Supplementary Appendix

To "Specification Tests for Temporal Heterogeneity in Spatial Panel Data Models with Fixed Effects, by Xu and Yang, 2019"

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Summary

In this Supplementary Appendix, we present details on (*i*) the derivations of the Hessian, expected Hessian, and the VC matrices of the AQS functions, (*ii*) the proofs of the four theorems, and (*iii*) the estimation of the 'typical' sub-models including the homogeneous models, the SLE models with homogeneity in spatial error coefficients, and the panel SE (spatial error) models with one-way or two-way FE.

Appendix A: Some Basic Lemmas

Lemma A.1 (Kelejian and Prucha, 1999; Lee, 2002): *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ matrices that are uniformly bounded in both row and column sums. Let C_n be a sequence of conformable matrices whose elements are uniformly bounded. Then*

(i) the sequence $\{A_n B_n\}$ are uniformly bounded in both row and column sums,

(ii) the elements of A_n are uniformly bounded and $tr(A_n) = O(n)$, and

(iii) the elements of $A_n C_n$ and $C_n A_n$ are uniformly bounded.

Lemma A.2 (Yang, 2015b, Lemma A.1, extended). *For $t = 1, 2$, let A_{nt} be $n \times n$ matrices and c_{nt} be $n \times 1$ vectors. Let ε_n be an $n \times 1$ ramdom vector of iid elements with mean zero, variance σ^2, and finite 3rd and 4th cumulants μ_3 and μ_4. Let a_{nt} be the vector of diagonal elements of A_{nt}. Define $Q_{nt} = c'_{nt}\varepsilon_n + \varepsilon'_n A_{nt}\varepsilon_n$, $t = 1, 2$. Then, for $t, s = 1, 2$,*

$$\begin{aligned}&\mathrm{Cov}(Q_{nt}, Q_{ns}) \equiv f(A_{nt}, c_{nt}; A_{ns}, c_{ns})\\ &= \sigma^4 \mathrm{tr}[(A'_{nt} + A_{nt})A_{ns})] + \mu_3 a'_{nt} c_{ns} + \mu_3 c'_{nt} a_{ns} + \mu_4 a'_{nt} a_{ns} + \sigma^2 c'_{nt} c_{ns}. \end{aligned} \tag{A.1}$$

Various useful special cases of (A.1) *are as follows:*

(*i*) $\mathrm{Cov}(c'_{n1}\varepsilon_n, Q_{n2}) = f(\mathbf{0}, c_{n1}; A_{n2}, c_{n2}) = \mu_3 c'_{n1} a_{n2} + \sigma^2 c'_{n1} c_{n2}$, *where c_{n1} can be an $n \times k$ matrix with $k \geq 1$;*

(*ii*) $\mathrm{Var}(Q_{n1}) = f(A_{n1}, c_{n1}; A_{n1}, c_{n1}) = \sigma^4 \mathrm{tr}[(A'_{n1} + A_{n1})A_{n1})] + 2\mu_3 a'_{n1} c_{n1} + \mu_4 a'_{n1} a_{n1} + \sigma^2 c'_{n1} c_{n1}$;

(*iii*) $\mathrm{Var}(\varepsilon'_n A_{n1}\varepsilon_n) = f(A_{n1}, \mathbf{0}; A_{n1}, \mathbf{0}) = \sigma^4 \mathrm{tr}[(A'_{n1} + A_{n1})A_{n1})] + \mu_4 a'_{n1} a_{n1}$.

Lemma A.3 (CLT for Linear-Quadratic Forms, Kelejian and Prucha, 2001). *Let A_n, a_n, c_n and ε_n be as in Lemma A.2. Assume (i) A_n is bounded uniformly in row and column sums, (ii) $n^{-1}\sum_{i=1}^n |c_{n,i}^{2+\eta_1}| < \infty, \eta_1 > 0$, and (iii) $E|\varepsilon_{n,i}^{4+\eta_2}| < \infty$, $\eta_2 > 0$. Then,*

$$\frac{\varepsilon'_n A_n \varepsilon_n + c'_n \varepsilon_n - \sigma^2 \mathrm{tr}(A_n)}{\{\sigma^4 \mathrm{tr}(A'_n A_n + A_n^2) + \mu_4 a'_n a_n + \sigma^2 c'_n c_n + 2\mu_3 a'_n c_n\}^{\frac{1}{2}}} \xrightarrow{D} N(0, 1).$$

Appendix B: Hessian, Expected Hessian and VC Matrices

Notation. For $t, s = 1, \ldots, T$, $\texttt{blkdiag}\{A_t\}$ forms a block-diagonal matrix by placing A_t diagonally, $\{A_t\}$ forms a matrix by stacking A_t horizontally, and $\{B_{ts}\}$ forms a matrix by the component matrices B_{ts}. The negative Hessian matrix $J_{\varpi}(\boldsymbol{\theta}_0)$, its expectation $I_{\varpi}(\boldsymbol{\theta}_0)$, and the VC matrix $\Sigma_{\varpi}(\boldsymbol{\theta}_0)$ of the AQS function, ϖ=SL1, SL2, SLE1, SLE2, are all partitioned according to the slope parameters $\boldsymbol{\beta}$, the spatial lag parameters $\boldsymbol{\lambda}$, spatial error parameters $\boldsymbol{\rho}$ (if existing in the model), and the error variance σ^2, with the sub-matrices denoted by, e.g., $J_{\boldsymbol{\beta\beta}}$, $J_{\boldsymbol{\beta\lambda}}$, $I_{\boldsymbol{\beta\beta}}$, $I_{\boldsymbol{\beta\lambda}}$, $\Sigma_{\boldsymbol{\beta\beta}}$, $\Sigma_{\boldsymbol{\beta\lambda}}$. Furthermore, $\texttt{diag}(\cdot)$ forms a diagonal matrix and $\texttt{diagv}(\cdot)$ a column vector, based on the diagonal elements of a square matrix.

Parametric quantities, e.g., $A_n(\lambda_{t0})$ and $B_n(\rho_{t0})$, evaluated at the true parameters are denoted as A_{nt} and B_{nt}. For a matrix A_n, denote $A_n^s = A_n + A_n'$. The bold $\mathbf{0}$ represents generically a vector or a matrix of zeros, to distinguish from the scalar 0.

B.1. Panel SL model with one-way FE. Letting $\eta_{nt} = G_{nt}(X_{nt}\beta_t + c_n)$ and $g_{nt} = \texttt{diagv}(G_{nt})$, the negative Hessian matrix, $J_{\texttt{SL1}}(\boldsymbol{\theta}_0)$, has the components:

$$
\begin{aligned}
J_{\boldsymbol{\beta\beta}} &= \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}X_{nt}'X_{nt}\big\} - \big\{\tfrac{1}{T\sigma_0^2}X_{nt}'X_{ns}\big\},\\
J_{\boldsymbol{\lambda\beta}} &= \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}(W_nY_{nt})'X_{nt}\big\} - \big\{\tfrac{1}{T\sigma_0^2}(W_nY_{nt})'X_{ns}\big\},\\
J_{\boldsymbol{\lambda\lambda}} &= \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}(W_nY_{nt})'(W_nY_{nt}) + \tfrac{T-1}{T}\mathrm{tr}(G_{nt}^2)\big\} - \big\{\tfrac{1}{T\sigma_0^2}(W_nY_{nt})'(W_nY_{ns})\big\},\\
J_{\sigma^2\boldsymbol{\beta}} &= \big\{\tfrac{1}{\sigma_0^4}\tilde{V}_{nt}'X_{nt}\big\},\ J_{\sigma^2\boldsymbol{\lambda}} = \big\{\tfrac{1}{\sigma_0^4}(W_nY_{nt})'\tilde{V}_{nt}\big\},\ J_{\sigma^2\sigma^2} = -\tfrac{n(T-1)}{2\sigma_0^4} + \tfrac{1}{\sigma_0^6}\textstyle\sum_{t=1}^T \tilde{V}_{nt}'\tilde{V}_{nt}.
\end{aligned}
$$

The expected negative Hessian matrix, $I_{\texttt{SL1}}(\boldsymbol{\theta}_0)$, has the components:

$$
\begin{aligned}
I_{\boldsymbol{\beta\beta}} &= J_{\boldsymbol{\beta\beta}},\ I_{\boldsymbol{\lambda\beta}} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}\eta_{nt}'X_{nt}\big\} - \big\{\tfrac{1}{T\sigma_0^2}\eta_{nt}'X_{ns}\big\},\\
I_{\boldsymbol{\lambda\lambda}} &= \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}\eta_{nt}'\eta_{nt} + \tfrac{T-1}{T}\mathrm{tr}(G_{nt}^sG_{nt})\big\} - \big\{\tfrac{1}{T\sigma_0^2}\eta_{nt}'\eta_{ns}\big\},\\
I_{\sigma^2\boldsymbol{\beta}} &= \mathbf{0},\ I_{\sigma^2\lambda} = \big\{\tfrac{T-1}{T\sigma_0^2}\mathrm{tr}(G_{nt})\big\},\ I_{\sigma^2\sigma^2} = \tfrac{n(T-1)}{2\sigma_0^4}.
\end{aligned}
$$

The VC matrix $\Sigma_{\texttt{SL1}}(\boldsymbol{\theta}_0) = I_{\texttt{SL1}}(\boldsymbol{\theta}_0) + \Omega_{\texttt{SL1}}(\boldsymbol{\theta}_0)$, where $\Omega_{\texttt{SL1}}(\boldsymbol{\theta}_0)$ has components:

$$
\begin{aligned}
\Omega_{\boldsymbol{\beta\beta}} &= 0_{tk\times tk},\ \Omega_{\boldsymbol{\lambda\beta}} = \texttt{bikdiag}\big\{\tfrac{T-1}{T\sigma_0}\gamma g_{nt}'X_{nt}\big\} - \big\{\tfrac{T-1}{T^2\sigma_0}\gamma g_{nt}'X_{ns}\big\},\\
\Omega_{\boldsymbol{\lambda\lambda}} &= \texttt{blkdiag}\big\{\tfrac{2(T-1)}{T\sigma_0}\gamma\eta_{nt}'g_{nt} + (\tfrac{T-1}{T})^2\kappa g_{nt}'g_{nt} - \tfrac{1}{T}\mathrm{tr}(G_{nt}G_{nt})\big\}\\
&\quad -\big\{\tfrac{T-1}{T^2\sigma_0}\gamma(\eta_{nt}'g_{ns} + g_{nt}'\eta_{ns}) - \tfrac{1}{T^2}\mathrm{tr}(G_{nt}G_{ns})\big\},\\
\Omega_{\sigma^2\boldsymbol{\beta}} &= \big\{0_{tk}'\big\},\ \Omega_{\sigma^2\boldsymbol{\lambda}} = \big\{\tfrac{(T-1)^2}{2T^2\sigma_0^2}\kappa\mathrm{tr}(G_{nt})\big\},\ \Omega_{\sigma^2\sigma^2} = \tfrac{n(T-1)^2}{4T\sigma_0^4}\kappa.
\end{aligned}
$$

where γ and κ are, respectively, the measures of skewness and excess kurtosis of $v_{i,t}$.

Alternatively, we can find the VC matrix $\Sigma_{n,T}(\boldsymbol{\theta}_0)$ by first expressing the AQS function $S_{\texttt{SL1}}^{\star}(\boldsymbol{\theta})$ at $\boldsymbol{\theta}_0$ in terms of $\mathbb{V}_N = (V_{n1}', \ldots, V_{nT}')'$, where $N = nT$, and then applying Lemma A.2. Let z_t be a $T \times 1$ vector of element 1 in the tth position and 0 elsewhere, and define $Z_{Nt} = z_t \otimes I_n$, $\bar{Z}_N = \frac{1}{T}(l_T \otimes I_n)$, and $Z_{Nt}^{\circ} = Z_{Nt} - \bar{Z}_N$. Thus,

$V_{nt} = Z'_{Nt}\mathbb{V}_N$ and $\widetilde{V}_{nt} = V_{nt} - \overline{V}_n = Z^{\circ\prime}_{Nt}\mathbb{V}_N$. The AQS function $S^{\star}_{\mathtt{SL1}}(\theta)$ at θ_0 takes the form:

$$S^{\star}_{\mathtt{SL1}}(\theta_0) = \begin{cases} \Pi'_{1t}\mathbb{V}_N,\ t = 1,\dots,T, \\ \Pi'_{2t}\mathbb{V}_N + \mathbb{V}'_N\Phi_t\mathbb{V}_N - \frac{T-1}{T}\mathrm{tr}(G_{nt}),\ t = 1,\dots,T, \\ \mathbb{V}'_N\Psi\mathbb{V}_N - \frac{n(T-1)}{2\sigma^2}, \end{cases} \tag{B.1}$$

where $\Pi_{1t} = \frac{1}{\sigma_0^2}Z^{\circ}_{Nt}X_{nt}$, $\Pi_{2t} = \frac{1}{\sigma_0^2}Z^{\circ}_{Nt}\eta_{nt}$, $\Phi_t = \frac{1}{\sigma_0^2}Z_{Nt}G'_{nt}Z^{\circ\prime}_{Nt}$, and $\Psi = \frac{1}{2\sigma^4}\sum_{t=1}^T Z^{\circ}_{Nt}Z^{\circ\prime}_{Nt}$.

Applying Lemma A.2 with ε, c_{nt} and A_{nt} replaced by $\mathbb{V}_N$, Π_{1t} and Π_{2t}, Φ_t, and Ψ, we obtain the VC matrix of the AQS function:

$$\Sigma_{\mathtt{SL1}}(\theta_0) = \begin{pmatrix} \{f(\mathbf{0},\Pi_{1t};\mathbf{0},\Pi_{1s})\}, & \{f(\mathbf{0},\Pi_{1t};\Phi_s,\Pi_{2s})\}, & \{f(\mathbf{0},\Pi_{1t};\Psi,\mathbf{0})\} \\ \sim, & \{f(\Phi_t,\Pi_{2t};\Phi_s,\Pi_{2s})\}, & \{f(\Phi_t,\Pi_{2t};\Psi,\mathbf{0})\} \\ \sim, & \sim, & f(\Psi,\mathbf{0};\Psi,\mathbf{0}) \end{pmatrix}.$$

This expression can be reduced to that given above, but it greatly simplifies the calculation.

B.2. Panel SL model with two-way FE. Letting $\eta^*_{nt} = G^*_{nt}(X^*_{nt}\beta_t + c^*_n)$ and $g^*_{nt} = \mathtt{diagv}(G^*_{nt})$, as the AQS function takes a similar form as that for 1FE panel SL model, the negative Hessian, $J_{\mathtt{SL2}}(\theta_0)$, also takes a similar form:

$$\begin{aligned}
J_{\beta\beta} &= \mathtt{blkdiag}\{\tfrac{1}{\sigma_0^2}X^{*\prime}_{nt}X^*_{nt}\} - \{\tfrac{1}{T\sigma_0^2}X^{*\prime}_{nt}X^*_{ns}\},\\
J_{\lambda\beta} &= \mathtt{blkdiag}\{\tfrac{1}{\sigma_0^2}(W^*_nY^*_{nt})'X^*_{nt}\} - \{\tfrac{1}{T\sigma_0^2}(W^*_nY^*_{nt})'X^*_{ns}\},\\
J_{\lambda\lambda} &= \mathtt{blkdiag}\{\tfrac{1}{\sigma_0^2}(W^*_nY^*_{nt})'(W^*_nY^*_{nt}) + \tfrac{T-1}{T}\mathrm{tr}(G^{*2}_{nt})\} - \{\tfrac{1}{T\sigma_0^2}(W^*_nY^*_{nt})'(W^*_nY^*_{ns})\},\\
J_{\sigma^2\beta} &= \{\tfrac{1}{\sigma_0^4}\tilde{V}^{*\prime}_{nt}X^*_{nt}\},\ J_{\sigma^2\lambda} = \{\tfrac{1}{\sigma_0^4}(W^*_nY^*_{nt})'\tilde{V}^*_{nt}\},\ J_{\sigma^2\sigma^2} = -\tfrac{(n-1)(T-1)}{2\sigma_0^4} + \tfrac{1}{\sigma_0^6}\textstyle\sum_{t=1}^T\tilde{V}^{*\prime}_{nt}\tilde{V}^*_{nt}.
\end{aligned}$$

As the derivation of the expected negative Hessian matrix involves only the first two moments of the transformed errors which are the same as the first two moments of the original error, the expected negative Hessian matrix, $I_{\mathtt{SL2}}(\theta_0)$, also takes a similar form as that of 1FE panel SL model and contains the following components:

$$\begin{aligned}
I_{\beta\beta} &= J_{\beta\beta},\ I_{\lambda\beta} = \mathtt{blkdiag}\{\tfrac{1}{\sigma_0^2}\eta^{*\prime}_{nt}X^*_{nt}\} - \{\tfrac{1}{T\sigma_0^2}\eta^{*\prime}_{nt}X^*_{ns}\},\\
I_{\lambda\lambda} &= \mathtt{blkdiag}\{\tfrac{1}{\sigma_0^2}\eta^{*\prime}_{nt}\eta^*_{nt} + \tfrac{T-1}{T}\mathrm{tr}(G^{*s}_{nt}G^*_{nt})\} - \{\tfrac{1}{T\sigma_0^2}\eta^{*\prime}_{nt}\eta^*_{ns}\},\\
I_{\sigma^2\beta} &= 0'_{tk},\ I_{\sigma^2\lambda} = \{\tfrac{T-1}{T\sigma_0^2}\mathrm{tr}(G^*_{nt})\},\ I_{\sigma^2\sigma^2} = \tfrac{(n-1)(T-1)}{2\sigma_0^4}.
\end{aligned}$$

The derivation of the VC matrix of the AQS function, however, is different from that of one-way panel SL model due to the involvement of 3rd and 4th moments of the errors. The elements of the transformed errors V^*_{nt} may not be totally independent unless the original errors are normal and their 3rd and 4th moments may not be constant. Thus, one needs to work with the original error vector V_{nt} through $V^*_{nt} = F'_{n,n-1}V_{nt}$. Using the results: (*i*) $(I_{n-1} - \lambda_t F'_{n,n-1}W_nF_{n,n-1})^{-1} = F'_{n,n-1}(I_n - \lambda_tW_n)^{-1}F_{n,n-1}$ (Lee and Yu, 2010, Lemma A.2), and (*ii*) for a row normalized W_n, $F'_{n,n-1}W_nJ_n = F'_{n,n-1}W_n$ and

$G_n^*(\lambda_t) = F'_{n,n-1} G_n(\lambda_t) F_{n,n-1}$ and $g^*_{nt} = \mathtt{diag}(F_{n,n-1} G^*_n(\lambda_t) F'_{n,n-1})$, we obtain the VC matrix $\Sigma_{\mathtt{SL2}}(\theta_0) = I_{\mathtt{SL2}}(\theta_0) + \Omega_{\mathtt{SL2}}(\theta_0)$, where $\Omega_{\mathtt{SL2}}(\theta_0)$ has components:

$$
\begin{aligned}
\Omega_{\beta\beta} &= 0_{tk\times tk},\ \Omega_{\lambda\beta} = \mathtt{bikdiag}\big\{\tfrac{T-1}{T\sigma_0}\gamma g^{*\prime}_{nt} F_{n,n-1} X^*_{nt}\big\} - \big\{\tfrac{T-1}{T^2\sigma_0}\gamma g^{*\prime}_{nt} F_{n,n-1} X^*_{ns}\big\},\\
\Omega_{\lambda\lambda} &= \mathtt{blkdiag}\big\{\tfrac{2(T-1)}{T\sigma_0}\gamma \eta^{*\prime}_{nt} F'_{n,n-1} g^*_{nt} + (\tfrac{T-1}{T})^2 \kappa g^{*\prime}_{nt} g^*_{nt} - \tfrac{1}{T}\mathrm{tr}(G^*_{nt} G^*_{nt})\big\}\\
&\quad - \big\{\tfrac{T-1}{T^2\sigma_0}\gamma(\eta^{*\prime}_{nt} F'_{n,n-1} g^*_{ns} + g^{*\prime}_{nt} F_{n,n-1}\eta^*_{ns}) - \tfrac{1}{T^2}\mathrm{tr}(G^*_{nt} G^*_{ns})\big\},\\
\Omega_{\sigma^2\beta} &= \big\{0'_{tk}\big\},\ \Omega_{\sigma^2\lambda} = \big\{\tfrac{(T-1)^2}{2T^2\sigma_0^2}\kappa \mathrm{diag}(J_n)\mathtt{diag}(F_{n,n-1} G^*_{nt} F'_{n,n-1})\big\},\ \Omega_{\sigma^2\sigma^2} = \tfrac{n(T-1)^2}{4T\sigma_0^4}\kappa.
\end{aligned}
$$

Similarly, $\Sigma_{\mathtt{SL2}}(\theta_0)$ can be obtained by first expressing $S^\star_{\mathtt{SL2}}(\theta_0)$ in $\mathbb{V}_N$, through $V^*_{nt} = F'_{n,n-1} Z'_{Nt}\mathbb{V}_N$ and $\widetilde{V}^*_{nt} = V^*_{nt} - \overline{V}^*_n = F'_{n,n-1} Z^{\circ\prime}_{Nt}\mathbb{V}_N$:

$$
S^\star_{\mathtt{SL2}}(\theta_0) = \begin{cases} \Pi'_{1t}\mathbb{V}_N,\ t = 1,\ldots,T,\\ \Pi'_{2t}\mathbb{V}_N + \mathbb{V}'_N \Phi_t \mathbb{V}_N - \frac{T-1}{T}\mathrm{tr}(G_{nt}),\ t = 1,\ldots,T,\\ \mathbb{V}'_N \Psi \mathbb{V}_N - \frac{(n-1)(T-1)}{2\sigma^2}, \end{cases} \tag{B.2}
$$

where $\Pi_{1t} = \frac{1}{\sigma_0^2} Z^\circ_{Nt} F_{n,n-1} X^*_{nt}$, $\Pi_{2t} = \frac{1}{\sigma_0^2} Z^\circ_{Nt} F_{n,n-1}\eta^*_{nt}$, $\Phi_t = \frac{1}{\sigma_0^2} Z_{Nt} F_{n,n-1} G^{*\prime}_{nt} F'_{n,n-1} Z^{\circ\prime}_{Nt}$, and $\Psi = \frac{1}{2\sigma^4}\sum_{t=1}^T Z^\circ_{Nt} F_{n,n-1} F'_{n,n-1} Z^{\circ\prime}_{Nt}$. Then, applying Lemma A.2 with ε, c_{nt} and A_{nt} replaced by $\mathbb{V}_N$, Π_{1t} and Π_{2t}, Φ_t, and Ψ to give $\Sigma_{\mathtt{SL2}}(\theta_0)$ in an identical form as $\Sigma_{\mathtt{SL1}}(\theta_0)$.

B.3. Panel SLE model with one-way FE. Let $\dot{D}_{nt} = -\frac{d}{d\rho_{t0}} D_{nt} = M'_n B_{nt} + B'_{nt} M_n$. We have the components of the negative Hessian matrix $J_{\mathtt{SEL1}}(\theta_0)$:

$$
\begin{aligned}
J_{\beta\beta} &= \mathtt{blkdiag}\big\{\tfrac{1}{\sigma_0^2} X'_{nt} D_{nt} X_{nt}\big\} - \big\{\tfrac{1}{\sigma_0^2} X'_{nt} D_{nt}\mathbb{D}_n^{-1} D_{ns} X_{ns}\big\};\\
J_{\beta\lambda} &= \mathtt{blkdiag}\big\{\tfrac{1}{\sigma_0^2} X'_{nt} D_{nt} W_n Y_{nt}\big\} - \big\{\tfrac{1}{\sigma_0^2} X'_{nt} D_{nt}\mathbb{D}_n^{-1} D_{ns} W_n Y_{ns}\big\};\\
J_{\beta\rho} &= \mathtt{blkdiag}\big\{\tfrac{1}{\sigma_0^2} X'_{nt}\dot{D}_{nt} B^{-1}_{nt}\widetilde{V}_{nt}\big\} - \big\{\tfrac{1}{\sigma_0^2} X'_{nt} D_{nt}\mathbb{D}_n^{-1}\dot{D}_{ns} B^{-1}_{ns}\widetilde{V}_{ns}\big\};\\
J_{\lambda\lambda} &= \mathtt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}(W_n Y_{nt})' D_{nt}(W_n Y_{nt}) + \mathrm{tr}(R_{nt} G^2_{nt})\big\}\\
&\quad - \big\{\tfrac{1}{\sigma_0^2}(W_n Y_{nt})' D_{nt}\mathbb{D}_n^{-1} D_{ns}(W_n Y_{ns})\big\};\\
J_{\lambda\rho} &= \mathtt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}(W_n Y_{nt})'\dot{D}_{nt} B^{-1}_{nt}\widetilde{V}_{nt} + \mathrm{tr}[\mathbb{D}_n^{-1}\dot{D}_{nt} G_{nt}]\big\}\\
&\quad - \big\{\tfrac{1}{\sigma_0^2}(W_n Y_{nt})' D_{nt}\mathbb{D}_n^{-1}\dot{D}_{ns} B^{-1}_{ns}\widetilde{V}_{ns} + \mathrm{tr}[\mathbb{D}_n^{-1} D_{nt} G_{nt}\mathbb{D}_n^{-1}\dot{D}_{ns}]\big\};\\
J_{\rho\lambda} &= \mathtt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}(W_n Y_{nt})'\dot{D}_{nt} B^{-1}_{nt}\widetilde{V}_{nt}\big\} - \big\{\tfrac{1}{\sigma_0^2}(W_n Y_{ns})' D_{ns}\mathbb{D}_n^{-1}\dot{D}_{nt} B^{-1}_{nt}\widetilde{V}_{nt}\big\};\\
J_{\rho\rho} &= \mathtt{blkdiag}\big\{\tfrac{1}{\sigma^2}\widetilde{V}'_{nt} H'_{nt} H_{nt}\widetilde{V}_{nt} + \mathrm{tr}(H^2_{nt} + \mathbb{D}_n^{-1} M'_n M_n)\big\}\\
&\quad - \big\{\tfrac{1}{\sigma^2}\widetilde{V}'_{nt} H^s_{nt} B_{nt}\mathbb{D}_n^{-1} B'_{ns} H^s_{ns}\widetilde{V}_{ns} + \mathrm{tr}(\mathbb{D}_n^{-1} B'_{nt} M_n \mathbb{D}_n^{-1}\dot{D}_{ns})\big\};\\
J_{\sigma^2\beta} &= \big\{\tfrac{1}{\sigma_0^4} X'_{nt} B'_{nt}\widetilde{V}_{nt}\big\};\quad J_{\sigma^2\lambda} = \big\{\tfrac{1}{\sigma_0^4}(W_n Y_{nt})' B'_{nt}\widetilde{V}_{nt}\big\};\\
J_{\sigma^2\rho} &= \big\{\tfrac{1}{\sigma_0^4}\widetilde{V}'_{nt} H_{nt}\widetilde{V}_{nt}\big\};\quad J_{\sigma^2\sigma^2} = -\tfrac{n(T-1)}{2\sigma^4} + \tfrac{1}{\sigma^6}\textstyle\sum_{t=1}^T \widetilde{V}'_{nt}\widetilde{V}_{nt}.
\end{aligned}
$$

Letting $\bar{G}_{nt} = B_{nt} G_{nt} B^{-1}_{nt}$, the expected negative Hessian matrix $I_{\mathtt{SEL1}}(\theta_0)$ has the components:

$I_{\beta\beta} = J_{\beta\beta},\;\; I_{\lambda\beta} = \texttt{blkdiag}\big\{\frac{1}{\sigma_0^2}\eta_t' D_{nt}X_{nt}\big\} - \big\{\frac{1}{\sigma_0^2}\eta_t' D_{nt}\mathbb{D}_n^{-1}D_{ns}X_{ns}\big\},\;\; I_{\rho\beta} = 0_{Tk}$

$I_{\lambda\lambda} = \texttt{blkdiag}\big\{\frac{1}{\sigma_0^2}\eta_{nt}' D_{nt}\eta_{nt} + \mathrm{tr}[S_{nt}(\rho)\bar{G}_{nt}^s\bar{G}_{nt}]\big\} - \big\{\frac{1}{\sigma_0^2}\eta_{nt}' D_{nt}\mathbb{D}_n^{-1}D_{ns}\eta_{ns}\big\},$

$I_{\lambda\rho} = \texttt{blkdiag}\big\{\mathrm{tr}[\bar{G}_{nt}' S_{nt}(\rho)H_{nt}^s]\big\};\;\; I_{\sigma^2\sigma^2} = -\frac{n(T-1)}{2\sigma_0^4} + \frac{1}{\sigma_0^4}\sum_{t=1}^T \mathrm{tr}S_{nt}(\rho)$

$I_{\rho\lambda} = \texttt{blkdiag}\big\{\mathrm{tr}[\bar{G}_{nt}' S_{nt}(\rho)H_{nt}^s S_{nt}(\rho)]\big\} - \big\{\mathrm{tr}[G_{ns}' D_{ns}\mathbb{D}_n^{-1}\dot{D}_{nt}\mathbb{D}_n^{-1}]\big\}$

$I_{\rho\rho} = \texttt{blkdiag}\big\{\mathrm{tr}[H_{nt}^s S_{nt}(\rho)H_{nt} - B_{nt}\mathbb{D}_n^{-1}\dot{D}_{nt}B_{nt}^{-1}H_{nt}]\big\} + \big\{\mathrm{tr}[B_{nt}\mathbb{D}_n^{-1}\dot{D}_{ns}\mathbb{D}_n^{-1}B_{nt}' H_{nt}]\big\}$

$I_{\sigma^2\beta} = 0_{tk}',\;\; I_{\sigma^2\lambda} = \big\{\frac{1}{\sigma_0^2}\mathrm{tr}[R_{nt}(\rho)G_{nt}]\big\},\;\; I_{\sigma^2\rho} = \frac{1}{\sigma_0^2}\mathrm{tr}(S_{nt}(\rho)H_{nt}).$

To derive $\Sigma_{\texttt{SLE1}}(\theta_0)$, we have, $\widetilde{V}_{nt} \equiv \widetilde{V}_{nt}(\beta_0, \lambda_0, \rho_0) = V_{nt} - B_{nt}\mathbb{D}_n^{-1}\sum_{s=1}^T B_{ns}' V_{ns} = Z_{Nt}^{\diamond\prime}\mathbb{V}_N$, where $Z_{Nt}^{\diamond\prime} = [Z_{Nt}' - B_{nt}\mathbb{D}_n^{-1}(l_T'\otimes I_n)\mathbb{B}_N]$ and $\mathbb{B}_N = \texttt{blkdiag}(B_{n1}, \ldots, B_{nT})$, and $W_nY_{nt} = G_{nt}(X_{nt}\beta_0 + c_n + B_{nt}^{-1}V_{nt}) = \eta_{nt} + G_{nt}B_{nt}^{-1}Z_{Nt}'\mathbb{V}_N$. These lead to,

$$S_{\texttt{SLE1}}^*(\theta_0) = \begin{cases} \Pi_{1t}'\mathbb{V}_N,\;\; t = 1, \ldots, T, \\ \Pi_{2t}'\mathbb{V}_N + \mathbb{V}_N'\Phi_{1t}\mathbb{V}_N - \mathrm{tr}(R_{nt}G_{nt}),\;\; t = 1, \ldots, T, \\ \mathbb{V}_N'\Phi_{2t}\mathbb{V}_N - \mathrm{tr}(S_{nt}H_{nt}),\;\; t = 1, \ldots, T, \\ \mathbb{V}_N'\Psi\mathbb{V}_N - \frac{n(T-1)}{2\sigma^2}, \end{cases} \tag{B.3}$$

where $\Pi_{1t} = \frac{1}{\sigma_0^2}Z_{Nt}^{\diamond}B_{nt}X_{nt}$, $\Pi_{2t} = \frac{1}{\sigma_0^2}Z_{Nt}^{\diamond}B_{nt}\eta_{nt}$, $\Phi_{1t} = \frac{1}{\sigma_0^2}Z_{Nt}B_{nt}^{-1\prime}G_{nt}'B_{nt}'Z_{Nt}^{\diamond\prime}$, $\Phi_{2t} = \frac{1}{\sigma_0^2}Z_{Nt}^{\diamond}H_{nt}Z_{Nt}^{\diamond\prime}$, and $\Psi = \frac{1}{2\sigma^4}\sum_{t=1}^T Z_{Nt}^{\diamond}Z_{Nt}^{\diamond\prime}$. Applying Lemma A.2 gives:

$$\Sigma_{\texttt{SLE1}}(\theta_0) =$$
$$\begin{pmatrix} \{f(\mathbf{0}, \Pi_{1t}; \mathbf{0}, \Pi_{1s})\}, & \{f(\mathbf{0}, \Pi_{1t}; \Phi_{1s}, \Pi_{2s})\}, & \{f(\mathbf{0}, \Pi_{1t}; \Phi_{2s}, \mathbf{0})\}, & \{f(\mathbf{0}, \Pi_{1t}; \Psi, \mathbf{0})\} \\ \sim, & \{f(\Phi_{1t}, \Pi_{2t}; \Phi_{1s}, \Pi_{2s})\}, & \{f(\Phi_{1t}, \Pi_{2t}; \Phi_{2s}, \mathbf{0})\}, & \{f(\Phi_{1t}, \Pi_{2t}; \Psi, \mathbf{0})\} \\ \sim, & \sim, & \{f(\Phi_{2t}, \mathbf{0}; \Phi_{2s}, \mathbf{0})\}, & \{f(\Phi_{2t}, \mathbf{0}; \Psi, \mathbf{0})\} \\ \sim, & \sim, & \sim, & f(\Psi, \mathbf{0}; \Psi, \mathbf{0}) \end{pmatrix}$$

B.4. Panel SLE model with two-way FE. Let $\dot{D}_{nt}^* = -\frac{d}{d\rho_{t0}}D_{nt}^* = M_n^{*\prime}B_{nt}^* + B_{nt}^{*\prime}M_n^*$. We have the components of the negative Hessian matrix $J_{\texttt{SEL2}}(\theta_0)$:

$$J_{\beta\beta} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}X_{nt}^{*\prime}D_{nt}^*X_{nt}^*\big\} - \big\{\tfrac{1}{\sigma_0^2}X{*\prime}_{nt}D_{nt}^*\mathbb{D}_n^{*-1}D_{ns}^*X_{ns}^*\big\};$$
$$J_{\beta\lambda} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}X_{nt}^{*\prime}D_{nt}^*W_n^*Y_{nt}^*\big\} - \big\{\tfrac{1}{\sigma_0^2}X_{nt}^{*\prime}D_{nt}^*\mathbb{D}_n^{*-1}D_{ns}^*W_n^*Y_{ns}^*\big\};$$
$$J_{\beta\rho} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}X_{nt}^{*\prime}\dot{D}_{nt}^*B_{nt}^{*-1}\widetilde{V}_{nt}^*\big\} - \big\{\tfrac{1}{\sigma_0^2}X_{nt}^{*\prime}D_{nt}^*\mathbb{D}_n^{*-1}\dot{D}_{ns}^*B_{ns}^{*-1}\widetilde{V}_{ns}^*\big\};$$
$$J_{\lambda\lambda} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}(W_n^*Y_{nt}^*)'D_{nt}^*(W_n^*Y_{nt}^*) + \mathrm{tr}(R_{nt}^*G_{nt}^{*2})\big\}$$
$$- \big\{\tfrac{1}{\sigma_0^2}(W_n^*Y_{nt}^*)'D_{nt}^*\mathbb{D}_n^{*-1}D_{ns}(W_n^*Y_{ns}^*)\big\};$$

$$J_{\boldsymbol{\lambda\rho}} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}(W_n^*Y_{nt}^*)'\dot{D}_{nt}^*B_{nt}^{*-1}\widetilde{V}_{nt}^* + \mathrm{tr}[\mathbb{D}_n^{*-1}\dot{D}_{nt}^*G_{nt}^*]\big\}$$
$$- \big\{\tfrac{1}{\sigma_0^2}(W_n^*Y_{nt}^*)'D_{nt}^*\mathbb{D}_n^{*-1}\dot{D}_{ns}^*B_{ns}^{*-1}\widetilde{V}_{ns}^* + \mathrm{tr}[\mathbb{D}_n^{*-1}D_{nt}^*G_{nt}^*\mathbb{D}_n^{*-1}\dot{D}_{ns}^*]\big\};$$
$$J_{\boldsymbol{\rho\lambda}} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}(W_n^*Y_{nt}^*)'\dot{D}_{nt}^*B_{nt}^{*-1}\widetilde{V}_{nt}^*\big\} - \big\{\tfrac{1}{\sigma_0^2}(W_n^*Y_{ns}^*)'D_{ns}^*\mathbb{D}_n^{*-1}\dot{D}_{nt}^*B_{nt}^{*-1}\widetilde{V}_{nt}^*\big\};$$
$$J_{\boldsymbol{\rho\rho}} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma^2}\widetilde{V}_{nt}^{*\prime}H_{nt}^{*\prime}H_{nt}^*\widetilde{V}_{nt}^* + \mathrm{tr}(H_{nt}^{*2} + \mathbb{D}_n^{*-1}M_n^{*\prime}M_n^*)\big\};$$
$$- \big\{\tfrac{1}{\sigma^2}\widetilde{V}_{nt}^{*\prime}H_{nt}^{*s}B_{nt}^*\mathbb{D}_n^{*-1}B_{ns}^{*\prime}H_{ns}^{*s}\widetilde{V}_{ns}^* + \mathrm{tr}(\mathbb{D}_n^{*-1}B_{nt}^{*\prime}M_n^*\mathbb{D}_n^{*-1}\dot{D}_{ns}^*)\big\}$$
$$J_{\sigma^2\boldsymbol{\beta}} = \big\{\tfrac{1}{\sigma_0^4}X_{nt}^{*\prime}B_{nt}^{*\prime}\widetilde{V}_{nt}^*\big\};\;\; J_{\sigma^2\boldsymbol{\lambda}} = \big\{\tfrac{1}{\sigma_0^4}(W_n^*Y_{nt}^*)'B_{nt}^{*\prime}\widetilde{V}_{nt}^*\big\};$$
$$J_{\sigma^2\boldsymbol{\rho}} = \big\{\tfrac{1}{\sigma_0^4}\widetilde{V}_{nt}^{*\prime}H_{nt}^*\widetilde{V}_{nt}^*\big\};\;\; J_{\sigma^2\sigma^2} = -\tfrac{(n-1)(T-1)}{2\sigma^4} + \tfrac{1}{\sigma^6}\textstyle\sum_{t=1}^T \widetilde{V}_{nt}^{*\prime}\widetilde{V}_{nt}^*.$$

The expected negative Hessian matrix, $I_{\texttt{SEL2}}(\boldsymbol{\theta}_0)$, has the components:

$$I_{\boldsymbol{\beta\beta}} = J_{\boldsymbol{\beta\beta}},\;\; I_{\boldsymbol{\lambda\beta}} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}\eta_t^{*\prime}D_{nt}^*X_{nt}^*\big\} - \big\{\tfrac{1}{\sigma_0^2}\eta_t^{*\prime}D_{nt}^*\mathbb{D}_n^{*-1}D_{ns}^*X_{ns}^*\big\};\;\; I_{\boldsymbol{\rho\beta}} = \mathbf{0};$$
$$I_{\boldsymbol{\lambda\lambda}} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}\eta_{nt}^{*\prime}D_{nt}^*\eta_{nt}^* + \mathrm{tr}[S_{nt}^*\bar{G}_{nt}^{*s}\bar{G}_{nt}^*]\big\} - \big\{\tfrac{1}{\sigma_0^2}\eta_{nt}^{*\prime}D_{nt}^*\mathbb{D}_n^{*-1}D_{ns}^*\eta_{ns}^*\big\};$$
$$I_{\boldsymbol{\lambda\rho}} = \texttt{blkdiag}\big\{\mathrm{tr}[\bar{G}*\prime_{nt}S_{nt}^*H_{nt}^{*s}]\big\};\;\; I_{\sigma^2\sigma^2} = -\tfrac{(n-1)(T-1)}{2\sigma_0^4} + \tfrac{1}{\sigma_0^4}\textstyle\sum_{t=1}^T \mathrm{tr}(S_{nt}^*);$$
$$I_{\boldsymbol{\rho\lambda}} = \texttt{blkdiag}\big\{\mathrm{tr}(\bar{G}_{nt}^{*\prime}S_{nt}^*H_{nt}^{*s}S_{nt}^*)\big\} - \big\{\mathrm{tr}(G_{ns}^{*\prime}D_{ns}^*\mathbb{D}_n^{*-1}\dot{D}_{nt}^*\mathbb{D}_n^{*-1})\big\};$$
$$I_{\boldsymbol{\rho\rho}} = \texttt{blkdiag}\big\{\mathrm{tr}(H_{nt}^{*s}S_{nt}^*H_{nt}^* - B_{nt}^*\mathbb{D}_n^{*-1}\dot{D}_{nt}^*B_{nt}^{*-1}H_{nt}^*)\big\} + \big\{\mathrm{tr}(B_{nt}^*\mathbb{D}_n^{*-1}\dot{D}_{ns}^*\mathbb{D}_n^{*-1}B_{nt}^{*\prime}H_{nt}^*)\big\};$$
$$I_{\sigma^2\boldsymbol{\beta}} = \mathbf{0};\;\; I_{\sigma^2\boldsymbol{\lambda}} = \big\{\tfrac{1}{\sigma_0^2}\mathrm{tr}(R_{nt}^*G_{nt}^*)\big\};\;\; I_{\sigma^2\boldsymbol{\rho}} = \big\{\tfrac{1}{\sigma_0^2}\mathrm{tr}(S_{nt}^*H_{nt}^*)\big\}.$$

To derive $\Sigma_{\texttt{SLE2}}(\boldsymbol{\theta}_0)$, $\widetilde{V}_{nt}^* \equiv \widetilde{V}_{nt}^*(\boldsymbol{\beta}_0, \boldsymbol{\lambda}_0, \boldsymbol{\rho}_0) = V_{nt}^* - B_{nt}^*\mathbb{D}_n^{*-1}\sum_{s=1}^T B_{ns}^{*\prime}V_{ns}^* = F_{n,n-1}'Z_{Nt}^{\diamond\prime}\mathbb{V}_N$, and $W_n^*Y_{nt}^* = G_{nt}^*(X_{nt}^*\beta_0 + c_n^* + B_{nt}^{-*1}V_{nt}^*) = \eta_{nt}^* + G_{nt}^*B_{nt}^{*-1}F_{n,n-1}'Z_{Nt}'\mathbb{V}_N$, leading to,

$$S_{\texttt{SLE2}}^*(\boldsymbol{\theta}_0) = \begin{cases} \Pi_{1t}'\mathbb{V}_N, \;\; t = 1, \ldots, T, \\ \Pi_{2t}'\mathbb{V}_N + \mathbb{V}_N'\Phi_{1t}\mathbb{V}_N - \mathrm{tr}(R_{nt}^*G_{nt}^*), \;\; t = 1, \ldots, T, \\ \mathbb{V}_N'\Phi_{2t}\mathbb{V}_N - \mathrm{tr}(S_{nt}^*H_{nt}^*), \;\; t = 1, \ldots, T, \\ \mathbb{V}_N'\Psi\mathbb{V}_N - \frac{(n-1)(T-1)}{2\sigma^2}, \end{cases} \tag{B.4}$$

where $\Pi_{1t} = \frac{1}{\sigma_0^2}Z_{Nt}^{\diamond *}B_{nt}^*X_{nt}^*$, $\Pi_{2t} = \frac{1}{\sigma_0^2}Z_{Nt}^{\diamond *}B_{nt}^*\eta_{nt}^*$, $\Phi_{1t} = \frac{1}{\sigma_0^2}Z_{Nt}^*B_{nt}^{*-1\prime}G_{nt}^{*\prime}B_{nt}^{*\prime}Z_{Nt}^{\diamond\prime}$, $\Phi_{2t} = \frac{1}{\sigma_0^2}Z_{Nt}^{\diamond *}H_{nt}^*Z_{Nt}^{\diamond *\prime}$, and $\Psi = \frac{1}{2\sigma^4}\sum_{t=1}^T Z_{Nt}^{\diamond *}Z_{Nt}^{\diamond *\prime}$, with $Z_{Nt}^* = Z_{Nt}F_{n,n-1}$ and $Z_{Nt}^{\diamond *} = Z_{Nt}^{\diamond}F_{n,n-1}$. Applying Lemma A.2 with ε, c_{nt} and A_{nt} replaced by $\mathbb{V}_N$, Π_{1t} and Π_{2t}, and Φ_{1t}, Φ_{2t} and Ψ, we obtain the VC matrix $\Sigma_{\texttt{SLE2}}(\boldsymbol{\theta}_0)$ taking identical form as $\Sigma_{\texttt{SLE1}}(\boldsymbol{\theta}_0)$ given above.

B.5. Panel SLE model with wto-way FE and homogeneous ρ. The quantities Y_{nt}^*, X_{nt}^*, c_n^*, W_n^*, M_n^* and V_{nt}^*, are defined similarly as in Model (3.11). Letting $\eta_{nt}^* = G_{nt}^*(X_{nt}^*\beta_t + c_n^*)$ and $g_{nt}^* = \texttt{diagv}(G_{nt}^*)$. The AQS function of $\boldsymbol{\theta} = (\boldsymbol{\beta}', \boldsymbol{\lambda}', \rho, \sigma^2)'$ is

$$S_{\texttt{SLE2}}^\star(\boldsymbol{\theta}) = \begin{cases} \frac{1}{\sigma^2}X_{nt}^{*\prime}B_n^{*\prime}(\rho)\widetilde{V}_{nt}^*(\boldsymbol{\beta}, \boldsymbol{\lambda}, \rho), \;\; t = 1, \ldots, T, \\ \frac{1}{\sigma^2}(W_n^*Y_{nt}^*)'B_n^{*\prime}(\rho)\widetilde{V}_{nt}^*(\boldsymbol{\beta}, \boldsymbol{\lambda}, \rho) - \frac{T-1}{T}\mathrm{tr}[G_n^*(\lambda_t)], \;\; t = 1, \ldots, T, \\ \frac{1}{\sigma^2}\sum_{t=1}^T \widetilde{V}_{nt}^{*\prime}(\boldsymbol{\beta}, \boldsymbol{\lambda}, \rho)H_n^*(\rho)\widetilde{V}_{nt}^*(\boldsymbol{\beta}, \boldsymbol{\lambda}, \rho) - (T-1)\mathrm{tr}[H_n^*(\rho)], \\ -\frac{(n-1)(T-1)}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=1}^T \widetilde{V}_{nt}^{*\prime}(\boldsymbol{\beta}, \boldsymbol{\lambda}, \rho)\widetilde{V}_{nt}^*(\boldsymbol{\beta}, \boldsymbol{\lambda}, \rho). \end{cases} \tag{B.5}$$

The expected negative Hessian matrix, $I_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)$, has the components:

$$\begin{aligned}
I_{\boldsymbol{\beta}\boldsymbol{\beta}} &= J_{\boldsymbol{\beta}\boldsymbol{\beta}},\ I_{\boldsymbol{\beta}\boldsymbol{\lambda}} = \mathtt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}X_{nt}^{*\prime}D_n^*\eta_{nt}^*\big\} - \big\{\tfrac{1}{T\sigma_0^2}X_{nt}^{*\prime}D_n^*\eta_{ns}^*\big\},\ I_{\rho\boldsymbol{\beta}} = \mathbf{0}\\
I_{\boldsymbol{\lambda}\boldsymbol{\lambda}} &= \mathtt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}\eta_{nt}^{*\prime}D_n^*\eta_{nt}^* + \tfrac{T-1}{T}\mathrm{tr}(\bar{G}_{nt}^{*s}\bar{G}_{nt}^*)\big\} - \big\{\tfrac{1}{T\sigma_0^2}\eta_{nt}^{*\prime}D_n^*\eta_{ns}^*\big\},\\
I_{\boldsymbol{\lambda}\rho} &= \big\{\tfrac{T-1}{T}\mathrm{tr}(\bar{G}_{nt}^{*\prime}H_n^{*s})\big\},\quad I_{\rho\rho} = (T-1)\mathrm{tr}(H_n^{*s}H_n^*),\ \ I_{\sigma^2\sigma^2} = \tfrac{n(T-1)}{2\sigma_0^4},\\
I_{\sigma^2\boldsymbol{\beta}} &= 0_{tk}',\ I_{\sigma^2\boldsymbol{\lambda}} = \big\{\tfrac{T-1}{T\sigma_0^2}\mathrm{tr}(G_{nt}^*)\big\},\ I_{\sigma^2\rho} = \tfrac{T-1}{\sigma_0^2}\mathrm{tr}(H_n^*)
\end{aligned}$$

The representations for AQS function at θ_0 in terms of $\mathbb{V}_N = (V_{n1}, ..., V_{nT})$ turn out to be very useful. They lead to a simple way for estimating the variance-covariance (VC) matrix of the AQS vector. Thus, one needs to work with the original error vector V_{nt} through $V_{nt}^* = F_{n,n-1}'V_{nt}$. Let z_t be a $T \times 1$ of element 1 in the tth position and 0 elsewhere, and define $Z_{Nt} = z_t \otimes I_n$, $\bar{Z}_N = \frac{1}{T}(l_T \otimes I_n)$, and $Z_{Nt}^\circ = Z_{Nt} - \bar{Z}_N$. the AQS function at θ_0 can be written as

$$S_{\mathtt{SLE2}}^*(\boldsymbol{\theta}_0) = \begin{cases}
\Pi_{1t}'\mathbb{V}_N, \quad t = 1, \ldots, T,\\
\Pi_{2t}'\mathbb{V}_N + \mathbb{V}_N'\Phi_{1t}\mathbb{V}_N - \frac{T-1}{T}\mathrm{tr}(G_{nt}^*), \quad t = 1, \ldots, T,\\
\mathbb{V}_N'\Phi_{2t}\mathbb{V}_N - (T-1)\mathrm{tr}(H_n^*),\\
\mathbb{V}_N'\Psi\mathbb{V}_N - \frac{(n-1)(T-1)}{2\sigma^2},
\end{cases} \tag{B.6}$$

where $\Pi_{1t} = \frac{1}{\sigma_0^2}Z_{Nt}^{\circ^*}B_{nt}^*X_{nt}^*$, $\Pi_{2t} = \frac{1}{\sigma_0^2}Z_{Nt}^{\circ^*}B_{nt}^*\eta_{nt0}^*$, $\Phi_{1t} = \frac{1}{\sigma_0^2}Z_{Nt}^*B_{nt}^{*-1\prime}G_{nt}^{*\prime}B_{nt}^{*\prime}Z_{Nt}^{\circ*\prime}$, $\Phi_2 = \frac{1}{\sigma_0^2}\sum_{t=1}^T Z_{Nt}^{\circ^*}H_n^*Z_{Nt}^{\circ*\prime}$, and $\Psi = \frac{1}{2\sigma_0^4}\sum_{t=1}^T Z_{Nt}^{\circ^*}Z_{Nt}^{\circ*\prime}$, with $Z_{Nt}^* = Z_{Nt}F_{n,n-1}$ and $Z_{Nt}^{\circ*} = Z_{Nt}^\circ F_{n,n-1}$. Applying Lemma A.2 with ε, c_{nt} and A_{nt} replaced by $\mathbb{V}_N$, Π_{1t} and Π_{2t}, Φ_{1t}, Φ_2 and Ψ, we obtain the VC matrix of the AQS function:

$$\Sigma_{\mathtt{SLE2}}(\boldsymbol{\theta}_0) = $$
$$\begin{pmatrix}
\{f(\mathbf{0},\Pi_{1t};\mathbf{0},\Pi_{1s})\}, & \{f(\mathbf{0},\Pi_{1t};\Phi_{1s},\Pi_{2s})\}, & \{f(\mathbf{0},\Pi_{1t};\Phi_2,\mathbf{0})\}, & \{f(\mathbf{0},\Pi_{1t};\Psi,\mathbf{0})\}\\
\sim, & \{f(\Phi_{1t},\Pi_{2t};\Phi_{1s},\Pi_{2s})\}, & \{f(\Phi_{1t},\Pi_{2t};\Phi_2,\mathbf{0})\}, & \{f(\Phi_{1t},\Pi_{2t};\Psi,\mathbf{0})\}\\
\sim, & \sim, & f(\Phi_2,\mathbf{0};\Phi_2,\mathbf{0}), & f(\Phi_2,\mathbf{0};\Psi,\mathbf{0})\\
\sim, & \sim, & \sim, & f(\Psi,\mathbf{0};\Psi,\mathbf{0})
\end{pmatrix}.$$

B.6. Panel SE model with two-way FE Let $\dot{D}_{nt} = -\frac{d}{d\rho_{t0}}D_{nt}^* = M_n^{*\prime}B_{nt}^* + B_{nt}^{*\prime}M_n^*$, and the quantities Y_{nt}^*, X_{nt}^*, c_n^*, M_n^* and V_{nt}^* be defined similarly as in Model (3.11). Let $\eta_{nt}^* = G_{nt}^*(X_{nt}^*\beta_t + c_n^*)$ and $g_{nt}^* = \mathtt{diagv}(G_{nt}^*)$. The AQS function of $\boldsymbol{\theta} = (\boldsymbol{\beta}', \boldsymbol{\rho}', \sigma^2)'$ is

$$S_{\mathtt{SE2}}^\star(\boldsymbol{\theta}) = \begin{cases}
\frac{1}{\sigma^2}X_{nt}^{*\prime}B_n^{*\prime}(\rho_t)\widetilde{V}_{nt}^*(\boldsymbol{\beta},\boldsymbol{\rho}), \quad t = 1, \ldots, T,\\
\frac{1}{\sigma^2}\widetilde{V}_{nt}^{*\prime}(\boldsymbol{\beta},\boldsymbol{\rho})H_n^*(\rho_t)\widetilde{V}_{nt}^*(\boldsymbol{\beta},\boldsymbol{\rho}) - \mathrm{tr}[S_{nt}^*(\boldsymbol{\rho})H_n^*(\rho_t)], \quad t = 1, \ldots, T,\\
-\frac{(n-1)(T-1)}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=1}^T \widetilde{V}_{nt}^{*\prime}(\boldsymbol{\beta},\boldsymbol{\rho})\widetilde{V}_{nt}^*(\boldsymbol{\beta},\boldsymbol{\rho}),
\end{cases} \tag{B.7}$$

where $\widetilde{V}_{nt}^*(\boldsymbol{\beta},\boldsymbol{\rho}) = B_n^*(\rho_t)U_{nt}^{\circ*}(\beta_t) - B_n^*(\rho_t)\mathbb{D}_n^{*-1}(\boldsymbol{\rho})\sum_{s=1}^T D_n^*(\rho_s)U_{ns}^{\circ*}(\beta_s)$ is defined similarly to that in $S^{\star}_{\texttt{SLE2}}(\boldsymbol{\theta})$ but with $U^{\circ *}_{nt}(\beta_t) = Y^*_{nt} - X^*_{nt}\beta_t$.

The expected negative Hessian matrix, $I_{\texttt{SE2}}(\boldsymbol{\theta}_0)$, has the components:

$$I_{\boldsymbol{\beta}\boldsymbol{\beta}} = \texttt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}X^{*\prime}_{nt}D^*_{nt}X^*_{nt}\big\} - \big\{\tfrac{1}{\sigma_0^2}X^{*\prime}_{nt}D^*_{nt}\mathbb{D}^{*-1}_n D^*_{ns}X^*_{ns}\big\}, \;\; I_{\boldsymbol{\rho}\boldsymbol{\beta}} = \mathbf{0};$$
$$I_{\boldsymbol{\rho}\boldsymbol{\rho}} = \texttt{blkdiag}\big\{\mathrm{tr}(H^{*s}_{nt}S^*_{nt}H^*_{nt} - B^*_{nt}\mathbb{D}^{*-1}_n\dot{D}_{nt}B^{*-1}_{nt}H^*_{nt})\big\} + \big\{\mathrm{tr}(B^*_{nt}\mathbb{D}^{*-1}_n\dot{D}_{ns}\mathbb{D}^{*-1}_n B^{*\prime}_{nt}H^*_{nt})\big\};$$
$$I_{\sigma^2\boldsymbol{\beta}} = \mathbf{0}; \;\; I_{\sigma^2\boldsymbol{\rho}} = \big\{\tfrac{1}{\sigma_0^2}\mathrm{tr}(S^*_{nt}H^*_{nt})\big\}; \;\; I_{\sigma^2\sigma^2} = -\tfrac{(n-1)(T-1)}{2\sigma_0^4} + \tfrac{1}{\sigma_0^4}\textstyle\sum_{t=1}^T \mathrm{tr}(S^*_{nt}).$$

Similarly, the AQS function at θ_0 can be expressed in terms of the vector of original errors $\mathbb{V}_N = (V_{n1}, \ldots, V_{nT})$, which turns out to be very useful in finding the analitical expression of its VC matrix. Thus, one needs to work with the original error vector V_{nt} through $V^*_{nt} = F'_{n,n-1}V_{nt}$. $\widetilde{V}^*_{nt} \equiv \widetilde{V}^*_{nt}(\boldsymbol{\beta}_0, \boldsymbol{\rho}_0) = V^*_{nt} - B^*_{nt}\mathbb{D}^{*-1}_n \sum_{s=1}^T B^{*\prime}_{ns}V^*_{ns} = F'_{n,n-1}Z^{\circ\prime}_{Nt}\mathbb{V}_N$. The AQS function at θ_0 can be written as

$$S^*_{\texttt{SE2}}(\boldsymbol{\theta}_0) = \begin{cases} \Pi'_{1t}\mathbb{V}_N, \;\; t = 1, \ldots, T, \\ \mathbb{V}'_N\Phi_t\mathbb{V}_N - \mathrm{tr}(S^*_{nt}H^*_{nt}), \;\; t = 1, \ldots, T, \\ \mathbb{V}'_N\Psi\mathbb{V}_N - \frac{(n-1)(T-1)}{2\sigma^2}, \end{cases} \tag{B.8}$$

where $\Pi_{1t} = \frac{1}{\sigma_0^2}Z^{\circ *}_{Nt}B^*_{nt}X^*_{nt}$, $\Phi_t = \frac{1}{\sigma_0^2}Z^{\circ *}_{Nt}H^*_{nt}Z^{\circ *\prime}_{Nt}$, and $\Psi = \frac{1}{2\sigma^4}\sum_{t=1}^T Z^{\circ *}_{Nt}Z^{\circ *\prime}_{Nt}$, with $Z^*_{Nt} = Z_{Nt}F_{n,n-1}$ and $Z^{\circ *}_{Nt} = Z^{\circ}_{Nt}F_{n,n-1}$. Applying Lemma A.2 with ε, c_{nt} and A_{nt} replaced by $\mathbb{V}_N$, Π_{1t} and Φ_{2t} and Ψ, we obtain the corresponding VC matrix $\Sigma_{\texttt{SE2}}(\boldsymbol{\theta}_0)$:

$$\Sigma_{\texttt{SE2}}(\boldsymbol{\theta}_0) = \begin{pmatrix} \big\{f(\mathbf{0}, \Pi_{1t}; \mathbf{0}, \Pi_{1s})\big\}, & \big\{f(\mathbf{0}, \Pi_{1t}; \Phi_s, \mathbf{0})\big\}, & \big\{f(\mathbf{0}, \Pi_{1t}; \Psi, \mathbf{0})\big\} \\ \sim, & \big\{f(\Phi_t, \mathbf{0}; \Phi_s, \mathbf{0})\big\}, & \big\{f(\Phi_t, \mathbf{0}; \Psi, \mathbf{0})\big\} \\ \sim, & \sim, & f(\Psi, \mathbf{0}; \Psi, \mathbf{0}) \end{pmatrix}.$$

Appendix C: Proof of the Theorems

Proof of Theorem 2.1. From (B.1), we see that the AQS function at the true parameters contains both linear and quadratic forms in the vector of original errors $\mathbb{V}_N$,

$$S^{\star}_{\mathtt{SL1}}(\theta_0) = \begin{cases} \Pi'_{1t}\mathbb{V}_N,\ t = 1,\ldots,T, \\ \Pi'_{2t}\mathbb{V}_N + \mathbb{V}'_N\Phi_t\mathbb{V}_N - \frac{T-1}{T}\mathrm{tr}(G_{nt}),\ t = 1,\ldots,T, \\ \mathbb{V}'_N\Psi\mathbb{V}_N - \frac{n(T-1)}{2\sigma^2}, \end{cases}$$

where $\Pi_{1t} = \frac{1}{\sigma_0^2}Z^{\circ}_{Nt}X_{nt}$, $\Pi_{2t} = \frac{1}{\sigma_0^2}Z^{\circ}_{Nt}\eta_{nt}$, $\Phi_t = \frac{1}{\sigma_0^2}Z_{Nt}G'_{nt}Z^{\circ\prime}_{Nt}$, $\Psi = \frac{1}{2\sigma^4}\sum_{t=1}^T Z^{\circ}_{Nt}Z^{\circ\prime}_{Nt}$, $Z_{Nt} = z_t \otimes I_n$, $Z^{\circ}_{Nt} = Z_{Nt} - \bar{Z}_N$, $\bar{Z}_N = \frac{1}{T}(l_T \otimes I_n)$, and z_t is a $T \times 1$ vector with tth element being 1 and other elements being zero.

First, as the elements of X_{nt} are non-stochastic and uniformly bounded (by Assumption 3), it is easy to see that the elements of Π_{1t} are uniformly bounded. By Assumption A.4 and Lemma A.1(i), G_{nt} is uniformly bounded in both row and column sums. Thus, the elements of $\eta_{nt} = G_{nt}(X_{nt}\beta_{t0} + c_n)$ are uniformly bounded by Assumption A3 and Lemma A.1(iii). It follows that the elements of Π_{2t} are uniformly bounded. Now, from the definition of Z_{Nt} and Z°_{Nt}, it is easy to see that $\Phi_t$and Ψ are uniformly bounded in both row and column sums. Thus, under Assumptions 1-4 the central limit theorem (CLT) of linear-quadratic (LQ) form of Kelejian and Prucha (2001) or its simplified version (under iid errors) given in Lemma A.3 can be applied to the elements of $S^{\star}_{\mathtt{SL1}}(\theta_0)$. Therefore, an application of Cramér-Wold device under a finite T leads to, as $N_0 \to \infty$, $\frac{1}{\sqrt{N_0}}S^{\star}_{\mathtt{SL1}}(\theta_0) \xrightarrow{D} N\big(0,\ \lim_{N_0\to\infty}\frac{1}{N_0}\Sigma_{\mathtt{SL1}}(\theta_0)\big)$. It follows that by (2.11) and (2.12),

$$C[\tfrac{1}{N_0}I_{\mathtt{SL1}}(\theta_0)]^{-1}\tfrac{1}{\sqrt{N_0}}S^{\star}_{\mathtt{SL1}}(\tilde{\theta}_{\mathtt{SL1}}) \xrightarrow{D} N\big(0,\ \lim_{N_0\to\infty}\Xi_{\mathtt{SL1}}(\theta_0)\big).$$

It left to show that $\frac{1}{N_0}[I_{\mathtt{SL1}}(\tilde{\theta}_{\mathtt{SL1}}) - I_{\mathtt{SL1}}(\theta_0)] \xrightarrow{p} \mathbf{0}$ and $\frac{1}{N_0}[\Sigma_{\mathtt{SL1}}(\tilde{\theta}_{\mathtt{SL1}}) - \Sigma_{\mathtt{SL1}}(\theta_0)] \xrightarrow{p} \mathbf{0}$. Under the $\sqrt{N_0}$-consistency of $\tilde{\theta}_{\mathtt{SL1}}$ and with the analytical expressions of $I_{\mathtt{SL1}}(\theta_0)$ and $\Sigma_{\mathtt{SL1}}(\theta_0)$ given in Appendix B.1, the proofs of these results are repeated applications of the mean value theorem (MVT) to each component of $\frac{1}{N_0}[I_{\mathtt{SL1}}(\tilde{\theta}_{\mathtt{SL1}}) - I_{\mathtt{SL1}}(\theta_0)]$ and each component of $\frac{1}{N_0}[\Sigma_{\mathtt{SL1}}(\tilde{\theta}_{\mathtt{SL1}}) - \Sigma_{\mathtt{SL1}}(\theta_0)]$, with the key results to note:

$$\tfrac{1}{N_0}(\tilde{c}_n\tilde{G}_{nt}\tilde{c}_n - c_nG_{nt}c_n) \xrightarrow{p} 0;\ \ \tilde{\gamma} - \gamma \xrightarrow{p} 0;\ \ \tilde{\kappa} - \kappa \xrightarrow{p} 0. \tag{C.1}$$

See the end of Section 2.1 for details. See the proof of Theorem 3.2 for details in a more general setup. ■

Proof of Theorem 2.2. From the derivations in Section 2.2 and further results in Appendix B2, we see that all the quantities in the 2FE panel SL model relate to the corresponding quantities in the 1FE panel SL model through the orthonormal transformation matrix $F_{n,n-1}$. Thus, the proof of Theorem 2.2 is carried out in a similar manner as that for 1FE panel SL model. For the results similar to those in (C.1), see the end of Section

2.2 for details. ■

Proof of Theorem 3.1. Again the AQS function at the true parameters can be expressed in terms of linear and quadratic forms in $\mathbb{V}_N$ as shown in (B.3),

$$S^*_{\texttt{SLE1}}(\theta_0) = \begin{cases} \Pi'_{1t}\mathbb{V}_N, \ \ t = 1, \ldots, T, \\ \Pi'_{2t}\mathbb{V}_N + \mathbb{V}'_N\Phi_{1t}\mathbb{V}_N - \text{tr}(R_{nt}G_{nt}), \ \ t = 1, \ldots, T, \\ \mathbb{V}'_N\Phi_{2t}\mathbb{V}_N - \text{tr}(S_{nt}H_{nt}), \ \ t = 1, \ldots, T, \\ \mathbb{V}'_N\Psi\mathbb{V}_N - \frac{n(T-1)}{2\sigma^2}, \end{cases}$$

where $\Pi_{1t} = \frac{1}{\sigma_0^2}Z^{\circ}_{Nt}B_{nt}X_{nt}$, $\Pi_{2t} = \frac{1}{\sigma_0^2}Z^{\circ}_{Nt}B_{nt}\eta_{nt}$, $\Phi_{1t} = \frac{1}{\sigma_0^2}Z_{Nt}B^{-1\prime}_{nt}G'_{nt}B'_{nt}Z^{\circ\prime}_{Nt}$, $\Phi_{2t} = \frac{1}{\sigma_0^2}Z^{\circ}_{Nt}H_{nt}Z^{\circ\prime}_{Nt}$, $\Psi = \frac{1}{2\sigma^4}\sum_{t=1}^T Z^{\circ}_{Nt}Z^{\circ\prime}_{Nt}$, $Z^{\circ\prime}_{Nt} = [Z'_{Nt} - B_{nt}\mathbb{D}_n^{-1}(l'_T \otimes I_n)\mathbb{B}_N]$ and $\mathbb{B}_N = \texttt{blkdiag}(B_{n1}, \ldots, B_{nT})$. Under Assumptions 1-5, it is easy to verify that each component of $S^*_{\texttt{SLE1}}(\theta_0)$ or a linear combination of the components of $S^*_{\texttt{SLE1}}(\theta_0)$ satisfies the conditions of Lemma A.3, leading to the asymptotic normality result:

$$C[\tfrac{1}{N_0}I_{\texttt{SLE1}}(\theta_0)]^{-1}\tfrac{1}{\sqrt{N_0}}S^{\star}_{\texttt{SLE1}}(\tilde{\theta}_{\texttt{SLE1}}) \xrightarrow{D} N\big(0, \ \lim_{N_0\to\infty}\Xi_{\texttt{SLE1}}(\theta_0)\big).$$

The proofs of $\frac{1}{N_0}[I_{\texttt{SLE1}}(\tilde{\theta}_{\texttt{SLE1}}) - I_{\texttt{SLE1}}(\theta_0)] \xrightarrow{p} \mathbf{0}$ and $\frac{1}{N_0}[\Sigma_{\texttt{SLE1}}(\tilde{\theta}_{\texttt{SLE1}}) - \Sigma_{\texttt{SLE1}}(\theta_0)] \xrightarrow{p} \mathbf{0}$ are again carried out by repeated applications of MVT under the $\sqrt{N_0}$-consistency of $\tilde{\theta}_{\texttt{SLE1}}$. For details on the estimation of c_n, the skewness γ and excess kurtosis κ for the 1FE panel SLE model, and the consistency of these estimates, see the end of Section 3.1. ■

Proof of Theorem 3.2. Consider the AQS function $S^{\star}_{\texttt{SLE2}}(\theta)$ given in (3.13). We need to show that $\frac{1}{\sqrt{N_0}}S^{\star}_{\texttt{SLE2}}(\theta_0) \xrightarrow{D} N\big(0, \ \lim_{N_0\to\infty}\frac{1}{N_0}\Sigma_{\texttt{SLE2}}(\theta_0)\big)$, as $N_0 \to \infty$. We have

$$\begin{aligned} &\widetilde{V}^*_{nt} \equiv \widetilde{V}^*_{nt}(\beta_0, \rho_0) = V^*_{nt} - B^*_{nt}\mathbb{D}^{*-1}_n \textstyle\sum_{s=1}^T B^{*\prime}_{ns}V^*_{ns} = F'_{n,n-1}Z^{\diamond\prime}_{Nt}\mathbb{V}_N, \ \text{and} \\ &W^*_nY^*_{nt} = G^*_{nt}(X^*_{nt}\beta_{t0} + c^*_n + B^{-*1}_{nt}V^*_{nt}) = \eta^*_{nt} + G^*_{nt}B^{*-1}_{nt}F'_{n,n-1}Z'_{Nt}\mathbb{V}_N. \end{aligned}$$

Hence, the AQS function at true θ_0 can be written as

$$S^*_{\texttt{SLE2}}(\theta_0) = \begin{cases} \Pi'_{1t}\mathbb{V}_N, \ \ t = 1, \ldots, T, \\ \Pi'_{2t}\mathbb{V}_N + \mathbb{V}'_N\Phi_{1t}\mathbb{V}_N - \text{tr}(R^*_{nt}G^*_{nt}), \ \ t = 1, \ldots, T, \\ \mathbb{V}'_N\Phi_{2t}\mathbb{V}_N - \text{tr}(S^*_{nt}H^*_{nt}), \ \ t = 1, \ldots, T, \\ \mathbb{V}'_N\Psi\mathbb{V}_N - \frac{(n-1)(T-1)}{2\sigma^2}, \end{cases} \tag{C.2}$$

where $\Pi_{1t} = \frac{1}{\sigma_0^2}Z^{\diamond*}_{Nt}B^*_{nt}X^*_{nt}$, $\Pi_{2t} = \frac{1}{\sigma_0^2}Z^{\diamond*}_{Nt}B^*_{nt}\eta^*_{nt}$, $\Phi_{1t} = \frac{1}{\sigma_0^2}Z^*_{Nt}B^{*-1\prime}_{nt}G^{*\prime}_{nt}B^{*\prime}_{nt}Z^{\diamond*\prime}_{Nt}$, $\Phi_{2t} = \frac{1}{\sigma_0^2}Z^{\diamond*}_{Nt}H^*_{nt}Z^{\diamond*\prime}_{Nt}$, and $\Psi = \frac{1}{2\sigma_0^4}\sum_{t=1}^T Z^{\diamond*}_{Nt}Z^{\diamond*\prime}_{Nt}$, with $Z^*_{Nt} = Z_{Nt}F_{n,n-1}$ and $Z^{\diamond*}_{Nt} = Z^{\diamond}_{Nt}F_{n,n-1}$; $Z_{Nt} = z_t \otimes I_n$ and z_t is a $T \times 1$ vector with tth element being 1 and other elements being zero; and $Z^{\diamond\prime}_{Nt} = [Z'_{Nt} - B_{nt}\mathbb{D}_n^{-1}(l'_T \otimes I_n)\mathbb{B}_N]$ and $\mathbb{B}_N = \texttt{blkdiag}(B_{n1}, \ldots, B_{nT})$.

First, as the elements of X_{nt} are non-stochastic and uniformly bounded (by Assumption 3), the row and column sums of B^*_{nt} are uniformly bounded in absolute values by Assumption 5 and Lemma A.1. It follows that the elements of Π_{1t} are uniformly bounded. By Assumption A.4 and Lemma A.1(i), G_{nt} is uniformly bounded in both row and column sums. By Lemma A.2 of Lee and Yu (2010),

$$(I_n - \lambda F'_{n,n-1} W_n F_{n,n-1})^{-1} = F'_{n,n-1}(I_n - \lambda W_n)^{-1} F_{n,n-1}. \tag{C.3}$$

We have $A^{*-1}_{nt} = F'_{n,n-1} A^{-1}_{nt} F_{n,n-1}$. Thus, G^*_{nt} is uniformly bounded in both row and column sums by Lemma A.1(iii), and the elements of $\eta^*_{nt} = G^*_{nt}(X^*_{nt}\beta_{t0} + c^*_n)$ are uniformly bounded by Assumption A3. It follows that the elements of Π_{2t} are uniformly bounded. Similarly, $B^{*-1}_{nt} = F'_{n,n-1} B^{-1}_{nt} F_{n,n-1}$, and therefore the elements of H^*_{nt} is uniformly bounded in both row and column sums. With these and the definitions of Z_{Nt} and $Z^{\diamond}_{Nt}$, it is easy to show that Φ_{1t}, Φ_{2t} and Ψ are uniformly bounded in both row and column sums. Thus, under Assumptions 1-5, the central limit theorem (CLT) of linear-quadratic (LQ) form of Kelejian and Prucha (2001) or its simplified version (under iid errors) given in Lemma A.3 can be applied to each element of $S^{\star}_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)$ to establish its asymptotic normality. Then, an application of Cramér-Wold device under a finite T gives, $\frac{1}{\sqrt{N_0}} S^{\star}_{\mathtt{SLE2}}(\boldsymbol{\theta}_0) \xrightarrow{D} N\big(0, \lim_{N_0\to\infty} \frac{1}{N_0}\Sigma_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)\big)$, as $N_0 \to \infty$. Then, by (2.11) and (2.12),

$$C[\tfrac{1}{N_0} I_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)]^{-1} \tfrac{1}{\sqrt{N_0}} S^{\star}_{\mathtt{SLE2}}(\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}) \xrightarrow{D} N\big(0, \lim_{N_0\to\infty} \boldsymbol{\Xi}_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)\big).$$

It left to show that, as $N_0 \to \infty$,

(a) $\frac{1}{N_0}[I_{\mathtt{SLE2}}(\tilde{\boldsymbol{\theta}}_{\mathtt{SL1}}) - I_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)] \xrightarrow{p} \mathbf{0}$,
(b) $\frac{1}{N_0}[\Sigma_{\mathtt{SLE2}}(\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}) - \Sigma_{\mathtt{SL1}}(\boldsymbol{\theta}_0)] \xrightarrow{p} \mathbf{0}$.

Under the $\sqrt{N_0}$-consistency of $\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}$ and with the analytical expressions of $I_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)$ and $\Sigma_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)$ given in Appendix B.4, the proofs of these results are repeated applications of the mean value theorem (MVT) to each component of $\frac{1}{N_0}[I_{\mathtt{SLE2}}(\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}) - I_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)]$ and each component of $\frac{1}{N_0}[\Sigma_{\mathtt{SLE2}}(\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}) - \Sigma_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)]$.

To show (a), we pick a typical element of $I_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)$ given in Appendix B.4,

$$I_{\boldsymbol{\lambda}\boldsymbol{\lambda}} = \mathtt{blkdiag}\big\{\tfrac{1}{\sigma_0^2}\eta^{*\prime}_{nt} D^*_{nt}\eta^*_{nt} + \mathrm{tr}(S^*_{nt}\bar{G}^{*s}_{nt}\bar{G}^*_{nt})\big\} - \big\{\tfrac{1}{\sigma_0^2}\eta^{*\prime}_{nt} D^*_{nt}\mathbb{D}^{*-1}_n D^*_{ns}\eta^*_{ns}\big\}$$

to show that $\frac{1}{N_0}(\widetilde{I}_{\boldsymbol{\lambda}\boldsymbol{\lambda}} - I_{\boldsymbol{\lambda}\boldsymbol{\lambda}}) \xrightarrow{p} 0$. The proofs for the other components follow similarly. Recall: $\eta^*_{nt} = G^*_{nt}(X^*_{nt}\beta_{t0} + c^*_n)$, $\mathbb{D}^*_n(\boldsymbol{\rho}) = \sum_{t=1}^T D^*_n(\rho_t)$, $D^*_n(\rho_t) = B^{*\prime}_n(\rho_t)B^*_n(\rho_t)$, $B^*_n(\rho_t) = I_{n-1} - \rho_t M^*_n$, $S^*_{nt}(\boldsymbol{\rho}) = I_{n-1} - B^*_{nt}(\rho_t)\mathbb{D}^{*-1}_n(\boldsymbol{\rho})B^{*\prime}_{nt}(\rho_t)$, and $\bar{G}^*_{nt} = B^*_{nt}G^*_{nt}B^{*-1}_{nt}$.

By Assumptions 4 and 5 and Lemma A.1(i), it is straightforward to show the two matrices, $D^*_n(\rho_t)$ and $\bar{G}^*_{nt}(\lambda_t, \rho_t)$, are uniformly bounded in both row and column sums in a neighborhood of $(\lambda_{t0}, \rho_{t0})$ for each t, and so are their derivatives. Clearly with the properties of $D^*_n(\rho_t)$ and a finite T, $\mathbb{D}^*_n(\boldsymbol{\rho})$ is uniformly bounded in both row and column sums in a neighborhood of $\boldsymbol{\rho}_0$, and so are its derivatives.

By Assumption 5 and Lemma A.1(i), $D^{*-1}_n(\rho_t)$ is uniformly bounded in both row and column sums in a neighborhood of ρ_{t0} for each t, and so are its derivatives. By a matrix result that for two invertible matrices A_n and B_n, $(A_n + B_n)^{-1} = A_n^{-1} + \frac{1}{1+c} A_n^{-1} B_n A_n^{-1}$, where $c = \mathrm{tr}(B_n A_n^{-1})$, we infer that for a finite T, $\mathbb{D}_n^*(\boldsymbol{\rho})$ is uniformly bounded in both row and column sums in a neighborhood of $\boldsymbol{\rho}_0$, and so are its derivatives. It follows that $S_{nt}^*(\boldsymbol{\rho})$ is uniformly bounded in both row and column sums in a neighborhood of $\boldsymbol{\rho}_0$, and so are its derivatives. Noting that $\tilde{I}_{\boldsymbol{\lambda\lambda}} = I_{\boldsymbol{\lambda\lambda}}(\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}})$ and $I_{\boldsymbol{\lambda\lambda}} = I_{\boldsymbol{\lambda\lambda}}(\boldsymbol{\theta}_0)$, we have by MVT, for each component of $I_{\boldsymbol{\lambda\lambda}}(\boldsymbol{\theta})$ denoted as $I_{\boldsymbol{\lambda\lambda},ts}(\boldsymbol{\theta})$, $t, s = 1, \ldots, T$,

$$\tfrac{1}{N_0} I_{\boldsymbol{\lambda\lambda},ts}(\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}) = \tfrac{1}{N_0} I_{\boldsymbol{\lambda\lambda},ts}(\boldsymbol{\theta}_0) + [\tfrac{1}{N_0} \tfrac{\partial}{\partial \boldsymbol{\theta}'} I_{\boldsymbol{\lambda\lambda},ts}(\bar{\boldsymbol{\theta}})](\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}} - \boldsymbol{\theta}_0),$$

where $\bar{\boldsymbol{\theta}}$ lies elementwise between $\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}$ and $\boldsymbol{\theta}_0$, with $\bar{\boldsymbol{\theta}}$ being $\sqrt{N_0}$-consistent as $\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}$ is. With the above argument and Lemma A.1(ii), we have $\frac{1}{N_0} \frac{\partial}{\partial \boldsymbol{\theta}'} I_{\boldsymbol{\lambda\lambda},ts}(\bar{\boldsymbol{\theta}}) = O_p(1)$. Therefore, $\frac{1}{N_0}[I_{\boldsymbol{\lambda\lambda},ts}(\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}) - I_{\boldsymbol{\lambda\lambda},ts}(\boldsymbol{\theta}_0) = o_p(1)$ for each (t,s), and $\frac{1}{N_0}[I_{\boldsymbol{\lambda\lambda}}(\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}) - I_{\boldsymbol{\lambda\lambda}}(\boldsymbol{\theta}_0) = o_p(1)$. Note that the easily proved results such as $\frac{1}{N_0}(\tilde{c}_n \tilde{G}_{nt} \tilde{c}_n - c_n G_{nt} c_n) \xrightarrow{p} 0$, has been used. The proofs of the other components of $\frac{1}{N_0}[I_{\mathtt{SLE2}}(\tilde{\boldsymbol{\theta}}_{\mathtt{SL1}}) - I_{\mathtt{SL1}}(\boldsymbol{\theta}_0)] \xrightarrow{p} \mathbf{0}$ proceeds similarly.

To show (b), we again choose the most complicated term, $f(\Phi_{1t}, \Pi_{2t}; \Phi_{1s}, \Pi_{2s})$ that corresponds to $\boldsymbol{\lambda}$, to show in details where the quantities involved are given at the end of Appendix B.4: $\Pi_{1t} = \frac{1}{\sigma_0^2} Z_{Nt}^{\diamond *} B_{nt}^* X_{nt}^*$, $\Pi_{2t} = \frac{1}{\sigma_0^2} Z_{Nt}^{\diamond *} B_{nt}^* \eta_{nt}^*$, $\Phi_{1t} = \frac{1}{\sigma_0^2} Z_{Nt}^* B_{nt}^{*-1\prime} G_{nt}^{*\prime} B_{nt}^{*\prime} Z_{Nt}^{\diamond *\prime}$, and $\Phi_{2t} = \frac{1}{\sigma_0^2} Z_{Nt}^{\diamond *} H_{nt}^* Z_{Nt}^{\diamond *\prime}$, where $Z_{Nt}^* = Z_{Nt} F_{n,n-1}$ and $Z_{Nt}^{\diamond *} = Z_{Nt}^{\diamond} F_{n,n-1}$.

By (A.1) in Lemma A.2, replace A_{nt} by Φ_{1t}, a_{nt} by $\phi_{1t} = \mathtt{diagv}(\Phi_{1t})$, and c_{nt} by Π_{2t} (similarly for the quantities with subscript s), and note $\mu_3 = \gamma$ and $\mu_4 = \kappa$, we have,

$$f(\Phi_{1t}, \Pi_{2t}; \Phi_{1s}, \Pi_{2s}) = \sigma_0^4 \mathrm{tr}[(\Phi_{1t}' + \Phi_{1t})\Phi_{1s}] + \gamma \phi_{1t}' \Pi_{2s} + \gamma \Pi_{2t}' \phi_{1s} + \kappa \phi_{1t}' \phi_{1s} + \sigma_0^2 \Pi_{2t}' \Pi_{2s}.$$

Applying MVT and following the similar arguments as in (a), the convergence of the relevant terms can easily be proved, e.g., $\frac{1}{N_0}\{\mathrm{tr}[(\widetilde{\Phi}_{1t}' + \widetilde{\Phi}_{1t})\widetilde{\Phi}_{1s}] - \mathrm{tr}[(\Phi_{1t}' + \Phi_{1t})\Phi_{1s}]\} = o_p(1)$, $\frac{1}{N_0}[\phi_{1t}' \Pi_{2s} - \phi_{1t}' \Pi_{2s}] = o_p(1)$, etc. Furthermore, $\tilde{\sigma}_{\mathtt{SLE2}}^2 - \sigma_0^2 = o_p(1)$, and hence $\tilde{\sigma}_{\mathtt{SLE2}}^4 - \sigma_0^4 = o_p(1)$; for the estimates obtained from Lemma 4.1(a) of Yang et al. (2016), it is easy to show that $\tilde{\gamma} - \gamma \xrightarrow{p} 0$ and $\tilde{\kappa} - \kappa \xrightarrow{p} 0$. It follows that

$$[\tilde{f}(\widetilde{\Phi}_{1t}, \widetilde{\Pi}_{2t}; \widetilde{\Phi}_{1s}, \widetilde{\Pi}_{2s}) - f(\Phi_{1t}, \Pi_{2t}; \Phi_{1s}, \Pi_{2s})] = o_p(1).$$

Similarly, the convergence of the other elements of $\frac{1}{N_0}[\Sigma_{\mathtt{SLE2}}(\tilde{\boldsymbol{\theta}}_{\mathtt{SLE2}}) - \Sigma_{\mathtt{SLE2}}(\boldsymbol{\theta}_0)]$ is proved.
■

Appendix D: Estimation of Null Models

Homogeneous Panel SL model with 1FE. The construction of the AQS tests require the estimation of the null model, which could be the homogeneous model as specified by $H_0^{\texttt{TH}}$ in (2.2), the model with homogeneity in β's only, the model with homogeneity in λ's only, or the model with change points as specified by $H_0^{\texttt{CP}}$ in (2.3), etc. Each null model can estimated by solving the simplified AQS equations by simplifying $S^{\star}_{\texttt{SL1}}(\boldsymbol{\theta})$ according to the null hypothesis, which is clearly inconvenient to the applied researchers. **To facilitate practical applications** of our methods, a general *Lagrange Multiplier* (LM) method is introduced. Let $l_{\texttt{SL1}}(\boldsymbol{\theta})$ be the objective function to be maximized subject to $C\boldsymbol{\theta}_0 = 0$, with $S^{\star}_{\texttt{SL1}}(\boldsymbol{\theta})$ given in (2.7) being its partial derivatives. Define the Lagrangian

$$\mathscr{L}_{\texttt{SL1}}(\boldsymbol{\theta}) = l_{\texttt{SL1}}(\boldsymbol{\theta}) - \phi'(C\boldsymbol{\theta}),$$

where ϕ is a $k_p \times 1$ vector of Lagrange multipliers. Taking partial derivatives and equating to 0, we have k_q equations $\frac{\partial \mathscr{L}_{\texttt{SL1}}}{\partial \boldsymbol{\theta}} = S^{*}_{\texttt{SL1}}(\boldsymbol{\theta}) - C'\phi = 0$. Together with the k_p constraints $C\boldsymbol{\theta} = 0$, we have $k_q + k_p$ equations for the $k_q + k_p$ unknowns $\boldsymbol{\theta}$ and ϕ, leading to

$$\begin{pmatrix} \tilde{\boldsymbol{\theta}}_{\texttt{SL1}} \\ \tilde{\phi}_{\texttt{SL1}} \end{pmatrix} = \arg \left\{ \begin{array}{l} S^{\star}_{\texttt{SL1}}(\boldsymbol{\theta}) - C'\phi = 0 \\ C\boldsymbol{\theta} = 0 \end{array} \right\} \tag{A.1}$$

To further aid the applications, we make the Matlab codes available upon request, or online at `http://www.mysmu.edu/faculty/zlyang/SubPages/research.htm`.

Finally, from the expressions of $I_{\texttt{SL1}}(\boldsymbol{\theta}_0)$ and $\Sigma_{\texttt{SL1}}(\boldsymbol{\theta}_0)$ given in Appendix B1, we see that they both contain c_n, which is estimated by plugging the null estimates $\tilde{\boldsymbol{\beta}}_{\texttt{SL1}}$ and $\tilde{\boldsymbol{\lambda}}_{\texttt{SL1}}$ into $\tilde{c}_n(\boldsymbol{\beta}, \boldsymbol{\lambda})$. Furthermore, in case of nonnormality, the VC matrix $\Sigma_{\texttt{SL1}}(\boldsymbol{\theta}_0)$ contains two additional parameters, the skewness γ and excess kurtosis κ of the idiosyncratic errors $V_{n,it}$, and their estimates are obtained by applying Lemma 4.1 (a) of Yang et al. (2016).

However, as the hypothesis $H_0^{\texttt{HT}}$ given in (2.2) and the corresponding homogeneous model plays an important role, we present some details to show how $S^{\star}_{\texttt{SL1}}(\boldsymbol{\theta})$ is simplified and how it leads to constrained AQS estimators with the desired asymptotic properties needed in the implementation of the AQS tests. This simplified AQS function is also useful in Monte Carlo simulation for computational efficiency. Let $\theta = (\beta', \lambda, \sigma^2)'$. The constrained estimate of c_n given (β, λ) becomes $\tilde{c}_n^{\circ}(\beta, \lambda) = A_n(\lambda)\bar{Y}_n - \bar{X}_n\beta$ where $\bar{Y}_n$ and $\bar{X}_n$ are the averages of $\{Y_{nt}\}$ and $\{X_{nt}\}$, respectively. Along the same line leading to (2.7), one can easily show that AQS function for the null model takes the form:

$$S^{\circ}_{\texttt{SL1}}(\theta) = \begin{cases} \frac{1}{\sigma^2}\sum_{t-1}^{T} X_{nt}^{\circ\prime}\widetilde{V}_{nt}^{\circ}(\beta, \lambda), \\ \frac{1}{\sigma^2}\sum_{t-1}^{T}(W_n Y_{nt}^{\circ})'\widetilde{V}_{nt}^{\circ}(\beta, \lambda) - (T-1)\mathrm{tr}[G_n(\lambda)], \\ -\frac{n(T-1)}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=1}^{T}\widetilde{V}_{nt}^{\circ\prime}(\beta, \lambda)\widetilde{V}_{nt}^{\circ}(\beta, \lambda), \end{cases} \tag{A.2}$$

$\widetilde{V}_{nt}^{\circ}(\beta, \lambda) = A_n(\lambda)Y_{nt} - X_{nt}\beta - \tilde{c}_n^{\circ}(\beta, \lambda) = A_n(\lambda)Y_{nt}^{\circ} - X_{nt}^{\circ}\beta$, where $Y_{nt}^{\circ} = Y_{nt} - \bar{Y}_n$ and

$X_{nt}^{\circ} = X_{nt} - \bar{X}_n$. Solving the estimating equations, $S_{\texttt{SL1}}^{\circ}(\theta) = 0$, gives the null estimator $\tilde{\theta}_{\texttt{SL1}}$ of θ. The process can be simplified by first solving the first set of equations and the last equation of (A.2), giving the constrained estimators of β and σ^2 (for a given λ) as

$$\tilde{\beta}_{\texttt{SL1}}(\lambda) = (\textstyle\sum_{t=1}^{T} X_{nt}^{\circ\prime} X_{nt}^{\circ})^{-1} \sum_{t=1}^{T} X_{nt}^{\circ} A_n(\lambda) Y_{nt}^{\circ},$$
$$\tilde{\sigma}_{\texttt{SL1}}^2(\lambda) = \tfrac{1}{n(T-1)} \textstyle\sum_{t=1}^{T} \widetilde{V}_{nt}^{\circ\prime}(\hat{\beta}_{\texttt{SL1}}(\lambda), \lambda) \widetilde{V}_{nt}^{\circ}(\hat{\beta}_{\texttt{SL1}}(\lambda), \lambda).$$

Substituting $\tilde{\beta}_{\texttt{SL1}}(\lambda)$ and $\tilde{\sigma}_{\texttt{SL1}}^2(\lambda)$ into the middle equation of (A.2) and solving the resulted concentrated estimating equation lead to the AQS estimator $\tilde{\lambda}_{\texttt{SL1}}$ of the common λ, which in turn gives the AQS estimator $\tilde{\beta}_{\texttt{SL1}} = \tilde{\beta}_{\texttt{SL1}}(\tilde{\lambda}_{\texttt{SL1}})$ of the common β, and the AQS estimator $\tilde{\sigma}_{\texttt{SL1}}^2 = \tilde{\sigma}_{\texttt{SL1}}^2(\tilde{\lambda}_{\texttt{SL1}})$ of σ^2. Finally, the AQS estimator of θ is $\tilde{\theta}_{\texttt{SL1}} = (\tilde{\beta}_{\texttt{SL1}}', \tilde{\lambda}_{\texttt{SL1}}, \tilde{\sigma}_{\texttt{SL1}}^2)'$. The proposed null estimator based on the AQS function provides an alternative to the direct and transformation approaches of Lee and Yu (2010). It can be shown to be asymptotically equivalent to the estimator based on an orthogonal transformation given in Lee and Yu (2010). Thus, $\tilde{\theta}_{\texttt{SL1}}$ is $\sqrt{n(T-1)}$-consistent for θ.

Homogeneous Panel SL model with 2FE. The LM procedure presented at the end of Sec. 2.1 for 1FE panel SL model directly applies to 2FE panel SL model to give constrained estimates of various null models. Again, the homogeneous model specified by $H_0^{\texttt{th}}$ in (2.2) and its AQS estimation play important roles in studying the asymptotic properties and performing Monte Carlo simulations, and therefore some details are given on the estimation procedures based on the simplified AQS function. The constrained estimate of c_n^*, given (β, λ), becomes $\tilde{c}_n^{*\circ}(\beta, \lambda) = A_n^*(\lambda)\bar{Y}_n^* - \bar{X}_n^*\beta$, where $\bar{Y}_n^*$ and $\bar{X}_n^*$ are the averages of $\{Y_{nt}^*\}$ and $\{X_{nt}^*\}$, respectively. Along the same line leading to (2.15), we have the AS or AQS function for the 2FE panel SL null model:

$$S_{\texttt{SL2}}^{\circ}(\theta) = \begin{cases} \frac{1}{\sigma^2} \sum_{t-1}^{T} X_{nt}^{*\circ\prime} \widetilde{V}_{nt}^{*\circ}(\beta, \lambda), \\ \frac{1}{\sigma^2} \sum_{t-1}^{T} (W_n^* Y_{nt}^{*\circ})' \widetilde{V}_{nt}^{*\circ}(\beta, \lambda) - (T-1)\mathrm{tr}[G_n^*(\lambda)], \\ -\frac{(n-1)(T-1)}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{t=1}^{T} \widetilde{V}_{nt}^{*\circ}{}'(\beta, \lambda) \widetilde{V}_{nt}^{*\circ}(\beta, \lambda), \end{cases} \tag{A.3}$$

where $\widetilde{V}_{nt}^{*\circ}(\beta, \lambda) = A_n^*(\lambda) Y_{nt}^* - X_{nt}^*\beta - \tilde{c}_n^{*\circ}(\beta, \lambda) = A_n(\lambda) Y_{nt}^{*\circ} - X_{nt}^{*\circ}\beta$, $Y_{nt}^{*\circ} = Y_{nt}^* - \bar{Y}_n^*$ and $X_{nt}^{*\circ} = X_{nt}^* - \bar{X}_n^*$. Solving the estimating equations, $S_{\texttt{SL2}}^{\circ}(\theta) = 0$, gives the null estimator $\tilde{\theta}_{\texttt{SL2}}$ of θ, which is obtained by first solving the first and last sets of equations of (A.3) to give the constrained estimators of β and σ^2, given λ, as

$$\tilde{\beta}_{\texttt{SL2}}(\lambda) = (\textstyle\sum_{t=1}^{T} X_{nt}^{\circ*\prime} X_{nt}^{\circ*})^{-1} \sum_{t=1}^{T} X_{nt}^{\circ*} A_n^*(\lambda) Y_{nt}^{\circ*},$$
$$\tilde{\sigma}_{\texttt{SL2}}^2(\lambda) = \tfrac{1}{n(T-1)} \textstyle\sum_{t=1}^{T} \widetilde{V}_{nt}^{\circ*\prime}(\hat{\beta}_{\texttt{SL2}}(\lambda), \lambda) \widetilde{V}_{nt}^{\circ*}(\hat{\beta}_{\texttt{SL2}}(\lambda), \lambda),$$

and then substituting $\tilde{\beta}_{\texttt{SL2}}(\lambda)$ and $\tilde{\sigma}_{\texttt{SL2}}^2(\lambda)$ into the middle equation of (A.3) and solving the resulted concentrated estimating equation to give the null AQS estimators $\tilde{\lambda}_{\texttt{SL2}}$ of the common λ, which in turn gives the AQS estimator $\tilde{\beta}_{\texttt{SL2}} = \tilde{\beta}_{\texttt{SL2}}(\tilde{\lambda}_{\texttt{SL2}})$ of the common β, and the AQS estimator $\tilde{\sigma}_{\texttt{SLE2}}^2 = \tilde{\sigma}_{\texttt{SL2}}^2(\tilde{\lambda}_{\texttt{SL2}})$ of σ^2. Denote the AQS estimator of the parameter vector θ in the null model as $\tilde{\theta}_{\mathtt{SL2}} = (\tilde{\beta}'_{\mathtt{SL2}}, \tilde{\lambda}_{\mathtt{SL2}}, \tilde{\sigma}^2_{\mathtt{SL2}})'$.

The proposed null estimator based on the AQS function provides an alternative to the transformation approaches of Lee and Yu (2010). It can be shown to be asymptotically equivalent to the estimator based on an orthogonal transformation given in Lee and Yu (2010). Thus, $\tilde{\theta}_{\mathtt{SL2}}$ is $\sqrt{(n-1)(T-1)}$-consistent for θ. However, the AQS approach is more general as it allows the estimation of a non-homogeneous model. Finally, the estimation of c_n and γ and κ contained in $I_{\mathtt{SL2}}(\boldsymbol{\theta}_0)$ and $\Sigma_{\mathtt{SL2}}(\boldsymbol{\theta}_0)$ proceeds similarly.

Homogeneous Panel SLE model with 1FE. The general LM procedure presented in Sec. 2.1 can be applied to estimate a null (1FE-SLE) model based on $S^{\star}_{\mathtt{SLE1}}(\boldsymbol{\theta})$ and a properly specified linear contrast matrix C. To estimate the homogeneous null model, whhich is important for the asymptotic arguments and for Monte Carlo simulation, let $\theta = (\beta', \lambda, \rho, \sigma^2)'$. Under H_0, the constrained estimate of c_n given (β, λ) becomes $\tilde{c}^{\circ}_n(\beta, \lambda) = A_n(\lambda)\bar{Y}_n - \bar{X}_n\beta$, and the error vector becomes $\widetilde{V}^{\circ}_{nt}(\beta, \lambda, \rho) = B_n(\rho)[A_n(\lambda)Y^{\circ}_{nt} - X^{\circ}_{nt}\beta]$, where $Y^{\circ}_{nt} = Y_{nt} - \bar{Y}_n$, $X^{\circ}_{nt} = X_{nt} - \bar{X}_n$, and $\bar{Y}_n = \frac{1}{T}\sum_{t=1}^T Y_{nt}$ and $\bar{X}_n = \frac{1}{T}\sum_{t=1}^T X_{nt}$. Along the same line leading to (3.7), one can easily show that AQS function for the null model takes the form:

$$S^{\circ}_{\mathtt{SLE1}}(\theta) = \begin{cases} \frac{1}{\sigma^2}\sum_{t=1}^T X^{\circ}_{nt} B'_n(\rho)\widetilde{V}^{\circ}_{nt}(\beta, \lambda, \rho), \\ \frac{1}{\sigma^2}\sum_{t=1}^T (W_n Y^{\circ}_{nt})' B'_n(\rho)\widetilde{V}^{\circ}_{nt}(\beta, \lambda, \rho) - (T-1)\mathrm{tr}[G_n(\lambda)], \\ \frac{1}{\sigma^2}\sum_{t=1}^T \widetilde{V}^{\circ\prime}_{nt}(\beta, \lambda, \rho) H_n(\rho)\widetilde{V}^{\circ}_{nt}(\beta, \lambda, \rho) - (T-1)\mathrm{tr}[H_n(\rho)], \\ -\frac{n(T-1)}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=1}^T \widetilde{V}^{\circ\prime}_{nt}(\beta, \lambda)\widetilde{V}^{\circ}_{nt}(\beta, \lambda). \end{cases} \tag{A.4}$$

Solving the estimating equations, $S^{\circ}_{\mathtt{SLE1}}(\theta) = 0$, gives the null estimator $\tilde{\theta}_{\mathtt{SLE1}}$ of θ. The process can be simplified by first solving the first set of equations and the last equation of (A.4), giving the constrained estimators of β and σ^2 (for given λ and ρ) as

$$\begin{aligned} \tilde{\beta}_{\mathtt{SLE1}}(\lambda, \rho) &= (\textstyle\sum_{t=1}^T X^{\circ\prime}_{nt} D_n(\rho) X^{\circ}_{nt})^{-1} \sum_{t=1}^T X^{\circ}_{nt} D_n(\rho) A_n(\lambda) Y^{\circ}_{nt}, \\ \tilde{\sigma}^2_{\mathtt{SLE1}}(\lambda, \rho) &= \tfrac{1}{n(T-1)} \textstyle\sum_{t=1}^T \widetilde{V}^{\circ\prime}_{nt}(\hat{\beta}_{\mathtt{SLE1}}(\lambda, \rho), \lambda, \rho)\widetilde{V}^{\circ}_{nt}(\hat{\beta}_{\mathtt{SLE1}}(\lambda, \rho), \lambda, \rho). \end{aligned}$$

Substituting $\tilde{\beta}_{\mathtt{SLE1}}(\lambda, \rho)$ and $\tilde{\sigma}^2_{\mathtt{SLE1}}(\lambda, \rho)$ into the middle two equations of (A.4) and solving the resulted concentrated estimating equations lead to the AQS estimators $(\tilde{\lambda}_{\mathtt{SLE1}}, \tilde{\rho}_{\mathtt{SLE1}})$ of (λ, ρ), which in turn give the AQS estimator $\tilde{\beta}_{\mathtt{SLE1}} = \tilde{\beta}_{\mathtt{SLE1}}(\tilde{\lambda}_{\mathtt{SLE1}}, \tilde{\rho}_{\mathtt{SLE1}})$ of β, and the AQS estimator $\tilde{\sigma}^2_{\mathtt{SLE1}} = \tilde{\sigma}^2_{\mathtt{SLE1}}(\tilde{\lambda}_{\mathtt{SLE1}}, \tilde{\rho}_{\mathtt{SLE1}})$ of σ^2. Finally, the AQS estimator of θ is $\tilde{\theta}_{\mathtt{SLE1}} = (\tilde{\beta}'_{\mathtt{SLE1}}, \tilde{\lambda}_{\mathtt{SLE1}}, \tilde{\sigma}^2_{\mathtt{SLE1}})'$, which is asymptotically equivalent to the estimator based on an orthogonal transformation given in Lee and Yu (2010), and thus is $\sqrt{n(T-1)}$-consistent. To estimate c_n, γ and κ, refer to the discussions at the end of Section 2.1.

Homogeneous Panel SLE model with 2FE. Again, the general LM procedure can be adapted to estimated a null (panel SLE-2FE) model based on the AQS function $S^{\star}_{\mathtt{SLE2}}(\theta)$ and a properly specified linear contrast matrix C. To estimate the homogeneous null model, let $\theta = (\beta', \lambda, \rho, \sigma^2)'$. Under H_0, the constrained estimate of c_n given (β, λ)

becomes $\tilde{c}_n^{\circ*}(\beta,\lambda) = A_n^*(\lambda)\bar{Y}_n^* - \bar{X}_n^*\beta$ where $\bar{Y}_n^*$ and $\bar{X}_n^*$ are the averages of $\{Y_{nt}^*\}$ and $\{X_{nt}^*\}$, respectively. Along the same line leading to (3.13), one can easily show that AQS function for the null homogeneous model takes the form:

$$S_{\mathtt{SLE2}}^{\circ*}(\theta) = \begin{cases} \frac{1}{\sigma^2}\sum_{t=1}^T X_{nt}^{\circ*\prime}B_n^{*\prime}(\rho)\widetilde{V}_{nt}^{\circ*}(\beta,\lambda,\rho), \\ \frac{1}{\sigma^2}\sum_{t=1}^T (W_n^*Y_{nt}^{\circ*})'B_n^{*\prime}(\rho)\widetilde{V}_{nt}^{\circ*}(\beta,\lambda,\rho) - (T-1)\mathrm{tr}[G_n^*(\lambda)], \\ \frac{1}{\sigma^2}\sum_{t=1}^T \widetilde{V}_{nt}^{\circ*\prime}(\beta,\lambda,\rho)H_n^*(\rho)\widetilde{V}_{nt}^{\circ*}(\beta,\lambda,\rho) - (T-1)\mathrm{tr}[H_n^*(\lambda)], \\ -\frac{(n-1)(T-1)}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=1}^T \widetilde{V}_{nt}^{\circ*\prime}(\beta,\lambda,\rho)\widetilde{V}_{nt}^{\circ*}(\beta,\lambda,\rho), \end{cases} \tag{A.5}$$

$\widetilde{V}_{nt}^{\circ*}(\beta,\lambda,\rho) = B_n^*(\rho)[A_n^*(\lambda)Y_{nt}^* - X_{nt}^*\beta - \tilde{c}_n^{\circ*}(\beta,\lambda)] = B_n^*(\rho)[A_n^*(\lambda)Y_{nt}^{\circ*} - X_{nt}^{\circ*}\beta]$, where $Y_{nt}^{\circ*} = Y_{nt}^* - \bar{Y}_n^*$ and $X_{nt}^{\circ*} = X_{nt}^* - \bar{X}_n^*$. Solving the estimating equations, $S_{\mathtt{SLE2}}^{\circ*}(\theta) = 0$, gives the null estimator $\tilde{\theta}_{\mathtt{SLE2}}$ of θ, which is obtained by first solving the first and last sets of equations of (A.5) to give the constrained estimators of β and σ^2, given λ and ρ, as

$$\tilde{\beta}_{\mathtt{SLE2}}(\lambda,\rho) = (\textstyle\sum_{t=1}^T X_{nt}^{\circ*\prime}B_n^{*\prime}(\rho)B_n^*(\rho)X_{nt}^{\circ*})^{-1}\sum_{t=1}^T X_{nt}^{\circ*}B_n^{*\prime}(\rho)B_n^*(\rho)A_n^*(\lambda)Y_{nt}^{\circ*},$$
$$\tilde{\sigma}_{\mathtt{SLE2}}^2(\lambda,\rho) = \tfrac{1}{n(T-1)}\textstyle\sum_{t=1}^T \widetilde{V}_{nt}^{\circ*\prime}(\hat{\beta}_{\mathtt{SLE2}}(\lambda,\rho),\lambda,\rho)\widetilde{V}_{nt}^{\circ*}(\hat{\beta}_{\mathtt{SLE2}}(\lambda,\rho),\lambda,\rho),$$

and then substituting $\tilde{\beta}_{\mathtt{SLE2}}(\lambda,\rho)$ and $\tilde{\sigma}_{\mathtt{SLE2}}^2(\lambda,\rho)$ into the middle equations of (A.5) and solving the resulted concentrated estimating equation to give the null AQS estimators $\tilde{\lambda}_{\mathtt{SLE2}}$ of the common λ and $\tilde{\rho}_{\mathtt{SLE2}}$ of the common ρ, which in turn gives the AQS estimator $\tilde{\beta}_{\mathtt{SLE2}} = \tilde{\beta}_{\mathtt{SLE2}}(\tilde{\lambda}_{\mathtt{SLE2}},\tilde{\rho}_{\mathtt{SLE2}})$ of the common β, and the AQS estimator $\tilde{\sigma}_{\mathtt{SLE2}}^2 = \tilde{\sigma}_{\mathtt{SLE2}}^2(\tilde{\lambda}_{\mathtt{SLE2}},\tilde{\rho}_{\mathtt{SLE2}})$ of σ^2. Finally, the AQS estimator of θ is $\tilde{\theta}_{\mathtt{SLE2}} = (\tilde{\beta}_{\mathtt{SLE2}}', \tilde{\lambda}_{\mathtt{SLE2}}, \tilde{\rho}_{\mathtt{SLE2}}, \tilde{\sigma}_{\mathtt{SLE2}}^2)'$. The proposed null estimator based on the AQS function provides an alternative to the direct and transformation approaches of Lee and Yu (2010). It can be shown to be asymptotically equivalent to the estimator based on an orthogonal transformation given in Lee and Yu (2010). Thus, $\tilde{\theta}_{\mathtt{SLE2}}$ is $\sqrt{(n-1)(T-1)}$-consistent for θ. Estimation of c_n, γ and κ proceeds similarly.