Heteroskedasticity Robust Estimation and Testing for Higher Order Spatial Autoregressive Models

Shew Fan Liu* and Zhenlin Yang†

School of Economics, Singapore Management University

May 20, 2017

Abstract

The likelihood-based estimation of the spatial autoregressive model of order p with spatial autoregressive errors of order q, or SARAR(p,q), is considered. We first present asymptotic results for the quasi maximum likelihood (QML) estimators developed under the homoskedasticity and normality assumptions, and then show that the QML estimators are in general not robust against unknown heteroskedasticity. We propose an *adjusted quasi score* (AQS) estimator and an *adjusted concentrated quasi score* (ACQS) estimator, both fully robust against heteroskedasticity and nonnormality, with latter possessing better finite sample properties. The consistency and asymptotic normality of the AQS and ACQS estimators are established. *Outer-product-of-martingale-difference* methods are introduced for consistent and robust estimation of their standard errors. To facilitate the model selection, we introduce a set of LM and robust LM tests for the SARAR(p,q) effects based on the joint scores, and a set of *standardised* LM and robust LM tests based on the concentrated scores. Extensive Monte Carlo experiments are conducted, and the results show excellent finite sample performance of the proposed ACQS estimators and standardised tests. The proposed methods are simple and can be easily applied by the applied researchers. Compared with the existing GMM estimators, the proposed ACQS estimator is less biased and more efficient. The ACQS estimator out performs the AQS estimator, and the standardised tests out perform the non-standardised tests.

Key Words: High order spatial dependence; Bias; Efficiency; Unknown heteroskedasticity; Nonnormality; Adjusted scores; Robust standard errors.

JEL Classification: C10, C13, C15, C21

*Corresponding author: 90 Stamford Road, Singapore 178903. Phone: +65-6828-9592; Fax: +65-6828-0833; email: shewfan.liu.2011@smu.edu.sg.

†email: zlyang@smu.edu.sg

1. Introduction

We consider a p-order spatial autoregressive (SAR) model with q-order SAR disturbances, commonly known as the SARAR(p, q) model, given below,

$$Y_n = \sum_{j=1}^{p} \lambda_j W_{\ell j} Y_n + X_n \beta + u_n, \text{ where } u_n = \sum_{k=1}^{q} \rho_k W_{ek} u_n + \varepsilon_n, \tag{1.1}$$

where Y_n is an $n \times 1$ vector of observations on the dependent variable, X_n is an $n \times k$ matrix of observations on the exogenous regressors with the $k \times 1$ regression parameter vector β, u_n is the $n \times 1$ vector of disturbances, $\{W_{\ell j}\}$ and $\{W_{ek}\}$ are the two sets of $n \times n$ spatial weights matrices which summarizes, respectively, the higher order *spatial lag* dependence of the dependent variable and the higher order *spatial error* dependence, with the associated spatial parameters $\lambda = (\lambda_1, \ldots, \lambda_p)'$ and $\rho = (\rho_1, \ldots, \rho_q)'$. The $W'_{\ell j}s$ are different and so are the $W'_{ek}s$ to account for different level or different type of spatial dependence, but some (or all) of the $W'_{\ell j}s$ can be the same as some (or all) of the $W'_{ek}s$. The elements of the error vector ε_n are independent with means zero, and variances constant or varying giving the so-called *homoskedastic* SARAR(p, q) model or *heteroskedastic* SARAR(p, q) model.

Model (1.1) provides an attractive way to study the effects of various types of spatial interactions that commonly exist in spatial data, e.g., geographical distance, social relationship, and peer effects. Also common for spatial data are the *heteroskedasticity* and *nonnormality* (Anselin 1988b). Thus, it is highly desirable to have a set of simple inference methods for the SARAR(p, q) model that are robust against unknown heteroskedasticity and nonnormality.

Lin and Lee (2010) proposed a heteroskedasticity robust GMM (generalised method of moments) estimator for SARAR(1,0) using both linear and quadratic moments. Kelejian and Prucha (2010) presented a three-step GMM-type estimator for SARAR(1,1) robust against unknown heteroskedasticity, which is generalised by Badinger and Egger (2011) to SARAR(p, q). Lee and Liu (2010) considered the efficient GMM estimation of SARAR(p, q) under homoskedasticity. Elhorst et al. (2012) considered issues on model specification and parameter space definition in higher order spatial econometric models. Jin and Lee (2012) presented asymptotic results for the quasi maximum likelihood (QML) estimators of the SARAR(1,1) model under homoskedasticity. Baltagi and Yang (2013) gave heteroskedasticity robust LM tests for SARAR(1,1) effects in linear and fixed effects panel data models, but not for SARAR(p, q) effects. Liu and Yang (2015) presented adjusted quasi score (AQS) estimators for the SARAR(1,0) and SARAR(1,1) models which are heteroskedasticity robust, but the asymptotic properties of the AQS estimators are studied only for the SARAR(1,0) model. Liu and Prucha (2018) proposed robust tests for network generated dependence of higher order, which generalised Moran's I test allowing heteroskedastic errors and endogenous regressors. Liu and Lee (2019) considered generalised empirical likelihood (GEL) estimation and tests for the SARAR(p, q) model allowing heteroskedasticity and nonnormality.

Detailed studies on the (quasi) likelihood-based estimation and testing for the $\texttt{SARAR}(p, q)$ have not been given under either homoskedasticity or heteroskedasticity, which are clearly desirable as often the likelihood-based inferences are simpler and more efficient than those based on the GMM-type and GEL-type approaches. Researchers often advocate the use of GMM and GEL approaches on the ground that the GEL and GMM approaches are computationally less demanding than the QML approach. However, we show in this paper that our proposed AQS method is computational equivalent to the GMM and GEL methods, if not less demanding than them, as the AQS method also works on a set of moment conditions. In terms of tests on the spatial effects, the tests given in Liu and Prucha (2018) and Jin and Lee (2019) test only the null hypothesis of no spatial effects. We introduce tests that cover all possible scenarios corresponding to a model reduction, e.g., from a high-order $\texttt{SARAR}(p, q)$ model down to a low-order $\texttt{SARAR}(1, 1)$ model.

In this paper, we consider in detail the likelihood-based inference methods (point estimation, standard error estimation and hypothesis testing) for the $\texttt{SARAR}(p, q)$ model. We first present asymptotic results for the QML estimators (QMLEs) developed under homoskedasticity and normality assumptions, and then study the robustness of the QMLEs when these assumptions, in particular the homoskedasticity assumption, are violated. These studies in part provide simpler and more efficient methods for the practitioners when the assumptions are met, and in part motivate the main studies of this paper: the heteroskedasticity robust estimation and testing of the $\texttt{SARAR}(p, q)$ model. We first proposed an *adjusted quasi score* (AQS) estimator by adjusting the joint quasi score function, and then an *adjusted concentrated quasi score* (ACQS) estimator by adjusting the concentrated quasi score function to improve the finite sample performance of the estimator. Both AQS and ACQS estimators are fully robust against unknown heteroskedasticity and nonnormality. Their consistency and asymptotic normality are established. For statistical inference, we introduce an *outer-product-of-martingale-difference* methods for consistent and robust estimation of the standard errors of the proposed estimators. Finally, we introduce LM tests, robust LM-type tests, and finite sample improved LM-type tests for the $\texttt{SARAR}(p, q)$ effects to facilitate the model selection.[1] To facilitate model selection, we introduce a set of LM and robust LM tests for the $\texttt{SARAR}(p, q)$ effects model based on the joint scores, and a set of *standardised* LM and robust LM tests based on the concentrated scores. Extensive Monte Carlo experiments are conducted, and the results show excellent finite sample performance of the proposed ACQS estimators and standardised tests. The proposed methods are simple and can be easily applied by the applied researchers. Compared with the existing GMM estimators, the proposed ACQS estimator is less biased and more efficient. The ACQS estimator out performs the AQS estimator, and the standardised tests out perform the non-standardised tests.

The rest of the paper goes as follows. Section 2 studies the regular QMLE for its asymp-

[1] The results presented in this paper contain, as special cases, the results of Jin and Lee (2012) for the QML estimation, the results of Liu and Yang (2015) for the ACQS estimation, and the results of Baltagi and Yang (2013) for the heteroskedasticity robust LM tests.

totic properties, and its robustness. Section 3 introduces the AQS and ACQS estimators, studies its asymptotic properties, and introduces a method for estimating the robust standard errors of these estimators. Section 4 presents various LM and LM-type tests for various orders of spatial specifications. Section 5 presents a set of standardised tests with improved finite sample property. Section 6 presents Monte Carlo results, and Section 7 concludes the paper. Proofs of all the theoretical results are given in Appendices.

2. QML Estimation: Asymptotic Properties and Robustness

In this section, we introduce the QML estimation of the general $\texttt{SARAR}(p, q)$ model given in (1.1), exam the asymptotic properties of the QMLEs under homoskedasticity and their robustness against unknown heteroskedasticity. The former fills in a gap in the literature, and the latter motivates the main study of the paper - heteroskedasticity robust estimation and testing for the higher order spatial autoregressive models.

Model (1.1) is written as,

$$Y_n = A_n^{-1}(\lambda)[X_n\beta + B_n^{-1}(\rho)\varepsilon_n], \tag{2.1}$$

where, $A_n(\lambda) = I_n - \sum_{j=1}^{p} \lambda_j W_{\ell j}$ and $B_n(\rho) = I_n - \sum_{k=1}^{q} \rho_k W_{ek}$. Denoting $\delta = (\lambda', \rho')'$ and $\theta = (\beta', \sigma^2, \lambda', \rho')'$, the Gaussian log-likelihood function for θ is,

$$\ell_n(\theta) = -\frac{n}{2}\ln(2\pi\sigma^2) + \ln|A_n(\lambda)| + \ln|B_n(\rho)| - \frac{1}{2\sigma^2}\varepsilon_n'(\beta,\delta)\varepsilon_n(\beta,\delta), \tag{2.2}$$

where $\varepsilon_n(\beta,\delta) = B_n(\rho)[A_n(\lambda)Y_n - X_n\beta]$ and $|\cdot|$ denotes the determinant of a matrix. Maximising (2.2) gives the maximum likelihood estimator (MLE) of θ_0 if the disturbances are indeed Gaussian, otherwise the quasi MLE (QMLE) if the disturbances are not Gaussian.

The above maximization process can be simplified by first maximising (2.2) with respect to β and σ^2, resulting in the constrained QMLEs for β and σ^2 as follows,

$$\hat\beta_n(\delta) = [X_n'(\rho)X_n(\rho)]^{-1}X_n'(\rho)Y_n(\delta) \;\text{ and }\; \hat\sigma_n^2(\delta) = \tfrac{1}{n}Y_n'(\delta)M_n(\rho)Y_n(\delta), \tag{2.3}$$

where $X_n(\rho) = B_n(\rho)X_n$, $Y_n(\delta) = B_n(\rho)A_n(\lambda)Y_n$, $M_n(\rho) = I_n - X_n(\rho)[X_n'(\rho)X_n(\rho)]^{-1}X_n'(\rho)$, and I_n is an $n\times n$ identity matrix. Substituting $\hat\beta_n(\delta)$ and $\hat\sigma_n^2(\delta)$ into (2.2) for β and σ^2, we get the concentrated log-likelihood function for δ,

$$\ell_n^c(\delta) = -\frac{n}{2}[\ln(2\pi)+1] - \frac{n}{2}\ln[\hat\sigma_n^2(\delta)] + \ln|A_n(\lambda)| + \ln|B_n(\rho)|. \tag{2.4}$$

Maximising $\ell_n^c(\delta)$ gives the unconstrained QMLE $\hat\delta_n$ of δ, and thus the unconstrained MLEs of β and σ^2 as $\hat\beta_n \equiv \hat\beta(\hat\lambda_n)$ and $\hat\sigma_n^2 \equiv \hat\sigma_n^2(\hat\lambda_n)$. The unconstrained QMLE of the entire parameter vector θ is thus $\hat\theta_n = (\hat\beta_n', \hat\sigma_n^2, \hat\delta_n')'$. The concentrated log-likelihood function $\ell_n^c(\delta)$ is also a crucial function for studying the asymptotic properties of $\hat\delta_n$, and hence of $\hat\theta_n$.[2] A formal

[2]For the asymptotic properties of the QMLEs under homoskedasticity, see Lee (2004) for $\texttt{SARAR}$(1,0), Jin

study on the asymptotic properties of the QMLE $\hat{\theta}_n$ of the general $\texttt{SARAR}(p,q)$ model is not given, and the robustness of $\hat{\theta}_n$ against unknown heteroskedasticity is not formally examined.

To proceed with these studies, some **notational conventions** are adhered to in the entire article. Let θ_0 denote the true parameter vector and view from now onwards that Model (1.1) holds only at θ_0. Any 'parametric' quantity evaluated at the true parameter is denoted with a suppressed variable argument, e.g. $A_n \equiv A_n(\lambda_0)$, $B_n \equiv B_n(\rho_0)$ and so on. Let $\text{tr}(\cdot)$, $|\cdot|$ and $\|\cdot\|$ denote, respectively, the trace, determinant, and Frobenius norm of a matrix. Let I_n be an $n\times n$ identity matrix, ι_n an n-vector of ones, $\texttt{diag}(\cdot)$ be a diagonal matrix formed by the elements of a given vector, or the diagonal elements of a given square matrix, $\texttt{diagv}(\cdot)$ be a vector formed by the diagonal elements of a given square matrix, and $\texttt{blkdiag}(\cdots)$ the block-diagonal matrix formed by the given set of matrices. The usual expectation operator, $\text{E}(\cdot)$, and the variance operator, $\text{Var}(\cdot)$, correspond to the true parameter θ_0.

First, let $\bar{\ell}_n(\theta) = \text{E}[\ell_n(\theta)]$, the population counterpart of the log-likelihood function. It is easy to see that, under some mild conditions, $\bar{\ell}_n(\theta)$ is partially maximised at

$$\bar{\beta}_n(\delta) = [X_n'(\rho)X_n(\rho)]^{-1}X_n'(\rho)\text{E}[Y_n(\delta)] \quad \text{and} \tag{2.5}$$

$$\bar{\sigma}_n^2(\delta) = \tfrac{1}{n}\text{E}\big\{[Y_n(\delta)-\text{E}(Y_n(\delta))]'[Y_n(\delta)-\text{E}(Y_n(\delta))]\big\} + \tfrac{1}{n}\text{E}[Y_n'(\delta)]M_n(\rho)\text{E}[Y_n(\delta)], \tag{2.6}$$

giving the concentrated version of $\bar{\ell}_n(\theta)$ after β and σ^2 being concentrated out,

$$\bar{\ell}_n^c(\delta) = -\frac{n}{2}[\ln(2\pi)+1] - \frac{n}{2}\ln[\bar{\sigma}_n^2(\delta)] + \ln|A_n(\lambda)| + \ln|B_n(\rho)|. \tag{2.7}$$

Consistency of $\hat{\delta}_n$ follows if (i) $\bar{\ell}_n^c(\delta)$ has an identifiably unique maximizer at δ_0 and (ii) $\frac{1}{n}[\ell_n^c(\delta) - \bar{\ell}_n^c(\delta)] = \frac{1}{2}[\ln(\bar{\sigma}_n^2(\delta)) - \ln(\hat{\sigma}_n^2(\delta))] \xrightarrow{p} 0$, uniformly in $\delta\in\Delta$, a parameter space. Consistency of $\hat{\beta}_n$ and $\hat{\sigma}_n^2$, and thus of $\hat{\theta}_n$, follow almost immediately from (2.3) and some conditions on the regressors. A sufficient set of regularity conditions are given as follows.

Assumption 1: *The true value δ_0 of the vector of spatial parameters is in the interior of a compact parameter set Δ.*[3]

Assumption 2: $\{\varepsilon_{n,i}\}$ are $\text{iid}(0,\sigma_0^2)$, and $E|\varepsilon_{n,i}|^{4+\epsilon} < \infty$ for some $\epsilon > 0$.

Assumption 3: *The elements of the $n\times k$ regressor matrix X_n are uniformly bounded for all n, X_n has the full rank k, and $\lim_{n\to\infty}\frac{1}{n}X_n'B_n'B_nX_n$ exists and is non-singular*

Assumption 4: *The spatial weights matrices $W_{\ell j}$ and W_{ek} are uniformly bounded in absolute value in both row and column sums and their diagonal elements are zero.*

and Lee (2012) for $\texttt{SARAR}$(1,1), and Liu and Yang (2015) for $\texttt{SARAR}$(0,1). For the asymptotic properties of the QMLEs under heteroskedasticity, see Liu and Yang (2015) for $\texttt{SARAR}$(1,0). For the GMM estimation under homoskedasticity, see Lee and Liu (2010) for $\texttt{SARAR}(p,q)$. For the GMM estimation under heteroskedasticity, see Lin and Lee (2010), Kelejian and Prucha (2010), and Badinger and Egger (2011).

[3]For QML estimation, the parameter space Δ must be such that $A_n(\lambda)$ and $B_n(\rho)$ are non-singular $\forall\delta\in\Delta$ for log-likelihood function can be well defined. Lee and Liu (2010) shows that since $\|\sum_{j=1}^p\lambda_jW_{\ell j}\| \le (\sum_{j=1}^p|\lambda_j|)\cdot\max_{j=1,\dots,p}\|W_{\ell j}\|$, a viable parameter space for λ_j is such that $\sum_{j=1}^p|\lambda_j| < (\max_{j=1,\dots,p}\|W_{\ell j}\|)^{-1}$, which simplifies to $\sum_{j=1}^p|\lambda_j|<1$ when $W_{\ell j}$ are row-normalised. However, Elhorst et al. (2012) argues that this parametrisation is too restrictive and gives an alternative procedure to determine exact boundaries which depends on the specification of $W_{\ell j}$. Similar arguments apply for the parameter space of $\{\rho_k\}$.

Assumption 5: *The matrix A_n and B_n are non-singular and A_n^{-1} and B_n^{-1} are uniformly bounded in absolute value in both row and column sums. Further, $A_n^{-1}(\lambda)$ and $B_n^{-1}(\rho)$ are uniformly bounded in either row or column sums, uniformly in $\delta \in \Delta$.*

Besides these standard regularity conditions applicable to all spatial linear regression models, the identification conditions for the higher-order spatial models are as follows.

Assumption 6: *Either (a): $\lim_{n\to\infty} \mathcal{H}_n(\rho)$ is non-singular $\forall \rho$ and $\lim_{n\to\infty} \mathcal{Q}_{1n}(\rho) \neq 0$ for $\rho \neq \rho_0$; or (b): $\lim_{n\to\infty} \mathcal{Q}_{2n}(\delta) \neq 0$ for $\delta \neq \delta_0$, where*
$\mathcal{H}_n(\rho) = \frac{1}{n}(X_n, F_{1n}X_n\beta_0, \ldots, F_{pn}X_n\beta_0)' B_n'(\rho)B_n(\rho)(X_n, F_{1n}X_n\beta_0, \ldots, F_{pn}X_n\beta_0)$,
$\mathcal{Q}_{1n}(\rho) = \frac{1}{n}\left(\ln|\sigma_0^2 B_n'^{-1}B_n^{-1}| - \ln|\sigma_{1n}^2(\rho)B_n'^{-1}(\rho)B_n^{-1}(\rho)|\right)$,
$\mathcal{Q}_{2n}(\delta) = \frac{1}{n}\left(\ln|\sigma_0^2 B_n'^{-1}A_n'^{-1}A_n^{-1}B_n^{-1}| - \ln|\sigma_{2n}^2(\delta)B_n'^{-1}(\rho)A_n'^{-1}(\lambda)A_n^{-1}(\lambda)B_n^{-1}(\rho)|\right)$,
$\sigma_{1n}^2(\rho) = \frac{\sigma_0^2}{n}\text{tr}[B_n'^{-1}B_n'(\rho)B_n(\rho)B_n^{-1}]$, and $\sigma_{2n}^2(\delta) = \frac{\sigma_0^2}{n}\text{tr}[B_n'^{-1}A_n'^{-1}A_n'(\lambda)B_n'(\rho)B_n(\rho)A_n(\lambda)A_n^{-1}B_n^{-1}]$.

To establish the asymptotic normality of $\hat{\theta}_n$, we have the score function, $S_n(\theta) = \frac{\partial}{\partial\theta}\ell_n(\theta)$:

$$S_n(\theta) = \begin{cases} \frac{1}{\sigma^2}X_n'B_n'(\rho)\varepsilon_n(\beta,\delta), \\ \frac{1}{2\sigma^4}\varepsilon_n'(\beta,\delta)\varepsilon_n(\beta,\delta) - \frac{n}{2\sigma^2}, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)B_n(\rho)W_{\ell j}Y_n - \text{tr}(F_{jn}(\lambda)), & j = 1,\cdots,p, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)G_{kn}(\rho)\varepsilon_n(\beta,\delta) - \text{tr}(G_{kn}(\rho)), & k = 1,\cdots,q, \end{cases} \tag{2.8}$$

where $F_{jn}(\lambda) = W_{\ell j}A_n^{-1}(\lambda)$ and $G_{kn}(\rho) = W_{ek}B_n^{-1}(\rho)$. At the true value θ_0,

$$S_n(\theta_0) = \begin{cases} \frac{1}{\sigma_0^2}X_n'B_n'\varepsilon_n \\ \frac{1}{2\sigma_0^4}(\varepsilon_n'\varepsilon_n - n\sigma^2) \\ \frac{1}{\sigma_0^2}\left(\varepsilon_n'\bar{F}_{jn}\varepsilon_n - \sigma_0^2\text{tr}(F_{jn}) + \varepsilon_n'\eta_{jn}\right), & j = 1,\ldots,p, \\ \frac{1}{\sigma_0^2}\left(\varepsilon_n'G_{kn}\varepsilon_n - \sigma_0^2\text{tr}(G_{kn})\right), & k = 1,\ldots,q, \end{cases} \tag{2.9}$$

where $\bar{F}_{jn} \equiv \bar{F}_{jn}(\delta_0) = B_n(\rho_0)F_{jn}(\lambda_0)B_n^{-1}(\rho_0)$ and $\eta_{jn} \equiv \eta_{jn}(\delta_0) = B_n(\rho_0)F_{jn}(\lambda_0)X_n\beta_0$. Thus, all the elements of $S_n(\theta_0)$ are linear or quadratic forms in ε_n, and the central limit theorem (CLT) for linear-quadratic (LQ) forms of Kelejian and Prucha (2001), reproduced in Lemma A.3 in Appendix A for easy reference, can be applied to show the asymptotic normality of $\frac{1}{\sqrt{n}}S_n(\theta_0)$. Now, by the mean value theorem applied to $S_n(\hat{\theta}_n) = 0$, we have,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \big(-\tfrac{1}{n}\tfrac{\partial}{\partial\theta'}S_n(\theta)|_{\theta=\tilde{\theta}}\big)^{-1}\big(\tfrac{1}{\sqrt{n}}S_n(\theta_0)\big), \tag{2.10}$$

where $\tilde{\theta}_n$ lies element-wise between $\hat{\theta}_n$ and θ_0. The asymptotic normality of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ follows from the asymptotic normality of $\frac{1}{\sqrt{n}}S_n(\theta_0)$, and the 'regular' behavior of $\frac{\partial}{\partial\theta'}S_n(\theta)|_{\theta=\tilde{\theta}}$.

Theorem 2.1. *Under Assumptions 1-6, we have, as $n \to \infty$, $\hat{\theta}_n \xrightarrow{p} 0$, and*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{D} N(0,\ \lim_{n\to\infty}\Sigma_n^{-1}\Omega_n\Sigma_n^{-1}),$$

where $\Sigma_n = -\frac{1}{n}E[\frac{\partial^2}{\partial\theta\partial\theta'}\ell_n(\theta_0)]$ and $\Omega_n = \frac{1}{n}\text{E}[S_n(\theta_0)S_n'(\theta_0)]$, with the limit of the former being a positive definite matrix, and that of the latter simply a constant matrix.

To facilitate direct applications of the results of Theorem 2.1, and for the construction of LM-type of tests in Sec. 4, we give the expressions of the two matrices here. The following notational conventions have been followed: $\{a_j\}$ forms a row vector based on the elements $a_j, j=1,\ldots,p$, $\{a_k\}$ forms a row vector based on the elements $a_k, k=1,\ldots,q$ and $A^s = A+A'$ for a square matrix A. The information matrix $\Sigma_n = \Sigma_n(\theta_0)$ has the form:

$$n\Sigma_n = \begin{pmatrix} \{\frac{1}{\sigma_0^2}X_n'B_n'B_nX_n\}, & 0, & \{\frac{1}{\sigma_0^2}X_n'B_n'\eta_{j'n}\}, & 0 \\ \sim, & \frac{n}{2\sigma_0^2}, & \{\frac{1}{\sigma_0^2}\mathrm{tr}(\bar{F}_{j'n})\}, & \{\frac{1}{\sigma_0^2}\mathrm{tr}(G_{k'n})\} \\ \sim, & \sim, & \{\frac{1}{\sigma_0^2}\eta_{jn}'\eta_{j'n} + \mathrm{tr}(\bar{F}_{jn}\bar{F}_{j'n}^s)\}, & \{\mathrm{tr}(\bar{F}_{jn}G_{k'n}^s)\} \\ \sim, & \sim, & \sim, & \{\mathrm{tr}(G_{kn}G_{k'n}^s)\} \end{pmatrix}, \quad (2.11)$$

where The variance-covariance (VC) matrix of the score $\Omega_n = \Sigma_n + \Gamma_n$, where

$$n\Gamma_n = \begin{pmatrix} 0_{k\times k}, & \frac{\gamma}{2\sigma_0^3}X_n'B_n'\iota_n, & \{\frac{\gamma}{\sigma_0}X_n'B_n'\bar{f}_{j'n}\}, & \{\frac{\gamma}{\sigma_0}X_n'B_n'g_{k'n}\} \\ \sim, & \frac{n\kappa}{4\sigma_0^4}, & \{\frac{\kappa}{2\sigma_0^2}\mathrm{tr}(\bar{F}_{j'n}) + \frac{\gamma}{2\sigma_0^3}\iota_n'\eta_{j'n}\}, & \{\frac{\kappa}{2\sigma_0^2}\mathrm{tr}(G_{k'n})\} \\ \sim, & \sim, & \{\kappa\bar{f}_{jn}'\bar{f}_{j'n} + \frac{2\gamma}{\sigma_0}\bar{f}_{jn}'\eta_{j'n}\}, & \{\kappa\bar{f}_{jn}'g_{k'n} + \frac{\gamma}{\sigma_0}\eta_{jn}'g_{k'n}\} \\ \sim, & \sim, & \sim, & \{\kappa g_{kn}'g_{k'n}\} \end{pmatrix}, \quad (2.12)$$

for $j, j' = 1,\ldots,p$ and $k, k' = 1,\ldots,q$, where $g_{kn} = \mathtt{diagv}(G_{kn})$ and $\bar{f}_{jn} = \mathtt{diagv}(\bar{F}_{jn})$, and γ and κ are respectively, the measures of skewness and excess kurtosis of ε_{ni}. In practical applications, Σ_n can be estimated by plugging-in $\hat{\theta}_n$ for θ_0, and Ω_n can be estimated by plugging-in $\hat{\theta}_n$ for θ_0 and $\hat{\gamma}_n$ and $\hat{\kappa}_n$ for γ and κ, where $\hat{\gamma}_n$ and $\hat{\kappa}_n$ are the sample skewness and excess kurtosis of the QMLE estimate $\hat{\varepsilon}_n$. When ε_{ni} are normal, $\gamma = \kappa = 0$, and $\Omega_n = \Sigma_n$.

Robustness of $\hat{\theta}_n$. The QMLE given above is consistent as long as the errors are iid, not necessarily normal. In other words, the QMLE is robust against nonnormality. Now, suppose Assumption 2 is violated in the sense that the errors are not iid, and instead they are independent but not identically (inid) distributed, i.e.,

Assumption 2*: $\varepsilon_n \sim (0, \sigma_0^2 H_n)$, *where* $H_n = \mathrm{diag}(h_{n,1},\ldots,h_{n,n})$, *such that* $\frac{1}{n}\sum_{i=1}^n h_{n,i} = 1$ *and* $h_{n,i} > 0, \forall i$, *and* $E|\varepsilon_{n,i}|^{4+\eta} < c$ *for some* $\eta > 0$ *and constant* c *for all* n *and* i.[4]

Does $\hat{\theta}_n$ continue to be consistent? It is well known that for any extremum estimator to be consistent such as $\hat{\theta}_n$ given above, it is necessary that $\mathrm{plim}_{n\to\infty}\frac{1}{n}S_n(\theta_0) = 0$ for an estimating function $S_n(\theta)$. This condition is satisfied for the derivatives with respect to (w.r.t.) β and σ^2, but not by these w.r.t. the spatial parameters, λ_j and ρ_k. Note that,

$$\begin{aligned} \tfrac{1}{n}\tfrac{\partial}{\partial\lambda_j}\ell_n(\theta_0) &= \tfrac{1}{n}\mathrm{tr}(H_n\bar{F}_{jn} - F_{jn}) + o_p(1) = \tfrac{1}{n}\textstyle\sum_{i=1}^n (h_{n,i}-1)(\bar{f}_{jn,i} - \bar{\bar{f}}_{jn}) + o_p(1) \\ &= \mathrm{Cov}(h_n, \bar{f}_{jn}) + o_p(1), \ j = 1,\ldots,p, \text{ and} \end{aligned} \quad (2.13)$$

[4]For generality, we allow $h_{n,i}$ to depend on n for each i, which is sensible as heteroskedasticity may depend on the degree of spatial interaction, e.g., number of neighbours This parametrisation is a non-parametric version of Breusch and Pagan (1979) and allows the estimation of the average scale parameter.

$$\frac{1}{n}\frac{\partial}{\partial\rho_k}\ell_n(\theta_0) = \frac{1}{n}\mathrm{tr}(H_nG_{kn} - G_{kn}) + o_p(1) = \frac{1}{n}\sum_{i=1}^n(h_{n,i}-1)(g_{kn,i}-\bar{g}_{kn})) + o_p(1)$$
$$= \mathrm{Cov}(h_n, g_{kn}) + o_p(1),\ k = 1,\ldots,q, \tag{2.14}$$

where $h_n = (h_{n,1},\ldots,h_{n,n})'$, $\bar{\bar{f}}_{jn} = \frac{1}{n}\sum_{i=1}^n \bar{f}_{jn,i}$, $\bar{g}_{kn} = \frac{1}{n}\sum_{i=1}^n g_{kn,i}$, and $\mathrm{Cov}(\cdot,\cdot)$ here denotes the sample covariance between two vectors.

As such, for $\hat{\theta}_n$ to be consistent under the relaxed condition, Assumption 2*, it is necessary that as $n\to\infty$, $\mathrm{Cov}(h_n,\bar{b}_{jn})\to 0$ and $\mathrm{Cov}(h_n,g_{kn})\to 0$, for $j=1,\ldots,p$ and $k=1,\ldots,q$. Further, similar to the case of simpler models such as SARAR(1,0) and SARAR(1,1) (Liu and Yang, 2015), satisfaction of these two conditions alone may not be sufficient to attain consistency. Additional restrictions on the relationship between the elements of $\bar{b}_{jn}$ and g_{kn} are needed. Thus, instead of spending effort to give a set of lengthy and potentially infeasible sufficient conditions, we endeavour to introduce a new set of estimators that are fully robust against unknown heteroskedasticity and nonnormality.

3. Heteroskedasticity Robust Estimation of SARAR(p,q) Model

In this section, we introduce an adjusted quasi score (AQS) method, and an adjusted concentrated quasi score (ACQS) method for estimating the parameters of the SARAR(p,q) model to give estimators that are fully robust against unknown heteroskedasticity (UH) and nonnormality (NN).

3.1. The AQS estimation

The AQS method simply modifies the full quasi score function given in (2.8) or (2.9), so that the resulted AQS function is zero in expected value at the true parameters θ_0. This is achieved as follows.

$$S_n^\circ(\theta) = \begin{cases} \frac{1}{\sigma^2}X_n'B_n'(\rho)\varepsilon_n(\beta,\delta), \\ \frac{1}{2\sigma^4}\varepsilon_n'(\beta,\delta)\varepsilon_n(\beta,\delta) - \frac{n}{2\sigma^2}, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)\bar{F}_{jn}^\circ(\delta)Y_n(\delta),\ j=1,\cdots,p, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)G_{kn}^\circ(\rho)\varepsilon_n(\beta,\delta),\ k=1,\cdots,q, \end{cases} \tag{3.1}$$

where $\bar{F}_{jn}^\circ(\delta) = \bar{F}_{jn}(\delta) - \mathrm{diag}(\bar{F}_{jn}(\delta))$ and $G_{kn}^\circ(\rho) = G_{kn}(\rho) - \mathrm{diag}(G_{kn}(\rho))$. It is easy to verify that, under UH, $\mathrm{E}[S_n^\circ(\theta_0)] = 0$ and $\frac{1}{n}S_n^\circ(\theta_0) \xrightarrow{p} 0$. Solving the estimating equation $S_n^\circ(\theta) = 0$ leads to NNUH-robust estimator of θ_0 for the full model, i.e.,

$$\hat{\theta}_n^\circ = \arg\{S_n^\circ(\theta) = 0\}. \tag{3.2}$$

Clearly, the AQS estimator $\hat{\theta}_n^\circ$ is an M-estimator (Huber, 1964; van der Vaart, 1998). It is also a method of moment (MM) estimator, and hence computationally it is no more demanding than the GMM and GEL types of estimators. The same applies to the estimators given latter.

The root-finding process can be simplified by first solving the first two sets of equations for β and σ^2 for a given δ, resulting in the constrained AQS estimators $\hat{\beta}_n(\delta)$ and $\hat{\sigma}_n^2(\delta)$ with the same expressions as the constrained QMLEs given in (2.3). Then, δ is estimated by solving the last two sets of equations after plugging in $\hat{\beta}_n(\delta)$ and $\hat{\sigma}_n^2(\delta)$. The consistency of $\hat{\theta}_n^{\circ}$ can be established by verifying the conditions of Theorem 5.9 of van der Vaart (1998), under Assumptions 1, 2*, 3-5, and the following modified identification condition.

Assumption 6*: *Either (a):* $\lim_{n\to\infty} \mathcal{H}_n(\rho)$ *is non-singular* $\forall \rho$ *and* $\lim_{n\to\infty} \mathcal{Q}_{1n}(\rho) \neq 0$ *for* $\rho \neq \rho_0$*; or (b):* $\lim_{n\to\infty} \mathcal{Q}_{2n}(\delta) \neq 0$ *for* $\delta \neq \delta_0$*, where*
$\mathcal{H}_n(\rho) \ = \frac{1}{n}(X_n, F_{1n}X_n\beta_0, \ldots, F_{pn}X_n\beta_0)' B_n'(\rho) B_n(\rho)(X_n, F_{1n}X_n\beta_0, \ldots, F_{pn}X_n\beta_0)$,
$\mathcal{Q}_{1n}(\rho) = \frac{1}{n}\left(\ln|\sigma_0^2 B_n'^{-1} B_n^{-1}| - \ln|\sigma_{1n}^2(\rho) B_n'^{-1}(\rho) B_n^{-1}(\rho)|\right)$,
$\mathcal{Q}_{2n}(\delta) = \frac{1}{n}\left(\ln|\sigma_0^2 B_n'^{-1} A_n'^{-1} A_n^{-1} B_n^{-1}| - \ln|\sigma_{2n}^2(\delta) B_n'^{-1}(\rho) A_n'^{-1}(\lambda) A_n^{-1}(\lambda) B_n^{-1}(\rho)|\right)$,
$\sigma_{1n}^2(\rho) \ = \frac{\sigma_0^2}{n}\mathrm{tr}[H_n B_n'^{-1} B_n'(\rho) B_n(\rho) B_n^{-1}]$, and
$\sigma_{2n}^2(\delta) \ = \frac{\sigma_0^2}{n}\mathrm{tr}[H_n B_n'^{-1} A_n'^{-1} A_n'(\lambda) B_n'(\rho) B_n(\rho) A_n(\lambda) A_n^{-1} B_n^{-1}]$.

The asymptotic normality of $\hat{\theta}_n^{\circ}$ can be established in a manner similar to that of the usual QMLEs, starting with a Taylor expansion of $S_n^{\circ}(\hat{\theta}_n^{\circ})$ around θ_0. We have the following theorem.

Theorem 3.1. *Under Assumptions 1, 2*, 3-5, and 6*, we have as* $n \to \infty$, $\hat{\theta}_n^{\circ} \xrightarrow{p} 0$, *and*

$$\sqrt{n}(\hat{\theta}_n^{\circ} - \theta_0) \xrightarrow{D} N(0,\ \lim_{n\to\infty} \Sigma_n^{\circ-1} \Omega_n^{\circ} \Sigma_n^{\circ-1}),$$

where $\Sigma_n^{\circ} = -\frac{1}{n}\mathrm{E}[\frac{\partial}{\partial\theta'} S_n^{\circ}(\theta_0)]$ *and* $\Omega_n^{\circ} = \frac{1}{n}\mathrm{E}[S_n^{\circ}(\theta_0) S_n^{\circ,\prime}(\theta_0)]$*, with the limit of the former being a positive definite matrix, and that of the latter simply a constant matrix.*

Practical applications of the proposed method requires the estimation of the VC matrix of the AQS estimator $\hat{\theta}_n^{\circ}$. Although the analytical expressions for Σ_n° and Ω_n° can easily be obtained, they may not lead to feasible plug-in estimators, unlike Σ_n and Ω_n of the homoskedastic model. As errors are allowed to be heteroskedastic in an unknown form, Σ_n° involves $h_{n,i}$ and Ω_n° involves $h_{n,i}$, $\gamma_{n,i}$, and $\kappa_{n,i}$, being the measures of heteroskedasticity, skewness and excess kurtosis of $\varepsilon_{n,i}$, respectively. In this case, although Σ_n° can still be estimated by the plug-in method (i.e., plugging-in the standardised AQS residuals $\hat{\varepsilon}_n^2/\hat{\sigma}_n^2$ for h_n in addition to plugging-in $\hat{\theta}_n$ for θ_0) or by its sample analogue,

$$\widehat{\Sigma}_n^{\circ} = -\frac{1}{n}\frac{\partial}{\partial\theta'} S_n^{\circ}(\hat{\theta}_n), \tag{3.3}$$

the plug-in method cannot be applied for the estimation of Ω_n° due to the involvements of the higher-order moments $\{\gamma_{n,i}\}$ and $\{\kappa_{n,i}\}$ which vary with i. To overcome this difficulty, we propose an *outer-product-of-martingale-difference* (OPMD) method for estimating Ω_n° similar in idea to Baltagi and Yang (2013) in hypothesis testing.

We start by decomposing a $n \times n$ non-stochastic matrix A_n, into upper triangular, lower triangular and diagonal matrices, so that $A_n = A_n^u + A_n^l + A_n^d$. Hence, any centred quadratic form in ε_n, $\varepsilon_n' A_n \varepsilon_n - \mathrm{E}(\varepsilon_n' A_n \varepsilon_n)$, can be written as

$$\begin{aligned}
\varepsilon_n' A_n \varepsilon_n - \mathrm{E}(\varepsilon_n' A_n \varepsilon_n) &= \varepsilon_n'(A_n^u + A_n^l + A_n^d)\varepsilon_n - \sigma_0^2 \mathrm{tr}(A_n) \\
&= \varepsilon_n'(A_n^{u\prime} + A_n^l)\varepsilon_n + \varepsilon_n' A_n^d \varepsilon_n - \sigma_0^2 \mathrm{tr}(A_n) \\
&= \varepsilon_n' \xi_n + \varepsilon_n' A_n^d \varepsilon_n - \sigma_0^2 \mathrm{tr}(A_n) \\
&= \textstyle\sum_{i=1}^n (\varepsilon_{n,i}\xi_{n,i} + (\varepsilon_{n,i}^2 - \sigma_0^2)a_{n,ii}) \equiv \sum_{i=1}^n s_{n,i},
\end{aligned}$$

where $\xi_n = (A_n^{u\prime} + A_n^l)\varepsilon_n$ and $a_{n,ii}$ are the diagonal elements of A_n. Clearly, $\{s_{n,i}\}$ form a martingale difference (MD) sequence with respect to the increasing σ-field $\mathcal{F}_{n,i}$ generated by $(\varepsilon_{n,1}, \ldots, \varepsilon_{n,i})$, and hence are uncorrelated. It follows that $\mathrm{Var}(\varepsilon_n' A_n \varepsilon_n) = \sum_{i=1}^n \mathrm{E}(s_{n,i}^2)$, and a consistent estimator of it would be $\sum_{i=1}^n \hat{s}_{n,i}^2$ in the sense that $\frac{1}{n}(\sum_{i=1}^n \hat{s}_{n,i}^2 - \mathrm{Var}(\varepsilon_n' A_n \varepsilon_n)) \xrightarrow{p} 0$ as $n \to 0$, where $\hat{s}_{n,i}$ is obtained from $s_{n,i}^2$ by replacing the parameters by their consistent estimates, and ε by the corresponding AQS residuals.

With this general idea, the AQS function (3.1) can be written as,

$$S_n^\circ(\theta_0) = \sum_{i=1}^n \mathbf{s}_{ni}^\circ(\theta_0), \quad \text{where,} \quad \mathbf{s}_{ni}^\circ(\theta_0) = \begin{cases} \frac{1}{\sigma_0^2} x_{bi}\varepsilon_{ni}, \\ \frac{1}{2\sigma_0^4}(\varepsilon_{ni}^2 - \sigma_0^2), \\ \frac{1}{\sigma_0^2}(\varepsilon_{ni}\xi_{jn,i}^\circ + \eta_{jn,i}^\circ \varepsilon_{ni}), \ j = 1, \cdots, p, \\ \frac{1}{\sigma_0^2}\varepsilon_{ni}\zeta_{kn,i}^\circ, \ k = 1, \cdots, q, \end{cases} \tag{3.4}$$

where x_{bi} is the ith column of $X_n' B_n'$, $\xi_{jn}^\circ = (\bar{F}_{jn}^{\circ u\prime} + \bar{F}_{jn}^{\circ l})\varepsilon_n$, $\zeta_{kn}^\circ = (G_{kn}^{\circ u\prime} + G_{kn}^{\circ l})\varepsilon_n$, and $\eta_{jn}^\circ = \bar{F}_{jn}^\circ B_n X_n \beta_0$. It is easy to see that $\{\mathbf{s}_{ni}^\circ(\theta_0)\}$ form a vector MD array with respect to $\mathcal{F}_{n,i}$, and hence $\widehat{\Omega}_n^\circ = \frac{1}{n}\sum_{i=1}^n \hat{\mathbf{s}}_{n,i}\hat{\mathbf{s}}_{n,i}'$ gives a consistent estimator of Ω_n°, where $\hat{\mathbf{s}}_{n,i}$ is a plug-in estimate of $\mathbf{s}_{ni}^\circ(\theta_0)$ by replacing θ_0 by $\hat{\theta}_n^\circ$ and ε_{ni} by its AQS estimate $\hat{\varepsilon}_{ni}^\circ$. We give the following results.

Corollary 3.1. *Under the assumptions of Theorem 3.1, we have, as* $n \to \infty$,

$$\widehat{\Sigma}_n^\circ - \Sigma_n^\circ \xrightarrow{p} 0 \ \text{ and } \ \widehat{\Omega}_n^\circ - \Omega_n^\circ \xrightarrow{p} 0.$$

Further, if the limit of Σ_n° *is positive definite, then,* $\widehat{\Sigma}_n^{\circ -1}\widehat{\Omega}_n^\circ \widehat{\Sigma}_n^{\circ -1} - \Sigma_n^{\circ -1}\Omega_n^\circ \Sigma_n^{\circ -1} \xrightarrow{p} 0$.

To facilitate practical applications, the Hessian of the AQS function is given below:

$$n\Sigma_n^\circ = \begin{pmatrix} \{\frac{1}{\sigma_0^2} X_n' B_n' B_n X_n\}, & 0, & \{\frac{1}{\sigma_0^2} X_n' B_n' \eta_{j'n}\}, & 0 \\ 0, & \frac{n}{2\sigma_0^2}, & \{\frac{1}{\sigma_0^2}\mathrm{tr}(H_n \bar{F}_{j'n})\}, & \{\frac{1}{\sigma_0^2}\mathrm{tr}(H_n G_{k'n})\} \\ \{\frac{1}{\sigma_0^2}\eta_{jn}^{\circ\prime} B_n X_n\}, & \{\frac{1}{\sigma_0^2}\mathrm{tr}(H_n \bar{F}_{jn}^\circ)\}, & \{\frac{1}{\sigma_0^2}\eta_{j'n}'\eta_{jn}^\circ + \mathrm{tr}(H_n T_{1n})\}, & \{\mathrm{tr}(H_n T_{2n})\} \\ 0, & \{\frac{1}{\sigma_0^2}\mathrm{tr}(H_n G_{kn}^\circ)\}, & \{\mathrm{tr}(H_n T_{3n})\}, & \{\mathrm{tr}(H_n T_{4n})\} \end{pmatrix},$$

for $j, j' = 1, \ldots, p$ and $k, k' = 1, \ldots, q$, where, $\eta_{jn}^\circ = \bar{F}_{jn}^\circ B_n X_n \beta_0$, $\eta_{jn} = \bar{F}_{jn} B_n X_n \beta_0$,
$T_{1n} = \bar{F}_{j'n}' \bar{F}_{jn}^\circ + \bar{F}_{jn}^\circ \bar{F}_{j'n} - \dot{\bar{F}}_{jj'n}^\circ$,
$T_{2n} = G_{k'n}' \bar{F}_{jn}^\circ + \bar{F}_{jn}^\circ G_{k'n} - \dot{\bar{F}}_{jkn}^\circ$,

$T_{3n} = \bar{F}'_{j'n} G^{\circ}_{kn} + G^{\circ}_{kn} \bar{F}_{j'n}$,
$T_{4n} = G'_{k'n} G^{\circ}_{kn} u_n + G^{\circ}_{kn} G_{k'n} - \dot{G}^{\circ}_{kk'n}$,
$\dot{\bar{F}}^{\circ}_{jj'n} = \frac{\partial}{\partial \lambda'} \bar{F}^{\circ}_{jn} = \bar{F}_{jn} \bar{F}_{j'n} - \text{diag}(\bar{F}_{jn} \bar{F}_{j'n})$,
$\dot{\bar{F}}^{\circ}_{jkn} = \frac{\partial}{\partial \rho'} \bar{F}^{\circ}_{jn} = \bar{F}_{jn} G_{k'n} - G_{k'n} \bar{F}_{jn} - \text{diag}(\bar{F}_{jn} G_{k'n} - G_{k'n} \bar{F}_{jn})$ and
$\dot{G}^{\circ}_{kk'n} = \frac{\partial}{\partial \rho'} \dot{G}^{\circ}_{kn} = G_{kn} G_{k'n} - \text{diag}(G_{kn} G_{k'n})$.

3.2. The ACQS estimation

As discussed in the introduction, the AQS estimator may not have good finite sample performance, since it does not take into account the variations stemming from the estimation of the linear and scale parameters in the model. To enhance the finite sample performance of the estimator, we introduce the ACQS method that modifies concentrated quasi score function for the spatial parameters. The basic methodology is inspired by the approach taken by Liu and Yang (2015) and is extended to suit the more general model considered in this paper. The concentrated score function take into account the variability caused by the estimation of β and σ^2 parameters, and thus can be expected to have a better finite sample performance. The Monte Carlo results confirm this. For robust inferences concerning the spatial or regression parameters, we again introduce an OPMD estimator of the VC matrix of the ACQS, to give robust estimates of the standard errors.

Consider the concentrated score function of δ, $S^c_n(\delta) = \frac{\partial}{\partial \delta} \ell^c_n(\delta)$, or equivalently the δ-components of $S_n(\theta)$ with β and σ^2 being replaced by $\hat{\beta}_n(\delta)$ and $\hat{\sigma}^2_n(\delta)$,

$$S^c_n(\delta) = \begin{cases} \dfrac{Y'_n(\delta) M_n(\rho) [\bar{F}_{jn}(\delta) - \frac{1}{n} \text{tr}(F_{jn}(\lambda) I_n)] Y_n(\delta)}{Y'_n(\delta) M_n(\rho) Y_n(\delta)}, & j = 1, \ldots, p, \\ \dfrac{Y'_n(\delta) M_n(\rho) [\bar{G}_{kn}(\rho) - \frac{1}{n} \text{tr}(G_{kn}(\rho) I_n)] Y_n(\delta)}{Y'_n(\lambda) M_n(\rho) Y_n(\delta)}, & k = 1, \ldots, q, \end{cases} \tag{3.5}$$

where $\bar{F}_{jn}$ and $M_n(\rho)$ are as defined in the last section, and $\bar{G}_{kn}(\rho) = G_{kn}(\rho) M_n(\rho)$.

Following the ideas given in Lin and Lee (2010) and Liu and Yang (2015), we can modify the numerator of the concentrated score function to attain the desired probability limit of zero at the true parameters under heteroskedasticity. The main problem of this approach lies in the fact that under heteroskedasticity, the expectation of the estimating function $\frac{1}{n} S^c_n(\delta_0)$ does not tend to zero thus rendering the estimator, $\hat{\delta}_n = \arg\{S^c_n(\delta) = 0\}$, inconsistent.[5] Adjusting the score or especially the concentrated score function to have a set of unbiased or nearly-unbiased estimating functions would also lead to an estimator that is less biased in finite samples. The adjustments to the ACQS function could be as simple as replacing $\frac{1}{n} \text{tr}(F_{jn}(\lambda)) I_n$ by $\text{diag}(\bar{F}_{jn}(\delta))$ and $\frac{1}{n} \text{tr}(G_{kn}(\lambda)) I_n$ by $\text{diag}(G_{kn}(\delta))$. These straightforward adjustments, however, may lead to an estimate that is still poor in finite samples. Better refined choices would be the pair $\text{diag}(M_n \bar{F}_{jn}(\delta))$ and $\text{diag}(M_n G_{kn}(\delta))$, but these choices miss out the effect

[5]Even under homoskedasticity, the expectation of $\frac{1}{n} S^c_n(\delta_0)$ is not zero although it tends to zero, and as a result the QMLE $\hat{\delta}_n$ would be biased in finite samples.

of an extra M_n term as, e.g., $\mathrm{E}(Y_n' M_n \bar{F}_{jn} Y_n) = \sigma_0^2 \mathrm{tr}(H_n M_n \bar{F}_{jn}) = \sigma_0^2 \mathrm{tr}(H_n \mathrm{diag}(M_n \bar{F}_{jn}))$, but $\mathrm{E}(Y_n' M_n \mathrm{diag}(M_n \bar{F}_{jn}) Y_n) = \sigma_0^2 \mathrm{tr}(H_n M_n \mathrm{diag}(M_n \bar{F}_{jn})$. Motivated by these arguments, the final choices for the adjustments are:

$$\begin{aligned}
\bar{F}_{jn}^{\circ}(\delta) &= \bar{F}_{jn}(\delta) - \mathrm{diag}(M_n(\rho))^{-1}\mathrm{diag}[M_n(\rho)\bar{F}_{jn}(\delta)],\ j = 1, \ldots, p, \\
\bar{G}_{kn}^{\circ}(\rho) &= \bar{G}_{kn}(\delta) - \mathrm{diag}(M_n(\rho))^{-1}\mathrm{diag}[M_n(\rho)\bar{G}_{kn}(\rho)],\ k = 1, \ldots, q.
\end{aligned}$$

The adjusted concentrated quasi score function thus takes the form:

$$S_n^{ac}(\delta) = \begin{cases} \dfrac{Y_n'(\delta) M_n(\rho) \bar{F}_{jn}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)}, & j = 1, \ldots, p, \\[2ex] \dfrac{Y_n'(\delta) M_n(\rho) \bar{G}_{kn}^{\circ}(\rho) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)}, & k = 1, \ldots, q, \end{cases} \tag{3.6}$$

and the ACQS estimator of δ_0 is defined to be,

$$\tilde{\delta}_n = \arg\{S_n^{ac}(\delta) = 0\}. \tag{3.7}$$

In a manner similar to the AQS estimator, we can show that the ACQS estimator $\tilde{\delta}_n$ is consistent and asymptotically normal in the presence of unknown heteroskedasticity. Some important remarks are in order.

The adjustment suggested here is in line with the ideas of Lin and Lee (2010)'s modification to the moment conditions of the SARAR(1,0) model so that the resulting GMM estimator is robust. They suggest to restrict a broad class of quadratic moments of the form $\varepsilon_n' P_n \varepsilon_n$ with $\mathrm{tr}(P_n)$ being zero to a smaller class with $\mathrm{diag}(P_n)$ being zero, so that the moment condition $\mathrm{E}(\varepsilon_n' P_n \varepsilon_n) = 0$ even under unknown heteroskedasticity. However, it is not clear how we can extend these ideas to construct a GMM estimator for a spatial model with spatial error terms, e.g., the SARAR(p, q) model. In contrast, it is much easier to apply these ideas to a likelihood-based estimation framework.

The adjustments leading to (3.6) are in line with the heteroskedasticity robust LM tests of Baltagi and Yang (2013) with finite sample corrections. Similar ideas can be applied to construct heteroskedasticity robust LM-type tests for spatial effects in the SARAR(p, q) model. This issue is formally treated in the next section.

Due to the non-linear manner in which spatial parameters enter the log-likelihood function, the resulting QMLEs are biased in finite samples. An estimator derived from an 'unbiased' estimating function is potentially bias corrected. In this sense, the proposed estimator is not only robust against heteroskedasticity and non-normality, but also performs well in finite samples. Thus combined with a robust estimator for the standard errors we have an improved basis for inference related matters as illustrated in Section 4.

In order to establish the consistency of the ACQS estimator $\tilde{\delta}_n$, we need to modify the identification condition given in Assumption 6 to suit the new model as follows.

Assumption 6****:** $\lim_{n\to\infty} \mathcal{R}_{1n}(\delta) \neq 0$ *and* $\lim_{n\to\infty} \mathcal{R}_{2n}(\delta) \neq 0$, $\forall \delta \neq \delta_0$, *where*

$\mathcal{R}_{1n}(\delta) = \frac{1}{n}\left[\beta_0' X_n' B_n'^{-1} A_n'^{-1} \Psi_{jn}(\delta) A_n^{-1} B_n^{-1} X_n \beta_0 + \sigma_0^2 \text{tr}\left(H_n B_n'^{-1} A_n'^{-1} \Psi_{jn}(\delta) A_n^{-1} B_n'^{-1}\right)\right]$,
$\mathcal{R}_{2n}(\delta) = \frac{\sigma_0^2}{n} \text{tr}\left(H_n B_n'^{-1} A_n'^{-1} \Omega_{kn}(\delta) A_n^{-1} B_n'^{-1}\right)$,
$\Psi_{jn}(\delta) = A_n'(\lambda) B_n'(\rho)[\bar{F}_{jn}(\delta) - \text{diag}(\bar{F}_{jn}(\delta))] B_n(\rho) A_n(\lambda),\ j = 1, \ldots, p$
$\Psi_{kn}(\delta) = A_n'(\lambda) B_n'(\rho)[\bar{G}_{kn}(\rho) - \text{diag}(\bar{G}_{kn}(\rho))] B_n(\rho) A_n(\lambda),\ k = 1, \ldots, q.$

The normalised ACQS function evaluated at δ_0 takes the form,

$$\sqrt{n} S_n^{ac} = \begin{cases} \frac{1}{\sqrt{n}\sigma_0^2}\left(\varepsilon_n' P_{jn} \varepsilon_n + c_{jn}' \varepsilon_n\right) + o_p(1), & j = 1, \ldots, p \\ \frac{1}{\sqrt{n}\sigma_0^2}\left(\varepsilon_n' Q_{kn} \varepsilon_n + c_{kn}' \varepsilon_n\right) + o_p(1), & k = 1, \ldots, q \end{cases} \tag{3.8}$$

where $P_{jn} = M_n \bar{F}_{jn}^{\circ}$, $Q_{kn} = M_n \bar{G}_{kn}^{\circ} M_n$, $c_{jn} = M_n \bar{F}_{jn}^{\circ} B_n X_n \beta_0$ and $c_{kn} = M_n \bar{G}_{kn}^{\circ} B_n X_n \beta_0$ because $\hat{\sigma}_n^{-2}(\delta_0) = \sigma_0^{-2} + o_p(1)$. Let $\tau_n^2(S_n^{ac}) = \text{Var}(\sqrt{n} S_n^{ac})$. Using the CLT for LQ forms of Kelejian and Prucha (2001), we give the following theorem.

Theorem 3.2. *Under Assumptions 1, 2^*, 3-5 and 6^{**}, the ACQS estimator $\tilde{\delta}_n$ is consistent and asymptotically normal, i.e., as $n \to \infty$, $\tilde{\delta}_n \xrightarrow{p} \delta_0$, and*

$$\sqrt{n}(\tilde{\delta}_n - \delta_0) \xrightarrow{D} N\left(0, \lim_{n\to\infty} \tau_n^2(\tilde{\delta}_n)\right),$$

where $\tau_n^2(\tilde{\delta}_n) = \Phi_n^{-1} \tau_n^2(S_n^{ac}) \Phi_n^{-1}$, where $\Phi_n = -E(\frac{\partial}{\partial \delta'} S_n^{ac}(\delta_0))$ or it's first order term.

To facilitate the practical applications, the detailed expression for Φ_n is given below:

$$\begin{aligned}
\Phi_{n,11} &= \text{tr}[H_n(\bar{F}_{jn}^{\circ} \bar{F}_{j'n} + \bar{F}_{j'n}' \bar{F}_{jn}^{\circ} - \dot{\bar{F}}_{jj',n}^{\circ})] + \tfrac{1}{n\sigma_0^2} c_{jn}' \eta_{j'n}, \\
\Phi_{n,12} &= \text{tr}[H_n(\bar{F}_{jn}^{\circ} G_{k'n} + \bar{G}_{k'n}' \bar{F}_{jn}^{\circ} - G_{k'n} \bar{F}_{jn}^{\circ} + \bar{G}_{k'n} \bar{F}_{jn}^{\circ} - \dot{\bar{F}}_{jk,n}^{\circ})] + \tfrac{1}{n\sigma_0^2} c_{jn}' \eta_{k'n}, \\
\Phi_{n,21} &= \text{tr}[H_n(\bar{G}_{kn}^{\circ} \bar{F}_{j'n} + \bar{F}_{j'n}' \bar{G}_{kn}^{\circ})] + \tfrac{1}{n\sigma_0^2} c_{kn}' \eta_{j'n}, \text{ and} \\
\Phi_{n,22} &= \text{tr}[H_n(\bar{G}_{kn}^{\circ} G_{k'n} + \bar{G}_{k'n}' \bar{G}_{kn}^{\circ} - G_{k'n} \bar{G}_{kn}^{\circ} + \bar{G}_{k'n} \bar{G}_{kn}^{\circ} - \dot{\bar{G}}_{kk',n}^{\circ})] + \tfrac{1}{n\sigma_0^2} c_{kn}' \eta_{k'n},
\end{aligned}$$

for $j, j' = 1, \ldots, p$ and $k, k' = 1, \ldots, q$, where, $\eta_{jn} = B_n F_{jn} X_n \beta_0$, $\eta_{kn} = \bar{G}_{kn} B_n X_n \beta_0$,
$\dot{\bar{F}}_{jj'n}^{\circ} = \frac{\partial}{\partial \lambda'} \bar{F}_{jn}^{\circ} = \bar{F}_{jn} \bar{F}_{j'n} - \text{diag}(M_n)^{-1} \text{diag}(M_n \bar{F}_{jn} \bar{F}_{j'n})$,
$\dot{\bar{F}}_{jkn}^{\circ} = \frac{\partial}{\partial \rho'} \bar{F}_{jn}^{\circ} = -G_{k'n} \bar{F}_{jn} - \bar{F}_{jn} G_{k'n} - \text{diag}(M_n^{-1} \dot{M}_{k'n} M_n^{-1}) \text{diag}(M_n \bar{F}_{jn})$
$\quad + \text{diag}(M_n)^{-1} \text{diag}(M_n G_{k'n} \bar{F}_{jn} + M_n \bar{F}_{jn} G_{k'n} - \dot{M}_{k'n} \bar{F}_{jn})$,
$\dot{\bar{G}}_{kk'n}^{\circ} = \frac{\partial}{\partial \rho'} \bar{G}_{kn}^{\circ} = G_{kn} \bar{G}_{k'n} + G_{kn} \dot{M}_{k'n} + \text{diag}(M_n^{-1} \dot{M}_{k'n} M_n^{-1}) \text{diag}(M_n \bar{G}_{kn})$
$\quad - \text{diag}(M_n)^{-1} \text{diag}(M_n G_{kn} \bar{G}_{k'n} + M_n G_{kn} \dot{M}_{k'n} + \dot{M}_{k'n} \bar{G}_{kn})$,
$\dot{M}_{kn} = \frac{\partial}{\partial \rho} M_n = M_n G_{kn} P_n + P_n G_{kn}' M_n$ and $P_n = I_n - M_n$.

Given $\tilde{\delta}_n$, the ACQS estimator for β is defined as,

$$\tilde{\beta}_n \equiv \hat{\beta}_n(\tilde{\delta}_n) = (X_n' \tilde{B}_n' \tilde{B}_n X_n)^{-1} X_n' \tilde{B}_n' \tilde{B}_n \tilde{A}_n Y_n,$$

where $\tilde{A}_n = A_n(\tilde{\lambda}_n)$ and $\tilde{B}_n = B_n(\tilde{\rho}_n)$. A Taylor expansion of $\hat{\beta}_n(\delta)$ around $\delta = \delta_0$ gives, $\hat{\beta}_n(\delta) = \hat{\beta}_n(\delta_0) + \dot{\beta}_n(\delta_0)(\delta - \delta_0) + O_p(\frac{1}{n})$, where $\dot{\beta}(\delta_0) = \frac{\partial}{\partial \delta'} \hat{\beta}_n(\delta_0)$. This can be used to derive the asymptotic distribution of $\tilde{\beta}_n$.

Theorem 3.3. *Under Assumptions 1, 2*, 3-5 and 6**, we have, as* $n \to \infty$, $\tilde{\beta}_n \xrightarrow{p} \beta_0$, $\tilde{\sigma}_n^2 \xrightarrow{p} \sigma_0^2$, *and*

$$\sqrt{n}(\tilde{\beta}_n - \beta_0) \xrightarrow{D} N\left(0, \lim_{n\to\infty} (X_n' B_n' B_n X_n)^{-1} X_n' B_n' \mathbb{A}_n B_n X_n (X_n' B_n' B_n X_n)^{-1}\right),$$

where,

$$\mathbb{A}_n = n\sigma_0^2 H_n + \eta_{jn}\tau_{jn}^2(\tilde{\lambda}_n)\eta_{jn}' + 2\sqrt{n}\begin{pmatrix} \sigma_0^{-2}P_{jn}^d s_n + H_n c_{jn} \\ \sigma_0^{-2}Q_{kn}^d s_n + H_n c_{kn}\end{pmatrix}' \Phi^{-1}\begin{pmatrix}\eta_{jn} \\ 0_n\end{pmatrix},$$

$\tau_{jn}^2(\tilde{\delta}_n)$ *is the VC matrix block corresponding to* λ_j, $s_n = \mathrm{E}(\varepsilon_n^3)$, $P_{jn}^d = \mathrm{diag}(P_{jn})$, $Q_{kn}^d = \mathrm{diag}(Q_{kn})$ *for* $j = 1, \ldots, p$ *and* $k = 1, \ldots, q$ *and* 0_n *is an* $n \times 1$ *vector of zeros.*[6]

Robust standard errors. It remains, for practical applications, to provide feasible methods for estimating the standard errors of the ACQS estimators. It is possible to use the OPMD methodology to estimate $\tau_n^2(S_n^{ac})$ as given in of Baltagi and Yang (2013). To that effect write the numerator of (3.8) as,

$$R_n(\varepsilon_n) = \begin{cases} \varepsilon_n' P_{jn}\varepsilon_n + c_{jn}'\varepsilon_n, & j = 1, \ldots, p \\ \varepsilon_n' Q_{kn}\varepsilon_n + c_{kn}'\varepsilon_n, & k = 1, \ldots, q \end{cases} \tag{3.9}$$

where both P_{jn} and Q_{kn} can be decomposed as a sum of upper triangular, lower triangular and diagonal matrices. Then we can estimate the variance of the score as,

$$\tilde{\tau}_n^2(S_n^{ac}) = \textstyle\sum_{i=1}^n \varepsilon_{ni}^2 \Upsilon_{ni}\Upsilon_{ni}', \tag{3.10}$$

where, $\Upsilon_{ni} = (\{\tilde{\zeta}_{jn,i} + \tilde{p}_{jn,ii}\tilde{\varepsilon}_{ni} + \tilde{c}_{jn,i}\}_{1\times p}, \{\tilde{\xi}_{kn,i} + \tilde{q}_{kn,ii}\tilde{\varepsilon}_{ni} + \tilde{c}_{kn,i}\}_{1\times q})'$, $\tilde{\zeta}_{jn} = (P_{jn}^{u\prime} + P_{jn}^{l})\varepsilon_n$, $\tilde{\xi}_{jn} = (Q_{kn}^{u\prime} + Q_{kn}^{l})\varepsilon_n$, and $\tilde{p}_{jn,ii}$ and $\tilde{q}_{kn,ii}$ are respectively the diagonal elements of P_{jn} and Q_{kn} for $j = 1, \ldots, p$ and $k = 1, \ldots, q$.

The estimator of $\tau_n^2(\tilde{\delta}_n)$ is defined as,

$$\tilde{\tau}_n^2(\tilde{\delta}_n) = \tilde{\Phi}_n^{-1}\tilde{\tau}_n^2(S_n^{ac})\tilde{\Phi}_n^{-1},$$

where $\tilde{\Phi}_n$ is the plug-in estimator of Φ_n. Further, using estimates $\tilde{\Phi}_n$, $\tilde{\tau}_n^2(S_n^{ac})$, $\tilde{\tau}_n^2(\tilde{\delta}_n)$, $\tilde{s}_n = \tilde{\varepsilon}_n^3$, $\tilde{H}_n = \tilde{\sigma}_n^{-2}\mathrm{diag}(\tilde{\varepsilon}_n^2)$, $\tilde{\mathbb{A}}_n$ and plug-in estimates for other quantities, we have a consistent estimator for $\tau_n^2(\tilde{\beta}_n)$ as,

$$\tilde{\tau}_n^2(\tilde{\beta}_n) = (X_n'\tilde{B}_n'\tilde{B}_n X_n)^{-1} X_n'\tilde{B}_n'\tilde{\mathbb{A}}_n\tilde{B}_n X_n (X_n'\tilde{B}_n'\tilde{B}_n X_n)^{-1}.$$

We give the following corollary.

Corollary 3.2. *If Assumptions 1, 2*, 3-5 and 6** hold, then we have as* $n \to \infty$, $\tilde{\tau}_n^2(\tilde{\delta}_n) - \tau_n^2(\tilde{\delta}_n) \xrightarrow{p} 0$ *and* $\tilde{\tau}_n^2(\tilde{\beta}_n) - \tau_n^2(\tilde{\beta}_n) \xrightarrow{p} 0$.

[6] The limiting distribution of $\tilde{\sigma}_n^2$ can be easily derived. However, it is not of use as any inference on $\tilde{\sigma}_n^2$ requires the consistent estimation of $\frac{1}{n}\sum_{i=1}^n \mathrm{Var}(\varepsilon_{n,i}^2) = \frac{\sigma^4}{n}\sum_{i=1}^n(\kappa_{n,i} + h_{n,i}^2)$ which cannot be done.

For the plug-in estimator of $\tilde{\Phi}_n$, one could use $-\frac{\partial}{\partial\delta_0'} S^{ac}(\delta_0)|_{\delta_0=\tilde{\delta}}$, or the estimate of the first order term of $-\mathrm{E}(\frac{\partial}{\partial\delta_0'} S^{ac}(\delta_0))$.

4. LM and Robust LM Tests for Spatial Effects

In this section, we present LM tests for the existence of various spatial effects in the $\texttt{SARAR}(p,q)$ model. Also presented are the LM-type tests robust against nonnormality (NN), and the LM-type tests robust against both nonnormality and unknown heteroskedasticity (UH). The following hypotheses are of primary interest, for $p, q \geq 2$:

(*a*) H_0^a: $\lambda = 0$ and $\rho = 0$, in $\texttt{SARAR}(p,q)$;
(*b*) H_0^b: $\lambda_2 = \cdots = \lambda_p = 0$ and $\rho_2 = \cdots = \rho_q = 0$, in $\texttt{SARAR}(p,q)$;
(*c*) H_0^c: $\lambda = 0$, in $\texttt{SARAR}(p,q)$;
(*d*) H_0^d: $\rho = 0$, in $\texttt{SARAR}(p,q)$;
(*e*) H_0^e: $\lambda = 0$, in $\texttt{SARAR}(p,0)$;
(*f*) H_0^f: $\rho = 0$, in $\texttt{SARAR}(0,q)$.

These tests are all tests of model reduction, i.e, from a larger SARAR model down to a smaller one. Other tests of model reduction may also be of interest, e.g., tests of $\texttt{SARAR}(1,0)$ vs. $\texttt{SARAR}(p,0)$, $\texttt{SARAR}(1,0)$ vs. $\texttt{SARAR}(p,q)$, $\texttt{SARAR}(0,1)$ vs. $\texttt{SARAR}(0,q)$, $\texttt{SARAR}(0,1)$ vs. $\texttt{SARAR}(p,q)$, etc., and can all be handled in a similar manner. The availability of robust, reliable and simple tests would no doubt benefit the applied researchers in choosing an appropriate spatial model.

To save notation, we now use Σ_n and Ω_n to denote the expected information matrix and the VC matrix of the score, which are, respectively, $n\Sigma_n$ and $n\Omega_n$ given by (2.11) and (2.12).

4.1. LM tests

For a parametric model with known error distributions, let α be the parameter vector in the null model, $\theta = (\alpha', \varphi')'$ parameter vector in the full model, and the null hypothesis specifies $\varphi = 0$. The score vector $S_n(\theta)$ has components $S_{n,\alpha}(\alpha,\varphi)$ and $S_{n,\varphi}(\alpha,\varphi)$ corresponding to α and φ, and the information matrix $\Sigma_n(\theta)$ has sub-matrices $\Sigma_{n,\alpha\alpha}(\alpha,\varphi)$, $\Sigma_{n,\varphi\varphi}(\alpha,\varphi)$, $\Sigma_{n,\alpha\varphi}(\alpha,\varphi)$, and $\Sigma_{n,\varphi\alpha}(\alpha,\varphi)$. Then, **the LM test** of $H_0 : \varphi = 0$ takes the form:

$$\mathrm{LM}_n = \tilde{S}_{n,\varphi}'(\tilde{\Sigma}_n^{-1})_{\varphi\varphi}\tilde{S}_{n,\varphi}, \tag{4.1}$$

with its limiting null distribution being $\chi^2_{\dim(\varphi)}$, where $\tilde{S}_{n,\varphi} = S_{n,\varphi}(\tilde{\alpha}_n, 0)$, $\tilde{\Sigma}_{n,\varphi\varphi} = \Sigma_n(\tilde{\alpha}_n, 0)$, $\tilde{\alpha}_n$ is the null estimate of α, and $(\,\cdot\,)_{\varphi\varphi}$ denotes the φ-φ block of the corresponding matrix.

The validity of the general principle given in (4.1) is due to the information matrix equality (IME), which can be seen more clearly from the derivation of the NN-robust LM tests given in the subsequent subsection. With this general principle, the LM tests concerning the spatial effects δ in the $\texttt{SARAR}(p,q)$ model with iid normal errors can easily be derived, based on the score function given in (2.8) and information matrix given in (2.11). First, we give a general form of the LM test for testing the hypotheses of (a)-(d):

$$\mathrm{LM}_{\texttt{SARAR}}(\delta) = \tilde{S}_{n,\delta}(\delta)' \begin{pmatrix} \tilde{J}_n(\delta) + K_n^{\ell\ell}(\delta), & K_n^{\ell e}(\delta) \\ K_n^{\ell e\prime}(\delta), & K_n^{ee}(\delta) \end{pmatrix}^{-1} \tilde{S}_{n,\delta}(\delta), \tag{4.2}$$

where $\tilde{J}_n(\delta) = \frac{1}{\tilde{\sigma}_n^2(\delta)}\{\tilde{\eta}_{jn}'(\delta)M_n(\rho)\tilde{\eta}_{j'n}(\delta)\}_{p\times p}$, $\tilde{\eta}_{jn}(\delta) = \eta_{jn}(\tilde{\beta}_n(\delta), \delta)$, $j = 1, \cdots, p$, and

$$\begin{aligned} K_n^{\ell\ell}(\delta) &= \{\mathrm{tr}(\bar{F}_{jn}\bar{F}_{j'n}^s) - 2\mathrm{tr}(\bar{F}_{jn})\mathrm{tr}(\bar{F}_{j'n})\}_{p\times p}, \\ K_n^{\ell e}(\delta) &= \{\mathrm{tr}(\bar{F}_{jn}G_{j'n}^s) - 2\mathrm{tr}(\bar{F}_{jn})\mathrm{tr}(G_{j'n})\}_{p\times q}, \\ K_n^{ee}(\delta) &= \{\mathrm{tr}(G_{jn}G_{j'n}^s) - 2\mathrm{tr}(G_{jn})\mathrm{tr}(G_{j'n})\}_{q\times q}. \end{aligned}$$

With the general expression (4.2), the LM test statistics for (a)-(d) are obtained by setting $\delta = 0$ for (a), $\delta = (\tilde{\lambda}_{1n}, 0_{p-1}', \tilde{\rho}_{1n}, 0_{q-1}')'$ for (b), $\delta = (0_p', \tilde{\rho}_n')$ for (c), and $\delta = (\tilde{\lambda}_n', 0_q')$ for (d), where the *tilded* parameters are the constrained QMLEs of the corresponding parameters under the respective null hypothesis. For easy reference, these LM tests are denoted by $\mathrm{LM}_{\texttt{SARAR}}^{(a)}$, $\mathrm{LM}_{\texttt{SARAR}}^{(b)}$, $\mathrm{LM}_{\texttt{SARAR}}^{(c)}$, and $\mathrm{LM}_{\texttt{SARAR}}^{(d)}$, respectively.

Of particular interest is $\mathrm{LM}_{\texttt{SARAR}}^{(a)}$ for testing H_0^a: $\lambda = 0$ and $\rho = 0$, which takes the form:

$$\mathrm{LM}_{\texttt{SARAR}}^{(a)} = \frac{1}{\tilde{\sigma}_n^4} \begin{pmatrix} \tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n \\ \tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n \end{pmatrix}' \begin{pmatrix} \tilde{J}_n + K_n^{\ell\ell}, & K_n^{\ell e} \\ K_n^{\ell e\prime}, & K_n^{ee} \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n \\ \tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n \end{pmatrix}, \tag{4.3}$$

where $\tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n$ denotes $(\tilde{\varepsilon}_n'W_{\ell 1}Y_n, \cdots, \tilde{\varepsilon}_n'W_{\ell p}Y_n)'$, $\tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n$ denotes $(\tilde{\varepsilon}_n'W_{e1}\tilde{\varepsilon}_n, \cdots, \tilde{\varepsilon}_n'W_{eq}\tilde{\varepsilon}_n)'$, $\tilde{\varepsilon}_n$, $\tilde{\beta}_n$ and $\tilde{\sigma}_n^2$ are from the simple OLS regression of Y_n on X_n, $\tilde{J}_n = \{\frac{1}{\tilde{\sigma}_n^2}\tilde{\eta}_{jn}'M_n^\circ\tilde{\eta}_{j'n}\}$, $\tilde{\eta}_{jn} = W_{\ell j}X_n\tilde{\beta}_n$, $K_n^{\ell\ell} = \{\mathrm{tr}(W_{\ell j}W_{\ell j'}^s)\}$, $K_n^{\ell e} = \{\mathrm{tr}(W_{\ell j}W_{ej'}^s)\}$, $K_n^{ee} = \{\mathrm{tr}(W_{ej}W_{ej'}^s)\}$, and $M_n^\circ = M_n(0)$. The LM test $\mathrm{LM}_{\texttt{SARAR}}^{(a)}$ can easily be simplified to give LM tests for $H_0^{(e)}$ and H_0^f:

$$\mathrm{LM}_{\texttt{SLD}}^{(e)} = \tilde{\sigma}_n^{-4}(\tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n)'(\tilde{J}_n + K_n^{\ell\ell})^{-1}(\tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n), \tag{4.4}$$

$$\mathrm{LM}_{\texttt{SED}}^{(f)} = \tilde{\sigma}_n^{-4}(\tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n)'(K_n^{ee})^{-1}(\tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n). \tag{4.5}$$

These tests generalize the tests given in Burridge (1980) and Anselin (1988a,b) for $\texttt{SARAR}(1,0)$, $\texttt{SARAR}(0,1)$, and $\texttt{SARAR}(1,1)$, and can be shown to be NN-robust by verifying that the 'variance' in, e.g., (4.3), is (asymptotically) equivalent to $\tilde{\Sigma}_{n,\varphi\varphi} - \tilde{\Sigma}_{n,\varphi\alpha}\tilde{\Sigma}_{n,\alpha\alpha}^{-1}\tilde{\Sigma}_{n,\alpha\varphi}$.

Theorem 4.1. *Under the Assumptions 1-6, we have, as* $n \to \infty$,

$$\mathrm{LM}_{\texttt{SARAR}}^{(m)}|_{H_0} \xrightarrow{D} \chi_{df}^2,$$

for $m = a, e, f$ *if errors are iid; and for* $m = b, c, d$ *if errors are iid normal.*

4.2. NN-robust LM tests

When the errors are iid but not normally distributed, the LM tests for (b), (c), and (d) given above are generally invalid, as the asymptotic variance used for approximating the variance of $S_{n,\varphi}(\tilde{\alpha}_n, 0)$ is invalid. Denote the sub-matrices of $\Omega_n = \text{Var}[S_n(\theta_0)]$ by $\Omega_{n,\alpha\alpha}(\alpha,\varphi)$, $\Omega_{n,\varphi\varphi}(\alpha,\varphi)$ and $\Omega_{n,\alpha\varphi}(\alpha,\varphi)$. Consider the Taylor expansion,

$$\tfrac{1}{\sqrt{n}} S_{n,\varphi}(\tilde{\alpha}_n, 0) = \tfrac{1}{\sqrt{n}} S_{n,\varphi}(\alpha_0, 0) - \tfrac{1}{\sqrt{n}} \Pi_n(\alpha_0) S_{n,\alpha}(\alpha_0, 0) + o_p(1), \tag{4.6}$$

where $\Pi_n(\alpha_0) = \Sigma_{n,\varphi\alpha}(\alpha_0, 0)\Sigma^{-1}_{n,\alpha\alpha}(\alpha_0, 0)$. This gives,

$$\begin{aligned} \text{Var}[\tfrac{1}{\sqrt{n}} S_{n,\varphi}(\tilde{\alpha}_n, 0)] &= \tfrac{1}{n}\big[\Omega_{n,\varphi\varphi}(\alpha_0, 0) - \Omega_{n,\varphi\alpha}(\alpha_0, 0)\Pi'_n(\alpha_0) - \Pi'_n(\alpha_0)\Omega_{n,\alpha\varphi}(\alpha_0, 0) \\ &\quad + \Pi_n(\alpha_0)\Omega_{n,\alpha\alpha}(\alpha_0, 0)\Pi'_n(\alpha_0)\big] + o(1). \end{aligned} \tag{4.7}$$

An NN-robust LM test, with its limiting null distribution being $\chi^2_{\dim(\varphi)}$, takes the form:

$$\text{LMN}^0_n = \tilde{S}'_{n,\varphi}\big(\tilde{\Omega}_{n,\varphi\varphi} - 2\tilde{\Omega}_{n,\varphi\alpha}\tilde{\Pi}'_n + \tilde{\Pi}_n\tilde{\Omega}_{n,\alpha\alpha}\tilde{\Pi}'_n\big)^{-1}\tilde{S}_{n,\varphi}, \tag{4.8}$$

where $\tilde{S}_{n,\varphi} = S_{n,\varphi}(\tilde{\alpha}_n, 0)$, $\tilde{\Omega}_{n,\varphi\varphi}$, $\tilde{\Omega}_{n,\varphi\alpha}$ and $\tilde{\Omega}_{n,\alpha\alpha}$ are the sub-matrices of $\Omega_n(\tilde{\alpha}_n, 0)$, and $\tilde{\Pi}_n = \Pi_n(\tilde{\alpha}_n)$. Clearly, when $\{\varepsilon_{ni}\}$ are iid normal, IME holds, i.e., $\Omega_n(\theta_0) = \Sigma_n(\theta_0)$, and LMN^0_n reduces to LM_n. When $\{\varepsilon_{ni}\}$ are iid but non-normal, the IME does not hold and the explicit expression of $\Omega_n(\alpha_0, 0)$ is required in order to implement LMN^0_n. It is seen that $\Omega_n(\alpha_0, 0)$ typically involves third and fourth moments of the model errors which need to be estimated based on the null residuals defined by $\tilde{\alpha}_n$.

An *alternative way* to estimate $\Omega_n(\alpha_0, 0)$ is via the OPMD method (Baltagi and Yang, 2013). If $S_n(\alpha_0, 0)$ has an MD representation: $S_n(\alpha_0, 0) = \sum_{i=1}^n \mathbf{s}_{ni}(\alpha_0)$, where $\{\mathbf{s}_{ni}(\alpha_0)\}$ form an MD sequence, then $\text{Var}[S_n(\theta_0, 0)] = \sum_{i=1}^n \text{E}[\mathbf{s}_{ni}(\alpha_0)\mathbf{s}'_{ni}(\alpha_0)]$. Hence, $\sum_{i=1}^n \tilde{\mathbf{s}}_{ni}\tilde{\mathbf{s}}'_{ni}$, the sum of the estimated OPMDs, gives a consistent estimate of $\text{Var}[S_n(\alpha_0, 0)]$ in the sense that $\frac{1}{n}[\sum_{i=1}^n \tilde{\mathbf{s}}_{ni}\tilde{\mathbf{s}}'_{ni} - \text{Var}[S_n(\alpha_0, 0)] \xrightarrow{p} 0$, where $\tilde{\mathbf{s}}_{ni} = \mathbf{s}_{ni}(\tilde{\alpha}_n)$. Denote by $\mathbf{s}_{ni,\alpha}$ and $\mathbf{s}_{ni,\varphi}$ the sub-vectors of $\mathbf{s}_{ni} \equiv \mathbf{s}_{ni}(\alpha_0)$. Replacing $\tilde{\Omega}_{n,\varphi\varphi}$, $\tilde{\Omega}_{n,\varphi\alpha}$ and $\tilde{\Omega}_{n,\alpha\alpha}$ in (4.8) by the corresponding sub-matrices of $\sum_{i=1}^n \tilde{\mathbf{s}}_{ni}\tilde{\mathbf{s}}'_{ni}$, we obtain an OPMD form of NN-robust LM test:

$$\text{LMN}_n = \tilde{S}'_{n,\varphi}\big[\textstyle\sum_{i=1}^n(\tilde{\mathbf{s}}_{ni,\varphi} - \tilde{\Pi}_n\tilde{\mathbf{s}}_{ni,\alpha})(\tilde{\mathbf{s}}_{ni,\varphi} - \tilde{\Pi}_n\tilde{\mathbf{s}}_{ni,\alpha})'\big]^{-1}\tilde{S}_{n,\varphi}. \tag{4.9}$$

Equivalently, (4.9) can be obtained as follows. By (4.6) and the MD representation $S_n(\theta_0, 0) = \sum_{i=1}^n \mathbf{s}_{ni}(\alpha_0)$, $S_{n,\alpha}(\tilde{\alpha}_n, 0)$ has the following asymptotic MD representation:

$$\tfrac{1}{\sqrt{n}} S_{n,\varphi}(\tilde{\alpha}_n, 0) = \tfrac{1}{\sqrt{n}} \textstyle\sum_{i=1}^n[\mathbf{s}_{ni,\varphi} - \Pi_n(\theta_0)\mathbf{s}_{ni,\theta}] + o_p(1), \tag{4.10}$$

where $\{\mathbf{s}_{ni,\varphi} - \Pi_n(\alpha_0)\mathbf{s}_{ni,\alpha}\}$ form an MD sequence. Thus,

$$\text{Var}[\tfrac{1}{\sqrt{n}} S_{n,\varphi}(\tilde{\alpha}_n, 0)] = \tfrac{1}{n}\textstyle\sum_{i=1}^n \text{E}[(\mathbf{s}_{ni,\varphi} - \Pi_n(\alpha_0)\mathbf{s}_{ni,\theta})(\mathbf{s}_{ni,\varphi} - \Pi_n(\theta_0)\mathbf{s}_{ni,\alpha})'] + o(1), \tag{4.11}$$

leading to (4.9).

The advantages of using the OPMD estimate of $\Omega_n(\alpha_0, 0)$ are: (i) it avoids the analytical expression of Ω_n containing higher order moments, and (ii) it is also robust against UH besides being robust against NN. These are crucial in developing LM tests that are both NN and UH robust, as seen below. With these, an OPMD alternative to the LM test (4.1) can be easily developed.

Now, for the $\texttt{SARAR}(p, q)$ model, Σ_n is given in (2.11) as $n\Sigma_n$. An MD representation for the score function (2.8) is, $S_n(\theta_0) = \sum_{i=1}^n \mathbf{s}_{ni}(\theta_0)$, where,

$$\mathbf{s}_{ni}(\theta_0) = \begin{cases} \frac{1}{\sigma_0^2} x_{bi}\varepsilon_{ni}, \\ \frac{1}{2\sigma_0^4}(\varepsilon_{ni}^2 - \sigma_0^2), \\ \frac{1}{\sigma_0^2}[\varepsilon_{ni}\xi_{jn,i} + \bar{F}_{jn,ii}(\varepsilon_{ni}^2 - \sigma_0^2) + \eta_{jn,i}\varepsilon_{ni}],\ j = 1, \cdots, p, \\ \frac{1}{\sigma_0^2}[\varepsilon_{ni}\zeta_{jn,i} + G_{jn,ii}(\varepsilon_{ni}^2 - \sigma_0^2)],\ j = 1, \cdots, q, \end{cases} \tag{4.12}$$

where x_{bi} is the ith column of $X_n' B_n'$, $\xi_{jn} = (\bar{F}_{jn}^{u\prime} + \bar{F}_{jn}^l)\varepsilon_n$, and $\zeta_{jn} = (G_{jn}^{u\prime} + G_{jn}^l)\varepsilon_n$. Thus, following general principles laid out by (4.9) and the discussions around it, we have the general form of NN-robust LM test:

$$\text{LMN}_{\texttt{SARAR}}^{(\texttt{m})} = \tilde{S}_{n,\varphi}' \big[\sum_{i=1}^n (\tilde{\mathbf{s}}_{ni,\varphi} - \tilde{\Pi}_n \tilde{\mathbf{s}}_{ni,\alpha})(\tilde{\mathbf{s}}_{ni,\varphi} - \tilde{\Pi}_n \tilde{\mathbf{s}}_{ni,\alpha})'\big]^{-1} \tilde{S}_{n,\varphi}, \tag{4.13}$$

where $\texttt{m} = a, b, c, d, e, f$, giving the NN-robust tests for the six hypotheses listed above. The $\tilde{\Pi}_n$ can be either the plug-in estimate of $\Pi_n = \Sigma_{n,\varphi\alpha}\Sigma_{n,\alpha\alpha}^{-1}$ based on Σ_n given in (2.11), or the estimate based on the Hessian matrix, with α and φ defined accordingly. For example, for testing $H_0^a : \delta = 0$, we have $\alpha = (\beta', \sigma^2)'$ and $\varphi = \delta$, $\Pi_n = \Sigma_{n,\varphi\alpha}\Sigma_{n,\alpha\alpha}^{-1}$ has non-zero elements only at the upper-left corner block: $\{\eta_{jn}' \mathbf{X}_n(\rho)[\mathbf{X}_n'(\rho)\mathbf{X}_n(\rho)]^{-1}\}$, and for tests in (e) and (f), we have $\theta = (\beta', \sigma^2)'$, and $\delta = \lambda$ for (e) and ρ for (f). The null asymptotic distribution of $\text{LMN}_{\texttt{SARAR}}^{(\texttt{m})}$ is chi-square with degrees of freedom being $\dim(\varphi)$.

Theorem 4.2. *Under the Assumptions 1-6, we have, as $n \to \infty$,*

$$\text{LMN}_{\texttt{SARAR}}^{(m)}|_{H_0} \xrightarrow{D} \chi^2_{\dim(\varphi)},$$

for $m = a, b, c, d, e, f$.

4.3. NNUH-robust LM tests

Neither LM_n nor LMN_n (or LMN_n^0) is robust against UH in model errors. To derive an LM-type test that is UH-robust, adjust $S_n(\theta)$ so that the adjusted score vector $S_n^\circ(\theta)$ is such that $\text{E}[S_n^\circ(\theta_0)|_{H_0}] = 0$ or $\frac{1}{n}S_n^\circ(\theta_0)|_{H_0} \xrightarrow{p} 0$ as $n \to \infty$ under UH. Let $\tilde{\alpha}_n^\circ = \arg\{S_{n,\varphi}^\circ(\alpha, 0) = 0\}$, the UH-robust estimator of the null model, let $\Sigma_n^\circ = -\text{E}[\frac{\partial}{\partial\theta'}S_n^\circ(\theta_0)]$ and $\Omega_n^\circ = \text{Var}[S_n^\circ(\theta_0)]$, partitioned similarly as Σ_n and Ω_n. Similar to $S_{n,\varphi}(\tilde{\alpha}_n, 0)$ in (4.10), $S_{n,\varphi}^\circ(\alpha_0, 0)$ has an MD representation referring to (3.4). Then, $S_{n,\varphi}^\circ(\tilde{\alpha}_n^\circ, 0)$ has the following asymptotic MD representation:

$$\tfrac{1}{\sqrt{n}} S^{\circ}_{n,\varphi}(\tilde{\alpha}_n, 0) = \tfrac{1}{\sqrt{n}} \textstyle\sum_{i=1}^{n} [\mathbf{s}^{\circ}_{ni,\varphi}(\alpha_0) - \Pi^{\circ}_n(\alpha_0)\mathbf{s}^{\circ}_{ni,\alpha}(\alpha_0)] + o_p(1),$$

where $\{\mathbf{s}^{\circ}_{ni,\varphi} - \Pi^{\circ}_n(\alpha_0)\mathbf{s}^{\circ}_{ni,\alpha}\}$ form an MD sequence, $(\mathbf{s}^{\circ\prime}_{ni,\alpha}, \mathbf{s}^{\circ\prime}_{ni,\varphi})' = \mathbf{s}^{\circ}_{ni} \equiv \mathbf{s}^{\circ}_{ni}(\alpha_0)$ and $\Pi^{\circ}_n(\alpha_0) = \Sigma^{\circ}_{n,\varphi\alpha}(\alpha_0, 0)\Sigma^{\circ-1}_{n,\alpha\alpha}(\alpha_0, 0)$.

An OPMD-based, NNUH-robust LM test takes a similar form:

$$\text{LMNH}_n = \tilde{S}^{\circ\prime}_{n,\varphi} \big[\textstyle\sum_{i=1}^{n} (\tilde{\mathbf{s}}^{\circ}_{ni,\varphi} - \tilde{\Pi}^{\circ}_n \tilde{\mathbf{s}}^{\circ}_{ni,\alpha})(\tilde{\mathbf{s}}^{\circ}_{ni,\varphi} - \tilde{\Pi}^{\circ}_n \tilde{\mathbf{s}}^{\circ}_{ni,\alpha})' \big]^{-1} \tilde{S}^{\circ}_{n,\varphi}, \tag{4.14}$$

where the *tilded* quantities are the estimates at H_0 of the corresponding quantities. Clearly, the key in developing the UH-robust LM tests is to find the AQS function $S^{\circ}_n(\alpha_0, 0)$ that is UH-robust, and hence the UH-robust estimate $\tilde{\alpha}^{\circ}_n = \arg\{S^{\circ}_{n,\alpha}(\alpha_0, 0) = 0\}$.

The above general principle can easily be applied to give OPMD-based and NNUH-robust LM tests for the various hypotheses placed on the $\texttt{SARAR}(p, q)$. The general AQS function $S^{\circ}_n(\theta_0)$ is given in (3.1), and its MD representation is given in (3.4). Letting $\tilde{\Pi}^{\circ}_n = [\frac{\partial}{\partial \alpha'} S^{\circ}_{n,\varphi}(\tilde{\alpha}^{\circ}_n, 0)][\frac{\partial}{\partial \alpha'} S^{\circ}_{n,\alpha}(\tilde{\alpha}^{\circ}_n, 0)]^{-1}$ be a feasible estimate of $\Pi^{\circ}_n = \Sigma^{\circ}_{n,\varphi\alpha}\Sigma^{\circ-1}_{n,\alpha\alpha}$, the LM tests fully robust against NN and UH take the general form as that given in (4.14):

$$\text{LMNH}^{(\text{m})}_{\texttt{SARAR}} = \tilde{S}^{\circ\prime}_{n,\varphi} \big[\textstyle\sum_{i=1}^{n} (\tilde{\mathbf{s}}^{\circ}_{ni,\varphi} - \tilde{\Pi}^{\circ}_n \tilde{\mathbf{s}}^{\circ}_{ni,\alpha})(\tilde{\mathbf{s}}^{\circ}_{ni,\varphi} - \tilde{\Pi}^{\circ}_n \tilde{\mathbf{s}}^{\circ}_{ni,\alpha})' \big]^{-1} \tilde{S}^{\circ}_{n,\varphi}, \tag{4.15}$$

where $\tilde{S}^{\circ}_{n,\varphi} = S^{\circ}_{n,\varphi}(\tilde{\alpha}^{\circ}_n, 0)$, $\tilde{\mathbf{s}}^{\circ}_{ni,\alpha}$ and $\tilde{\mathbf{s}}^{\circ}_{ni,\varphi}$ are the sub-vectors of $\mathbf{s}^{\circ}_{ni}(\tilde{\alpha}^{\circ}_n, 0)$, and $\texttt{m} = a, b, c, d, e, f$ corresponding to the six tests defined at the beginning of this section with relevant choice of α and φ and the related quantities.

Theorem 4.3. *Under the Assumptions 1, 2^*, 3-5 and 6^*, we have, as $n \to \infty$,*

$$\text{LMNH}^{(m)}_{\texttt{SARAR}}|_{H_0} \xrightarrow{D} \chi^2_{\dim(\varphi)},$$

for $m = a, b, c, d, e, f$.

5. Finite Sample Improved LM and Robust LM Tests

The LM and robust LM tests developed in the early section are simple, but may not perform well in finite sample. As in Baltagi and Yang (2013), working with concentrated (quasi) score (CQS), or the adjusted concentrated quasi score (ACQS) potentially lead tests with finite sample improvements. Paralleled with the developments of Sec. 4, we introduce a set of 'standardised' LM and robust LM tests, based on the CQS and AQS functions. Let $\delta = (\alpha', \varphi')'$, the parameter vector of the CQS/ACQS function (after concentrating out the linear and scale parameters β and σ^2). Then the null hypothesis given at the beginning of Section 4 specifies, $\varphi = 0$.

5.1. Adjusted LM tests

Consider the numerators of the concentrated score function given in (3.5):

$$S_n^c(\delta) = \begin{cases} Y_n'(\delta)M_n(\rho)[\bar{F}_{jn}(\delta) - \frac{1}{n}\text{tr}(F_{jn}(\lambda)I_n)]Y_n(\delta), & j = 1, \ldots, p, \\ Y_n'(\delta)M_n(\rho)[\bar{G}_{kn}(\rho) - \frac{1}{n}\text{tr}(G_{kn}(\rho)I_n)]Y_n(\delta), & k = 1, \ldots, q, \end{cases} \tag{5.1}$$

where $\bar{F}_{jn}$ and $M_n(\rho)$ are defined in the last section, and $\bar{G}_{kn}(\rho) = G_{kn}(\rho)\mathcal{M}_n(\rho)$. Finding the mean of $S_n^c(\delta)$, recentering and rescaling lead immediately to an adjusted LM tests for the hypothesis (a), which can be simplified to give the adjusted LM tests for the hypotheses (e) and (f). These three tests have a common feature with null model being an OLS regression.

For the adjusted tests for (b)-(d), the variance of $S_n^c(\tilde{\delta}_n)$ is desired, where $\tilde{\delta}_n$ is the constrained estimator under the null, e.g., $\tilde{\delta}_n = (\tilde{\lambda}_n', 0_q')'$ for (b), $\tilde{\delta}_n = (0_p', \tilde{\rho}_n')'$, etc.

5.2. Adjusted NN-robust LM tests

When errors are non-normal, $S_n^c(\delta)$ is the concentrated *quasi* score (CQS) function and as before, the LM tests for (b), (c), and (d) given above are generally invalid, since the asymptotic variance used for approximating the variance of $S_{n,\varphi}^c(\tilde{\alpha}_n, 0)$ is invalid. Denote the sub-matrices of $\tau_n^2 = \text{Var}[S_n^c(\delta_0)]$ by $\tau_{n,\alpha\alpha}^2(\alpha,\varphi)$, $\tau_{n,\varphi\varphi}^2(\alpha,\varphi)$ and $\tau_{n,\alpha\varphi}^2(\alpha,\varphi)$. Consider the Taylor expansion,

$$\tfrac{1}{\sqrt{n}} S_{n,\varphi}^c(\tilde{\alpha}_n, 0) = \tfrac{1}{\sqrt{n}} S_{n,\varphi}^c(\alpha_0, 0) - \tfrac{1}{\sqrt{n}} \Pi_n^c(\alpha_0) S_{n,\alpha}^c(\alpha_0, 0) + o_p(1), \tag{5.2}$$

where $\Pi_n^c(\alpha_0) = \Phi_{n,\varphi\alpha}(\alpha_0, 0)\Phi_{n,\alpha\alpha}^{-1}(\alpha_0, 0)$ and $\Phi_n = -E(\frac{\partial}{\partial\delta'} S_n^c(\delta_0))$. This gives,

$$\begin{aligned} \text{Var}[\tfrac{1}{\sqrt{n}} S_{n,\varphi}^c(\tilde{\alpha}_n, 0)] &= \tfrac{1}{n}\big[\tau_{n,\varphi\varphi}^2(\alpha_0, 0) - \tau_{n,\varphi\alpha}^2(\alpha_0, 0)\Pi_n^{c\prime}(\alpha_0) - \Pi_n^{c\prime}(\alpha_0)\tau_{n,\alpha\varphi}^2(\alpha_0, 0) \\ &\quad + \Pi_n^c(\alpha_0)\tau_{n,\alpha\alpha}^2(\alpha_0, 0)\Pi_n^{c\prime}(\alpha_0)\big] + o(1). \end{aligned} \tag{5.3}$$

A finite sample improved, standardised, NN-robust LM test, with its limiting null distribution being $\chi^2_{\dim(\varphi)}$, takes the form:

$$\text{LMNF}_n = \tilde{S}_{n,\varphi}^{c\prime}\big(\tilde{\tau}_{n,\varphi\varphi}^2 - 2\tilde{\tau}_{n,\varphi\alpha}^2\tilde{\Pi}_n^{c\prime} + \tilde{\Pi}_n^c\tilde{\tau}_{n,\alpha\alpha}^2\tilde{\Pi}_n^{c\prime}\big)^{-1}\tilde{S}_{n,\varphi}^c, \tag{5.4}$$

where $\tilde{S}_{n,\varphi}^c = S_{n,\varphi}^c(\tilde{\alpha}_n, 0)$, $\tilde{\tau}_{n,\varphi\varphi}^2$, $\tilde{\tau}_{n,\varphi\alpha}^2$ and $\tilde{\tau}_{n,\alpha\alpha}^2$ are the sub-matrices of $\tau_n^2(\tilde{\alpha}_n, 0)$, and $\tilde{\Pi}_n^c = \Pi_n^c(\tilde{\alpha}_n)$. When $\{\varepsilon_{ni}\}$ are iid normal, IME holds, i.e., $\tau_n^2(\delta_0) = \Phi_n(\delta_0)$, and LMNF_n reduces to LMF_n. When $\{\varepsilon_{ni}\}$ are iid but non-normal, the IME does not hold and the explicit expression of $\tau_n^2(\alpha_0, 0)$ is required in order to implement LMNF_n. It is seen that $\tau_n^2(\alpha_0, 0)$ typically involves third and fourth moments of the model errors which need to be estimated based on the null residuals defined by $\tilde{\alpha}_n$.

As before, the OPMD method (Baltagi and Yang, 2013) can be employed to estimate $\tau_n^2(\alpha_0, 0)$. If $S_n^c(\alpha_0, 0)$ has an MD representation: $S_n^c(\alpha_0, 0) = \sum_{i=1}^n \mathbf{s}_{ni}^c(\alpha_0)$, where $\{\mathbf{s}_{ni}^c(\alpha_0)\}$ form an MD sequence, then $\text{Var}[S_n^c(\delta_0, 0)] = \sum_{i=1}^n \text{E}[\mathbf{s}_{ni}^c(\alpha_0)\mathbf{s}_{ni}^{c\prime}(\alpha_0)]$. Hence, $\sum_{i=1}^n \tilde{\mathbf{s}}_{ni}^c\tilde{\mathbf{s}}_{ni}^{c\prime}$, the sum of the estimated OPMDs, gives a consistent estimate of $\text{Var}[S_n^c(\alpha_0, 0)]$ in the sense that $\frac{1}{n}[\sum_{i=1}^n \tilde{\mathbf{s}}_{ni}^c \tilde{\mathbf{s}}_{ni}^{c\prime} - \text{Var}[S_n^c(\alpha_0, 0)] \stackrel{p}{\longrightarrow} 0$, where $\tilde{\mathbf{s}}_{ni}^c = \mathbf{s}_{ni}^c(\tilde{\alpha}_n)$. Denote by $\mathbf{s}_{ni,\alpha}^c$ and $\mathbf{s}_{ni,\varphi}^c$ the sub-vectors of $\mathbf{s}_{ni}^c \equiv \mathbf{s}_{ni}^c(\alpha_0)$. Replacing $\tilde{\tau}_{n,\varphi\varphi}^2$, $\tilde{\tau}_{n,\varphi\alpha}^2$ and $\tilde{\tau}_{n,\alpha\alpha}^2$ in (5.4) by the corresponding sub-matrices of $\sum_{i=1}^n \tilde{\mathbf{s}}_{ni}^c \tilde{\mathbf{s}}_{ni}^{c\prime}$, we obtain an OPMD form of finite sample improved, standardised, NN-robust LM test:

$$\text{LMNF}_n = \tilde{S}_{n,\varphi}^{c\prime} \big[\sum_{i=1}^n (\tilde{\mathbf{s}}_{ni,\varphi}^c - \tilde{\Pi}_n^c \tilde{\mathbf{s}}_{ni,\alpha}^c)(\tilde{\mathbf{s}}_{ni,\varphi}^c - \tilde{\Pi}_n^c \tilde{\mathbf{s}}_{ni,\alpha}^c)' \big]^{-1} \tilde{S}_{n,\varphi}^c. \tag{5.5}$$

Equivalently, (5.5) can be obtained as follows. By (5.2) and the MD representation $S_n^c(\delta_0, 0) = \sum_{i=1}^n \mathbf{s}_{ni}^c(\alpha_0)$, $S_{n,\alpha}^c(\tilde{\alpha}_n, 0)$ has the following asymptotic MD representation:

$$\tfrac{1}{\sqrt{n}} S_{n,\varphi}^c(\tilde{\alpha}_n, 0) = \tfrac{1}{\sqrt{n}} \sum_{i=1}^n [\mathbf{s}_{ni,\varphi}^c - \Pi_n^c(\delta_0)\mathbf{s}_{ni,\theta}^c] + o_p(1), \tag{5.6}$$

where $\{\mathbf{s}_{ni,\varphi}^c - \Pi_n^c(\alpha_0)\mathbf{s}_{ni,\alpha}^c\}$ form an MD sequence. Thus,

$$\text{Var}[\tfrac{1}{\sqrt{n}} S_{n,\varphi}^c(\tilde{\alpha}_n, 0)] = \tfrac{1}{n} \sum_{i=1}^n \text{E}[(\mathbf{s}_{ni,\varphi}^c - \Pi_n^c(\alpha_0)\mathbf{s}_{ni,\theta}^c)(\mathbf{s}_{ni,\varphi}^c - \Pi_n^c(\delta_0)\mathbf{s}_{ni,\alpha}^c)'] + o(1), \tag{5.7}$$

leading to (5.5).

With these, an OPMD alternative to the LM test (5.4) can be developed.

Now, for the $\texttt{SARAR}(p, q)$ model an MD representation for the numerator of concentrated score function (3.5) is, $S_n^c(\delta_0) = \sum_{i=1}^n \mathbf{s}_{ni}^c(\delta_0)$, where,

$$\mathbf{s}_{ni}^c(\delta_0) = \begin{cases} \frac{1}{\sigma_0^2}[\varepsilon_{ni}\xi_{jn,i} + \bar{F}_{jn,ii}(\varepsilon_{ni}^2 - \sigma_0^2) + \eta_{jn,i}\varepsilon_{ni}], \ j = 1, \cdots, p, \\ \frac{1}{\sigma_0^2}[\varepsilon_{ni}\zeta_{jn,i} + G_{jn,ii}(\varepsilon_{ni}^2 - \sigma_0^2)], \ j = 1, \cdots, q, \end{cases} \tag{5.8}$$

$\xi_{jn} = (\bar{F}_{jn}^{w\prime} + \bar{F}_{jn}^l)\varepsilon_n$, and $\zeta_{jn} = (G_{jn}^{w\prime} + G_{jn}^l)\varepsilon_n$. Thus, following general principles laid out by (5.5) and the discussions around it, we have the general form of NN-robust LM test:

$$\text{LMNF}_{\texttt{SARAR}}^{(\texttt{m})} = \tilde{S}_{n,\varphi}^{c\prime} \big[\sum_{i=1}^n (\tilde{\mathbf{s}}_{ni,\varphi}^c - \tilde{\Pi}_n^c \tilde{\mathbf{s}}_{ni,\alpha}^c)(\tilde{\mathbf{s}}_{ni,\varphi}^c - \tilde{\Pi}_n^c \tilde{\mathbf{s}}_{ni,\alpha}^c)' \big]^{-1} \tilde{S}_{n,\varphi}^c, \tag{5.9}$$

where $\texttt{m} = a, b, c, d, e, f$, giving the NN-robust tests for the six hypotheses listed above. The $\tilde{\Pi}_n^c$ can be the estimate based on the Hessian matrix, with α and φ defined accordingly.

Theorem 5.1. *Under the Assumptions 1-6, we have, as $n \to \infty$,*

$$\text{LMNF}_{\texttt{SARAR}}^{(m)}|_{H_0} \stackrel{D}{\longrightarrow} \chi_{\dim(\varphi)}^2,$$

for $m = a, b, c, d, e, f$.

5.3. Adjusted NNUH-robust LM tests

Consider the adjusted concentrated quasi score function given in (3.6). As $S_n^{ac}(\delta)$ is already centred, simply rescaling using the OPMD method leads to the adjusted LM tests for (a), (e) and (f), which are heteroskedasticity robust. For the adjusted tests for (b)-(d), the variance of $S_n^{ac}(\delta_0)$ is desired, where $\tilde{\delta}_n$ is the constrained estimator under the null, e.g.,

$\tilde{\delta}_n = (\tilde{\lambda}'_n, 0'_q)'$ for (c), $\tilde{\delta}_n = (0'_p, \tilde{\rho}'_n)'$ for (d), etc.

Let $\tilde{\alpha}_n^{ac} = \arg\{S_n^{ac}(\delta) = 0\}$, the finite sample corrected, standardised, NNUH-robust estimator of the null model. Let $\tau_n^2(S_n^{ac}) = \text{Var}(S_n^{ac}(\delta_0))$ and $\Phi_n = -\text{E}(\frac{\partial}{\partial \delta'} S_n^{ac}(\delta_0))$. Given that the ACQS function under under the null has the MD representation $S_{n,\varphi}^{ac}(\alpha_0, 0) = \sum_{i=1}^n \varepsilon_{ni} \Upsilon_{ni}(\alpha_0)$, where $\{\varepsilon_{ni} \Upsilon_{ni}(\alpha_0)\}$ form an MD sequence referring to (3.10), $S_{n,\varphi}^{ac}$ has the following asymptotic MD representation:

$$\frac{1}{\sqrt{n}} S_{n,\varphi}^{ac}(\tilde{\alpha}_n^{ac}, 0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n [\mathbf{s}_{ni,\varphi}^{ac}(\alpha_0) - \Pi_n^{ac}(\alpha_0) \mathbf{s}_{ni,\alpha}^{ac}(\alpha_0)] + o_p(1),$$

where $\mathbf{s}_{ni,\varphi}^{ac}(\alpha_0) = \varepsilon_{ni} \Upsilon_{ni}(\alpha_0)$, $\{\mathbf{s}_{ni,\varphi}^{ac} - \Pi_n^{ac}(\alpha_0) \mathbf{s}_{ni,\alpha}^{ac}\}$ form an MD sequence, $(\mathbf{s}_{ni,\alpha}^{c\prime}, \mathbf{s}_{ni,\varphi}^{c\prime})' = \mathbf{s}_{ni}^{ac} \equiv \mathbf{s}_{ni}^{ac}(\alpha_0)$, $\Pi_n^{ac}(\alpha_0) = \Phi_{n,\varphi\alpha}(\alpha_0, 0) \Phi_{n,\alpha\alpha}^{-1}(\alpha_0, 0)$ and $\Phi_n = -E(\frac{\partial}{\partial \delta'} S_n^{ac}(\delta_0))$. A finite sample improved, standardised, OPMD-based, NNUH-robust LM test takes a similar form:

$$\text{LMNHF}_n = \tilde{S}_{n,\varphi}^{ac\prime} \big[\sum_{i=1}^n (\tilde{\mathbf{s}}_{ni,\varphi}^{ac} - \tilde{\Pi}_n^{ac} \tilde{\mathbf{s}}_{ni,\alpha}^{ac})(\tilde{\mathbf{s}}_{ni,\varphi}^{ac} - \tilde{\Pi}_n^{ac} \tilde{\mathbf{s}}_{ni,\alpha}^{ac})' \big]^{-1} \tilde{S}_{n,\varphi}^{ac}, \tag{5.10}$$

where the *tilded* quantities are the estimates at H_0 of the corresponding quantities.

As before, the above general principle can be applied to give OPMD-based, finite sample improved, standardised, NNUH-robust LM tests for the various hypotheses placed on $\texttt{SARAR}(p, q)$. Letting $\tilde{\Pi}_n^{ac} = [\frac{\partial}{\partial \alpha'} S_{n,\varphi}^{ac}(\tilde{\alpha}_n^{ac}, 0)][\frac{\partial}{\partial \alpha'} S_{n,\alpha}^{ac}(\tilde{\alpha}_n^{ac}, 0)]^{-1}$ be a feasible estimate of $\Pi_n^{ac} = \Phi_{n,\varphi\alpha} \Phi_{n,\alpha\alpha}^{-1}$, the finite sample improved LM tests fully robust against NN and UH take the general form as that given in (5.10):

$$\text{LMNHF}_{\texttt{SARAR}}^{(\text{m})} = \tilde{S}_{n,\varphi}^{ac\prime} \big[\sum_{i=1}^n (\tilde{\mathbf{s}}_{ni,\varphi}^{ac} - \tilde{\Pi}_n^{ac} \tilde{\mathbf{s}}_{ni,\alpha}^{ac})(\tilde{\mathbf{s}}_{ni,\varphi}^{ac} - \tilde{\Pi}_n^{ac} \tilde{\mathbf{s}}_{ni,\alpha}^{ac})' \big]^{-1} \tilde{S}_{n,\varphi}^{ac}, \tag{5.11}$$

where $\tilde{S}_{n,\varphi}^{ac} = S_{n,\varphi}^{ac}(\tilde{\alpha}_n^{ac}, 0)$, $\tilde{\mathbf{s}}_{ni,\alpha}^{ac}$ and $\tilde{\mathbf{s}}_{ni,\varphi}^{ac}$ are the sub-vectors of $\mathbf{s}_{ni}^{ac}(\tilde{\alpha}_n^{ac}, 0)$, and $\texttt{m} = a, b, c, d, e, f$ corresponding to the six tests defined at the beginning of this section with relevant choice of α and φ and the related quantities.

Theorem 5.2. *Under the Assumptions 1, 2^*, 3-5 and 6^{**}, we have, as $n \to \infty$,*

$$\text{LMNHF}_{\texttt{SARAR}}^{(m)}|_{H_0} \xrightarrow{D} \chi^2_{\dim(\varphi)},$$

for $m = a, b, c, d, e, f$.

6. Monte Carlo Experiments

We consider a $\texttt{SARAR}(3,3)$ specification with $W_j = M_j$ for $j = 1, 2, 3$:

$$Y_n = \sum_{j=1}^3 \lambda_j W_{jn} Y_n + \beta_0 \iota_n + \beta_1 X_{1n} + \beta_2 X_{2n} + u_n \text{ where } u_n = \sum_{j=1}^3 \rho_j W_{jn} u_n + \varepsilon_n, \tag{6.12}$$

where ι_n is an $n \times 1$ vector of ones corresponding to the intercept term, X_{1n} and X_{2n} are the $n \times 1$ vectors containing the values of two fixed regressors generated as random draws from a standard normal distribution and $\varepsilon_n = \sigma H_n e_n$. The regression coefficients β is set to $(3,1,1)'$, σ is set to 1 and n take values from $\{100, 250, 500\}$. The ways of generating the values for the spatial weights matrix W_{jn}, the heteroskedasticity measure H_n, and the idiosyncratic errors e_n are described below. Each set of Monte Carlo results is based on $1,000$ Monte Carlo samples.

Spatial Weight Matrix: We use two different spatial layouts: (i) Circular Neighbours and (ii) Queen Contiguity. In (i), neighbours occur in the positions immediately ahead and behind a particular spatial unit. For example, for the ith spatial unit with 6 neighbours, the ith row of W_n matrix has non-zero elements in the positions: $i-3, i-2, i-1, i+1, i+2$, and $i+3$. The initial weights matrix we consider W_{0n} has 2, 4, 6, 8 and 10 neighbours with equal proportion. Then we decompose W_{0n} into three distinct matrices s.t. $W_{1n}+W_{2n}+W_{3n} = W_{0n}$, where W_{1n} contains 2 and 4 band of neighbours of W_{0n}, W_{2n} contains 6 and 8 band of neighbours of W_{0n} and W_{3n} contains 10 band of neighbours of W_{0n}. In (ii), neighbours could occur in the eight cardinal and ordinal positions of each unit. To ensure the heteroskedasticity effect does not fade as n increases (so that the regular QMLE remains inconsistent), the degree of spatial dependence is fixed with respect to n. This is attained by fixing the number of neighbours behind and ahead in the Circular Neighbours scheme. The degree of spatial dependence is naturally bounded in the Queen Contiguity weights matrix. To analyse the performance of the original QMLE when it is robust against heteroskedasticity, we use Queen Contiguity scheme. All individual final weights matrices are row normalised.

Heteroskedasticity: $h_{n,i}$ is generated as the ratio of the total number of neighbours to the average number of neighbours for each i of W_{0n}. For the Queen Contiguity schemes, we use $h_{n,i} = n[\sum_{i=1}^{n}(|X_{1n,i}| + |X_{2n,i}|)]^{-1}(|X_{1n,i}| + |X_{2n,i}|)$.

Error Distribution: To generate the e_n component of the disturbance term, three DGPs are considered: DGP1: $\{e_{n,i}\}$ are iid standard normal, DGP2: $\{e_{n,i}\}$ are iid standardised normal mixture with 10% of values from $N(0,4)$ and the remaining from $N(0,1)$ and DGP3: $\{e_{n,i}\}$ iid standardised log-normal with parameters 0 and 1. Thus, the error distribution from DGP2 is leptokurtic, and that of DGP3 is both skewed and leptokurtic.

Given row normalised weights matrices, the parameter space for λ and ρ must satisfy, $0 \le \sum_{j=1}^{3} |\lambda_j| < 1$ and $0 \le \sum_{j=1}^{3} |\rho_j| < 1$. We follow the following parameter constellations:

Constellation	λ_1	λ_2	λ_3	ρ_1	ρ_2	ρ_3
1	.5	.3	.1	.4	.2	.1
2	.4	.2	.1	.5	.3	.1
3	.2	.2	.2	.2	.2	.2
4	0	0	0	0	0	0
5	.5	.3	.1	0	0	0
6	0	0	0	.5	.3	.1

In case 1, the spatial dependence in the dependent variable Y is at least as strong as the spatial dependence in the disturbance u_n while the opposite holds in case 2. In case 3 the spatial dependence is equal in all the spatially dependent variables. Cases 4-6 are sub-models developed in $\texttt{SARAR}(p, q)$ beginning with no spatial dependence in case 4 which is the general linear regression model, $\texttt{SARAR}(3,0)$ in case 5 and $\texttt{SARAR}(0,3)$ in case 6.

From the Monte Carlo results, we observe that the ACQSE of δ performs well in all cases considered, and it generally outperforms QMLE in terms of bias and rmse. Further, in the case where QMLE is consistent, ACQSE can be less biased than QMLE, and is as efficient as QMLE. The relative performance of various estimators of β is much less contrasting than that of various estimators of δ, although it can be seen that ACQSE of β is slightly less biased and more efficient than the QMLE.

Additional Monte Carlo experiments were carried out to test the performance of robust LM tests.

7. Conclusion

We extend the results of Liu and Yang (2015) to give a heteroskedasticity-robust estimator for the high order spatial autoregressive models with autoregressive errors, or the $\texttt{SARAR}(p, q)$ model, as well as robust standard errors for model inferences. The results presented in the paper show a great potential of the heteroskedasticity-robust method proposed in Liu and Yang (2015). Its simplicity and good finite sample property render the proposed set of inference methods for the general $\texttt{SARAR}(p, q)$ model an important set of tools for the applied spatial econometricians, urban economists and regional scientists. We also give general LM tests for the existence of various spatial effects in the $\texttt{SARAR}(p, q)$ model that are robust against nonnormality and unknown heteroskedasticity. Since these general LM tests may not perform well in finite samples, ‘standardised’ LM and robust LM tests were also given.

Appendix A: Some Useful Lemmas

The following lemmas are the extended versions of the selected lemmas from Lee (2004), Kelejian and Prucha (2001) and Lin and Lee (2010), required in the proofs of the main results.

Lemma A.1. *Under Assumptions 1,3 and 4, for $B_n(\rho)$ and X_n defined in Section 2, the projection matrices $P_n(\rho) = B_n(\rho)X_n[X_n'B_n'(\rho)B_n(\rho)X_n]^{-1}X_n'B_n'(\rho)$ and $M_n(\rho) = I_n - P_n(\rho)$ are uniformly bounded in both row and column sums.*

Lemma A.2. *Let A_n be an $n \times n$ matrix, uniformly bounded in both row and column sums. Then for M_n defined in Lemma A.1,*

(i) $\mathrm{tr}(A_n^m) = O(n)$ *for* $m \geq 1$,
(ii) $\mathrm{tr}(A_n'A_n) = O(n)$,
(iii) $\mathrm{tr}((M_nA_n)^m) = \mathrm{tr}(A_n^m) + O(1)$ *for* $m \geq 1$ *and*
(iv) $\mathrm{tr}((A_n'M_nA_n)^m) = \mathrm{tr}((A_n'A_n)^m) + O(1)$ *for* $m \geq 1$.

Let B_n be another $n \times n$ matrix, uniformly bounded in both row and column sums. Then,

(iv) A_nB_n *is uniformly bounded in both row and column sums,*
(v) $\mathrm{tr}(A_nB_n) = \mathrm{tr}(B_nA_n) = O(n)$ *uniformly.*

Lemma A.3. *(Moments and Limiting Distribution of Quadratic Forms): For a given process of innovations $\{\varepsilon_{n,i}\}$, assume $\varepsilon_{n,i} \sim inid(0, \sigma_0^2 h_{n,i})$, where $h_{n,i} > 0$ and $\sum_{i=1}^n h_{n,i} = n$. Let $H_n = diag(h_{n,1}, \ldots, h_{n,n})$, D_n be an $n \times n$ matrix of elements $d_{n,ij}$, and c_n an $n \times 1$ vector of elements $c_{n,i}$. For $Q_{rn} = \varepsilon_n' D_{rn}\varepsilon_n + c_{rn}'\varepsilon_n$ where $r = 1, \ldots, R$, we have,*

(i) $\mathrm{E}(Q_{rn}) = \sigma_0^2\mathrm{tr}(H_nD_{rn})$,
(ii) $\mathrm{Var}(Q_{rn}) = \sigma_0^4 tr[H_nD_{rn}(H_nD_{rn} + D_{rn}'H_n)] + \sigma_0^2 c_{rn}'H_nc_{rn}$
$\quad + \sum_{i=1}^n(\sigma_{n,i}^4 d_{rn,ii}^2 h_{n,i}\kappa_{n,i} + 2\sigma_{n,i}^3 d_{rn,ii}c_{n,i}h_{n,i}^{3/2}\gamma_{n,i})$,
(iii) $\mathrm{Cov}(Q_{rn}, Q_{sn}) = 2\sigma_0^2\mathrm{tr}(H_nD_{rn}H_nD_{sn}) + \sigma_0^2 c_{rn}'H_nc_{sn}$
$\quad + \sum_{i=1}^n[\sigma_0^4 d_{rn,ii}d_{sn,ii}h_{n,i}^2\kappa_{n,i} + \sigma_0^3(d_{rn,ii}c_{sn} + d_{sn,ii}c_{rn})h_{n,i}^{3/2}\gamma_{n,i}]$,

where $\gamma_{n,i}$ and $\kappa_{n,i}$ are, respectively, the skewness and excess kurtosis of $\varepsilon_{n,i}$. Now, if D_{rn} is uniformly bounded in either row or column sums, then we have, for $r = 1, \ldots, R$,

(iv) $\mathrm{E}(Q_{rn}) = O(n)$,
(v) $\mathrm{Var}(Q_{rn}) = O(n)$,
(vi) $Q_{rn} = O_p(n)$,
(vi) $\frac{1}{n}Q_{rn} - \frac{1}{n}\mathrm{E}(Q_{rn}) = O_p(n^{-\frac{1}{2}})$,
(viii) $\mathrm{Var}(\frac{1}{n}Q_{rn}) = O(n^{-1})$.

Let $Q_n = (Q_{1n}, \ldots, Q_{Rn})'$ and $\Sigma_n = \big\{\mathrm{Cov}(Q_{rn}, Q_{sn})\big\}_{R\times R}$. If D_{rn} is uniformly bounded in both row and column sums and Assumption 2 holds, then,

(ix) $\frac{Q_{rn}-\mathrm{E}(Q_{rn})}{\sqrt{\mathrm{Var}(Q_{rn})}} \xrightarrow{D} \mathcal{N}(0,1)$; $r = 1, \ldots, R$,
(x) $\Sigma_n^{1/2}(Q_n - \mathrm{E}(Q_n)) \xrightarrow{D} \mathcal{N}(0, I_n)$,

where I_n is the $n \times n$ identity matrix, and $\Sigma_n^{1/2}$ is a square root matrix of Σ_n.

Appendix B: Proofs of Theorems and Corollaries

Proof of Theorem 2.1: Details of the proof of Theorem 2.1, that shows the consistency and the asymptotic normality of the QMLEs of θ under homoskedastic assumption is similar to the proofs of Theorem 3.1 and 3.2 of Lee (2004) after making adjustments for the autoregressive disturbances. As such we omit the proof of Theorem 2.1 and focus on the case where the errors are heteroskedastic.

Proof of Theorem 3.1 and Corollary 3.1 are omitted since the practical application of the Theorem requires the estimation of the higher order moments of the heteroskedastic errors $\epsilon_{n,i}$ which cannot be done.

Proof of Theorem 3.2: Let $\mathrm{E}(S_n^{ac}(\delta)) = \bar{S}_n^{ac}(\delta)$. By Theorem 5.9 of van der Vaart (1998), the proof of consistency of $\tilde{\delta}_n$ requires, (a) Uniform convergence: $\sup_{\delta\in\Delta}||S_n^{ac}(\delta) - \bar{S}^{ac}(\delta)|| = o_p(1)$ and (b) Identification uniqueness: for $\varepsilon > 0$, $\inf_{\delta:d(\delta,\delta_0)\geq\varepsilon}||\bar{S}^{ac}(\delta)| > 0 = |\bar{S}^{ac}(\delta_0)|$.

By virtue of Theorem 2.1 we have that, $\hat{\sigma}_n^2(\beta,\lambda)$ is bounded away from 0 with probability one for large enough n. Thus, the ACQSE $\tilde{\delta}_n = \arg\{S_n^{ac}(\delta) = 0\}$ is equivalently defined as,

$$\tilde{\delta}_n = \arg\left\{\begin{array}{l} Y_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)Y_n(\delta) = 0,\ j = 1,\ldots,p \\ Y_n'(\delta)M_n(\rho)\bar{G}_{kn}^{\circ}(\rho)Y_n(\delta) = 0,\ k = 1,...,q \end{array}\right\},$$

suggesting that we can work purely with the numerators of $S_n^{ac}(\delta)$ in order to establish consistency of $\tilde{\delta}_n$. Let $R_{jn}(\delta) = T_{jn}(\delta) - S_{jn}(\delta)$ where $T_{jn}(\delta) = Y_n'(\delta)M_n(\rho)\bar{F}_{jn}(\delta)Y_n(\delta)$ and $S_{jn}(\delta) = Y_n'(\delta)M_n(\rho)\mathrm{diag}(M_n(\rho))^{-1}\mathrm{diag}[M_n(\rho)\bar{F}_{jn}(\delta)]Y_n(\delta)$ for $j = 1,\ldots,p$.

Condition (a): Given $B_nA_nY_n = X_n(\rho_0)\beta_0 + \varepsilon_n$, we have, $Y_n(\delta) = B_n(\rho)A_n(\lambda)Y_n = B_n(\rho)A_n(\lambda)A_n^{-1}B_n^{-1}[X_n(\rho_0)\beta_0 + \varepsilon_n] = \eta_n(\delta) + D_n(\delta)\varepsilon_n$, where $D_n(\delta) = B_n(\rho)A_n(\lambda)A_n^{-1}B_n^{-1}$ and $\eta_n(\delta) = D_n(\delta)X_n(\rho_0)\beta_0$. This implies that,

$$\begin{aligned} Y_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)Y_n(\delta) &= [\eta_n(\delta) + D_n(\delta)\varepsilon_n]'M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)[\eta_n(\delta) + D_n(\delta)\varepsilon_n] \\ &= \eta_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)\eta_n(\delta) + \eta_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)D_n(\delta)\varepsilon_n \\ &\quad + \varepsilon_n'D_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)\eta_n(\delta) + \varepsilon_n'D_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)D_n(\delta)\varepsilon_n \text{ and} \end{aligned}$$

$$\mathrm{E}[Y_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)Y_n(\delta)] = \eta_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)\eta_n(\delta) + \sigma_0^2\mathrm{tr}[D_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)D_n(\delta)H_n].$$

Hence,

$$\begin{aligned} &\tfrac{1}{n}[Y_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)Y_n(\delta) - \mathrm{E}[Y_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)Y_n(\delta)]] \\ &\quad = \tfrac{1}{n}[\varepsilon_n'D_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)D_n(\delta)\varepsilon_n - \sigma_0^2\mathrm{tr}[D_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)D_n(\delta)H_n] \\ &\quad + \tfrac{1}{n}\eta_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)D_n(\delta)\varepsilon_n + \tfrac{1}{n}\varepsilon_n'D_n'(\delta)M_n(\rho)\bar{F}_{jn}^{\circ}(\delta)\eta_n(\delta) = o_p(1) \text{ uniformly by} \end{aligned}$$

Assumptions 3-6 and Lemma B.3.

Condition (b): Once again we prove the condition for the component of the adjusted quasi score with respect to λ noting that the proof for the component with respect to ρ follows in a similar manner.

First, by design, we have $\mathrm{E}[R_{jn}(\delta_0)] = 0$ and

$$\begin{aligned} \mathrm{E}[R_{jn}(\delta)] &= \beta_0' X_n' A_n'^{-1} A_n'(\lambda) B_n'(\rho) M_n(\rho) \bar{F}_n^{\circ}(\delta) B_n(\rho) A_n(\lambda) A_n^{-1} X_n \beta_0 \\ &\quad + \sigma_0^2 \mathrm{tr}(H_n B_n'^{-1} A_n'^{-1} A_n'(\lambda) B_n'(\rho) M_n(\rho) \bar{F}_n^{\circ}(\delta) B_n(\rho) A_n(\lambda) A_n^{-1} B_n^{-1}). \end{aligned} \tag{C-1}$$

By Assumption 6* and Lemma B.2, $\mathrm{E}[R_{jn}(\lambda)] \neq 0$, for any $\lambda \neq \lambda_0$. It follows that the conditions of Theorem 5.9 of van der Vaart (1998) hold, and thus the consistency of $\tilde{\lambda}_n$.

To prove asymptotic normality, we have, by the mean value theorem,

$$\mathbf{0} = \sqrt{n} S_n^{ac}(\tilde{\delta}_n) = \sqrt{n} S_n^{ac}(\delta_0) + \tfrac{\partial}{\partial \delta'} S_n^{ac}(\bar{\delta}_n) \sqrt{n}(\tilde{\delta}_n - \delta_0), \tag{C-2}$$

where $\bar{\delta}_n$ lies in the segment formed by $\tilde{\delta}_n$ and δ_0. It suffices to show that (i) $\frac{\partial}{\partial \delta'} S_n^{ac}(\bar{\delta}_n) - \frac{\partial}{\partial \delta'} S_n^{ac}(\delta_0) = o_p(1)$, (ii) $\frac{\partial}{\partial \delta'} S_n^{ac}(\delta_0) - \mathrm{E}\big(\frac{\partial}{\partial \delta'} S_n^{ac}(\delta_0)\big) = o_p(1)$, and (iii) $\mathrm{E}\big(\frac{\partial}{\partial \delta'} S_n^{ac}(\delta_0)\big) \neq 0$ for large enough n. Let,

$$\frac{\partial}{\partial \delta'} S_n^{ac}(\delta) = \begin{pmatrix} S_{jj',n}^{ac}(\delta) & S_{jk,n}^{ac}(\delta) \\ S_{kj,n}^{ac}(\delta) & S_{kk',n}^{ac}(\delta) \end{pmatrix}, \tag{C-3}$$

where $j, j' = 1, \ldots, p$ and $k, k' = 1, \ldots, q$. Then,

$$\begin{aligned} S_{jj',n}^{ac}(\delta) &= \tfrac{Y_n'(\delta) M_n(\rho) \dot{\bar{F}}_{jj',n}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} - \tfrac{Y_n'(\delta) M_n(\rho) \bar{F}_{jn}^{\circ}(\delta) \bar{F}_{j'n}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} - \tfrac{Y_n'(\delta) \bar{F}_{j'n}'(\delta) M_n(\rho) \bar{F}_{jn}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} \\ &\quad + 2 \tfrac{Y_n'(\delta) M_n(\rho) \bar{F}_{j'n}(\delta) Y_n(\delta) \cdot Y_n'(\delta) M_n(\rho) \bar{F}_{jn}^{\circ}(\delta) Y_n(\delta)}{[Y_n'(\delta) M_n(\rho) Y_n(\delta)]^2}, \end{aligned}$$

$$\begin{aligned} S_{jk,n}^{ac}(\delta) &= \tfrac{Y_n'(\delta) M_n(\rho) \dot{\bar{B}}_{jk,n}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} - \tfrac{Y_n'(\delta) M_n(\rho) \bar{F}_{jn}^{\circ}(\delta) G_{kn}(\rho) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} - \tfrac{Y_n'(\delta) \bar{G}_{kn}'(\rho) M_n(\rho) \bar{F}_{jn}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} \\ &\quad + \tfrac{Y_n'(\delta) M_n(\rho) G_{kn}(\rho) \bar{F}_{jn}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} - \tfrac{Y_n'(\delta) M_n(\rho) \bar{G}_{kn}(\rho) \bar{F}_{jn}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} \\ &\quad + 2 \tfrac{Y_n'(\delta) M_n(\rho) \bar{G}_{kn}(\delta) Y_n(\delta) \cdot Y_n'(\delta) M_n(\rho) \bar{F}_{jn}^{\circ}(\delta) Y_n(\delta)}{[Y_n'(\delta) M_n(\rho) Y_n(\delta)]^2}, \end{aligned}$$

$$\begin{aligned} S_{kj,n}^{ac}(\delta) &= -\tfrac{Y_n'(\delta) \bar{F}_{jn}'(\delta) M_n(\rho) \bar{G}_{kn}^{\circ}(\rho) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} - \tfrac{Y_n'(\delta) M_n(\rho) \bar{G}_{kn}^{\circ}(\rho) \bar{F}_{jn}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} \\ &\quad + 2 \tfrac{Y_n'(\delta) M_n(\rho) \bar{F}_{jn}(\delta) Y_n(\delta) \cdot Y_n'(\delta) M_n(\rho) \bar{G}_{kn}^{\circ}(\rho) Y_n(\delta)}{[Y_n'(\delta) M_n(\rho) Y_n(\delta)]^2} \text{ and} \end{aligned}$$

$$\begin{aligned} S_{kk',n}^{ac}(\delta) &= \tfrac{Y_n'(\delta) M_n(\rho) \dot{\bar{G}}_{kk',n}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} - \tfrac{Y_n'(\delta) M_n(\rho) \bar{G}_{kn}^{\circ}(\delta) G_{k'n}(\rho) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} - \tfrac{Y_n'(\delta) \bar{G}_{k'n}'(\rho) M_N(\rho) \bar{G}_{kn}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} \\ &\quad + \tfrac{Y_n'(\delta) M_n(\rho) G_{k'n}(\rho) \bar{G}_{kn}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} - \tfrac{Y_n'(\delta) M_n(\rho) \bar{G}_{k'n}(\rho) \bar{G}_{kn}^{\circ}(\delta) Y_n(\delta)}{Y_n'(\delta) M_n(\rho) Y_n(\delta)} \\ &\quad + 2 \tfrac{Y_n'(\delta) M_n(\rho) \bar{G}_{k'n}(\delta) Y_n(\delta) \cdot Y_n'(\delta) M_n(\rho) \bar{G}_{kn}^{\circ}(\delta) Y_n(\delta)}{[Y_n'(\delta) M_n(\rho) Y_n(\delta)]^2}, \end{aligned}$$

where,

$$\begin{aligned} \dot{\bar{F}}_{jj',n}^{\circ}(\delta) &= \tfrac{\partial}{\partial \lambda'} \dot{\bar{F}}_{jn}^{\circ}(\delta) \\ &= B_n(\rho) F_{jn}(\lambda) F_{j'n}(\lambda) B_n^{-1}(\rho) - \mathrm{diag}[M_n(\rho)]^{-1} \mathrm{diag}[M_n(\rho) B_n(\rho) F_{jn}(\lambda) F_{j'n}(\lambda) B_n^{-1}(\rho)], \\ \dot{\bar{F}}_{jk,n}^{\circ}(\delta) &= \tfrac{\partial}{\partial \rho'} \dot{B}_{j,n}^{\circ}(\delta) = \bar{F}_{jn}(\delta) G_{kn}(\rho) - G_{kn}(\rho) \bar{F}_{jn}(\delta) \\ &\quad + \mathrm{diag}[M_n^{-1}(\rho) \dot{M}_{kn}(\rho) M_n^{-1}(\rho)] \mathrm{diag}[M_n(\rho) \bar{F}_{jn}(\delta)] - \mathrm{diag}[M_n(\rho)]^{-1} \mathrm{diag}[\dot{M}_{kn}(\rho) \bar{F}_{jn}(\delta)] \\ &\quad - \mathrm{diag}[M_n(\rho)]^{-1} \mathrm{diag}[M_n(\rho) \bar{F}_{jn}(\delta) G_{kn}(\rho)] + \mathrm{diag}[M_n(\rho)]^{-1} \mathrm{diag}[M_n(\rho) G_{kn}(\rho) \bar{F}_{jn}(\delta)], \\ \dot{\bar{G}}_{kk',n}^{\circ}(\rho) &= \tfrac{\partial}{\partial \rho'} \dot{G}_{k,n}^{\circ}(\rho) = G_{kn}(\rho) \bar{G}_{k'n}(\rho) + G_{kn}(\rho) \dot{M}_{k'n}(\rho) \end{aligned}$$

$+\text{diag}[M_n^{-1}(\rho)\dot{M}_{k'n}(\rho)M_n^{-1}(\rho)]\text{diag}[M_n(\rho)\bar{G}_{kn}(\rho)] - \text{diag}[M_n(\rho)]^{-1}\text{diag}[\dot{M}_{k'n}(\rho)\bar{G}_{kn}(\rho)]$
$-\text{diag}[M_n(\rho)]^{-1}\text{diag}[M_n(\rho)G_{kn}(\rho)\bar{G}_{k'n}(\rho)]-\text{diag}[M_n(\rho)]^{-1}\text{diag}[M_n(\rho)G_{kn}(\rho)\dot{M}_{k'n}(\rho)]$ and $\dot{M}_{kn}(\rho) = M_n(\rho)G_{kn}(\rho)\mathcal{P}_n(\rho) + \mathcal{P}_n(\rho)G'_{kn}(\rho)M_n(\rho)$.

Condition (i)**:** First note that the common term in the denominator of the components in $\frac{\partial}{\partial\delta'}S_n^{ac}(\bar{\delta}_n)$ can be written as $Y_n'(\delta)M_n(\rho)Y_n(\delta) = n\hat{\sigma}_n^2(\bar{\delta}_n)$, where $\hat{\sigma}_n^2(\bar{\delta}_n) = \hat{\sigma}_n^2(\delta_0) + o_p(1)$.
By Assumptions 4 and 5 and continuous mapping theorem, we have, $\bar{F}_{jn}^{\circ}(\bar{\delta}_n) = \bar{F}_{jn}^{\circ}(\delta_0) + o_p(1)$, $\dot{\bar{F}}_n^{\circ}(\bar{\delta}_n) = \dot{\bar{F}}_n^{\circ}(\delta_0) + o_p(1)$, $\bar{G}_{kn}^{\circ}(\bar{\delta}_n) = \bar{G}_{kn}^{\circ}(\delta_0) + o_p(1)$ and $\dot{\bar{G}}_{kn}^{\circ}(\bar{\delta}_n) = \dot{\bar{G}}_{kn}^{\circ}(\delta_0) + o_p(1)$. Then, using a Taylor expansion, terms of the sort $T_{1n}(\bar{\delta}) = \frac{1}{n}Y_n'(\bar{\delta})\bar{F}'_{j'n}(\bar{\delta})M_n(\bar{\rho})\bar{F}_{jn}^{\circ}(\bar{\delta})Y_n(\bar{\delta})$ can be written as, $T_{1n}(\delta_0) + (\bar{\delta} - \delta_0)'\frac{\partial}{\partial\delta}T_{1n}(\delta_0)$, where together with the continuous mapping theorem, Lemma B.2 and Assumptions 3-6 and some tedious algebra, we have $T_{1n}(\bar{\delta}) = T_{1n}(\delta_0) + o_p(1)$.

Condition (ii)**:** The result follows from a direct application of Lemmas B.2 and B.3 using Assumptions 3-6. See Liu and Yang (2015) for further details.

Condition (iii)**:** By Assumptions 3-6 and Lemmas B.2 and B.3, it is easy to see that Φ_n is non-singular for large enough n, and thus $\text{E}\big(\frac{\partial}{\partial\delta'}S_n^{ac}(\delta_0)\big)$ is non-singular for large enough N.

Proof of Theorem 3.3: Recall $\tilde{\beta}_n = (X_n'B_n'(\tilde{\rho}_n)B_n(\tilde{\rho}_n)X_n)^{-1}X_n'B_n'(\tilde{\rho}_n)B_n(\tilde{\rho}_n)A_n(\tilde{\lambda}_n)Y_n$. Then by a Taylor expansion, we have,

$$\sqrt{n}(\tilde{\beta}_n - \beta_0) = (\tfrac{1}{n}X_n'B_n'B_nX_n)^{-1}\tfrac{1}{\sqrt{n}}X_n'B_n'\varepsilon_n - \tfrac{\partial}{\partial\delta}\hat{\beta}_n\Big|_{\delta=\delta_0}\sqrt{n}(\tilde{\delta}_n - \delta_0) + O_p\big(\tfrac{1}{\sqrt{n}}\big), \quad \text{(C-4)}$$

where $\frac{\partial}{\partial\delta}\hat{\beta}_n\Big|_{\delta=\delta_0} = (\mathcal{Q}_n\eta_{jn} + o_p(1),\ \mathcal{Q}_nG_{kn}^sB_n(I_n - X_n\mathcal{Q}_nB_n)X_n\beta_0 + o_p(1))'$, for $\eta_{jn} = B_nF_{jn}X_n\beta_0$, $\mathcal{Q}_n = (X_n'B_n'B_nX_n)^{-1}X_n'B_n'$, and $G_{kn}^s = G'_{kn} + G_{kn}$. By Assumptions 3-5, the asymptotic order of the second component of $\frac{\partial}{\partial\delta}\hat{\beta}_n\Big|_{\delta=\delta_0}$ is $o_p(1)$.

This shows that each component of $\sqrt{n}(\tilde{\beta}_n - \beta_0)$ is a linear-quadratic form in ε_n. Thus, Cramèr-Wold device and the CLT for linear-quadratic form of Kelejian and Prucha (2001) lead to the asymptotic normality of $\sqrt{n}(\tilde{\beta}_n - \beta_0)$. Let The asymptotic mean of $\sqrt{n}(\tilde{\beta}_n - \beta_0)$ is zero and the first-order variance of it can be found using (C-4):

$$\begin{aligned}\tau^2(\tilde{\beta}_n) &= \mathcal{Q}_n\text{Var}(\varepsilon_n)\mathcal{Q}_n' + \mathcal{Q}_n\eta_{jn}\tau^2(\tilde{\delta}_n)\eta_{jn}'\mathcal{Q}_n' - 2\sigma_0^{-2}\mathcal{Q}_n\text{Cov}(\varepsilon_n, R_n(\varepsilon_n))\Phi_n^{-1}\eta_{jn}'\mathcal{Q}_n' \\ &= \mathcal{Q}_n\mathbb{A}_n\mathcal{Q}_n',\end{aligned}$$

where $\mathbb{A}_n = n\sigma_0^2H_n + \eta_{jn}\tau_{jn}^2(\tilde{\delta}_n)\eta_{jn}' + 2\sqrt{n}(\sigma_0^{-2}P_{jn}^ds_n + H_nc_{jn},\ \sigma_0^{-2}Q_{kn}^ds_n + H_nc_{kn})\Phi^{-1}(\eta_{jn}, 0_n)'$, $s_n = \text{E}(\varepsilon_n^3)$ and $R_n(\varepsilon_n)$ is as defined in (3.9).

The limiting distribution of $\sqrt{n}(\tilde{\sigma}_n^2 - \sigma_0^2)$ can be found in a similar manner from

$$\begin{aligned}\sqrt{n}(\tilde{\sigma}_n^2 - \sigma_0^2) &= \sqrt{n}[\tfrac{1}{n}Y_n'(\tilde{\delta}_n)M_n(\tilde{\rho}_n)Y_n(\tilde{\delta}_n) - \sigma_0^2] \\ &= \tfrac{1}{\sqrt{n}}(\varepsilon_n'\varepsilon_n - n\sigma_0^2) + 2\sqrt{n}(\tilde{\delta}_n - \delta_0)'\tfrac{1}{n}\tfrac{\partial}{\partial\delta}Y_n'(\tilde{\delta}_n)M_n(\tilde{\rho}_n)Y_n(\tilde{\delta}_n)\Big|_{\delta=\delta_0} + o_p(1),\end{aligned}$$

which has a limiting mean of zero and first-order variance that can be easily derived but not needed in light of Footnote 4. Thus by the consistency of $\tilde{\delta}_n$ from Theorem 3.2, we have the consistency of $\tilde{\sigma}_n^2$, in particular, it is $\sqrt{n}$-consistent.

Proof of Corollary 3.2: To prove the consistency of $\tilde{\tau}_n^2(\tilde{\delta}_n)$ as an estimator of $\tau_n^2(\tilde{\delta}_n)$, we need to prove (a) $\tilde{\Phi}_n - \Phi_n = o_p(1)$, and (b) $\tilde{\tau}_n^2(S_n^{ac}) - \tau_n^2(S_n^{ac}) = o_p(1)$.

First, (a) follows from the proof of Theorem 3.2 (the asymptotic normality part). To prove (b), as $\tilde{\sigma}_n^2 = \sigma_0^2 + o_p(1)$ by Theorem 3.3, it suffices to show that, by the consistency of $\tilde{\theta}_n$ and referring to (3.10),

$$\tfrac{1}{n}\textstyle\sum_{i=1}^n \big(\varepsilon_{n,i}^2 \Upsilon_{sn,i}^2 - \mathrm{Var}(\varepsilon_{n,i}\Upsilon_{sn,i})\big) = o_p(1),$$

where $\Upsilon_{sn,i}$ are elements from the vector $\Upsilon_{n,i} = (\tilde{\zeta}_{jn,i}+\tilde{p}_{jn,ii}\tilde{\varepsilon}_{n,i}+\tilde{c}_{jn,i},\ \tilde{\xi}_{kn,i}+\tilde{q}_{kn,ii}\tilde{\varepsilon}_{n,i}+\tilde{c}_{kn,i})'$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$. This follows immediately by the Central Limit theorem for quadratic forms by Kelejian and Prucha (2001) and the poof of Theorem 1 of Baltagi and Yang (2013).

The consistency of $\tilde{\tau}_n^2(\tilde{\beta}_n)$ follows that of $\tilde{\tau}_n^2(\tilde{\lambda}_n)$ and the consistency of $\tilde{\theta}_n$.

Proof of Theorem 4.1:

Proof of Theorem 4.2:

Proof of Theorem 4.3:

Proof of Theorem 5.1:

Proof of Theorem 5.2:

References

[1] Anselin, L., 1988a. Lagrange multiplier test diagnostics for spatial dependence and heterogeneity. *Geographical Analysis* 20, 1-17.

[2] Anselin, L., 1988b. *Spatial Econometrics: Methods and Models.* The Netherlands: Kluwer Academic Publishers.

[3] Badinger, H., Egger, P., 2011. Estimation of higher-order spatial autoregressive cross-section models with heteroskedastic disturbances. *Papers in Regional Science* 90, 213-235.

[4] Baltagi, B., Yang, Z. L., 2013. Heteroskedasticity and non-normality robust LM tests of spatial dependence. *Regional Science and Urban Economics* 43, 725-739.

[5] Breusch, T., Pagan, A., 1979. A simple test for heteroskedasticity and random coefficient variation. *Econometrica* 47, 1287-1294.

[6] Burridge, P., 1980. On the Cliff-Ord test for spatial correlation. *Journal of the Regional Statistical Society B* 42, 107-108.

[7] Elhorst, J. P., Lacombe, D. J., Piras, G., 2012. On model specification and parameter space definitions in higher order spatial econometric models. *Regional Science and Urban Economics* 42, 211-220.

[8] Huber, P. J., 1964. Robust estimation of a location parameter. *Annals of Mathematical Statistics* 35, 73-101.

[9] Jin, F., Lee, L. F., 2012. Approximated likelihood and root estimators for spatial interaction in spatial autoregressive models. *Regional Science and Urban Economics* 42, 446-458.

[10] Jin, F., Lee, L. F., 2019. GEL estimation and tests for spatial autoregressive models. *Journal of Econometrics* 208, 585-612.

[11] Kelejian H. H., Prucha, I. R., 2001. On the asymptotic distribution of the Moran I test statistic with applications. *Journal of Econometrics* 104, 219-257

[12] Kelejian H. H., Prucha, I. R., 2010. Specification and estimation of spatial autoregressive models with autoregressive and heteroskedastic disturbances. *Journal of Econometrics* 157, 53-67.

[13] Lee, L. F., 2004. Asymptotic distributions of quasi-maximum likelihood estimators for spatial autoregressive models. *Econometrica* 72, 1899-1925.

[14] Lee, L. F., Liu, X., 2010. Efficient GMM estimation of high order spatial autoregressive models with autoregressive disturbances. *Econometric Theory* 26, 187-230.

[15] Lin, X., Lee, L. F., 2010. GMM estimation of spatial autoregressive models with unknown heteroskedasticity. *Journal of Econometrics* 157, 34-52.

[16] Liu, X., Prucha, I. R., 2018. A robust test for network generated dependence. *Journal of Econometrics* 207, 92-113.

[17] Liu, S. F., Yang, Z. L., 2015. Modified QML Estimation of Spatial Autoregressive Models with Unknown Heteroskedasticity and Nonnormality. *Regional Science and Urban Economics* 52, 50-70.

[18] van der Vaart, A. W., 1998. *Asymptotic Statistics.* Cambridge University Press.

Table 1a. Empirical Mean(rmse)[sd] of Estimators of δ of `SARAR`(3,3)
Case when the regular QMLE is inconsistent under heteroskedasticity
$n = 100,\ \beta = (3, 1, 1)',\ \sigma = 1$, `Circular Neighbours, REG-1`

δ_0	QMLE	MQMLE	QMLE	MQMLE	QMLE	MQMLE
	Normal Errors		Mixed Normal Errors		Log-Normal Errors	
.5	.322[.192](.142)	.432[.231](.178)	.334[.229](.224)	.435[.231](.227)	.339[.254](.196)	.441[.231](.226)
.3	.172[.213](.170)	.275[.176](.146)	.187[.215](.183)	.320[.227](.225)	.174[.221](.181)	.290[.259](.235)
.1	.069[.182](.123)	.043[.221](.187)	.052[.227](.225)	.126[.231](.192)	.055[.267](.218)	.125[.230](.228)
.4	.066[.197](.191)	.327[.177](.175)	.058[.193](.186)	.421[.180](.178)	.175[.180](.177)	.363[.179](.178)
.2	.163[.225](.222)	.188[.180](.180)	.159[.144](.140)	.170[.152](.152)	.158[.222](.218)	.180[.162](.161)
.1	.258[.191](.165)	.097[.210](.201)	.246[.190](.168)	.140[.173](.171)	.263[.194](.167)	.120[.168](.156)
.4	.289[.263](.239)	.387[.231](.228)	.296[.225](.223)	.431[.192](.178)	.315[.227](.211)	.430[.227](.225)
.2	.142[.176](.167)	.144[.159](.153)	.142[.189](.180)	.166[.156](.154)	.153[.176](.170)	.235[.225](.223)
.1	.020[.219](.204)	.069[.184](.177)	.022[.222](.208)	.055[.177](.174)	.038[.203](.193)	.127[.167](.157)
.5	.054[.198](.187)	.479[.180](.174)	.048[.194](.183)	.516[.182](.175)	.078[.191](.180)	.446[.178](.174)
.3	.190[.254](.229)	.255[.178](.169)	.184[.253](.225)	.283[.183](.168)	.173[.163](.123)	.324[.145](.144)
.1	.119[.142](.142)	.086[.194](.194)	.120[.124](.123)	.079[.193](.193)	.097[.141](.141)	.125[.148](.147)
.2	.123[.238](.225)	.157[.266](.262)	.100[.196](.178)	.172[.167](.164)	.088[.161](.136)	.243[.169](.163)
.2	.161[.167](.163)	.180[.226](.225)	.150[.180](.173)	.184[.216](.215)	.135[.190](.178)	.163[.229](.226)
.2	.162[.166](.161)	.177[.222](.221)	.148[.185](.177)	.180[.212](.211)	.132[.189](.176)	.162[.221](.217)
.2	.035[.198](.169)	.267[.190](.176)	.303[.193](.193)	.218[.183](.171)	.168[.192](.184)	.168[.183](.175)
.2	.101[.218](.194)	.192[.236](.235)	.105[.210](.187)	.184[.235](.230)	.109[.219](.199)	.210[.249](.236)
.2	.105[.193](.177)	.198[.192](.179)	.109[.168](.153)	.186[.173](.155)	.120[.185](.174)	.211[.183](.173)
.0	-.070[.245](.235)	-.051[.228](.227)	-.077[.159](.147)	-.050[.181](.176)	-.075[.148](.137)	-.058[.152](.146)
.0	-.026[.154](.152)	-.035[.223](.223)	-.025[.169](.167)	-.029[.231](.230)	-.025[.171](.169)	-.045[.220](.215)
.0	-.029[.156](.153)	-.035[.226](.224)	-.027[.173](.171)	-.027[.228](.227)	-.026[.181](.179)	-.040[.207](.203)
.0	-.061[.195](.165)	-.028[.188](.177)	-.562[.201](.195)	-.043[.182](.170)	-.084[.200](.187)	-.056[.184](.176)
.0	-.051[.181](.173)	-.059[.216](.210)	-.038[.177](.173)	-.053[.221](.216)	-.038[.182](.178)	-.028[.198](.197)
.0	-.086[.197](.184)	-.042[.176](.166)	-.073[.192](.183)	-.090[.168](.157)	-.087[.232](.231)	-.077[.174](.166)
.5	.273[.232](.225)	.474[.237](.194)	.287[.232](.224)	.468[.239](.231)	.301[.297](.220)	.530[.235](.229)
.3	.126[.241](.168)	.250[.228](.224)	.134[.243](.178)	.344[.198](.154)	.138[.259](.203)	.254[.229](.225)
.1	.130[.232](.221)	.092[.234](.183)	.111[.231](.231)	.085[.235](.196)	.099[.131](.124)	.076[.232](.183)
.0	-.102[.199](.198)	.064[.184](.183)	-.160[.198](.197)	.012[.187](.186)	-.038[.184](.184)	.096[.183](.182)
.0	.067[.214](.204)	.031[.202](.200)	.046[.208](.203)	.034[.243](.242)	.053[.203](.203)	.045[.197](.195)
.0	.268[.297](.234)	.015[.228](.207)	.246[.158](.152)	.012[.151](.150)	.239[.159](.154)	.044[.182](.160)
.0	-.053[.205](.201)	-.004[.209](.209)	-.044[.226](.223)	-.010[.206](.205)	-.045[.170](.166)	.000[.179](.179)
.0	.027[.169](.167)	.032[.237](.235)	.030[.195](.193)	.028[.233](.232)	.015[.176](.175)	.020[.226](.225)
.0	.015[.181](.181)	.024[.236](.235)	.023[.196](.195)	.025[.226](.224)	.001[.194](.194)	.010[.221](.221)
.5	.066[.204](.192)	.560[.187](.170)	.036[.201](.186)	.465[.186](.171)	.006[.193](.178)	.519[.186](.172)
.3	.125[.263](.196)	.324[.184](.146)	.125[.265](.199)	.335[.238](.234)	.125[.270](.206)	.338[.238](.234)
.1	.021[.215](.190)	.090[.212](.187)	.024[.217](.197)	.095[.192](.171)	.012[.220](.200)	.124[.194](.173)

Table 1b. Empirical Mean(rmse)[sd] of Estimators of δ of SARAR(3,3)
Case when the regular QMLE is inconsistent under heteroskedasticity
$n = 250,\ \beta = (3, 1, 1)',\ \sigma = 1$, Circular Neighbours, REG-1

δ_0	QMLE	MQMLE	QMLE	MQMLE	QMLE	MQMLE
	Normal Errors		Mixed Normal Errors		Log-Normal Errors	
.5	.343[.127](.122)	.501[.134](.128)	.346[.169](.121)	.503[.133](.127)	.360[.124](.120)	.503[.129](.125)
.3	.213[.154](.127)	.303[.190](.175)	.214[.109](.103)	.303[.104](.101)	.216[.117](.114)	.307[.118](.117)
.1	.056[.127](.122)	.090[.126](.124)	.052[.167](.120)	.090[.127](.125)	.033[.125](.122)	.089[.125](.124)
.4	.471[.145](.145)	.414[.141](.139)	.474[.136](.129)	.405[.140](.139)	.455[.140](.140)	.409[.138](.137)
.2	.184[.148](.147)	.204[.128](.128)	.175[.105](.105)	.209[.135](.135)	.176[.117](.116)	.202[.123](.123)
.1	.287[.145](.141)	.109[.138](.137)	.273[.143](.140)	.104[.140](.139)	.236[.142](.140)	.107[.141](.140)
.4	.326[.122](.121)	.424[.127](.125)	.321[.126](.112)	.408[.127](.124)	.337[.098](.088)	.409[.125](.123)
.2	.186[.119](.118)	.199[.119](.118)	.175[.106](.104)	.205[.109](.109)	.178[.103](.103)	.207[.108](.108)
.1	.010[.122](.120)	.091[.125](.124)	.010[.124](.105)	.003[.125](.124)	.004[.123](.121)	.106[.134](.128)
.5	.420[.146](.145)	.472[.138](.138)	.452[.137](.134)	.509[.140](.140)	.422[.097](.089)	.508[.098](.098)
.3	.194[.188](.155)	.266[.132](.131)	.201[.108](.108)	.303[.131](.131)	.197[.118](.115)	.307[.138](.137)
.1	.190[.363](.352)	.104[.144](.143)	.185[.151](.140)	.106[.146](.145)	.216[.141](.139)	.110[.140](.130)
.2	.145[.191](.183)	.190[.187](.187)	.130[.104](.101)	.202[.105](.103)	.135[.109](.109)	.197[.106](.106)
.2	.180[.098](.095)	.201[.134](.134)	.169[.107](.102)	.190[.132](.132)	.173[.109](.106)	.201[.103](.103)
.2	.179[.108](.106)	.200[.147](.147)	.164[.123](.107)	.187[.144](.143)	.172[.102](.102)	.202[.104](.104)
.2	.043[.157](.155)	.203[.145](.142)	.060[.141](.122)	.204[.143](.140)	.080[.148](.146)	.206[.142](.139)
.2	.125[.138](.116)	.203[.138](.128)	.138[.104](.109)	.208[.123](.122)	.123[.141](.117)	.208[.139](.131)
.2	.156[.169](.163)	.215[.122](.121)	.165[.117](.108)	.206[.121](.121)	.149[.106](.106)	.204[.110](.102)
.0	-.051[.120](.192)	-.014[.119](.119)	-.045[.121](.106)	-.012[.108](.104)	-.049[.123](.122)	-.019[.108](.107)
.0	-.008[.096](.096)	.000[.113](.113)	-.005[.096](.096)	-.001[.127](.127)	-.006[.104](.103)	-.009[.106](.106)
.0	-.009[.105](.105)	-.001[.114](.114)	-.005[.107](.107)	.000[.135](.135)	-.008[.106](.106)	-.002[.102](.102)
.0	-.202[.163](.160)	-.017[.146](.143)	-.212[.063](.088)	-.008[.146](.142)	-.182[.157](.154)	-.006[.142](.139)
.0	-.021[.118](.116)	-.025[.122](.121)	-.016[.112](.111)	-.003[.103](.100)	-.024[.102](.102)	-.009[.110](.109)
.0	-.045[.117](.116)	-.056[.122](.122)	-.041[.107](.107)	-.005[.125](.118)	-.042[.099](.095)	-.007[.125](.122)
.5	.187[.142](.127)	.511[.152](.134)	.232[.172](.158)	.506[.148](.134)	.243[.156](.146)	.506[.147](.132)
.3	.148[.121](.115)	.313[.120](.116)	.163[.106](.104)	.308[.112](.104)	.158[.112](.107)	.310[.116](.107)
.1	.142[.134](.123)	.097[.128](.122)	.121[.132](.124)	.107[.129](.124)	.118[.133](.124)	.089[.130](.124)
.0	.420[.174](.160)	.025[.173](.151)	.386[.169](.157)	.047[.170](.139)	.372[.167](.156)	.049[.167](.152)
.0	.084[.117](.114)	.068[.126](.125)	.075[.104](.105)	.080[.127](.126)	.080[.118](.116)	.063[.126](.124)
.0	.309[.154](.144)	.015[.134](.131)	.266[.151](.129)	.015[.140](.136)	.246[.148](.142)	.019[.142](.138)
.0	-.070[.126](.125)	.012[.123](.123)	-.059[.161](.154)	.021[.128](.127)	-.032[.120](.119)	.019[.127](.119)
.0	.018[.099](.097)	.035[.116](.116)	.024[.106](.103)	.043[.116](.116)	.021[.108](.106)	.034[.106](.102)
.0	.014[.110](.109)	.036[.117](.117)	.018[.120](.118)	.038[.107](.107)	.016[.109](.108)	.032[.105](.102)
.5	.437[.143](.142)	.510[.141](.136)	.424[.131](.124)	.503[.140](.136)	.404[.139](.137)	.508[.139](.135)
.3	.176[.117](.112)	.216[.131](.129)	.171[.108](.106)	.308[.132](.129)	.176[.118](.113)	.309[.129](.127)
.1	.030[.118](.117)	.091[.125](.123)	.031[.108](.107)	.102[.124](.123)	.031[.119](.118)	.102[.125](.123)

Table 1c. Empirical Mean(rmse)[sd] of Estimators of δ of SARAR(3,3)

Case when the regular QMLE is inconsistent under heteroskedasticity

$n = 500$, $\beta = (3, 1, 1)'$, $\sigma = 1$, `Circular Neighbours, REG-1`

δ_0	QMLE	MQMLE	QMLE	MQMLE	QMLE	MQMLE
	Normal Errors		Mixed Normal Errors		Log-Normal Errors	
.5	.304[.110](.098)	.506[.089](.089)	.333[.118](.095)	.507[.063](.063)	.322[.128](.088)	.508[.063](.063)
.3	.210[.140](.074)	.292[.041](.041)	.224[.130](.062)	.304[.054](.054)	.220[.114](.061)	.309[.051](.051)
.1	.115[.053](.044)	.098[.097](.097)	.096[.093](.093)	.099[.096](.096)	.088[.094](.094)	.094[.094](.094)
.4	.625[.073](.051)	.410[.088](.088)	.596[.098](.078)	.408[.083](.083)	.587[.068](.064)	.405[.092](.092)
.2	.200[.051](.051)	.208[.082](.081)	.183[.101](.096)	.207[.076](.074)	.185[.091](.081)	.203[.079](.077)
.1	.453[.103](.090)	.108[.084](.084)	.396[.055](.046)	.103[.036](.035)	.375[.124](.065)	.104[.056](.056)
.4	.356[.092](.086)	.402[.072](.072)	.348[.094](.086)	.404[.072](.072)	.360[.072](.065)	.406[.072](.072)
.2	.206[.088](.087)	.198[.081](.081)	.200[.088](.088)	.193[.080](.072)	.200[.093](.093)	.204[.075](.072)
.1	.070[.074](.074)	.100[.096](.096)	.054[.068](.066)	.100[.092](.092)	.047[.129](.097)	.091[.092](.092)
.5	.484[.063](.056)	.509[.076](.070)	.490[.096](.095)	.505[.090](.087)	.486[.128](.080)	.507[.090](.090)
.3	.182[.073](.068)	.303[.095](.092)	.183[.102](.092)	.303[.043](.043)	.183[.116](.081)	.303[.051](.050)
.1	.299[.125](.075)	.101[.084](.084)	.276[.107](.071)	.109[.085](.075)	.278[.112](.072)	.104[.074](.074)
.2	.146[.066](.066)	.198[.050](.042)	.147[.062](.053)	.199[.099](.099)	.147[.114](.065)	.192[.053](.051)
.2	.184[.065](.063)	.196[.080](.080)	.183[.065](.063)	.199[.075](.075)	.180[.073](.070)	.200[.080](.080)
.2	.183[.065](.063)	.195[.078](.077)	.183[.066](.064)	.197[.075](.075)	.180[.071](.068)	.198[.082](.082)
.2	.184[.068](.067)	.196[.066](.066)	.188[.071](.069)	.198[.088](.088)	.199[.072](.072)	.195[.025](.024)
.2	.142[.101](.083)	.191[.034](.033)	.143[.101](.083)	.198[.088](.088)	.136[.107](.086)	.197[.046](.043)
.2	.177[.101](.085)	.196[.032](.031)	.170[.109](.104)	.197[.081](.081)	.173[.101](.077)	.190[.043](.041)
.0	-.015[.113](.112)	.000[.098](.098)	-.020[.081](.064)	-.007[.092](.091)	-.017[.043](.042)	-.003[.081](.081)
.0	-.001[.060](.060)	.002[.080](.080)	-.004[.061](.061)	-.003[.079](.078)	.002[.066](.066)	.003[.083](.083)
.0	.000[.059](.059)	.002[.076](.076)	-.005[.062](.062)	-.004[.077](.076)	.001[.067](.067)	.001[.079](.079)
.0	-.134[.100](.077)	-.009[.081](.081)	-.122[.089](.070)	-.009[.072](.071)	-.113[.124](.094)	-.003[.062](.062)
.0	-.012[.083](.082)	-.005[.056](.054)	-.013[.080](.079)	-.005[.049](.047)	-.019[.082](.079)	-.002[.048](.044)
.0	-.018[.081](.081)	-.001[.092](.092)	-.017[.104](.091)	-.009[.040](.039)	-.025[.112](.093)	-.005[.033](.030)
.5	.160[.097](.086)	.501[.068](.068)	.154[.094](.071)	.507[.034](.034)	.195[.134](.088)	.502[.088](.088)
.3	.162[.077](.069)	.300[.065](.065)	.163[.085](.075)	.296[.079](.079)	.179[.079](.077)	.295[.092](.092)
.1	-.195[.084](.076)	.095[.073](.073)	.193[.095](.093)	.093[.080](.080)	.139[.107](.093)	-.096[.073](.073)
.0	.554[.094](.046)	.007[.077](.077)	.587[.093](.082)	.010[.075](.068)	.549[.134](.085)	.004[.075](.075)
.0	.102[.074](.074)	.010[.075](.060)	.096[.109](.097)	.003[.079](.079)	.083[.068](.067)	.005[.035](.033)
.0	.496[.125](.053)	.005[.098](.098)	.486[.105](.983)	.010[.034](.030)	.357[.090](.069)	.002[.076](.076)
.0	-.076[.098](.098)	-.001[.093](.089)	-.086[.115](.098)	-.010[.066](.065)	-.044[.084](.079)	-.009[.063](.062)
.0	.016[.064](.062)	.008[.091](.090)	.015[.063](.062)	.008[.093](.092)	.019[.073](.070)	.001[.098](.097)
.0	.013[.066](.065)	.008[.090](.089)	.011[.068](.067)	.008[.093](.092)	.017[.078](.076)	.010[.100](.100)
.5	.526[.077](.076)	.507[.068](.068)	.544[.069](.065)	.497[.082](.082)	.491[.067](.066)	.493[.072](.072)
.3	.193[.138](.087)	.298[.080](.075)	.198[.103](.085)	.292[.080](.076)	.192[.105](.097)	.295[.093](.087)
.1	.057[.116](.082)	.097[.088](.088)	.059[.081](.051)	.093[.036](.033)	.053[.103](.069)	.095[.043](.040)

Table 2a. Empirical Mean(rmse)[sd] of Estimators of δ of SARAR(3,3)
Case when the regular QMLE is consistent under heteroskedasticity
$n = 100,\ \beta = (3, 1, 1)',\ \sigma = 1$, Queen Contiguity, REG-1

δ_0	QMLE	MQMLE	QMLE	MQMLE	QMLE	MQMLE
	Normal Errors		Mixed Normal Errors		Log-Normal Errors	
.5	.412[.210](.190)	.463[.200](.188)	.490[.210](.189)	.484[.190](.181)	.452[.207](.187)	.495[.181](.174)
.3	.298[.230](.212)	.253[.238](.226)	.320[.223](.213)	.272[.223](.220)	.262[.214](.200)	.313[.215](.204)
.1	.100[.225](.224)	.058[.224](.224)	.065[.225](.224)	.077[.224](.224)	.095[.224](.224)	.076[.223](.223)
.4	.395[.224](.224)	.394[.226](.226)	.397[.224](.224)	.386[.224](.223)	.405[.223](.223)	.398[.210](.209)
.2	.200[.192](.192)	.222[.195](.194)	.214[.182](.181)	.203[.174](.172)	.199[.183](.183)	.223[.171](.170)
.1	.093[.235](.234)	.104[.229](.225)	.099[.234](.234)	.073[.199](.186)	.066[.232](.230)	.093[.193](.183)
.4	.366[.216](.213)	.385[.190](.189)	.395[.207](.200)	.388[.192](.189)	.386[.196](.193)	.398[.184](.183)
.2	.186[.222](.221)	.174[.224](.223)	.176[.209](.206)	.176[.210](.209)	.187[.207](.207)	.183[.209](.208)
.1	.090[.224](.224)	.088[.224](.224)	.082[.224](.224)	.077[.224](.224)	.087[.223](.223)	.097[.223](.223)
.5	.497[.224](.223)	.465[.222](.212)	.487[.223](.221)	.454[.207](.201)	.497[.193](.193)	.496[.201](.196)
.3	.295[.193](.181)	.265[.168](.166)	.323[.172](.164)	.277[.158](.156)	.264[.228](.227)	.286[.226](.226)
.1	.097[.234](.233)	.096[.231](.231)	.009[.182](.182)	.013[.172](.172)	.105[.233](.232)	.088[.230](.230)
.2	.167[.206](.204)	.188[.198](.197)	.195[.191](.187)	.175[.189](.187)	.163[.183](.179)	.183[.172](.171)
.2	.200[.233](.225)	.185[.224](.223)	.193[.223](.222)	.154[.190](.187)	.184[.224](.217)	.160[.225](.220)
.2	.167[.168](.163)	.165[.175](.170)	.152[.165](.161)	.197[.166](.163)	.187[.224](.224)	.170[.224](.224)
.2	.196[.150](.146)	.165[.149](.147)	.162[.224](.224)	.177[.224](.224)	.165[.224](.224)	.196[.229](.226)
.2	.178[.171](.170)	.198[.178](.178)	.188[.177](.176)	.186[.167](.167)	.167[.226](.226)	.200[.226](.226)
.2	.186[.218](.214)	.198[.214](.213)	.194[.176](.176)	.197[.197](.197)	.187[.187](.185)	.173[.178](.177)
.0	-.005[.185](.185)	.001[.183](.183)	-.013[.185](.184)	-.012[.185](.185)	-.011[.173](.173)	-.012[.170](.170)
.0	-.040[.229](.224)	-.006[.225](.224)	-.040[.194](.194)	-.034[.170](.170)	-.047[.219](.215)	-.047[.224](.223)
.0	-.031[.174](.173)	-.034[.182](.180)	-.036[.178](.176)	-.032[.180](.178)	-.027[.224](.224)	-.021[.225](.225)
.0	-.069[.170](.161)	-.047[.161](.156)	-.050[.226](.225)	-.021[.174](.174)	-.047[.224](.223)	-.016[.220](.219)
.0	-.031[.198](.197)	.004[.207](.207)	-.032[.171](.169)	-.004[.173](.173)	-.027[.172](.170)	.002[.179](.179)
.0	-.052[.239](.235)	-.017[.234](.234)	-.023[.177](.177)	.000[.192](.192)	-.053[.230](.230)	-.021[.231](.231)
.5	.498[.228](.190)	.474[.224](.209)	.584[.187](.187)	.477[.224](.205)	.486[.191](.191)	.498[.224](.221)
.3	.316[.227](.214)	.266[.228](.224)	.315[.226](.221)	.278[.228](.226)	.291[.194](.194)	.301[.225](.223)
.1	.090[.184](.166)	.086[.175](.168)	.091[.227](.226)	.001[.226](.225)	.070[.268](.248)	.104[.262](.256)
.0	.010[.207](.190)	.011[.216](.195)	.010[.208](.208)	.122[.210](.184)	.039[.195](.181)	.017[.185](.185)
.0	.017[.215](.193)	.010[.155](.140)	.014[.197](.197)	.009[.165](.155)	.011[.188](.188)	.036[.234](.234)
.0	.015[.155](.155)	.003[.159](.159)	.024[.177](.169)	.009[.140](.140)	.027[.173](.161)	.005[.185](.184)
.0	.042[.226](.222)	.022[.224](.223)	.055[.233](.226)	.034[.221](.218)	.043[.210](.205)	.024[.207](.206)
.0	.040[.225](.224)	.012[.224](.224)	.060[.224](.223)	.009[.188](.186)	.044[.229](.220)	.012[.217](.217)
.0	.051[.228](.228)	-.004[.225](.225)	.052[.176](.170)	-.004[.154](.154)	.044[.170](.167)	.003[.192](.191)
.5	.492[.225](.223)	.464[.225](.223)	.538[.226](.223)	.494[.224](.223)	.497[.198](.198)	.450[.223](.221)
.3	.315[.202](.202)	.303[.170](.161)	.265[.186](.186)	.277[.194](.194)	.274[.280](.241)	.294[.245](.235)
.1	.096[.236](.221)	.066[.198](.195)	.050[.184](.184)	.095[.191](.187)	.093[.190](.190)	.095[.176](.171)

Table 2b. Empirical Mean(rmse)[sd] of Estimators of δ of `SARAR`(3,3)
Case when the regular QMLE is consistent under heteroskedasticity
$n = 250,\ \beta = (3,1,1)',\ \sigma = 1,$ `Queen Contiguity, REG-1`

δ_0	QMLE	MQMLE	QMLE	MQMLE	QMLE	MQMLE
	Normal Errors		Mixed Normal Errors		Log-Normal Errors	
.5	.433[.118](.116)	.503[.115](.114)	.443[.171](.155)	.506[.145](.139)	.463[.164](.149)	.503[.143](.141)
.3	.256[.122](.120)	.307[.117](.116)	.304[.120](.119)	.305[.160](.157)	.243[.120](.119)	.308[.117](.116)
.1	.087[.118](.118)	.090[.116](.116)	.076[.185](.180)	.099[.157](.157)	.063[.117](.116)	.099[.116](.116)
.4	.422[.117](.117)	.404[.115](.115)	.403[.117](.117)	.403[.115](.115)	.413[.116](.116)	.406[.115](.115)
.2	.236[.126](.126)	.209[.121](.121)	.233[.125](.125)	.204[.121](.121)	.226[.125](.124)	.206[.120](.120)
.1	.089[.125](.125)	.108[.123](.123)	.106[.125](.125)	.094[.121](.121)	.094[.123](.123)	.091[.122](.122)
.4	.376[.116](.116)	.407[.115](.115)	.379[.167](.165)	.406[.154](.152)	.396[.148](.148)	.394[.144](.144)
.2	.199[.119](.119)	.194[.119](.119)	.198[.120](.120)	.198[.119](.119)	.189[.119](.119)	.191[.119](.119)
.1	.092[.119](.119)	.096[.116](.116)	.098[.178](.178)	.094[.168](.167)	.087[.117](.117)	.094[.117](.117)
.5	.498[.116](.116)	.478[.115](.115)	.477[.163](.160)	.500[.149](.148)	.496[.152](.147)	.467[.146](.142)
.3	.288[.123](.123)	.289[.123](.122)	.277[.125](.125)	.308[.123](.123)	.265[.123](.123)	.300[.122](.122)
.1	.087[.126](.126)	.096[.122](.122)	.097[.125](.125)	.099[.121](.121)	.095[.123](.123)	.103[.123](.123)
.2	.198[.142](.140)	.196[.140](.139)	.177[.152](.149)	.194[.153](.150)	.173[.142](.139)	.198[.139](.137)
.2	.197[.118](.117)	.191[.118](.118)	.166[.117](.117)	.197[.117](.117)	.187[.117](.117)	.198[.116](.116)
.2	.176[.118](.117)	.197[.117](.117)	.197[.118](.118)	.198[.117](.117)	.163[.117](.117)	.198[.116](.116)
.2	.187[.117](.116)	.195[.116](.116)	.202[.117](.117)	.199[.117](.117)	.196[.117](.117)	.196[.116](.116)
.2	.188[.120](.120)	.196[.120](.120)	.192[.119](.119)	.196[.119](.119)	.193[.119](.119)	.199[.119](.119)
.2	.188[.121](.121)	.196[.121](.121)	.188[.121](.121)	.198[.119](.119)	.203[.120](.120)	.192[.119](.119)
.0	-.002[.135](.135)	-.007[.139](.139)	-.009[.138](.138)	-.010[.142](.142)	-.011[.134](.134)	-.012[.139](.138)
.0	-.013[.117](.117)	-.014[.118](.118)	-.017[.116](.116)	-.011[.117](.117)	-.016[.116](.116)	-.016[.117](.117)
.0	-.004[.117](.117)	-.006[.116](.116)	-.011[.116](.116)	-.009[.116](.116)	-.014[.116](.116)	-.011[.116](.116)
.0	-.023[.117](.117)	-.011[.117](.117)	-.018[.117](.117)	-.005[.117](.117)	-.020[.117](.117)	-.008[.117](.117)
.0	-.021[.120](.120)	-.008[.120](.120)	-.021[.120](.120)	-.005[.120](.120)	-.017[.120](.119)	-.002[.120](.120)
.0	-.025[.122](.122)	-.018[.121](.121)	-.026[.121](.121)	-.014[.120](.120)	-.020[.120](.120)	-.009[.120](.120)
.5	.484[.123](.119)	.508[.123](.120)	.539[.121](.118)	.502[.122](.119)	.498[.120](.118)	.505[.119](.118)
.3	.296[.119](.116)	.309[.119](.117)	.320[.118](.117)	.305[.118](.117)	.307[.118](.117)	.210[.183](.117)
.1	.167[.119](.117)	.091[.117](.117)	.124[.118](.116)	.091[.116](.115)	.073[.170](.156)	.098[.150](.150)
.0	.027[.127](.124)	.006[.128](.125)	.022[.126](.123)	.001[.127](.124)	.011[.126](.124)	.006[.125](.123)
.0	.008[.123](.122)	.012[.124](.123)	.020[.124](.122)	.002[.123](.122)	.009[.124](.122)	.007[.123](.122)
.0	.014[.124](.123)	.011[.122](.122)	.050[.123](.123)	.010[.121](.121)	.039[.123](.122)	.009[.120](.120)
.0	.055[.119](.118)	.013[.117](.117)	.015[.119](.118)	.004[.117](.117)	.050[.118](.117)	.008[.116](.116)
.0	.066[.119](.118)	.009[.118](.118)	.051[.120](.119)	.002[.117](.117)	.051[.119](.119)	.009[.116](.116)
.0	.024[.118](.117)	-.005[.116](.116)	.045[.184](.112)	-.005[.116](.116)	.040[.173](.169)	-.001[.147](.147)
.5	.485[.117](.116)	.504[.115](.114)	.474[.117](.116)	.508[.141](.137)	.492[.160](.144)	.507[.136](.132)
.3	.299[.122](.119)	.302[.118](.118)	.254[.121](.119)	.303[.117](.117)	.287[.121](.119)	.300[.117](.116)
.1	.095[.123](.121)	.091[.118](.118)	.093[.123](.121)	.091[.118](.118)	.094[.121](.120)	.093[.117](.117)

Table 2c. Empirical Mean(rmse)[sd] of Estimators of δ of SARAR(3,3)
Case when the regular QMLE is consistent under heteroskedasticity
$n = 500,\ \beta = (3, 1, 1)',\ \sigma = 1$, Queen Contiguity, REG-1

δ_0	QMLE	MQMLE	QMLE	MQMLE	QMLE	MQMLE
	Normal Errors		Mixed Normal Errors		Log-Normal Errors	
.5	.495[.033](.021)	.506[.022](.022)	.494[.055](.055)	.509[.022](.022)	.503[.081](.081)	.498[.061](.061)
.3	.313[.086](.075)	.308[.040](.040)	.323[.086](.074)	.301[.086](.086)	.295[.073](.068)	.298[.069](.069)
.1	.075[.097](.088)	.099[.040](.040)	.096[.072](.064)	.099[.080](.080)	.090[.064](.058)	.099[.081](.081)
.4	.424[.035](.032)	.401[.055](.055)	.456[.087](.063)	.399[.077](.075)	.423[.055](.055)	.394[.098](.098)
.2	.214[.075](.060)	.208[.075](.075)	.255[.072](.060)	.199[.077](.077)	.226[.042](.042)	.204[.083](.082)
.1	.147[.057](.052)	.100[.084](.084)	.133[.079](.079)	.099[.093](.093)	.137[.082](.082)	.105[.089](.089)
.4	.395[.068](.065)	.399[.079](.074)	.395[.045](.044)	.398[.041](.035)	.387[.091](.091)	.399[.079](.079)
.2	.199[.078](.078)	.195[.078](.076)	.191[.069](.069)	.198[.076](.075)	.197[.089](.087)	.198[.065](.064)
.1	.087[.066](.066)	.092[.053](.052)	.100[.058](.058)	.095[.064](.063)	.085[.054](.054)	.098[.067](.067)
.5	.497[.081](.081)	.499[.024](.022)	.474[.073](.073)	.500[.078](.078)	.487[.042](.042)	.499[.042](.042)
.3	.288[.049](.045)	.304[.090](.090)	.288[.100](.100)	.305[.103](.103)	.309[.092](.092)	.507[.099](.099)
.1	.099[.022](.022)	.103[.091](.091)	.076[.101](.095)	.102[.060](.060)	.089[.072](.067)	.100[.071](.071)
.2	.188[.064](.053)	.196[.032](.030)	.188[.111](.110)	.199[.109](.108)	.199[.043](.043)	.197[.111](.110)
.2	.189[.093](.088)	.199[.023](.022)	.186[.079](.069)	.195[.070](.070)	.192[.033](.029)	.196[.049](.048)
.2	.199[.087](.083)	.200[.022](.022)	.187[.087](.080)	.199[.060](.060)	.197[.089](.089)	.197[.041](.041)
.2	.200[.031](.031)	.200[.090](.090)	.197[.098](.098)	.196[.087](.087)	.198[.051](.051)	.198[.063](.062)
.2	.200[.044](.044)	.199[.056](.055)	.200[.072](.072)	.199[.082](.082)	.191[.074](.072)	.196[.063](.061)
.2	.199[.053](.052)	.196[.045](.044)	.199[.078](.075)	.193[.075](.073)	.200[.053](.053)	.198[.045](.044)
.0	.000[.044](.044)	.000[.053](.053)	-.004[.033](.032)	-.005[.060](.054)	-.007[.039](.037)	-.006[.025](.024)
.0	-.001[.044](.037)	-.013[.048](.041)	-.005[.046](.045)	-.005[.081](.072)	-.017[.064](.051)	-.009[.061](.061)
.0	-.008[.036](.032)	-.010[.058](.054)	-.016[.022](.021)	-.016[.070](.062)	-.009[.051](.051)	-.008[.051](.051)
.0	-.011[.031](.030)	-.006[.070](.060)	-.013[.082](.076)	-.007[.081](.079)	-.011[.091](.091)	-.006[.051](.051)
.0	-.002[.052](.052)	.001[.051](.051)	-.013[.055](.054)	-.005[.090](.085)	-.004[.044](.043)	.003[.043](.043)
.0	-.002[.073](.071)	.000[.085](.085)	-.006[.045](.045)	.001[.045](.045)	-.009[.053](.051)	-.003[.044](.044)
.5	.423[.071](.052)	.500[.070](.081)	.422[.070](.070)	.496[.071](.071)	.492[.074](.069)	.498[.073](.073)
.3	.263[.061](.040)	.307[.080](.092)	.288[.071](.071)	.296[.056](.060)	.294[.066](.066)	.298[.081](.081)
.1	.092[.065](.059)	.098[.069](.063)	.078[.081](.081)	.099[.061](.061)	.092[.097](.097)	.099[.031](.031)
.0	.005[.022](.020)	.043[.070](.064)	.006[.067](.067)	.003[.073](.068)	.048[.077](.077)	.010[.070](.066)
.0	.080[.099](.091)	.028[.068](.066)	.049[.096](.096)	.007[.064](.062)	.057[.057](.057)	.004[.066](.065)
.0	.007[.103](.098)	-.003[.080](.080)	.027[.082](.090)	.002[.054](.054)	.058[.097](.088)	-.002[.049](.049)
.0	.049[.057](.049)	.006[.082](.081)	.040[.072](.072)	.007[.081](.081)	.014[.061](.055)	.008[.082](.081)
.0	.042[.054](.049)	-.008 [.060](.058)	.042[.084](.084)	.005[.065](.064)	.003[.052](.047)	-.003[.054](.054)
.0	.033[.045](.039)	-.009[.065](.061)	.044[.071](.071)	.001[.056](.056)	.045[.038](.030)	-.003[.091](.091)
.5	.465[.036](.023)	.508[.103](.101)	.456[.093](.093)	.500[.078](.078)	.484[.096](.096)	.499[.100](.099)
.3	.255[.053](.041)	.294[.023](.023)	.288[.091](.091)	.293[.091](.091)	.294[.055](.045)	.298[.051](.051)
.1	.063[.069](.052)	.099[.031](.030)	.098[.092](.092)	.099[.081](.081)	.083[.092](.092)	.095[.021](.020)

Table 3. Empirical Mean(rmse)[sd] of Estimators of β when the regular QMLE is inconsistent under heteroskedasticity, `Circular Neighbours, REG-1`

n	Const.	β_0	QMLE	MQMLE	QMLE	MQMLE	QMLE	MQMLE
			Normal Errors		Mixed Normal Errors		Log-Normal Errors	
100	1	3	2.635[0.929](0.759)	2.984[1.148](1.035)	2.947[0.963](0.835)	2.904[1.146](1.078)	3.132[0.922](0.766)	2.635[1.106](1.008)
		1	0.968[0.145](0.141)	0.963[0.146](0.142)	0.970[0.145](0.142)	0.970[0.145](0.142)	0.971[0.150](0.147)	0.973[0.150](0.148)
		1	0.992[0.169](0.169)	0.976[0.170](0.169)	0.986[0.174](0.174)	0.970[0.172](0.169)	0.987[0.169](0.169)	0.978[0.169](0.167)
	2	3	3.229[0.748](0.712)	3.182[0.756](0.585)	3.122[0.710](0.675)	3.071[0.539](0.479)	3.147[0.662](0.639)	2.925[0.945](0.910)
		1	0.986[0.139](0.139)	0.975[0.141](0.139)	0.986[0.143](0.142)	0.979[0.146](0.145)	0.995[0.145](0.145)	0.988[0.145](0.145)
		1	0.991[0.166](0.166)	0.976[0.164](0.162)	0.988[0.172](0.171)	0.973[0.164](0.162)	1.000[0.172](0.172)	0.977[0.172](0.171)
	3	3	3.114[0.619](0.604)	3.075[0.386](0.353)	3.180[0.667](0.422)	3.061[0.798](0.795)	3.042[0.702](0.659)	3.147[0.874](0.862)
		1	0.998[0.138](0.138)	0.996[0.141](0.141)	0.991[0.140](0.140)	0.990[0.140](0.139)	0.993[0.145](0.145)	0.993[0.146](0.146)
		1	0.999[0.161](0.161)	0.985[0.153](0.152)	1.005[0.169](0.169)	0.996[0.161](0.161)	1.006[0.163](0.163)	0.997[0.163](0.163)
250	1	3	2.763[0.699](0.589)	3.195[0.840](0.776)	2.983[0.724](0.614)	3.053[0.866](0.811)	2.764[0.739](0.642)	2.833[0.827](0.777)
		1	0.980[0.101](0.099)	0.979[0.102](0.100)	0.976[0.100](0.098)	0.973[0.103](0.099)	0.986[0.094](0.093)	0.984[0.096](0.095)
		1	0.980[0.101](0.098)	0.978[0.104](0.101)	0.987[0.102](0.101)	0.983[0.102](0.101)	0.982[0.103](0.101)	0.977[0.103](0.101)
	2	3	3.005[0.501](0.497)	3.135[0.759](0.745)	3.075[0.576](0.566)	3.116[0.584](0.407)	3.076[0.529](0.524)	2.931[0.736](0.722)
		1	0.989[0.093](0.092)	0.978[0.103](0.101)	0.991[0.097](0.096)	0.981[0.103](0.102)	0.988[0.093](0.093)	0.979[0.100](0.098)
		1	0.990[0.102](0.101)	0.979[0.109](0.107)	0.985[0.100](0.099)	0.973[0.108](0.105)	0.996[0.095](0.095)	0.985[0.101](0.100)
	3	3	3.038[0.407](0.398)	2.996[0.427](0.426)	3.125[0.425](0.425)	3.005[0.530](0.528)	3.011[0.446](0.434)	2.989[0.531](0.531)
		1	0.998[0.094](0.094)	0.996[0.096](0.096)	1.000[0.094](0.094)	0.998[0.094](0.094)	0.995[0.092](0.092)	0.994[0.092](0.091)
		1	0.990[0.098](0.097)	0.986[0.099](0.098)	0.994[0.098](0.098)	0.991[0.100](0.099)	0.998[0.101](0.101)	0.995[0.103](0.103)
500	1	3	3.388[0.601](0.459)	3.134[0.578](0.544)	3.113[0.525](0.451)	3.220[0.660](0.623)	3.233[0.595](0.489)	3.122[0.658](0.623)
		1	0.972[0.077](0.071)	0.981[0.075](0.073)	0.980[0.074](0.072)	0.984[0.077](0.075)	0.977[0.080](0.076)	0.980[0.080](0.078)
		1	0.969[0.077](0.071)	0.982[0.073](0.071)	0.975[0.076](0.072)	0.985[0.074](0.073)	0.974[0.078](0.073)	0.982[0.074](0.071)
	2	3	2.988[0.441](0.441)	3.159[0.571](0.548)	2.998[0.218](0.217)	3.134[0.630](0.612)	3.000[0.338](0.338)	3.149[0.636](0.605)
		1	0.984[0.072](0.070)	0.978[0.081](0.078)	0.984[0.076](0.074)	0.974[0.086](0.082)	0.985[0.071](0.069)	0.975[0.084](0.080)
		1	0.979[0.075](0.072)	0.981[0.080](0.077)	0.979[0.072](0.069)	0.975[0.082](0.078)	0.980[0.071](0.068)	0.975[0.080](0.076)
	3	3	3.022[0.246](0.238)	3.018[0.295](0.295)	3.059[0.242](0.235)	3.003[0.274](0.274)	3.045[0.269](0.258)	3.003[0.297](0.297)
		1	0.999[0.065](0.065)	0.999[0.065](0.065)	0.996[0.066](0.066)	0.996[0.068](0.067)	0.999[0.065](0.065)	0.999[0.065](0.065)
		1	0.998[0.062](0.062)	0.997[0.062](0.062)	1.001[0.062](0.062)	1.000[0.062](0.062)	1.000[0.061](0.061)	0.999[0.061](0.061)

Table 4. Empirical Mean(rmse)[sd] of Estimators of β when the regular QMLE is consistent under heteroskedasticity, `Queen Contiguity, REG-1`

n	Const.	β_0	QMLE	MQMLE	QMLE	MQMLE	QMLE	MQMLE
			Normal Errors		Mixed Normal Errors		Log-Normal Errors	
100	1	3	2.558[2.292](1.562)	2.781[2.327](1.362)	2.603[2.268](1.508)	2.693[2.099](1.223)	2.647[2.243](1.454)	2.605[1.870](1.083)
		1	0.955[0.171](0.165)	0.965[0.169](0.166)	0.953[0.175](0.168)	0.965[0.173](0.169)	0.951[0.178](0.171)	0.965[0.176](0.173)
		1	0.954[0.170](0.164)	0.958[0.171](0.165)	0.954[0.171](0.165)	0.959[0.170](0.165)	0.953[0.172](0.166)	0.960[0.170](0.165)
	2	3	2.755[1.361](1.295)	2.599[1.138](1.094)	3.091[1.542](1.409)	2.817[1.776](1.705)	2.708[1.424](1.367)	2.954[1.084](1.034)
		1	0.957[0.169](0.164)	0.964[0.168](0.164)	0.972[0.163](0.160)	0.980[0.162](0.160)	0.970[0.157](0.154)	0.976[0.160](0.158)
		1	0.968[0.166](0.163)	0.971[0.162](0.160)	0.963[0.159](0.154)	0.969[0.155](0.152)	0.977[0.156](0.155)	0.983[0.153](0.152)
	3	3	3.075[2.031](1.834)	2.972[1.384](1.242)	3.194[1.980](1.731)	2.823[1.270](1.165)	3.098[1.952](1.740)	2.739[1.995](1.902)
		1	0.957[0.166](0.161)	0.960[0.167](0.162)	0.957[0.168](0.163)	0.964[0.168](0.164)	0.965[0.158](0.154)	0.974[0.158](0.156)
		1	0.969[0.168](0.165)	0.971[0.171](0.168)	0.967[0.162](0.158)	0.973[0.162](0.160)	0.970[0.161](0.158)	0.975[0.162](0.160)
250	1	3	3.194[1.667](1.317)	3.016[1.302](1.101)	3.375[1.539](0.965)	3.425[1.053](0.768)	3.469[1.513](0.978)	2.842[0.878](0.831)
		1	0.968[0.114](0.110)	0.979[0.112](0.109)	0.971[0.116](0.112)	0.984[0.109](0.108)	0.975[0.110](0.107)	0.984[0.109](0.108)
		1	0.967[0.122](0.117)	0.981[0.117](0.115)	0.969[0.115](0.111)	0.984[0.109](0.107)	0.974[0.116](0.113)	0.985[0.115](0.114)
	2	3	2.510[1.429](1.391)	3.491[1.392](1.356)	3.267[1.446](1.435)	2.506[1.167](1.126)	3.272[1.426](1.415)	2.592[1.304](1.250)
		1	0.981[0.108](0.106)	0.987[0.106](0.105)	0.976[0.114](0.112)	0.981[0.110](0.108)	0.976[0.121](0.118)	0.977[0.119](0.117)
		1	0.975[0.115](0.112)	0.980[0.113](0.111)	0.980[0.116](0.114)	0.983[0.114](0.113)	0.977[0.118](0.116)	0.977[0.119](0.117)
	3	3	2.615[1.070](0.976)	3.338[1.060](1.032)	2.727[1.190](1.066)	3.491[1.180](1.124)	2.706[1.179](1.062)	2.511[1.243](1.184)
		1	0.979[0.111](0.109)	0.983[0.113](0.111)	0.975[0.111](0.108)	0.979[0.111](0.109)	0.978[0.111](0.108)	0.984[0.110](0.109)
		1	0.984[0.114](0.113)	0.989[0.113](0.113)	0.977[0.118](0.115)	0.982[0.117](0.115)	0.986[0.115](0.114)	0.992[0.115](0.114)
500	1	3	2.843[0.289](0.289)	2.751[0.294](0.294)	3.024[1.246](0.930)	2.542[0.731](0.715)	2.876[1.022](0.877)	2.973[1.421](1.167)
		1	0.972[0.095](0.090)	0.986[0.084](0.082)	0.976[0.091](0.088)	0.988[0.084](0.083)	0.974[0.094](0.090)	0.983[0.089](0.087)
		1	0.972[0.092](0.087)	0.986[0.081](0.080)	0.972[0.088](0.084)	0.986[0.085](0.083)	0.974[0.089](0.085)	0.984[0.083](0.082)
	2	3	3.240[0.723](0.723)	3.236[0.753](0.753)	3.138[0.675](0.535)	2.881[1.004](0.965)	2.986[1.144](1.089)	3.135[1.406](1.396)
		1	0.979[0.092](0.090)	0.982[0.091](0.089)	0.984[0.093](0.091)	0.984[0.094](0.092)	0.982[0.090](0.088)	0.988[0.088](0.087)
		1	0.984[0.088](0.086)	0.986[0.085](0.084)	0.983[0.084](0.083)	0.982[0.087](0.085)	0.986[0.086](0.085)	0.992[0.083](0.083)
	3	3	3.286[0.728](0.704)	3.256[0.773](0.754)	3.130[0.767](0.740)	3.235[0.716](0.698)	3.138[1.706](1.677)	3.126[1.695](1.681)
		1	0.989[0.082](0.081)	0.991[0.082](0.082)	0.992[0.077](0.077)	0.994[0.078](0.078)	0.991[0.084](0.083)	0.993[0.084](0.084)
		1	0.990[0.078](0.078)	0.992[0.078](0.078)	0.990[0.082](0.082)	0.992[0.082](0.082)	0.990[0.083](0.083)	0.992[0.082](0.081)