047)	0.499 [.047](.047)	0.500 [.047](.047)
	0.8	0.821 [.081](.078)	0.800 [.076](.076)	0.799 [.076](.076)	0.800 [.076](.076)
	1.0	1.027 [.101](.097)	1.000 [.094](.094)	1.000 [.094](.094)	1.001 [.094](.094)
	2.0	2.057 [.200](.192)	2.002 [.187](.187)	2.002 [.187](.187)	2.003 [.187](.187)
	5.0	5.121 [.497](.482)	4.986 [.470](.470)	4.984 [.470](.470)	4.989 [.470](.469)
		$p = 0.3$			
10	0.5	0.952 [.767](.619)	0.453 [.300](.296)	0.358 [.275](.235)	0.506 [.343](.343)
	0.8	1.519 [1.25](1.02)	0.723 [.495](.489)	0.571 [.452](.389)	0.807 [.546](.546)
	1.0	1.934 [1.60](1.30)	0.921 [.625](.620)	0.726 [.562](.491)	1.028 [.736](.735)
	2.0	3.762 [2.99](2.42)	1.790 [1.17](1.15)	1.413 [1.09](.915)	1.998 [1.33](1.33)
	5.0	9.395 [7.44](6.00)	4.470 [2.90](2.85)	3.526 [2.69](2.26)	4.989 [3.24](3.24)
20	0.5	0.653 [.267](.219)	0.492 [.165](.165)	0.477 [.162](.160)	0.502 [.168](.168)
	0.8	1.038 [.422](.348)	0.782 [.264](.263)	0.757 [.258](.255)	0.797 [.268](.268)
	1.0	1.304 [.536](.441)	0.983 [.333](.333)	0.951 [.326](.322)	1.002 [.339](.339)
	2.0	2.608 [1.06](.873)	1.965 [.658](.658)	1.902 [.644](.636)	2.003 [.670](.670)
	5.0	6.563 [2.71](2.21)	4.946 [1.67](1.67)	4.786 [1.63](1.62)	5.040 [1.70](1.70)
50	0.5	0.550 [.115](.103)	0.498 [.094](.094)	0.496 [.093](.093)	0.499 [.094](.094)
	0.8	0.883 [.183](.164)	0.799 [.148](.148)	0.795 [.148](.147)	0.801 [.148](.148)
	1.0	1.104 [.235](.211)	0.999 [.191](.191)	0.995 [.190](.190)	1.002 [.191](.191)
	2.0	2.206 [.459](.410)	1.997 [.372](.372)	1.989 [.370](.370)	2.003 [.372](.372)
	5.0	5.500 [1.15](1.04)	4.978 [.938](.937)	4.956 [.935](.934)	4.993 [.940](.940)
100	0.5	0.524 [.071](.067)	0.500 [.064](.064)	0.499 [.064](.064)	0.500 [.064](.064)
	0.8	0.839 [.115](.108)	0.799 [.103](.103)	0.799 [.103](.103)	0.800 [.103](.103)
	1.0	1.049 [.143](.135)	1.000 [.128](.128)	0.999 [.128](.128)	1.001 [.128](.128)
	2.0	2.101 [.287](.269)	2.002 [.256](.256)	2.000 [.256](.256)	2.004 [.256](.256)
	5.0	5.246 [.715](.671)	5.000 [.641](.641)	4.995 [.640](.640)	5.004 [.641](.641)

Table 9. Empirical mean [rmse](se) of MLE-type estimators of β, Type-II data, $k = 8$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$n = 0.7$			
10	0.5	0.576 [.107](.075)	0.494 [.065](.065)	0.491 [.065](.064)	0.501 [.065](.065)
	0.8	0.920 [.169](.120)	0.788 [.104](.103)	0.783 [.104](.102)	0.800 [.104](.104)
	1.0	1.151 [.213](.150)	0.987 [.130](.129)	0.980 [.130](.128)	1.002 [.131](.131)
	2.0	2.299 [.423](.299)	1.971 [.258](.256)	1.958 [.258](.255)	2.001 [.260](.260)
	5.0	5.759 [1.07](.755)	4.937 [.651](.647)	4.904 [.650](.643)	5.012 [.657](.657)
20	0.5	0.535 [.058](.047)	0.499 [.044](.044)	0.498 [.044](.044)	0.501 [.044](.044)
	0.8	0.854 [.092](.074)	0.796 [.069](.069)	0.795 [.069](.069)	0.799 [.069](.069)
	1.0	1.069 [.117](.094)	0.997 [.088](.088)	0.996 [.087](.087)	1.001 [.088](.088)
	2.0	2.133 [.230](.188)	1.989 [.176](.176)	1.986 [.176](.175)	1.996 [.176](.176)
	5.0	5.338 [.576](.467)	4.978 [.436](.436)	4.971 [.436](.435)	4.997 [.437](.437)
50	0.5	0.514 [.031](.028)	0.500 [.027](.027)	0.500 [.027](.027)	0.501 [.027](.027)
	0.8	0.822 [.050](.045)	0.800 [.044](.044)	0.800 [.043](.043)	0.801 [.044](.044)
	1.0	1.027 [.062](.055)	1.000 [.054](.054)	1.000 [.054](.054)	1.001 [.054](.054)
	2.0	2.053 [.124](.112)	2.000 [.109](.109)	1.999 [.109](.109)	2.001 [.109](.109)
	5.0	5.134 [.309](.278)	5.001 [.271](.271)	5.000 [.271](.271)	5.004 [.271](.271)
100	0.5	0.506 [.020](.019)	0.500 [.019](.019)	0.500 [.019](.019)	0.500 [.019](.019)
	0.8	0.810 [.033](.031)	0.800 [.031](.031)	0.800 [.031](.031)	0.800 [.031](.031)
	1.0	1.013 [.040](.038)	1.000 [.038](.038)	1.000 [.038](.038)	1.000 [.038](.038)
	2.0	2.027 [.081](.077)	2.001 [.076](.076)	2.001 [.076](.076)	2.001 [.076](.076)
	5.0	5.065 [.203](.193)	4.999 [.191](.191)	4.999 [.191](.191)	5.000 [.191](.191)
		$p = 0.5$			
10	0.5	0.622 [.161](.105)	0.481 [.084](.082)	0.472 [.085](.080)	0.499 [.085](.085)
	0.8	0.998 [.261](.170)	0.772 [.134](.132)	0.758 [.136](.129)	0.801 [.136](.136)
	1.0	1.249 [.327](.212)	0.966 [.168](.164)	0.949 [.169](.161)	1.002 [.170](.170)
	2.0	2.500 [.654](.422)	1.934 [.334](.327)	1.899 [.337](.321)	2.006 [.338](.338)
	5.0	6.242 [1.63](1.06)	4.829 [.835](.818)	4.740 [.844](.803)	5.007 [.846](.846)
20	0.5	0.554 [.082](.061)	0.496 [.055](.055)	0.494 [.055](.055)	0.500 [.055](.055)
	0.8	0.886 [.130](.098)	0.793 [.088](.088)	0.791 [.088](.088)	0.800 [.088](.088)
	1.0	1.110 [.166](.123)	0.994 [.111](.111)	0.991 [.111](.110)	1.002 [.111](.111)
	2.0	2.218 [.327](.245)	1.985 [.220](.219)	1.978 [.219](.218)	2.001 [.221](.221)
	5.0	5.541 [.820](.617)	4.960 [.555](.553)	4.943 [.554](.551)	5.000 [.557](.557)
50	0.5	0.521 [.041](.035)	0.500 [.034](.034)	0.499 [.034](.034)	0.500 [.034](.034)
	0.8	0.833 [.065](.056)	0.800 [.054](.054)	0.799 [.054](.053)	0.801 [.054](.054)
	1.0	1.040 [.080](.069)	0.998 [.067](.067)	0.998 [.067](.067)	0.999 [.067](.067)
	2.0	2.082 [.163](.141)	1.998 [.135](.135)	1.997 [.135](.135)	2.001 [.135](.135)
	5.0	5.203 [.405](.351)	4.995 [.337](.337)	4.993 [.337](.337)	5.001 [.337](.337)
100	0.5	0.510 [.026](.024)	0.500 [.023](.023)	0.500 [.023](.023)	0.500 [.023](.023)
	0.8	0.816 [.042](.039)	0.800 [.038](.038)	0.799 [.038](.038)	0.800 [.038](.038)
	1.0	1.020 [.052](.048)	1.000 [.047](.047)	1.000 [.047](.047)	1.000 [.047](.047)
	2.0	2.040 [.103](.095)	2.000 [.093](.093)	2.000 [.093](.093)	2.001 [.093](.093)
	5.0	5.098 [.260](.241)	4.997 [.237](.237)	4.997 [.237](.237)	4.999 [.237](.237)
		$p = 0.3$			
10	0.5	0.770 [.333](.195)	0.431 [.130](.110)	0.389 [.149](.099)	0.498 [.126](.126)
	0.8	1.239 [.542](.318)	0.692 [.208](.178)	0.625 [.238](.161)	0.802 [.206](.206)
	1.0	1.551 [.680](.398)	0.867 [.260](.223)	0.783 [.296](.202)	1.004 [.258](.258)
	2.0	3.094 [1.36](.799)	1.730 [.523](.448)	1.562 [.597](.405)	2.002 [.517](.517)
	5.0	7.734 [3.36](1.96)	4.325 [1.29](1.10)	3.906 [1.48](.992)	5.005 [1.27](1.27)
20	0.5	0.608 [.144](.095)	0.489 [.078](.077)	0.482 [.078](.076)	0.502 [.079](.079)
	0.8	0.972 [.229](.151)	0.781 [.123](.122)	0.771 [.124](.120)	0.802 [.125](.125)
	1.0	1.215 [.285](.187)	0.976 [.152](.151)	0.963 [.153](.149)	1.003 [.155](.154)
	2.0	2.421 [.564](.376)	1.946 [.307](.302)	1.920 [.309](.298)	1.999 [.310](.310)
	5.0	6.056 [1.41](.934)	4.869 [.764](.753)	4.804 [.768](.743)	5.001 [.772](.772)
50	0.5	0.538 [.063](.050)	0.499 [.046](.046)	0.498 [.046](.046)	0.501 [.047](.047)
	0.8	0.860 [.099](.079)	0.798 [.073](.073)	0.796 [.073](.073)	0.801 [.073](.073)
	1.0	1.074 [.123](.098)	0.996 [.091](.091)	0.994 [.091](.091)	1.000 [.091](.091)
	2.0	2.150 [.248](.197)	1.993 [.183](.183)	1.990 [.183](.183)	2.001 [.183](.183)
	5.0	5.380 [.622](.492)	4.989 [.457](.457)	4.981 [.457](.456)	5.007 [.458](.458)
100	0.5	0.518 [.037](.032)	0.499 [.031](.031)	0.499 [.031](.031)	0.500 [.031](.031)
	0.8	0.828 [.060](.053)	0.799 [.051](.051)	0.799 [.051](.051)	0.800 [.051](.051)
	1.0	1.036 [.075](.066)	0.999 [.064](.064)	0.999 [.064](.064)	1.000 [.064](.064)
	2.0	2.070 [.147](.129)	1.997 [.125](.125)	1.996 [.125](.125)	1.998 [.125](.125)
	5.0	5.180 [.375](.329)	4.997 [.318](.318)	4.995 [.318](.318)	5.001 [.318](.318)

Table 10. Empirical mean [rmse](se) of MLE-type estimators of β, random censoring, $k = 1$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE-I}}$	$\hat{\beta}_{\text{MMLE-II}}$
		$p = 0.7$				
10	0.5	0.577 [.200](.185)	0.490 [.157]	0.495 [.159]	0.519 [.162](.161)	0.490 [.163](.163)
	0.8	0.918 [.316](.293)	0.781 [.249]	0.788 [.251]	0.823 [.256](.255)	0.783 [.257](.257)
	1.0	1.156 [.392](.360)	0.982 [.306]	0.991 [.309]	1.031 [.313](.312)	0.991 [.315](.314)
	2.0	2.338 [.782](.705)	1.981 [.599]	2.000 [.604]	2.044 [.611](.609)	2.028 [.615](.614)
	5.0	5.865 [1.97](1.77)	4.970 [1.51]	5.016 [1.52]	5.099 [1.54](1.53)	5.099 [1.54](1.53)
20	0.5	0.532 [.114](.110)	0.495 [.102]	0.496 [.102]	0.511 [.104](.103)	0.493 [.104](.103)
	0.8	0.853 [.179](.171)	0.792 [.159]	0.794 [.159]	0.817 [.162](.161)	0.791 [.161](.161)
	1.0	1.066 [.222](.212)	0.990 [.197]	0.992 [.198]	1.019 [.200](.199)	0.991 [.200](.200)
	2.0	2.147 [.431](.406)	1.991 [.377]	1.995 [.378]	2.023 [.379](.378)	2.009 [.381](.381)
	5.0	5.372 [1.11](1.05)	4.980 [.972]	4.992 [.974]	5.032 [.980](.980)	5.032 [.980](.980)
50	0.5	0.511 [.064](.063)	0.497 [.061]	0.498 [.061]	0.504 [.062](.062)	0.496 [.061](.061)
	0.8	0.819 [.102](.100)	0.796 [.097]	0.796 [.097]	0.807 [.099](.098)	0.795 [.098](.098)
	1.0	1.025 [.126](.124)	0.997 [.120]	0.997 [.120]	1.011 [.122](.122)	0.997 [.121](.121)
	2.0	2.055 [.244](.238)	1.997 [.231]	1.998 [.231]	2.015 [.231](.231)	2.003 [.232](.232)
	5.0	5.133 [.607](.592)	4.987 [.576]	4.989 [.577]	5.006 [.577](.577)	5.006 [.577](.577)
100	0.5	0.506 [.045](.044)	0.499 [.044]	0.499 [.044]	0.503 [.044](.044)	0.499 [.044](.044)
	0.8	0.809 [.070](.069)	0.798 [.068]	0.798 [.068]	0.804 [.069](.069)	0.798 [.068](.068)
	1.0	1.011 [.085](.084)	0.997 [.083]	0.997 [.083]	1.005 [.084](.084)	0.997 [.083](.083)
	2.0	2.029 [.166](.164)	2.001 [.161]	2.001 [.162]	2.012 [.162](.162)	2.004 [.162](.162)
	5.0	5.072 [.411](.405)	5.000 [.400]	5.001 [.400]	5.010 [.400](.400)	5.010 [.400](.400)
		$p = 0.5$				
10	0.5	0.575 [.204](.190)	0.489 [.162]	0.495 [.163]	0.524 [.171](.169)	0.478 [.167](.166)
	0.8	0.922 [.331](.308)	0.783 [.262]	0.794 [.264]	0.843 [.280](.277)	0.761 [.269](.266)
	1.0	1.152 [.417](.389)	0.977 [.329]	0.992 [.332]	1.055 [.357](.353)	0.953 [.339](.336)
	2.0	2.337 [.843](.773)	1.977 [.652]	2.003 [.660]	2.124 [.711](.700)	1.977 [.666](.666)
	5.0	5.852 [1.97](1.78)	4.950 [1.50]	5.005 [1.52]	5.230 [1.62](1.60)	5.038 [1.54](1.54)
20	0.5	0.533 [.119](.114)	0.495 [.106]	0.497 [.106]	0.513 [.111](.110)	0.488 [.108](.107)
	0.8	0.852 [.195](.188)	0.791 [.175]	0.794 [.175]	0.821 [.182](.181)	0.779 [.177](.176)
	1.0	1.064 [.240](.231)	0.988 [.215]	0.991 [.215]	1.026 [.224](.223)	0.974 [.218](.216)
	2.0	2.147 [.468](.444)	1.990 [.412]	1.996 [.413]	2.069 [.433](.428)	1.987 [.416](.416)
	5.0	5.383 [1.12](1.05)	4.986 [.976]	5.001 [.979]	5.145 [1.02](1.01)	5.021 [.984](.983)
50	0.5	0.511 [.068](.067)	0.498 [.065]	0.498 [.065]	0.504 [.066](.066)	0.494 [.066](.065)
	0.8	0.821 [.110](.108)	0.798 [.105]	0.799 [.105]	0.809 [.107](.106)	0.793 [.105](.105)
	1.0	1.024 [.135](.133)	0.996 [.130]	0.996 [.130]	1.010 [.132](.132)	0.990 [.130](.130)
	2.0	2.054 [.262](.257)	1.995 [.249]	1.996 [.250]	2.027 [.255](.254)	1.994 [.250](.250)
	5.0	5.161 [.628](.607)	5.013 [.590]	5.015 [.591]	5.090 [.606](.599)	5.025 [.592](.592)
100	0.5	0.506 [.047](.047)	0.500 [.046]	0.500 [.046]	0.503 [.047](.046)	0.498 [.046](.046)
	0.8	0.810 [.076](.075)	0.799 [.074]	0.799 [.074]	0.804 [.075](.075)	0.796 [.074](.074)
	1.0	1.012 [.095](.094)	0.998 [.093]	0.998 [.093]	1.005 [.093](.093)	0.995 [.093](.093)
	2.0	2.027 [.178](.176)	1.998 [.173]	1.999 [.173]	2.014 [.175](.175)	1.998 [.174](.174)
	5.0	5.079 [.422](.414)	5.006 [.409]	5.007 [.409]	5.047 [.414](.412)	5.013 [.409](.409)
		$p = 0.3$				
10	0.5	0.574 [.213](.200)	0.488 [.170]	0.496 [.172]	0.526 [.183](.181)	0.467 [.175](.171)
	0.8	0.925 [.366](.344)	0.782 [.290]	0.800 [.295]	0.853 [.322](.317)	0.735 [.298](.291)
	1.0	1.164 [.462](.433)	0.980 [.362]	1.004 [.369]	1.074 [.408](.401)	0.920 [.371](.363)
	2.0	2.355 [.953](.884)	1.962 [.721]	2.009 [.737]	2.185 [.849](.829)	1.894 [.748](.741)
	5.0	6.046 [2.41](2.17)	4.973 [1.71]	5.077 [1.76]	5.637 [2.15](2.05)	5.034 [1.79](1.79)
20	0.5	0.533 [.125](.121)	0.496 [.112]	0.497 [.113]	0.514 [.117](.116)	0.484 [.113](.112)
	0.8	0.855 [.209](.201)	0.792 [.187]	0.796 [.187]	0.823 [.196](.194)	0.769 [.189](.186)
	1.0	1.075 [.269](.258)	0.995 [.239]	1.000 [.240]	1.035 [.252](.250)	0.966 [.241](.239)
	2.0	2.152 [.527](.504)	1.983 [.463]	1.993 [.465]	2.079 [.496](.490)	1.951 [.470](.467)
	5.0	5.449 [1.27](1.19)	4.998 [1.08]	5.021 [1.09]	5.283 [1.19](1.15)	5.021 [1.10](1.10)
50	0.5	0.511 [.071](.070)	0.497 [.068]	0.498 [.068]	0.504 [.069](.069)	0.493 [.069](.068)
	0.8	0.820 [.120](.119)	0.797 [.115]	0.798 [.115]	0.808 [.117](.117)	0.788 [.116](.115)
	1.0	1.028 [.151](.149)	0.998 [.144]	0.999 [.144]	1.012 [.147](.147)	0.986 [.145](.144)
	2.0	2.058 [.290](.284)	1.995 [.275]	1.996 [.276]	2.030 [.283](.281)	1.982 [.277](.277)
	5.0	5.165 [.676](.656)	5.000 [.634]	5.004 [.634]	5.103 [.657](.649)	5.008 [.637](.637)
100	0.5	0.505 [.049](.049)	0.498 [.048]	0.498 [.048]	0.501 [.049](.049)	0.495 [.048](.048)
	0.8	0.809 [.082](.082)	0.798 [.081]	0.798 [.081]	0.803 [.081](.081)	0.793 [.081](.081)
	1.0	1.013 [.104](.103)	0.999 [.101]	0.999 [.101]	1.006 [.102](.102)	0.993 [.102](.101)
	2.0	2.031 [.201](.199)	2.001 [.196]	2.001 [.196]	2.018 [.199](.198)	1.994 [.196](.196)
	5.0	5.085 [.451](.443)	5.005 [.436]	5.006 [.436]	5.055 [.444](.440)	5.009 [.436](.436)

Note: for $\hat{\beta}^{bc2}_{\text{MLE}}$ and $\hat{\beta}^{bc3}_{\text{MLE}}$, the empirical sds are almost identical to the rmses and hence are not reported to conserve space.

Table 11. Empirical mean [rmse](se) of MLE-type estimators of β, random censoring, $k = 2$

n	β	$\hat{\beta}_{\mathrm{MLE}}$	$\hat{\beta}^{bc2}_{\mathrm{MLE}}$	$\hat{\beta}^{bc3}_{\mathrm{MLE}}$	$\hat{\beta}_{\mathrm{MMLE-I}}$	$\hat{\beta}_{\mathrm{MMLE-II}}$
		$p = 0.7$				
10	0.5	0.552 [.130](.119)	0.495 [.107]	0.494 [.106]	0.510 [.109](.109)	0.489 [.109](.109)
	0.8	0.881 [.201](.184)	0.790 [.165]	0.788 [.165]	0.814 [.168](.168)	0.784 [.168](.167)
	1.0	1.104 [.252](.229)	0.989 [.205]	0.987 [.205]	1.017 [.208](.207)	0.986 [.209](.208)
	2.0	2.222 [.492](.440)	1.987 [.394]	1.983 [.394]	2.017 [.395](.395)	2.001 [.397](.397)
	5.0	5.587 [1.26](1.12)	4.993 [1.00]	4.982 [.999]	5.042 [1.01](1.01)	5.042 [1.01](1.01)
20	0.5	0.522 [.078](.075)	0.497 [.071]	0.497 [.071]	0.504 [.072](.072)	0.493 [.072](.071)
	0.8	0.837 [.123](.117)	0.796 [.111]	0.796 [.111]	0.808 [.113](.113)	0.792 [.112](.112)
	1.0	1.047 [.151](.143)	0.996 [.136]	0.995 [.136]	1.011 [.138](.138)	0.992 [.137](.137)
	2.0	2.105 [.298](.278)	1.999 [.264]	1.998 [.264]	2.016 [.265](.264)	2.004 [.265](.265)
	5.0	5.251 [.729](.685)	4.985 [.651]	4.983 [.651]	5.003 [.653](.653)	5.003 [.653](.653)
50	0.5	0.509 [.046](.045)	0.499 [.044]	0.499 [.044]	0.502 [.044](.044)	0.497 [.044](.044)
	0.8	0.815 [.072](.071)	0.799 [.069]	0.799 [.069]	0.804 [.070](.070)	0.797 [.070](.069)
	1.0	1.019 [.089](.087)	1.000 [.085]	1.000 [.085]	1.006 [.086](.086)	0.998 [.085](.085)
	2.0	2.039 [.170](.166)	1.999 [.162]	1.999 [.162]	2.009 [.163](.163)	2.000 [.163](.163)
	5.0	5.103 [.419](.406)	5.002 [.399]	5.002 [.399]	5.009 [.399](.399)	5.009 [.399](.399)
100	0.5	0.504 [.031](.031)	0.499 [.031]	0.499 [.031]	0.500 [.031](.031)	0.498 [.031](.031)
	0.8	0.807 [.050](.049)	0.799 [.049]	0.799 [.049]	0.801 [.049](.049)	0.798 [.049](.049)
	1.0	1.009 [.060](.060)	1.000 [.059]	1.000 [.059]	1.003 [.059](.059)	0.999 [.059](.059)
	2.0	2.019 [.116](.115)	1.999 [.114]	1.999 [.114]	2.006 [.114](.114)	2.000 [.114](.114)
	5.0	5.047 [.285](.282)	4.997 [.279]	4.997 [.279]	5.000 [.279](.279)	5.000 [.279](.279)
		$p = 0.5$				
10	0.5	0.546 [.129](.121)	0.491 [.108]	0.490 [.108]	0.506 [.112](.112)	0.477 [.112](.109)
	0.8	0.877 [.211](.196)	0.788 [.176]	0.787 [.176]	0.813 [.183](.183)	0.763 [.181](.177)
	1.0	1.098 [.262](.243)	0.985 [.218]	0.984 [.218]	1.019 [.227](.226)	0.957 [.224](.219)
	2.0	2.222 [.521](.472)	1.988 [.422]	1.985 [.421]	2.064 [.443](.438)	1.967 [.426](.424)
	5.0	5.585 [1.28](1.14)	4.990 [1.02]	4.980 [1.02]	5.149 [1.07](1.06)	5.006 [1.03](1.03)
20	0.5	0.522 [.082](.079)	0.497 [.075]	0.497 [.075]	0.504 [.076](.076)	0.490 [.076](.075)
	0.8	0.834 [.131](.127)	0.794 [.121]	0.794 [.121]	0.804 [.123](.123)	0.780 [.122](.121)
	1.0	1.046 [.163](.156)	0.995 [.149]	0.995 [.149]	1.009 [.152](.151)	0.980 [.150](.149)
	2.0	2.099 [.312](.296)	1.994 [.281]	1.994 [.281]	2.029 [.289](.287)	1.982 [.283](.282)
	5.0	5.271 [.764](.715)	5.004 [.678]	5.002 [.678]	5.091 [.697](.691)	5.007 [.680](.680)
50	0.5	0.508 [.048](.047)	0.499 [.046]	0.499 [.046]	0.501 [.046](.046)	0.495 [.046](.046)
	0.8	0.814 [.077](.076)	0.799 [.074]	0.799 [.074]	0.802 [.075](.075)	0.793 [.075](.074)
	1.0	1.018 [.097](.096)	0.999 [.094]	0.999 [.094]	1.004 [.095](.094)	0.993 [.094](.094)
	2.0	2.040 [.181](.177)	2.000 [.174]	2.000 [.174]	2.013 [.175](.175)	1.995 [.174](.174)
	5.0	5.102 [.431](.419)	5.001 [.411]	5.001 [.411]	5.038 [.415](.413)	5.002 [.411](.411)
100	0.5	0.504 [.033](.033)	0.499 [.032]	0.499 [.032]	0.500 [.032](.032)	0.497 [.032](.032)
	0.8	0.807 [.053](.053)	0.799 [.052]	0.799 [.052]	0.801 [.052](.052)	0.796 [.052](.052)
	1.0	1.009 [.066](.065)	1.000 [.065]	1.000 [.065]	1.002 [.065](.065)	0.996 [.065](.065)
	2.0	2.019 [.123](.122)	1.999 [.120]	1.999 [.120]	2.006 [.121](.121)	1.997 [.120](.120)
	5.0	5.054 [.294](.289)	5.004 [.287]	5.004 [.287]	5.023 [.288](.287)	5.005 [.286](.286)
		$p = 0.3$				
10	0.5	0.546 [.133](.125)	0.491 [.112]	0.490 [.112]	0.506 [.116](.116)	0.470 [.116](.112)
	0.8	0.881 [.231](.216)	0.792 [.193]	0.792 [.193]	0.816 [.202](.201)	0.748 [.198](.191)
	1.0	1.100 [.292](.274)	0.987 [.245]	0.988 [.244]	1.019 [.256](.256)	0.931 [.252](.242)
	2.0	2.232 [.594](.547)	1.988 [.482]	1.988 [.482]	2.078 [.520](.514)	1.912 [.494](.486)
	5.0	5.695 [1.48](1.31)	5.017 [1.14]	5.010 [1.14]	5.339 [1.28](1.23)	5.001 [1.15](1.15)
20	0.5	0.521 [.084](.082)	0.497 [.078]	0.497 [.078]	0.502 [.079](.079)	0.486 [.079](.078)
	0.8	0.837 [.142](.137)	0.797 [.131]	0.797 [.131]	0.806 [.133](.133)	0.774 [.132](.130)
	1.0	1.044 [.178](.173)	0.994 [.164]	0.994 [.164]	1.006 [.167](.167)	0.965 [.167](.163)
	2.0	2.098 [.347](.333)	1.990 [.315]	1.990 [.315]	2.027 [.325](.323)	1.952 [.319](.315)
	5.0	5.307 [.844](.787)	5.009 [.738]	5.009 [.738]	5.147 [.779](.765)	4.999 [.742](.742)
50	0.5	0.509 [.050](.049)	0.500 [.048]	0.500 [.048]	0.502 [.048](.048)	0.495 [.048](.048)
	0.8	0.813 [.083](.082)	0.798 [.081]	0.798 [.081]	0.801 [.081](.081)	0.789 [.081](.080)
	1.0	1.017 [.104](.103)	0.998 [.101]	0.998 [.101]	1.002 [.102](.102)	0.986 [.102](.101)
	2.0	2.039 [.203](.200)	1.998 [.195]	1.998 [.195]	2.011 [.198](.197)	1.983 [.196](.195)
	5.0	5.110 [.465](.452)	4.999 [.442]	4.999 [.442]	5.049 [.450](.447)	4.994 [.442](.442)
100	0.5	0.504 [.035](.034)	0.500 [.034]	0.500 [.034]	0.501 [.034](.034)	0.497 [.034](.034)
	0.8	0.807 [.057](.057)	0.800 [.056]	0.800 [.056]	0.801 [.056](.056)	0.795 [.056](.056)
	1.0	1.008 [.073](.072)	0.999 [.072]	0.999 [.072]	1.001 [.072](.072)	0.993 [.072](.071)
	2.0	2.022 [.140](.138)	2.002 [.137]	2.002 [.137]	2.008 [.138](.137)	1.994 [.137](.137)
	5.0	5.059 [.319](.313)	5.005 [.310]	5.005 [.310]	5.029 [.313](.311)	5.002 [.310](.310)

Note: for $\hat{\beta}^{bc2}_{\mathrm{MLE}}$ and $\hat{\beta}^{bc3}_{\mathrm{MLE}}$, the empirical sds are almost identical to the rmses and hence are not reported to conserve space.

Table 12. Empirical mean [rmse](se) of MLE-type estimators of β, random censoring, $k = 8$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE-I}}$	$\hat{\beta}_{\text{MMLE-II}}$
		$p = 0.7$				
10	0.5	0.534 [.065](.055)	0.497 [.051]	0.497 [.051]	0.497 [.052](.052)	0.488 [.053](.052)
	0.8	0.856 [.103](.087)	0.797 [.081]	0.795 [.081]	0.797 [.081](.081)	0.785 [.082](.081)
	1.0	1.070 [.128](.107)	0.996 [.099]	0.995 [.099]	0.998 [.100](.100)	0.985 [.101](.099)
	2.0	2.158 [.261](.207)	2.005 [.193]	2.001 [.192]	2.011 [.192](.192)	1.998 [.192](.192)
	5.0	5.401 [.653](.515)	5.015 [.479]	5.005 [.478]	5.008 [.477](.477)	5.008 [.477](.477)
20	0.5	0.515 [.039](.036)	0.499 [.035]	0.499 [.035]	0.498 [.035](.035)	0.494 [.036](.035)
	0.8	0.824 [.062](.057)	0.798 [.055]	0.798 [.055]	0.797 [.055](.055)	0.791 [.056](.055)
	1.0	1.032 [.077](.070)	0.998 [.068]	0.998 [.068]	0.998 [.068](.068)	0.991 [.069](.068)
	2.0	2.072 [.152](.134)	2.002 [.129]	2.002 [.129]	2.005 [.129](.129)	1.997 [.129](.129)
	5.0	5.187 [.383](.334)	5.012 [.323]	5.010 [.323]	5.004 [.322](.322)	5.004 [.322](.322)
50	0.5	0.506 [.023](.022)	0.500 [.022]	0.500 [.022]	0.499 [.022](.022)	0.497 [.022](.022)
	0.8	0.809 [.035](.034)	0.799 [.034]	0.799 [.034]	0.798 [.034](.034)	0.796 [.034](.034)
	1.0	1.012 [.044](.043)	1.000 [.042]	0.999 [.042]	0.999 [.042](.042)	0.996 [.042](.042)
	2.0	2.028 [.087](.082)	2.001 [.081]	2.001 [.081]	2.002 [.081](.081)	1.999 [.081](.081)
	5.0	5.074 [.215](.202)	5.007 [.200]	5.007 [.200]	5.003 [.199](.199)	5.003 [.199](.199)
100	0.5	0.503 [.016](.015)	0.500 [.015]	0.500 [.015]	0.499 [.015](.015)	0.499 [.015](.015)
	0.8	0.805 [.025](.024)	0.800 [.024]	0.800 [.024]	0.800 [.024](.024)	0.799 [.024](.024)
	1.0	1.006 [.030](.030)	1.000 [.029]	1.000 [.029]	0.999 [.029](.029)	0.998 [.029](.029)
	2.0	2.014 [.059](.057)	2.001 [.057]	2.001 [.057]	2.002 [.057](.057)	2.000 [.057](.057)
	5.0	5.034 [.143](.139)	5.001 [.139]	5.001 [.139]	4.999 [.139](.139)	4.999 [.139](.139)
		$p = 0.5$				
10	0.5	0.530 [.064](.057)	0.496 [.053]	0.495 [.053]	0.492 [.054](.053)	0.480 [.057](.053)
	0.8	0.850 [.106](.093)	0.795 [.087]	0.794 [.087]	0.789 [.088](.087)	0.768 [.092](.086)
	1.0	1.063 [.132](.116)	0.994 [.108]	0.992 [.108]	0.988 [.109](.109)	0.962 [.114](.108)
	2.0	2.148 [.265](.219)	2.001 [.204]	1.997 [.203]	2.004 [.205](.205)	1.966 [.206](.203)
	5.0	5.402 [.663](.528)	5.019 [.491]	5.010 [.490]	5.048 [.496](.494)	4.984 [.489](.489)
20	0.5	0.514 [.041](.038)	0.499 [.037]	0.499 [.037]	0.496 [.037](.037)	0.490 [.038](.037)
	0.8	0.823 [.065](.061)	0.798 [.059]	0.798 [.059]	0.794 [.060](.059)	0.784 [.061](.059)
	1.0	1.030 [.082](.076)	0.998 [.074]	0.998 [.074]	0.994 [.074](.074)	0.981 [.076](.074)
	2.0	2.071 [.162](.145)	2.004 [.141]	2.004 [.141]	2.003 [.141](.141)	1.985 [.141](.140)
	5.0	5.188 [.393](.345)	5.014 [.334]	5.012 [.334]	5.023 [.335](.334)	4.993 [.333](.333)
50	0.5	0.506 [.024](.023)	0.500 [.023]	0.500 [.023]	0.498 [.023](.023)	0.496 [.023](.023)
	0.8	0.809 [.039](.038)	0.800 [.037]	0.800 [.037]	0.797 [.037](.037)	0.793 [.038](.037)
	1.0	1.012 [.048](.047)	1.000 [.046]	1.000 [.046]	0.998 [.046](.046)	0.993 [.047](.046)
	2.0	2.026 [.091](.087)	2.001 [.086]	2.001 [.086]	2.000 [.086](.086)	1.993 [.086](.086)
	5.0	5.070 [.219](.207)	5.005 [.205]	5.004 [.205]	5.007 [.205](.205)	4.996 [.204](.204)
100	0.5	0.503 [.017](.016)	0.500 [.016]	0.500 [.016]	0.499 [.016](.016)	0.498 [.016](.016)
	0.8	0.804 [.026](.026)	0.800 [.026]	0.800 [.026]	0.799 [.026](.026)	0.797 [.026](.026)
	1.0	1.006 [.033](.033)	1.000 [.032]	1.000 [.032]	0.999 [.032](.032)	0.997 [.032](.032)
	2.0	2.013 [.063](.062)	2.001 [.062]	2.001 [.062]	2.000 [.062](.062)	1.997 [.062](.061)
	5.0	5.036 [.150](.145)	5.004 [.145]	5.004 [.145]	5.005 [.145](.144)	4.999 [.144](.144)
		$p = 0.3$				
10	0.5	0.529 [.066](.059)	0.496 [.055]	0.495 [.055]	0.490 [.056](.055)	0.475 [.060](.055)
	0.8	0.847 [.109](.099)	0.794 [.092]	0.793 [.092]	0.784 [.093](.092)	0.752 [.102](.090)
	1.0	1.060 [.137](.124)	0.994 [.115]	0.992 [.115]	0.981 [.117](.115)	0.939 [.129](.113)
	2.0	2.138 [.281](.244)	1.993 [.226]	1.989 [.225]	1.989 [.230](.229)	1.911 [.241](.224)
	5.0	5.444 [.731](.580)	5.027 [.531]	5.015 [.529]	5.103 [.556](.547)	4.954 [.532](.530)
20	0.5	0.514 [.041](.039)	0.499 [.038]	0.499 [.038]	0.495 [.038](.038)	0.487 [.040](.038)
	0.8	0.823 [.070](.066)	0.799 [.064]	0.799 [.064]	0.792 [.064](.064)	0.777 [.067](.063)
	1.0	1.029 [.088](.083)	0.998 [.081]	0.998 [.080]	0.991 [.081](.080)	0.970 [.085](.080)
	2.0	2.067 [.176](.162)	2.001 [.156]	2.000 [.156]	1.997 [.158](.158)	1.960 [.161](.156)
	5.0	5.206 [.427](.374)	5.018 [.359]	5.016 [.359]	5.049 [.366](.363)	4.981 [.359](.358)
50	0.5	0.505 [.025](.024)	0.500 [.024]	0.500 [.024]	0.498 [.024](.024)	0.495 [.024](.024)
	0.8	0.809 [.041](.040)	0.800 [.040]	0.800 [.040]	0.797 [.040](.040)	0.791 [.041](.040)
	1.0	1.011 [.052](.051)	0.999 [.050]	0.999 [.050]	0.996 [.050](.050)	0.988 [.051](.050)
	2.0	2.024 [.100](.098)	1.999 [.096]	1.999 [.096]	1.996 [.097](.096)	1.982 [.098](.096)
	5.0	5.081 [.235](.220)	5.010 [.217]	5.009 [.217]	5.021 [.219](.218)	4.995 [.217](.217)
100	0.5	0.503 [.017](.017)	0.500 [.017]	0.500 [.017]	0.499 [.017](.017)	0.497 [.017](.017)
	0.8	0.804 [.029](.028)	0.800 [.028]	0.800 [.028]	0.798 [.028](.028)	0.795 [.028](.028)
	1.0	1.005 [.036](.036)	1.000 [.035]	1.000 [.035]	0.998 [.035](.035)	0.994 [.036](.035)
	2.0	2.012 [.068](.067)	1.999 [.067]	1.999 [.067]	1.998 [.067](.067)	1.991 [.067](.067)
	5.0	5.038 [.159](.154)	5.003 [.153]	5.003 [.153]	5.008 [.153](.153)	4.996 [.153](.153)

Note: for $\hat{\beta}^{bc2}_{\text{MLE}}$ and $\hat{\beta}^{bc3}_{\text{MLE}}$, the empirical sds are almost identical to the rmses and hence are not reported to conserve space.