Supplementary Material

for "Spatial Dynamic Panel Data Models with Interactive Fixed Effects: M-Estimation and Inference with Small T"

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In this Supplementary Material, we provide the detailed proof of Theorem 4.1, and a more comprehensive set of Monte Carlo results.

I. Proof of Theorem 4.1

First, the result $H_{nT}(\hat{\boldsymbol{\psi}})-H_{nT}(\boldsymbol{\psi}_0) \xrightarrow{p} 0$ is implied by result **(b)** in the proof of Theorem 3.1. Next, the result $\hat{\Sigma}_{nT} - \Sigma_{nT}(\boldsymbol{\psi}_0) \xrightarrow{p} 0$ follows from

(a) $\frac{1}{nT}\sum_{i=1}^n[\hat{\mathbf{g}}_i\hat{\mathbf{g}}_i' - \mathrm{E}(\mathbf{g}_i\mathbf{g}_i')] = o_p(1)$, and **(b)** $\frac{1}{nT}[\Upsilon(\hat{\boldsymbol{\psi}}) - \Upsilon(\boldsymbol{\psi}_0)] = o_p(1)$.

By the expression of Υ presented in Section 4, the proof of (**b**) is straightforward by the MVT and consistency of $\hat{\boldsymbol{\psi}}_{\mathtt{M}}$. We focus on the proof of (**a**), which follows if (i) $\frac{1}{nT}\sum_{i=1}^n(\hat{\mathbf{g}}_i\hat{\mathbf{g}}_i' - \mathbf{g}_i^*\mathbf{g}_i^{*\prime}) \xrightarrow{p} 0$, (ii) $\sum_{i=1}^n \mathbf{g}_i^*\mathbf{g}_i^{*\prime} = \sum_{i=1}^n \mathbf{g}_i\mathbf{g}_i'$, and (iii) $\frac{1}{nT}\sum_{i=1}^n[\mathbf{g}_i\mathbf{g}_i' - \mathrm{E}(\mathbf{g}_i\mathbf{g}_i')] \xrightarrow{p} 0$. The proof of (i) is straightforward by MVT. We focus on the proof of (ii) and (iii).

Proof of (ii): Recall that $g_{ri}^* = g_{\Pi i}^*, g_{\Psi i}^*, g_{\Phi i}^*$ is obtained by replacing v_{it} by z_{it}^* in $g_{ri} = g_{\Pi i}, g_{\Psi i}, g_{\Phi i}$ presented in (4.3), (4.4) and (4.6). It suffices to show that, for $r = 1, \ldots, 4$, $\nu = 1, 2, 3$, and $\iota = 1, \ldots, 5 + k_\phi$,

$$\sum_{i=1}^n g_{\kappa,i}^* g_{\varpi,i}^{*\prime} = \sum_{i=1}^n g_{\kappa,i} g_{\varpi,i}', \quad \text{for } \kappa, \varpi = \Pi_r, \Psi_\nu, \Phi_\iota.$$

First, assuming without loss of generality that Π_{it} are scalars, we show that for $r, \nu = 1, \ldots, 4$, $\sum_{i=1}^n g_{\Pi_r,i}^* g_{\Pi_\nu,i}^* = \sum_{i=1}^n g_{\Pi_r,i} g_{\Pi_\nu,i}$, we have by (4.3),

$$g_{\Pi_r,i}^* = \sum_{t=1}^T \Pi_{r,it} v_{it} + \sum_{t=1}^T \Pi_{r,it} b_i'(\Gamma_0 f_{t0}) = g_{\Pi_r,i} + \sum_{t=1}^T \Pi_{r,it} b_i'(\Gamma_0 f_{t0}),$$

where b_i' is the ith row of B_{30}. Denote the vector of the diagonal elements of a matrix $\mathrm{diag}(A) = (a_{11}, a_{22}, \ldots, a_{nn})'$, we can write

$$\begin{aligned}
&\sum_{i=1}^n g_{\Pi_r,i}^* g_{\Pi_\nu,i}^* = \sum_{i=1}^n g_{\Pi_r,i} g_{\Pi_\nu,i} + \sum_{i=1}^n g_{\Pi_r,i}[\sum_{t=1}^T \Pi_{\nu,it} b_i'(\Gamma_0 f_{t0})] \\
&\quad + \sum_{i=1}^n g_{\Pi_\nu,i}[\sum_{t=1}^T \Pi_{r,it} b_i'(\Gamma_0 f_{t0})] + \sum_{i=1}^n[\sum_{t=1}^T \Pi_{r,it} b_i'(\Gamma_0 f_{t0})][\sum_{s=1}^T \Pi_{\nu,it} b_i'(\Gamma_0 f_{s0})] \\
= \;& \sum_{i=1}^n (g_{\Pi_r,i} g_{\Pi_\nu,i}) + g_{\Pi_r}' \mathrm{diag}(\mathbb{D}_{\Pi,\nu} F_0 \Gamma_0' B_{30}') + g_{\Pi_\nu}' \mathrm{diag}(\mathbb{D}_{\Pi,r} F_0 \Gamma_0' B_{30}') \\
&\quad + \mathrm{diag}(\mathbb{D}_{\Pi,r} F_0 \Gamma_0' B_{30}')' \mathrm{diag}(\mathbb{D}_{\Pi,\nu} F_0 \Gamma_0' B_{30}'),
\end{aligned}$$

where $\mathbb{D}_{\Pi,r} = (\Pi_{r,1}, \Pi_{r,2}, \ldots, \Pi_{r,T})$ is a $n \times T$ matrix whose tth column corresponds to $\Pi_{r,t}$, the subvectors of Π_r corresponding to $t = 1, \ldots, T$. According to the expressions of Π_r in (3.2), $\mathbb{D}_{\Pi,r}$ can be written as $\mathbb{D}_{\Pi,r} = \mathbb{K}_r M_{F0}$, where $\mathbb{K}_r$ are some $n \times T$ matrices constructed from $\mathbf{X}, W_\ell, \ell = 1,2,3$ and $\boldsymbol{\psi}_0$. Therefore we have $\mathbb{D}_{\Pi,r} F_0 \Gamma_0' B_{30}' = \mathbb{K}_r M_{F0} F_0 \Gamma_0' B_{30}' = \mathbf{0}_{n\times n}$. Hence the result $\sum_{i=1}^n g^*_{\Pi_r,i} g^*_{\Pi_\nu,i} = \sum_{i=1}^n g_{\Pi_r,i} g_{\Pi_\nu,i}$ follows.

Second, we show that $\sum_{i=1}^n g^*_{\Psi_r,i} g^*_{\Psi_\nu,i} = \sum_{i=1}^n g_{\Psi_r,i} g_{\Psi_\nu,i}$, for $r, \nu = 1,2,3$. By (4.4), the bilinear term $g^*_{\Psi_r,i}$ can be written as,

$$g^*_{\Psi_r,i} = \textstyle\sum_{t=1}^T \xi_{r,it} v_{it} + \sum_{t=1}^T \xi_{r,it} b_i'(\Gamma_0 f_{t0}) = g_{\Psi_r,i} + \sum_{t=1}^T \xi_{r,it} b_i'(\Gamma_0 f_{t0}).$$

So, we can write $\sum_{i=1}^n g^*_{\Psi_r,i} g^*_{\Psi_\nu,i}$ as

$$\begin{aligned} &\textstyle\sum_{i=1}^n g^*_{\Psi_r,i} g^*_{\Psi_\nu,i} = \sum_{i=1}^n g_{\Psi_r,i} g_{\Psi_\nu,i} + \sum_{i=1}^n g_{\Psi_r,i} [\sum_{t=1}^T \xi_{\nu,it} b_i'(\Gamma_0 f_{t0})] \\ &\quad + \textstyle\sum_{i=1}^n g_{\Psi_\nu,i} [\sum_{t=1}^T \xi_{r,it} b_i'(\Gamma_0 f_{t0})] + \sum_{i=1}^n [\sum_{t=1}^T \xi_{r,it} b_i'(\Gamma_0 f_{t0})][\sum_{s=1}^T \xi_{\nu,it} b_i'(\Gamma_0 f_{s0})] \\ = &\textstyle\sum_{i=1}^n g_{\Psi_r,i} g_{\Psi_\nu,i} + g'_{\Psi_r} \mathrm{diag}(\mathbb{D}_{\xi,\nu} F_0 \Gamma_0' B_{30}') + g'_{\Psi_\nu} \mathrm{diag}(\mathbb{D}_{\xi,r} F_0 \Gamma_0' B_{30}') \\ &+ \mathrm{diag}(\mathbb{D}_{\xi,r} F_0 \Gamma_0' B_{30}')' \mathrm{diag}(\mathbb{D}_{\xi,\nu} F_0 \Gamma_0' B_{30}'), \end{aligned}$$

where $\mathbb{D}_{\xi,r}$ is a $n \times T$ matrix whose t-th column is $\xi_{r,t} = \Psi_{r,t+} y_0$. According to the expressions of Ψ_r given in (3.2), $\mathbb{D}_{\xi,r}$ can also be written as $\mathbb{K}_r M_{F0}$, where $\mathbb{K}_r$ are some $n \times T$ matrices constructed from $y_0, \mathbf{X}, W_\ell, \ell = 1,2,3$ and $\boldsymbol{\psi}_0$. Therefore we have $\mathbb{D}_{\Psi,r} F_0 \Gamma_0' B_{30}' = \mathbf{0}_{n\times n}$, and the result $\sum_{i=1}^n g^*_{\Psi_r,i} g^*_{\Psi_\nu,i} = \sum_{i=1}^n g_{\Psi_r,i} g_{\Psi_\nu,i}$ follows.

Third, we show that $\sum_{i=1}^n g^*_{\Phi_r,i} g^*_{\Phi_\nu,i} = \sum_{i=1}^n g_{\Phi_r,i} g_{\Phi_\nu,i}$ for $r = 1, \ldots, 5 + k_\gamma$. By (4.6), the quadratic term $g^*_{\Phi_r,i}$can be written as

$$\begin{aligned} &g^*_{\Phi_r,i} = \textstyle\sum_{t=1}^T z^*_{it} \varphi_{r,it} + \sum_{t=1}^T (z^*_{it} z^d_{r,it} - d_{it}) \\ = &\textstyle\sum_{t=1}^T v_{it} \varphi_{r,it} + \sum_{t=1}^T (v_{it} z^d_{r,it} - d_{it}) + \sum_{t=1}^T b_i'(\Gamma_0 f_{t0})(\varphi_{r,it} + z^d_{r,it}) \\ = &g_{\Phi_r,i} + \textstyle\sum_{t=1}^T b_i'(\Gamma_0 f_{t0})(\varphi_{r,it} + z^d_{r,it}) = g_{\Phi_r,i} + \sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \varphi^*_{r,it} \end{aligned}$$

where $\varphi^*_{r,it} = \varphi_{r,it} + z^d_{r,it}$. Then, we can write

$$\begin{aligned} &\textstyle\sum_{i=1}^n g^*_{\Phi_r,i} g^*_{\Phi_\nu,i} = \sum_{i=1}^n g_{\Phi_r,i} g_{\Phi_\nu,i} + \sum_{i=1}^n [g_{\Phi_r,i} \sum_{s=1}^T b_i'(\Gamma_0 f_{s0}) \varphi^*_{\nu,is}] \\ &+ \textstyle\sum_{i=1}^n [g_{\Phi_\nu,i} \sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \varphi^*_{r,it}] + \sum_{i=1}^n [\sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \varphi^*_{r,it}][\sum_{s=1}^T b_i'(\Gamma_0 f_{s0}) \varphi^*_{\nu,is}] \\ = &\textstyle\sum_{i=1}^n g_{\Phi_r,i} g_{\Phi_\nu,i} + g'_{\Phi_r} \mathrm{diag}(\mathbb{D}_{\varphi,\nu} F_0 \Gamma_0' B_{30}') + g'_{\Phi_\nu} \mathrm{diag}(\mathbb{D}_{\varphi,r} F_0 \Gamma_0' B_{30}') \\ &+ \mathrm{diag}(\mathbb{D}_{\varphi,r} F_0 \Gamma_0' B_{30}')' \mathrm{diag}(\mathbb{D}_{\varphi,\nu} F_0 \Gamma_0' B_{30}') = \textstyle\sum_{i=1}^n g_{\Phi_r,i} g_{\Phi_\nu,i} \end{aligned}$$

where $\mathbb{D}_{\varphi,r}$ is a $n \times T$ matrix whose tth column is $\varphi_{r,t} = \sum_{s=1}^T \Phi_{r,ts} z^*_s$. Similarly, according to expressions of Φ_r in (3.2), we have $\mathbb{D}_{\varphi,r} F_0 \Gamma_0' B_{30}' = \mathbf{0}_{n\times n}$. Therefore the result

$\sum_{i=1}^n g^*_{\Phi_r,i} g^*_{\Phi_\nu,i} = \sum_{i=1}^n g_{\Phi_r,i} g_{\Phi_\nu,i}$ follows.

Fourth, we examine the cross-product terms. Similarly to the early cases, we have

$$
\begin{aligned}
& \textstyle\sum_{i=1}^n g^*_{\Pi_r,i} g^*_{\Psi_\nu,i} \\
= & \textstyle\sum_{i=1}^n g_{\Pi_r,i} g_{\Psi_\nu,i} + \sum_{i=1}^n g_{\Pi_r,i}[\sum_{t=1}^T \xi_{\nu,it} b_i'(\Gamma_0 f_{t0})] \\
& + \textstyle\sum_{i=1}^n g_{\Psi_\nu,i}[\sum_{t=1}^T \Pi_{r,it} b_i'(\Gamma_0 f_{t0})] + \sum_{i=1}^n [\sum_{t=1}^T \Pi_{r,it} b_i'(\Gamma_0 f_{t0})][\sum_{s=1}^T \xi_{\nu,it} b_i'(\Gamma_0 f_{s0})] \\
= & \textstyle\sum_{i=1}^n g_{\Pi_r,i} g_{\Psi_\nu,i} + g'_{\Pi_r} \mathrm{diag}(\mathbb{D}_{\xi,\nu} F_0 \Gamma_0' B_{30}') + g'_{\Psi_\nu} \mathrm{diag}(\mathbb{D}_{\Pi,r} F_0 \Gamma_0' B_{30}') \\
& + \mathrm{diag}(\mathbb{D}_{\Pi,r} F_0 \Gamma_0' B_{30}')' \mathrm{diag}(\mathbb{D}_{\xi,\nu} F_0 \Gamma_0' B_{30}') = \textstyle\sum_{i=1}^n g_{\Pi_r,i} g_{\Psi_\nu,i}.
\end{aligned}
$$

$$
\begin{aligned}
& \textstyle\sum_{i=1}^n g^*_{\Pi_r,i} g^*_{\Phi_\nu,i} \\
= & \textstyle\sum_{i=1}^n g_{\Pi_r,i} g_{\Phi_\nu,i} + \sum_{i=1}^n g_{\Pi_r,i}[\sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \varphi^*_{\nu,it}] \\
& + \textstyle\sum_{i=1}^n g_{\Phi_\nu,i}[\sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \Pi_{r,it}] + \sum_{i=1}^n [\sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \Pi_{r,it}][\sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \varphi^*_{\nu,it}] \\
= & \textstyle\sum_{i=1}^n g_{\Pi_r,i} g_{\Phi_\nu,i} + g'_{\Pi_r} \mathrm{diag}(\mathbb{D}_{\varphi,\nu} F_0 \Gamma_0' B_{30}') + g'_{\Phi_\nu} \mathrm{diag}(\mathbb{D}_{\Pi,r} F_0 \Gamma_0' B_{30}') \\
& + \mathrm{diag}(\mathbb{D}_{\Pi,r} F_0 \Gamma_0' B_{30}') \mathrm{diag}(\mathbb{D}_{\varphi,\nu} F_0 \Gamma_0' B_{30}') = \textstyle\sum_{i=1}^n g_{\Pi_r,i} g_{\Phi_\nu,i},
\end{aligned}
$$

$$
\begin{aligned}
& \textstyle\sum_{i=1}^n g^*_{\Psi_r,i} g^*_{\Phi_\nu,i} \\
= & \textstyle\sum_{i=1}^n g_{\Psi_r,i} g_{\Phi_\nu,i} + \sum_{i=1}^n g_{\Psi_r,i}[\sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \varphi^*_{\nu,it}] \\
& + \textstyle\sum_{i=1}^n g_{\Phi_\nu,i}[\sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \xi_{r,it}] + \sum_{i=1}^n [\sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \xi_{r,it}][\sum_{t=1}^T b_i'(\Gamma_0 f_{t0}) \varphi^*_{\nu,it}] \\
= & \textstyle\sum_{i=1}^n g_{\Psi_r,i} g_{\Phi_\nu,i} + g'_{\Psi_r}) \mathrm{diag}(\mathbb{D}_{\varphi,\nu} F_0 \Gamma_0' B_{30}') + g'_{\Phi_\nu}) \mathrm{diag}(\mathbb{D}_{\xi,r} F_0 \Gamma_0' B_{30}') \\
& + \mathrm{diag}(\mathbb{D}_{\xi,r} F_0 \Gamma_0' B_{30}')' \mathrm{diag}(\mathbb{D}_{\varphi,\nu} F_0 \Gamma_0' B_{30}') = \textstyle\sum_{i=1}^n g_{\Psi_r,i} g_{\Phi_\nu,i}.
\end{aligned}
$$

Summarizing all the results above, we have $\sum_{i=1}^n \mathbf{g}_i^* \mathbf{g}_i^{*\prime} = \sum_{i=1}^n \mathbf{g}_i \mathbf{g}_i'$.

Proof of (iii). To show $\frac{1}{nT}\sum_{i=1}^n [\mathbf{g}_i \mathbf{g}_i' - \mathrm{E}(\mathbf{g}_i \mathbf{g}_i')] \xrightarrow{p} 0$, it suffices to show that

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^n [g_{\kappa,i} g'_{\varpi,i} - \mathrm{E}(g_{\kappa,i} g'_{\varpi,i})], \quad \text{for } \kappa, \varpi = \Pi_r, \Psi_\nu, \Phi_\iota,$$

where $r = 1, \ldots, 4$, $\nu = 1, 2, 3$, and $\iota = 1, \ldots, 5 + k_\phi$.

First, we show $\frac{1}{nT}\sum_{i=1}^n [g_{\Pi_r i} g_{\Pi_\nu i} - \mathrm{E}(g_{\Pi_r i} g_{\Pi_\nu i})] \xrightarrow{p} 0$. Letting $v_{i\cdot} = (v_{i1}, v_{i2}, \ldots, v_{iT})'$ be the subvector of $\mathbf{v}$ that picks up the elements with the same i, and $\Pi_{r,i\cdot}$ be defined similarly, we can write

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^n [g_{\Pi_r i} g_{\Pi_\nu i} - \mathrm{E}(g_{\Pi_r i} g_{\Pi_\nu i})] = \tfrac{1}{nT}\sum_{i=1}^n \Pi'_{r,i\cdot}(v_{i\cdot} v'_{i\cdot} - \sigma^2_{v0} I_T)\Pi_{\nu,i\cdot} \equiv \tfrac{1}{nT}\sum_{i=1}^n U_{n,i}$$

By Assumptions A and B, $U_{n,i}$ are independent across i. Elements of $\Pi_r, r = 1, \ldots, 4$ are uniformly bounded by Assumptions C, D, E, and Lemma A.1. Then, it is straightforward to show that $\sum_{i=1}^n U_{n,i} = o_p(1)$ by Chebyshev's inequality.

Second, we show $\frac{1}{nT}\sum_{i=1}^{n}[g_{\Psi_r i}g_{\Psi_\nu i}-\mathrm{E}(g_{\Psi_r i}g_{\Psi_\nu i})] \xrightarrow{p} 0$, $r,\nu=1,2,3$. By (4.4), we have

$$\begin{aligned} & \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}[g_{\Psi_r i}g_{\Psi_\nu i}-\mathrm{E}(g_{\Psi_r i}g_{\Psi_\nu i})] \\ = & \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\xi_{r,i\cdot}'(v_{i\cdot}v_{i\cdot}'-\sigma_{v0}^2 I_T)\xi_{\nu,i\cdot}+\frac{\sigma_{v0}^2}{nT}\sum_{i=1}^{n}[\xi_{r,i\cdot}'\xi_{\nu,i\cdot}-\mathrm{E}(\xi_{r,i\cdot}'\xi_{\nu i\cdot})] \\ = & \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}U_{1n,i}+\frac{1}{nT}\sum_{i=1}^{n}U_{2n,i}. \end{aligned}$$

Let $\{\mathcal{G}_{n,i}\}$ be the increasing sequence of σ -fields generated by $(v_{j1},\ldots,v_{jT},j=1,\ldots,i)$, $i=1,\ldots,n,n\geq 1$. Let $\mathcal{F}_{n,0}$ be the σ -field generated by $(v_0,\Delta y_0)$, and define $\mathcal{F}_{n,i}=\mathcal{F}_{n,0}\otimes\mathcal{G}_{n,i}$. Clearly, $\mathcal{F}_{n,i-1}\subseteq\mathcal{F}_{n,i}$ for each $n\geq 1$, i.e., $\{\mathcal{F}_{n,i}\}_{i=1}^{n}$ is an increasing sequence of σ-fields. As $\xi_{r,i\cdot}'$ is $\mathcal{F}_{n,i-1}$ -measurable, $\mathrm{E}\left(\mathrm{U_{1n,i}}\mid\mathcal{F}_{\mathrm{n,i-1}}\right)=0$. Thus, $\{U_{1n,i},\mathcal{F}_{n,i}\}$ form a M.D. array. Using Assumptions A, B, E, and F, it is easy to see that $\mathrm{E}\left|\mathrm{U_{1n,i}^{1+\epsilon}}\right|\leq\mathrm{K_v}<\infty$, for some $\epsilon>0$. Thus, $\{U_{1n,i}\}$ is uniformly integrable. With constant coefficients $\frac{1}{nT}$, the other two conditions of weak law of large numbers (WLLN) for MD array of Theorem 19.7 of Davidson (1994, p .299) are satisfied. Thus, $\frac{1}{nT}\sum_{i=1}^{n}U_{1n,i}\xrightarrow{p}0$. The convergence of the second term $\frac{1}{nT}\sum_{i=1}^{n}U_{2n,i}\xrightarrow{p}0$ follows from Assumption F.

Third, we show $\frac{1}{nT}\sum_{i=1}^{n}[g_{\Phi_r i}g_{\Phi_\nu i}-\mathrm{E}(g_{\Phi_r i}g_{\Phi_\nu i})]\xrightarrow{p}0$, $r,\nu=1,\ldots,5+k_\phi$, without loss of generality we show $\frac{1}{nT}\sum_{i=1}^{n}[g_{\Phi_r i}^2-\mathrm{E}(g_{\Phi_r i}^2)]\xrightarrow{p}0$, for $r=1,\ldots,5+k_\phi$. Recall expression (4.7), $g_{\Phi,i}=\sum_{t=1}^{T}v_{it}\varphi_{it}+\sum_{t=1}^{T}(v_{it}z_{it}^d-d_{it})$, where $\{\varphi_{it}\}=\varphi_t=\sum_{s=1}^{T}\left(\Phi_{ts}^u+\Phi_{ts}^\ell\right)z_s^*$, and $\left\{z_{it}^d\right\}=z_t^d=\sum_{s=1}^{T}\Phi_{ts}^d z_s^*$, further recall that $z_t^*=v_t+B_{30}\Gamma f_t$, we can write,

$$\begin{aligned} g_{\Phi_r i} = & \textstyle\sum_{t=1}^{T}v_{it}\varphi_{r,it}+\sum_{t=1}^{T}(v_{it}z_{r,it}^d-d_{r,it}) \\ = & \textstyle\sum_{t=1}^{T}v_{it}\varphi_{r,it}^v+\sum_{t=1}^{T}(v_{it}v_{r,it}^*-d_{r,it})+\sum_{t=1}^{T}v_{it}c_{r,it} \\ = & v_{i\cdot}'\varphi_{r,i\cdot}^v+v_{i\cdot}'v_{r,i\cdot}^*-1_T'd_{r,i\cdot}+v_{i\cdot}'c_{r,i\cdot} \end{aligned}$$

where $\{\varphi_{r,it}^v\}=\varphi_t^v=\sum_{s=1}^{T}\left(\Phi_{r,ts}^u+\Phi_{r,ts}^\ell\right)v_s$, $\{v_{r,it}^*\}=v_t^*=\sum_{s=1}^{T}\Phi_{ts}^d v_s$, and $\{c_{r,it}\}=c_{r,t}=\sum_{s=1}^{T}\Phi_{r,ts}B_{30}\Gamma f_s$. It follows that for $r=1,\ldots,5+k_\gamma$,

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}[g_{\Phi_r i}^2-\mathrm{E}(g_{\Phi_r i}^2)]=\sum_{k=1}^{9}U_k,$$

where $U_9=\frac{2}{nT}\sum_{i=1}^{n}\{(v_{i\cdot}'v_{r,i\cdot}^*)(v_{i\cdot}'c_{r,i\cdot})-\mathrm{E}[(v_{i\cdot}'v_{r,i\cdot}^*)(v_{i\cdot}'c_{r,i\cdot})]\}$,

$$\begin{aligned} & U_1=\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\{(v_{i\cdot}'\varphi_{r,i\cdot}^v)^2-\mathrm{E}[(v_{i\cdot}'\varphi_{r,i\cdot}^v)^2]\}, && U_2=\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\{(v_{i\cdot}'v_{r,i\cdot}^*)^2-\mathrm{E}[(v_{i\cdot}'v_{r,i\cdot}^*)^2]\}, \\ & U_3=\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\{(v_{i\cdot}'c_{r,i\cdot})^2-\mathrm{E}[(v_{i\cdot}'c_{r,i\cdot})^2]\}, && U_4=\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}(v_{i\cdot}'\varphi_{r,i\cdot}^v)(v_{i\cdot}'v_{r,i\cdot}^*), \\ & U_5=-\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}(v_{i\cdot}'\varphi_{r,i\cdot}^v)(1_T'd_{r,i\cdot}), && U_6=\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}(v_{i\cdot}'\varphi_{r,i\cdot}^v)(v_{i\cdot}'c_{r,i\cdot}) \\ & U_7=-\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}(v_{i\cdot}'v_{r,i\cdot}^*)(1_T'd_{r,i\cdot}), && U_8=-\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}(v_{i\cdot}'c_{r,i\cdot})(1_T'd_{r,i\cdot}). \end{aligned}$$

For U_1, we can write $(v_{i\cdot}'\varphi_{i\cdot}^v)^2=(\sum_{t=1}^{T}v_{it}\varphi_{it}^v)^2=\sum_{t=1}^{T}(v_{it}\varphi_{it}^v)^2+\sum_{t=1}^{T}\sum_{s\neq t}v_{it}\varphi_{it}^v v_{is}\varphi_{is}^v$.

The second term can be written as $\sum_{t=1}^{T} v_{it}\kappa_{it}$, where $\kappa_{it} = \sum_{s\neq t} \varphi_{it}^{v} v_{is} \varphi_{is}^{v}$. By Assumptions A and B, κ_{it} is independent of v_{it}. Recall that a'_{its} is the ith row of the $n \times n$ matrix $\Phi_{ts}^{u} + \Phi_{ts}^{\ell}$, we have $\mathrm{E}(\kappa_{it}^2) = \sigma_{v0}^6 \sum_t \sum_s a'_{its} a_{its}$, which equals the (i,i) element of matrix $A = (\Phi^u + \Phi^\ell)(\Phi^u + \Phi^\ell)'$. By Assumption E and Lemma A.1, A is uniformly bounded in both row and column sums with elements of uniform order $O\left(h_n^{-1}\right)$. So, by Lemma A.4, we have $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} v_{it}\kappa_{it} = o_p(1)$. For the first term, as v_{it} is independent of φ_{it}^{v}, we have,

$$
\begin{aligned}
& \textstyle\sum_{t=1}^{T}\{(v_{it}\varphi_{it}^{v})^2 - \mathrm{E}[(v_{it}\varphi_{it}^{v})^2]\} \\
= & \textstyle\sum_{t=1}^{T}\{v_{it}^2(\phi_{it}^{u} + \phi_{it}^{\ell})^2 - \mathrm{E}[(v_{it}\varphi_{it}^{v})^2]\} \\
= & \textstyle\sum_{t=1}^{T}\{v_{it}^2(\phi_{it}^{u^2} + \phi_{it}^{\ell 2} + 2\phi_{it}^{u^2}\phi_{it}^{\ell 2}) - \sigma_{v0}^2 \mathrm{E}(\varphi_{it}^{v^2})\} \\
= & \textstyle\sum_{t=1}^{T}(v_{it}^2 - \sigma_{v0}^2)\phi_{it}^{u^2} + \sum_{t=1}^{T}(v_{it}^2 - \sigma_{v0}^2)\phi_{it}^{\ell 2} + 2\sum_{t=1}^{T} v_{it}^2 \phi_{it}^{u}\phi_{it}^{\ell} \\
& \textstyle + \sum_{t=1}^{T}[\phi_{it}^{u^2} - \mathrm{E}(\phi_{it}^{u^2})] + \sum_{t=1}^{T}[\phi_{it}^{\ell 2} - \mathrm{E}(\phi_{it}^{\ell 2})] \\
\equiv & H_{1n,i} + H_{2n,i} + 2H_{3n,i} + H_{4n,i} + H_{5n,i},
\end{aligned}
$$

where $\phi_{it}^{u} = \sum_{s=1}^{T} a_{its}^{u\prime} v_s$, $\phi_{it}^{\ell} = \sum_{s=1}^{T} a_{its}^{\ell\prime} v_s$, and $a_{its}^{u\prime}$, and $a_{its}^{\ell\prime}$ are respectively the ith row of Φ_{ts}^{u}, and Φ_{ts}^{ℓ}.

First consider $H_{1n,i}$. By Assumptions A and B, we have $\mathrm{E}(H_{1n,i}) = 0$, and

$$
\mathrm{E}(H_{1n,i}H_{1n,j}) = \mathrm{E}[\phi_{i\cdot}^{u\prime}(v_{i\cdot}v_{i\cdot}' - \sigma_{v0}^2 I_T)\phi_{i\cdot}^{u}][\phi_{j\cdot}^{u\prime}(v_{j\cdot}v_{j\cdot}' - \sigma_{v0}^2 I_T)\phi_{j\cdot}^{u}] = 0.
$$

Therefore, $\{H_{1n,i}\}$ are uncorrelated across i with expectation 0. Moreover, by Assumptions A and B, we have

$$
\begin{aligned}
& \textstyle \mathrm{E}(H_{1n,i}^2) = \sum_{t=1}^{T}\mathrm{E}[(v_{it}^2 - \sigma_{v0}^2)^2\phi_{it}^{u^4}] = \sum_{t=1}^{T}\{\mathrm{E}[(v_{it}^2 - \sigma_{v0}^2)^2]\mathrm{E}(\phi_{it}^{u^4})\} \\
= & \textstyle (\mu_{v0}^{(4)} - \sigma_{v0}^2)\sum_{t=1}^{T}\mathrm{E}[(\sum_{s=1}^{T} a_{its}^{u\prime} v_s)^4] \\
= & \textstyle (\mu_{v0}^{(4)} - \sigma_{v0}^2)\sum_{t=1}^{T}\mathrm{E}[\sum_{p=1}^{T}\sum_{q=1}^{T}(a_{itp}^{u\prime}v_p)^2(a_{itq}^{u\prime}v_q)^2 + \sum_{s=1}^{T}(a_{its}^{u\prime}v_s)^4] \\
= & \textstyle (\mu_{v0}^{(4)} - \sigma_{v0}^2)\sum_{t=1}^{T}\{\sigma_{v0}^4(\sum_{p=1}^{T} a_{itp}^{u\prime}a_{itp}^{u})(\sum_{q=1}^{T} a_{itq}^{u\prime}a_{itq}^{u}) + \sum_{s=1}^{T}\mathrm{E}[(a_{its}^{u\prime}v_s)^4]\} \\
= & \textstyle (\mu_{v0}^{(4)} - \sigma_{v0}^2)\sum_{t=1}^{T}\{\sigma_{v0}^4(\sum_{p=1}^{T} a_{itp}^{u\prime}a_{itp}^{u})^2 + \sum_{s=1}^{T}[\sigma_{v0}^4(a_{its}^{u\prime}a_{its}^{u})^2 + \mu_{v0}^{(4)}(\sum_{j=1}^{n} a_{its,j}^{u^4})]\} \\
= & \textstyle (\mu_{v0}^{(4)} - \sigma_{v0}^2)\sigma_{v0}^4\sum_{t=1}^{T}(\sum_{s=1}^{T} a_{its}^{u\prime}a_{its}^{u})^2 + (\mu_{v0}^{(4)} - \sigma_{v0}^2)\sigma_{v0}^4\sum_{t=1}^{T}\sum_{s=1}^{T}(a_{its}^{u\prime}a_{its}^{u})^2 \\
& \textstyle + (\mu_{v0}^{(4)} - \sigma_{v0}^2)\mu_{v0}^{(4)}\sum_{t=1}^{T}\sum_{s=1}^{T}(\sum_{j=1}^{n} a_{its,j}^{u^4}),
\end{aligned}
$$

where $a_{its,j}^{u}$ is the jth element of $a_{its,j}^{u}$, which is, by Assumption E and Lemma A.1 uniformly bounded. $a_{its}^{u\prime}a_{its}^{u}$ is the (i,i) element of $\Phi_{ts}^{u}\Phi_{ts}^{u\prime}$, which is, by Assumption E and Lemma A.1, uniformly bounded. So, as T is fixed and small, we have $\sum_{t=1}^{T}(\sum_{s=1}^{T} a_{its}^{u\prime}a_{its}^{u})^2 \leq C < \infty$, $\sum_{t=1}^{T}\sum_{s=1}^{T}(a_{its}^{u\prime}a_{its}^{u})^2 \leq C < \infty$, and $\sum_{j=1}^{n} a_{its,j}^{u^4} \leq \max_j |a_{its,j}^2| \sum_{j=1}^{n} a_{its,j}^{u^2} = \max_j |a_{its,j}^2|(a_{its}^{u\prime}a_{its}^{u}) \leq C < \infty$. Thus, we have $\mathrm{E}(H_{1n,i}^2) \leq C < \infty$. Therefore, by the

WLLN we have $\frac{1}{nT}\sum_{i=1}^{n} H_{1n,i} = o_p(1)$.

Next, we consider $H_{2n,i} = \sum_{t=1}^{T}(v_{it}^2 - \sigma_{v0}^2)\phi_{it}^{\ell 2}$. As $\phi_{it}^{\ell} = \sum_s a^{\ell\prime} v_s$ is $\mathcal{G}_{n,i-1}$-measurable, we have $\mathrm{E}(H_{2n,i}|\mathcal{G}_{n,i-1}) = 0$. Thus $\{H_{2n,i}, \mathcal{G}_{n,i}\}$ form a M.D. array. Similar as $H_{1n,i}$, we can show that $\mathrm{E}(H_{2n,i}^2) \le C < \infty$. With constant coefficients $\frac{1}{nT}$, the other two conditions of WLLN for MD array of Theorem 19.7 of Davidson (1994, p .299) are satisfied. So, we have $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} H_{2n,i} = o_p(1)$.

For $H_{3n,i}$, we can write $H_{3n,i} = \sum_{t=1}^{T} v_{it}^2(\sum_{p=1}^{T} a_{itp}^{u\prime} v_p)(\sum_{s=1}^{T} a_{its}^{\ell\prime} v_s) = \sum_{s=1}^{T} v_s' \kappa_{is}$, where $\kappa_{is} = \sum_{t=1}^{T}\sum_{p=1}^{T} a_{its}^{\ell} a_{itp}^{u\prime} v_p v_{it}^2$. So we can write $\frac{1}{nT}\sum_{i=1}^{n} H_{3n,i} = \sum_{t=1}^{T} v_t'(\sum_{i=1}^{n}\kappa_{it})$, which is a bilinear form. By Assumptions A, B, E and Lemma A.1, we can verify the conditions of Lemma A.4 (vi) holds. Therefore we have $\frac{1}{nT}\sum_{i=1}^{n} H_{3n,i} = o_p(1)$.

Finally, the proof for convergence of $H_{4n,i}$ and $H_{5n,i}$ are the same. So, we only show the proof for $H_{4n,i}$. Write,

$$\begin{aligned}\frac{1}{nT}\sum_{i=1}^{n} H_{4n,i} = & \ \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(\sum_{p=1}^{T} a_{itp}^{u\prime} v_p)(\sum_{q=1}^{T} a_{itq}^{u\prime} v_q) \\ = & \ \frac{1}{nT}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{q=1}^{T} v_p' \sum_{i=1}^{n}(a_{itp}^{u} a_{itq}^{u\prime}) v_q \\ = & \ \frac{1}{nT}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{q=1}^{T} v_p' \Phi_{tp}^{u\prime}\Phi_{tq}^{u} v_q = \frac{1}{nT}\mathbf{v}'\Phi^{u\prime}\Phi^{u}\mathbf{v}\end{aligned}$$

By Lemma A.1 and Assumption E, $\Phi^{u\prime}\Phi^{u}$ is uniformly bounded in either row or column sums. Thus, the result $\frac{1}{nT}\sum_{i=1}^{n} H_{4n,i} = o_p(1)$, and $\frac{1}{nT}\sum_{i=1}^{n} H_{5n,i} = o_p(1)$ follow from Lemma A.4. Combining the results above, we have $U_1 = o_p(1)$

$U_r, r = 2, 3, 7, 8, 9$ are the means of n independent terms, therefore their convergence can be shown using WLLN similarly as in the proof of $\frac{1}{nT}\sum_{i=1}^{n} H_{1n,i} = o_p(1)$ in U_1.

The proof of convergence of $U_r, r = 4, 5, 6$ are similar. Here we present the proof for U_4. We can write

$$\begin{aligned} U_4 = & \ \frac{2}{nT}\sum_{i=1}^{n}(v_{i\cdot}'\varphi_{i\cdot}^{v})(v_{i\cdot}'v_{i\cdot}^{*}) = \frac{2}{nT}\sum_{i=1}^{n}[v_{i\cdot}'(\phi_{i\cdot}^{u} + \phi_{i\cdot}^{\ell})](v_{i\cdot}'v_{i\cdot}^{*}) \\ = & \ \frac{2}{nT}\sum_{i=1}^{n}(v_{i\cdot}'\phi_{i\cdot}^{u})(v_{i\cdot}'v_{i\cdot}^{*}) + \frac{2}{nT}\sum_{i=1}^{n}(v_{i\cdot}'\phi_{i\cdot}^{\ell})(v_{i\cdot}'v_{i\cdot}^{*}) \\ = & \ \frac{2}{nT}\sum_{i=1}^{n}\phi_{i\cdot}^{u\prime}(v_{i\cdot}v_{i\cdot}'v_{i\cdot}^{*} - \mu_{v0}^{(3)}d_{i\cdot}) + \frac{2}{nT}\sum_{i=1}^{n}\phi_{i\cdot}^{\ell\prime}(v_{i\cdot}v_{i\cdot}'v_{i\cdot}^{*} - \mu_{v0}^{(3)}d_{i\cdot}) + \frac{2\mu_{v0}^{(3)}}{nT}\sum_{i=1}^{n}\varphi_{i\cdot}^{v\prime}d_{i\cdot}.\end{aligned}$$

The first term is the mean of n uncorrelated terms, its convergence can be shown using WLLN similarly as in the proof of $\frac{1}{nT}\sum_{i=1}^{n} H_{1n,i} = o_p(1)$ in U_1. The second term is the mean of a M.D. array , its convergence can be shown using WLLN for MD array similarly as in the proof of $\frac{1}{nT}\sum_{i=1}^{n} H_{2n,i} = o_p(1)$ in U_1. The convergence of the third term can be shown similarly as in the proof of $\frac{1}{nT}\sum_{i=1}^{n} H_{4n,i} = o_p(1)$ in U_1.

Subsequently, the cross-product terms: $\frac{1}{nT}\sum_{i=1}^{n}[g_{\Pi i}g_{\Phi i} - \mathrm{E}(g_{\Pi i}g_{\Phi i})]$, $\frac{1}{nT}\sum_{i=1}^{n}[g_{\Pi i}g_{\Psi i} -$

$\mathrm{E}(g_{\Pi i} g_{\Psi i})]$ and $\frac{1}{nT}\sum_{i=1}^{n}[g_{\Psi i} g_{\Phi i} - \mathrm{E}(g_{\Psi i} g_{\Phi i})]$ can all be decomposed in a similar manner, and the convergence of each of the decomposed terms can be proved in a similar way. These complete the proof of Theorem 4.1.

II. Additional Monte Carlo Results

In this section, we provide an extended set of Monte Carlo results including those unreported but involved in the discussions in the main text. Tables I-VI extends Tables 1-6 in the main text by including the estimated robust standard errors ($\widehat{\mathrm{rse}}$) for the BC-CQMLE. Table I-a to Table VI-a contain estimated robust and no-robust standard errors for the M-estimator.

Table I. Empirical Mean(sd)[$\widehat{\text{rse}}$] of BC-CQMLE and M-Estimator: DGP1, $T = 3$, $m = 10$
$W_1 = W_2 = W_3$: `Rook Contiguity`, $r_0 = 1$, $r = 1$

	Normal Error		Normal Mixture		Chi-Square	
ψ	BC-CQMLE	M-Est	BC-CQMLE	M-Est	BC-CQMLE	M-Est
	$n = 50$					
1	.9746(.103)[.075]	.9982(.100)[.100]	.9770(.103)[.073]	.9998(.100)[.098]	.9736(.104)[.075]	.9955(.104)[.098]
1	.9744(.106)[.078]	.9925(.103)[.099]	.9691(.110)[.077]	.9890(.107)[.099]	.9782(.112)[.077]	.9965(.109)[.099]
1	.5801(.088)[−]	.9007(.141)[.132]	.5674(.167)[[−]	.8832(.212)[.202]	.5752(.123)[−]	.8930(.200)[.177]
.3	.2427(.072)[.047]	.2959(.062)[.062]	.2407(.083)[.045]	.2930(.068)[.065]	.2435(.073)[.046]	.2953(.060)[.061]
.2	.1766(.141)[.099]	.1929(.129)[.124]	.1741(.136)[.097]	.1936(.121)[.127]	.1720(.133)[.098]	.1904(.125)[.120]
.2	.2219(.076)[.057]	.2028(.079)[.077]	.2224(.075)[.057]	.2062(.077)[.078]	.2210(.078)[.057]	.2032(.078)[.080]
.2	.1881(.200)[.135]	.1931(.195)[.181]	.1835(.195)[.135]	.1869(.187)[.187]	.1896(.191)[.135]	.1892(.190)[.180]
	$n = 100$					
1	.9994(.076)[.057]	.9988(.075)[.073]	1.0005(.078)[.056]	1.0006(.078)[.072]	1.0009(.080)[.057]	1.0012(.078)[.074]
1	.9929(.073)[.058]	.9984(.072)[.072]	.9892(.075)[.057]	.9960(.073)[.072]	.9934(.076)[.058]	.9993(.075)[.073]
1	.6306(.065)[−]	.9497(.099)[.095]	.6196(.134)[−]	.9341(.204)[.185]	.6300(.098)[−]	.9493(.150)[.141]
.3	.3117(.058)[.030]	.2996(.046)[.047]	.3146(.064)[.030]	.2990(.050)[.051]	.3137(.062)[.030]	.3016(.049)[.050]
.2	.1956(.092)[.072]	.1936(.091)[.091]	.2062(.085)[.070]	.2023(.084)[.087]	.1960(.093)[.072]	.1947(.092)[.089]
.2	.1869(.079)[.056]	.1989(.073)[.073]	.1829(.081)[.055]	.1959(.075)[.076]	.1877(.079)[.055]	.1994(.074)[.074]
.2	.1921(.133)[.101]	.1971(.133)[.132]	.1799(.127)[.101]	.1899(.127)[.130]	.1939(.134)[.101]	.1983(.133)[.130]
	$n = 200$					
1	.9851(.051)[.041]	1.0003(.053)[.052]	.9852(.052)[.040]	1.0002(.054)[.052]	.9811(.051)[.041]	.9963(.053)[.051]
1	.9792(.051)[.040]	.9997(.052)[.051]	.9798(.053)[.040]	.9995(.054)[.051]	.9812(.054)[.040]	1.0014(.054)[.051]
1	.6252(.046)[−]	.9756(.075)[.072]	.6210(.092)[−]	.9688(.143)[.140]	.6262(.072)[−]	.9773(.119)[.107]
.3	.2571(.031)[.024]	.3003(.034)[.033]	.2583(.034)[.024]	.3002(.036)[.036]	.2577(.033)[.024]	.3009(.037)[.035]
.2	.1874(.065)[.054]	.1974(.065)[.064]	.1903(.067)[.053]	.2000(.064)[.064]	.1937(.065)[.054]	.2012(.064)[.064]
.2	.2007(.048)[.037]	.1996(.052)[.050]	.1995(.047)[.037]	.1983(.050)[.050]	.1990(.049)[.037]	.1997(.052)[.050]
.2	.1993(.091)[.074]	.1980(.090)[.089]	.1980(.091)[.073]	.1976(.090)[.089]	.1960(.088)[.074]	.1960(.087)[.089]
	$n = 400$					
1	.9951(.036)[.029]	.9985(.036)[.036]	.9964(.036)[.029]	.9997(.036)[.035]	.9971(.036)[.029]	.9999(.036)[.036]
1	.9861(.037)[.029]	1.0005(.037)[.036]	.9858(.038)[.029]	1.0000(.037)[.036]	.9837(.037)[.029]	.9980(.036)[.037]
1	.6425(.032)[−]	.9899(.050)[.051]	.6367(.068)[−]	.9891(.105)[.105]	.6424(.051)[−]	.9880(.081)[.078]
.3	.2593(.027)[.018]	.2999(.023)[.023]	.2595(.031)[.018]	.3001(.027)[.027]	.2595(.029)[.018]	.3000(.025)[.024]
.2	.1993(.048)[.040]	.1994(.048)[.048]	.1985(.049)[.040]	.1999(.048)[.047]	.2008(.049)[.040]	.2002(.048)[.048]
.2	.2057(.030)[.024]	.2001(.031)[.031]	.2046(.030)[.023]	.1998(.032)[.032]	.2041(.030)[.024]	.1999(.031)[.032]
.2	.1954(.065)[.054]	.1995(.066)[.066]	.1994(.068)[.054]	.2035(.066)[.066]	.1915(.066)[.054]	.1982(.066)[.066]

Note: 1. $\psi = (\beta', \sigma_v^2, \rho, \lambda')'$; 2. r_0 = true number of factor, r = assumed number of factor.

Table I-a. Empirical sd and average of estimated standard errors of M-Estimator
Parameter configurations as in Table I.

	Normal Error				Normal Mixture				Chi-Square			
ψ	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$
						$n = 50$						
1	.100	.100	.098	.112	.100	.098	.095	.125	.104	.098	.097	.117
1	.103	.099	.098	.113	.107	.099	.097	.127	.109	.099	.098	.117
1	.141	.132	.133	.157	.272	.222	.131	.122	.200	.177	.132	.132
.3	.062	.062	.059	.067	.068	.065	.058	.075	.060	.061	.058	.072
.2	.129	.122	.118	.135	.121	.127	.116	.152	.126	.120	.117	.142
.2	.079	.077	.074	.084	.077	.078	.073	.095	.078	.080	.074	.087
.2	.195	.181	.170	.182	.187	.187	.170	.200	.190	.180	.170	.191
						$n = 100$						
1	.075	.073	.072	.077	.078	.072	.071	.084	.078	.074	.072	.080
1	.072	.072	.072	.076	.073	.072	.071	.085	.075	.073	.072	.080
1	.099	.095	.097	.106	.204	.185	.095	.069	.150	.141	.097	.085
.3	.046	.047	.042	.043	.050	.051	.041	.047	.049	.050	.042	.044
.2	.091	.091	.089	.095	.084	.087	.087	.105	.094	.089	.088	.100
.2	.073	.073	.070	.074	.075	.076	.069	.082	.074	.074	.070	.079
.2	.133	.132	.127	.131	.127	.130	.126	.143	.133	.130	.126	.136
						$n = 200$						
1	.053	.052	.051	.053	.054	.052	.051	.056	.053	.051	.051	.055
1	.052	.051	.051	.053	.054	.051	.051	.056	.054	.051	.051	.055
1	.075	.072	.073	.076	.149	.140	.072	.048	.119	.107	.073	.057
.3	.034	.033	.033	.034	.038	.036	.033	.035	.037	.035	.033	.034
.2	.065	.064	.064	.066	.067	.064	.063	.070	.064	.064	.064	.068
.2	.052	.050	.050	.051	.050	.050	.049	.055	.052	.050	.050	.053
.2	.092	.089	.088	.090	.090	.089	.088	.095	.087	.089	.088	.093
						$n = 400$						
1	.036	.036	.036	.036	.036	.035	.035	.038	.036	.036	.036	.037
1	.037	.036	.036	.037	.037	.036	.036	.038	.036	.037	.036	.038
1	.050	.051	.051	.052	.108	.105	.050	.028	.081	.078	.051	.036
.3	.023	.022	.022	.023	.027	.026	.022	.022	.025	.024	.022	.022
.2	.048	.048	.047	.048	.048	.047	.047	.050	.048	.047	.047	.049
.2	.031	.032	.032	.032	.032	.032	.032	.033	.031	.032	.032	.033
.2	.066	.066	.065	.066	.066	.066	.065	.068	.066	.066	.065	.067

Table II. Empirical Mean(sd)[$\widehat{\text{rse}}$] of BC-CQMLE and M-Estimator: DGP1, $T = 3$, $m = 10$
$W_1 = W_2$: Group Interaction; W_3: Queen Contiguity, $r_0 = 1$, $r = 1$

	Normal Error		Normal Mixture		Chi-Square	
ψ	BC-CQMLE	M-Est	BC-CQMLE	M-Est	BC-CQMLE	M-Est
	$n = 50$					
1	.9637(.109)[.088]	.9988(.116)[.116]	.9624(.108)[.087]	.9971(.116)[.112]	.9599(.112)[.088]	.9934(.119)[.112]
1	.9811(.109)[.084]	1.0012(.109)[.107]	.9753(.108)[.083]	.9973(.109)[.102]	.9745(.111)[.084]	.9964(.111)[.104]
1	.6111(.096)[−]	.9072(.144)[.137]	.6096(.179)[−]	.9064(.237)[.232]	.6114(.135)[−]	.9134(.201)[.185]
.3	.2601(.064)[.049]	.3011(.070)[.069]	.2590(.067)[.049]	.3005(.072)[.069]	.2555(.065)[.049]	.2977(.071)[.068]
.2	.1403(.135)[.082]	.1716(.126)[.132]	.1366(.137)[.082]	.1768(.130)[.119]	.1360(.136)[.082]	.1685(.121)[.118]
.2	.2141(.099)[.069]	.2062(.092)[.093]	.2093(.099)[.069]	.2004(.091)[.086]	.2100(.101)[.070]	.2028(.091)[.087]
.2	.1477(.162)[.130]	.1908(.178)[.188]	.1547(.155)[.130]	.1797(.187)[.180]	.1451(.154)[.130]	.1849(.174)[.179]
	$n = 100$					
1	.9691(.079)[.061]	.9987(.081)[.079]	.9729(.080)[.060]	1.0017(.085)[.081]	.9674(.079)[.060]	.9956(.083)[.080]
1	.9577(.083)[.063]	.9973(.080)[.079]	.9594(.087)[.063]	.9966(.082)[.079]	.9601(.083)[.063]	.9995(.080)[.079]
1	.6444(.068)[−]	.9554(.104)[.099]	.6406(.135)[−]	.9497(.209)[.181]	.6447(.100)[−]	.9557(.1545)[.141]
.3	.2638(.050)[.035]	.3005(.053)[.052]	.2637(.059)[.034]	.2993(.061)[.060]	.2648(.055)[.035]	.3010(.059)[.058]
.2	.0689(.084)[.075]	.1881(.089)[.086]	.0686(.085)[.075]	.1867(.090)[.085]	.0687(.089)[.075]	.1816(.085)[.086]
.2	.3473(.086)[.066]	.2174(.083)[.080]	.3447(.085)[.066]	.2190(.089)[.082]	.3403(.086)[.066]	.2110(.085)[.082]
.2	.2132(.117)[.091]	.1917(.127)[.125]	.2093(.122)[.091]	.1845(.125)[.124]	.2212(.110)[.091]	.1887(.123)[.124]
	$n = 200$					
1	.9918(.042)[.035]	.9988(.041)[.042]	.9909(.044)[.035]	.9980(.043)[.043]	.9933(.043)[.035]	1.0002(.043)[.042]
1	.9935(.052)[.041]	.9979(.050)[.049]	.9957(.050)[.041]	.9999(.049)[.049]	.9939(.050)[.041]	.9989(.049)[.049]
1	.6683(.048)[−]	.9744(.071)[.069]	.6708(.097)[−]	.9779(.136)[.134]	.6694(.075)[−]	.9780(.103)[.100]
.3	.3105(.031)[.020]	.3001(.028)[.028]	.3118(.044)[.020]	.3004(.039)[.037]	.3096(.036)[.020]	.2992(.032)[.032]
.2	.0408(.037)[.059]	.1894(.063)[.062]	.0392(.039)[.059]	.1894(.065)[.062]	.0414(.035)[.059]	.1894(.061)[.062]
.2	.3381(.031)[.051]	.2095(.056)[.056]	.3378(.032)[.051]	.2082(.059)[.057]	.3367(.032)[.051]	.2115(.057)[.057]
.2	.2178(.096)[.065]	.1948(.085)[.085]	.2194(.096)[.065]	.1898(.087)[.085]	.2161(.092)[.065]	.1927(.086)[.085]
	$n = 400$					
1	.9561(.036)[.027]	1.0005(.036)[.036]	.9607(.036)[.027]	.9997(.036)[.036]	.9558(.036)[.027]	.9989(.036)[.036]
1	.9481(.041)[.029]	1.0001(.037)[.036]	.9508(.046)[.029]	.9991(.037)[.036]	.9467(.043)[.029]	.9997(.037)[.036]
1	.6382(.033)[−]	.9866(.051)[.050]	.6315(.066)[−]	.9797(.110)[.109]	.6375(.049)[−]	.9898(.083)[.082]
.3	.1532(.047)[.015]	.2999(.023)[.023]	.1618(.064)[.015]	.3001(.028)[.027]	.1527(.054)[.014]	.2995(.025)[.024]
.2	.1161(.048)[.048]	.2004(.057)[.056]	.1181(.048)[.047]	.1981(.054)[.055]	.1126(.047)[.048]	.1979(.057)[.056]
.2	.2611(.063)[.040]	.2001(.043)[.043]	.2664(.060)[.039]	.2008(.045)[.045]	.2551(.060)[.039]	.2007(.046)[.046]
.2	.1880(.083)[.047]	.1997(.060)[.059]	.1968(.078)[.047]	.1989(.058)[.059]	.1843(.083)[.048]	.1977(.060)[.059]

Note: 1. $\psi = (\beta', \sigma_v^2, \rho, \lambda')'$; 2. r_0 = true number of factor, r = assumed number of factor.

Table II-a. Empirical sd and average of estimated standard errors of M-Estimator Parameter configurations as in Table II.

	`Normal Error`				`Normal Mixture`				`Chi-Square`			
ψ	`sd`	$\widehat{\mathrm{rse}}$	$\widehat{\mathrm{se}}$	$\widetilde{\mathrm{se}}$	`sd`	$\widehat{\mathrm{rse}}$	$\widehat{\mathrm{se}}$	$\widetilde{\mathrm{se}}$	`sd`	$\widehat{\mathrm{rse}}$	$\widehat{\mathrm{se}}$	$\widetilde{\mathrm{se}}$
	$n = 50$											
1	.116	.116	.110	.124	.116	.112	.109	.139	.119	.112	.110	.131
1	.109	.107	.103	.117	.109	.102	.102	.133	.111	.104	.103	.125
1	.144	.137	.136	.161	.277	.232	.136	.126	.211	.183	.137	.136
.3	.070	.069	.065	.075	.074	.069	.065	.083	.071	.068	.066	.079
.2	.126	.132	.111	.119	.130	.119	.109	.128	.124	.117	.111	.125
.2	.092	.093	.084	.095	.091	.086	.084	.108	.093	.087	.085	.102
.2	.178	.188	.168	.171	.187	.180	.169	.180	.174	.179	.168	.176
	$n = 100$											
1	.083	.079	.078	.082	.085	.081	.077	.089	.083	.080	.077	.086
1	.080	.079	.078	.082	.083	.079	.077	.089	.080	.079	.078	.085
1	.104	.099	.100	.108	.209	.181	.099	.073	.155	.141	.100	.084
.3	.053	.052	.049	.050	.061	.060	.048	.051	.059	.058	.048	.050
.2	.089	.086	.085	.088	.090	.085	.084	.094	.085	.086	.084	.091
.2	.085	.080	.078	.083	.089	.082	.077	.088	.085	.082	.078	.085
.2	.127	.125	.119	.120	.125	.124	.120	.126	.123	.124	.119	.123
	$n = 200$											
1	.041	.042	.042	.044	.043	.043	.042	.046	.043	.042	.042	.045
1	.051	.049	.049	.051	.049	.049	.049	.055	.049	.049	.049	.053
1	.071	.069	.069	.073	.142	.134	.070	.043	.110	.100	.070	.059
.3	.028	.028	.025	.025	.039	.037	.025	.022	.032	.032	.025	.023
.2	.063	.062	.062	.063	.065	.062	.062	.066	.061	.062	.062	.065
.2	.056	.056	.055	.056	.059	.057	.055	.058	.057	.057	.055	.057
.2	.085	.085	.083	.084	.087	.085	.084	.087	.086	.085	.083	.085
	$n = 400$											
1	.036	.036	.036	.036	.036	.036	.035	.038	.037	.036	.036	.037
1	.037	.036	.036	.037	.040	.036	.036	.038	.037	.036	.036	.037
1	.052	.050	.050	.051	.113	.109	.050	.027	.084	.082	.050	.036
.3	.023	.023	.022	.022	.033	.026	.023	.023	.026	.024	.023	.023
.2	.057	.056	.056	.056	.053	.055	.055	.059	.058	.056	.056	.058
.2	.043	.045	.044	.044	.045	.046	.044	.046	.046	.046	.044	.046
.2	.060	.059	.059	.059	.058	.059	.058	.060	.060	.059	.059	.059

Table III. Empirical Mean(sd)[$\widehat{\text{rse}}$] of BC-CQMLE and M-Estimator: DGP1, $T = 3$, $m = 10$
$W_1 = W_2 = W_3$: Rook Contiguity, $r_0 = 2$, $r = 2$

	Normal Error		Normal Mixture		Chi-Square	
ψ	BC-CQMLE	M-Est	BC-CQMLE	M-Est	BC-CQMLE	M-Est
	$n = 50$					
1	.7616(.106)[.064]	1.0212(.200)[.165]	.7876(.132)[.062]	1.0404(.208)[.167]	.7760(.122)[.063]	1.0350(.205)[.155]
1	.6464(.141)[.072]	.9876(.172)[.159]	.6812(.173)[.070]	.9923(.178)[.160]	.6705(.145)[.072]	.9965(.177)[.155]
1	.2120(.046)[−]	.7848(.176)[.170]	.2017(.061)[−]	.7028(.184)[.177]	.2104(.052)[−]	.7481(.189)[.179]
.3	-.1793(.108)[.045]	.2698(.110)[.098]	-.1395(.170)[.043]	.2616(.115)[.110]	-.1591(.132)[.045]	.2520(.119)[.113]
.2	.2638(.177)[.090]	.1941(.190)[.188]	.2488(.168)[.088]	.1931(.198)[.190]	.2487(.164)[.091]	.1898(.197)[.190]
.2	.2348(.151)[.085]	.2133(.143)[.142]	.2282(.147)[.079]	.2266(.143)[.140]	.2174(.154)[.083]	.2166(.147)[.141]
.2	.0139(.272)[.133]	.1574(.329)[.303]	.0476(.263)[.131]	.1521(.310)[.294]	.0374(.266)[.134]	.1706(.311)[.297]
	$n = 100$					
1	.7475(.135)[.066]	.9699(.141)[.144]	.7750(.149)[.063]	.9817(.142)[.142]	.7556(.143)[.065]	.9759(.144)[.147]
1	.7796(.104)[.051]	.9863(.105)[.109]	.7989(.119)[.050]	.9729(.106)[.109]	.7881(.109)[.051]	.9724(.110)[.114]
1	.2053(.031)[−]	.8981(.115)[.121]	.1968(.041)[−]	.9023(.149)[.146]	.2024(.036)[−]	.7435(.133)[.136]
.3	-.0547(.123)[.043]	.2906(.097)[.093]	-.0094(.169)[.040]	.2991(.101)[.092]	-.0454(.137)[.042]	.2837(.100)[.102]
.2	.1294(.254)[.095]	.1964(.160)[.163]	.1234(.241)[.089]	.1950(.163)[.166]	.1123(.237)[.094]	.1833(.164)[.167]
.2	.1771(.208)[.075]	.2011(.098)[.095]	.1797(.194)[.069]	.2024(.114)[.110]	.1675(.199)[.073]	.1987(.114)[.117]
.2	.1992(.302)[.109]	.1902(.202)[.201]	.2117(.288)[.105]	.1845(.204)[.207]	.2263(.287)[.108]	.1951(.212)[.215]
	$n = 200$					
1	.9759(.176)[.037]	1.0102(.087)[.087]	1.0022(.167)[.036]	1.0021(.088)[.087]	.9866(.168)[.037]	1.0014(.089)[.088]
1	.9668(.137)[.039]	1.0071(.071)[.072]	.9769(.123)[.038]	1.0055(.070)[.072]	.9739(.131)[.038]	1.0087(.074)[.075]
1	.2973(.029)[−]	.9489(.083)[.087]	.2837(.046)[−]	.9640(.096)[.099]	.2920(.036)[−]	.9348(.103)[.104]
.3	.2091(.192)[.022]	.3011(.050)[.048]	.2346(.181)[.021]	.3061(.051)[.049]	.2251(.184)[.022]	.3175051(.051)[.049]
.2	.1786(.103)[.052]	.1982(.083)[.084]	.1808(.094)[.050]	.1983(.084)[.084]	.1754(.106)[.051]	.1981(.090)[.091]
.2	.1900(.063)[.034]	.1993(.060)[.059]	.1858(.063)[.033]	.1994(.061)[.062]	.1843(.068)[.033]	.1982(.067)[.069]
.2	.1933(.139)[.073]	.1994(.125)[.123]	.1839(.126)[.072]	.1980(.127)[.129]	.1926(.138)[.073]	.1978(.131)[.147]
	$n = 400$					
1	.9289(.047)[.019]	.9996(.028)[.028]	.9290(.048)[.018]	.9989(.031)[.031]	.9301(.048)[.018]	.9984(.029)[.030]
1	.8905(.091)[.029]	.9963(.049)[.050]	.8978(.089)[.029]	.9983(.051)[.051]	.8925(.089)[.029]	.9865(.048)[.049]
1	.3138(.022)[−]	.9893(.071)[.071]	.3073(.034)[−]	.9888(.084)[.085]	.3095(.027)[−]	.9833(.083)[.083]
.3	.1682(.180)[.020]	.2996(.030)[.030]	.1970(.185)[.019]	.2988(.031)[.032]	.1807(.182)[.019]	.2983(.034)[.034]
.2	.1662(.043)[.026]	.1994(.031)[.031]	.1680(.046)[.025]	.1960(.032)[.033]	.1662(.044)[.026]	.1973(.032)[.033]
.2	.2073(.032)[.018]	.2003(.026)[.028]	.1999(.035)[.018]	.1970(.028)[.029]	.2041(.032)[.018]	.2000(.027)[.028]
.2	.1910(.078)[.045]	.1996(.074)[.075]	.1982(.076)[.045]	.1961(.075)[.076]	.1930(.078)[.045]	.1962(.076)[.076]

Note: 1. $\psi = (\beta', \sigma_v^2, \rho, \lambda')'$; 2. r_0 = true number of factor, r = assumed number of factor.

Table III-a. Empirical sd and average of estimated standard errors of M-Estimator
Parameter configurations as in Table III.

	Normal Error				Normal Mixture				Chi-Square			
ψ	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$
	$n = 50$											
1	.206	.129	.144	.155	.218	.119	.148	.147	.205	.127	.148	.155
1	.188	.138	.155	.158	.178	.132	.162	.158	.177	.135	.160	.155
1	.176	.164	.200	.170	.284	.152	.171	.207	.227	.159	.189	.189
.3	.130	.068	.074	.087	.115	.064	.076	.093	.119	.066	.076	.083
.2	.216	.162	.182	.198	.202	.144	.180	.169	.197	.157	.187	.190
.2	.153	.112	.120	.143	.143	.103	.123	.130	.147	.108	.121	.140
.2	.329	.238	.243	.303	.310	.232	.251	.294	.311	.236	.250	.297
	$n = 100$											
1	.144	.120	.122	.146	.142	.110	.119	.142	.144	.115	.121	.147
1	.116	.100	.104	.118	.106	.094	.104	.113	.113	.097	.104	.120
1	.115	.127	.139	.137	.159	.111	.108	.149	.133	.121	.127	.156
.3	.097	.071	.071	.088	.101	.062	.066	.082	.100	.068	.069	.102
.2	.183	.145	.149	.178	.163	.132	.147	.166	.174	.138	.148	.168
.2	.151	.115	.114	.145	.136	.099	.109	.124	.144	.108	.112	.149
.2	.214	.185	.186	.222	.204	.180	.191	.217	.212	.182	.188	.215
	$n = 200$											
1	.097	.067	.066	.087	.088	.064	.068	.087	.089	.066	.068	.088
1	.081	.070	.071	.081	.070	.067	.074	.077	.074	.069	.072	.080
1	.083	.094	.100	.095	.146	.088	.069	.143	.115	.092	.085	.117
.3	.049	.038	.037	.048	.055	.035	.037	.046	.054	.037	.038	.049
.2	.089	.081	.081	.098	.084	.078	.085	.094	.091	.081	.086	.099
.2	.060	.056	.056	.067	.061	.054	.057	.066	.067	.056	.057	.069
.2	.139	.123	.122	.148	.127	.122	.128	.144	.135	.123	.126	.147
	$n = 400$											
1	.028	.027	.026	.031	.031	.027	.027	.034	.029	.027	.027	.032
1	.049	.049	.050	.052	.051	.049	.051	.054	.048	.049	.050	.053
1	.071	.069	.072	.071	.114	.069	.048	.110	.089	.069	.061	.088
.3	.052	.033	.024	.058	.062	.031	.023	.069	.057	.032	.023	.064
.2	.031	.031	.032	.033	.032	.032	.034	.033	.032	.031	.034	.033
.2	.026	.026	.026	.028	.028	.026	.027	.029	.027	.026	.026	.028
.2	.074	.072	.072	.076	.074	.072	.074	.076	.076	.072	.074	.076

Table IV. Empirical Mean(sd)[$\widehat{\text{rse}}$] of BC-CQMLE and M-Estimator: DGP1, $T = 10$, $m = 10$
$W_1 = W_2 = W_3$: `Rook Contiguity`, $r_0 = 1$, $r = 1$

	Normal Error		Normal Mixture		Chi-Square	
ψ	BC-CQMLE	M-Est	BC-CQMLE	M-Est	BC-CQMLE	M-Est
	$n = 25$					
1	.9958(.069)[.062]	.9957(.069)[.065]	.9995(.070)[.062]	.9994(.070)[.065]	.9991(.069)[.062]	.9990(.069)[.066]
1	.9966(.070)[.063]	.9967(.070)[.066]	.9926(.072)[.062]	.9927(.072)[.065]	.9995(.069)[.062]	.9996(.069)[.065]
1	.8256(.078)[−]	.9176(.087)[.085]	.8199(.186)[−]	.9112(.207)[.170]	.8265(.133)[−]	.9186(.147)[.136]
.3	.2979(.038)[.034]	.2987(.038)[.035]	.3018(.037)[.034]	.3015(.037)[.035]	.2986(.038)[.034]	.2994(.038)[.035]
.2	.1941(.076)[.069]	.1940(.076)[.073]	.1971(.074)[.069]	.1971(.074)[.071]	.1978(.072)[.069]	.1976(.072)[.071]
.2	.2020(.064)[.058]	.2011(.064)[.061]	.1982(.063)[.057]	.1974(.063)[.061]	.1976(.063)[.057]	.1968(.063)[.061]
.2	.2064(.120)[.101]	.2017(.121)[.117]	.1983(.113)[.101]	.2033(.115)[.113]	.1975(.110)[.101]	.2028(.112)[.113]
	$n = 50$					
1	.9978(.044)[.041]	.9979(.044)[.042]	.9992(.045)[.041]	.9993(.045)[.043]	.9996(.046)[.041]	.9997(.046)[.043]
1	.9985(.046)[.045]	.9985(.046)[.047]	.9997(.048)[.045]	.9997(.048)[.046]	1.0007(.049)[.045]	1.0007(.049)[.047]
1	.8610(.059)[−]	.9568(.066)[.064]	.8686(.136)[−]	.9653(.141)[.139]	.8649(.098)[−]	.9611(.103)[.101]
.3	.2985(.026)[.024]	.2994(.026)[.026]	.2991(.027)[.024]	.3000(.027)[.027]	.2979(.026)[.024]	.2998(.026)[.026]
.2	.1973(.060)[.055]	.1974(.060)[.059]	.1980(.060)[.055]	.1981(.060)[.059]	.1952(.059)[.055]	.1983(.059)[.059]
.2	.2000(.042)[.038]	.1997(.042)[.041]	.1986(.044)[.038]	.1988(.044)[.042]	.2017(.042)[.038]	.2013(.042)[.042]
.2	.1984(.087)[.078]	.2012(.088)[.086]	.1963(.087)[.078]	.2011(.087)[.085]	.1987(.084)[.078]	.2013(.084)[.084]
	$n = 100$					
1	.9995(.029)[.028]	.9995(.029)[.030]	1.0000(.032)[.028]	1.0000(.032)[.032]	1.0009(.031)[.028]	1.0009(.031)[.030]
1	1.0013(.033)[.031]	1.0004(.033)[.033]	1.0016(.034)[.031]	1.0016(.034)[.033]	.9981(.033)[.031]	.9971(.033)[.033]
1	.8837(.041)[−]	.9841(.046)[.046]	.8843(.098)[−]	.9848(.107)[.105]	.8821(.071)[−]	.9882(.078)[.075]
.3	.2997(.018)[.017]	.3002(.018)[.018]	.2985(.019)[.017]	.2992(.019)[.018]	.2999(.018)[.017]	.3001(.018)[.018]
.2	.1961(.038)[.035]	.1986(.038)[.037]	.1990(.038)[.035]	.1989(.038)[.037]	.1998(.038)[.035]	.1997(.038)[.037]
.2	.2014(.029)[.027]	.2001(.029)[.029]	.2006(.029)[.027]	.2001(.029)[.029]	.1983(.029)[.027]	.1989(.029)[.029]
.2	.2006(.056)[.053]	.2006(.056)[.057]	.1982(.058)[.053]	.2002(.058)[.057]	.1982(.058)[.053]	.2003(.058)[.057]
	$n = 200$					
1	.9990(.023)[.022]	.9998(.023)[.023]	1.0005(.024)[.022]	1.0003(.024)[.024]	.9997(.023)[.022]	1.0002(.023)[.023]
1	.9990(.022)[.022]	.9997(.022)[.023]	.9996(.023)[.022]	.9998(.023)[.023]	1.0009(.023)[.022]	1.0001(.023)[.023]
1	.8901(.030)[−]	.9989(.033)[.033]	.8905(.070)[−]	.9981(.076)[.076]	.8886(.051)[−]	.9880(.057)[.054]
.3	.2978(.013)[.012]	.2999(.013)[.013]	.2971(.014)[.012]	.2999(.014)[.014]	.2975(.014)[.012]	.2998(.014)[.014]
.2	.2006(.028)[.027]	.2001(.028)[.028]	.1991(.029)[.027]	.1988(.029)[.029]	.1985(.029)[.027]	.1982(.029)[.029]
.2	.2003(.021)[.020]	.1999(.021)[.021]	.2015(.021)[.020]	.2001(.021)[.021]	.2001(.021)[.020]	.1996(.021)[.021]
.2	.1974(.042)[.040]	.1997(.042)[.042]	.2006(.043)[.040]	.2003(.043)[.043]	.2011(.043)[.040]	.2002(.043)[.043]

Note: 1. $\psi = (\beta', \sigma_v^2, \rho, \lambda')'$; 2. r_0 = true number of factor, r = assumed number of factor.

Table IV-a. Empirical sd and average of estimated standard errors of M-Estimator Parameter configurations as in Table IV.

	Normal Error				Normal Mixture				Chi-Square			
ψ	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$
	$n = 50$											
1	.069	.066	.118	.065	.070	.066	.134	.065	.069	.066	.127	.066
1	.070	.067	.116	.066	.072	.066	.129	.065	.069	.066	.122	.065
1	.087	.088	.152	.085	.207	.087	.101	.170	.148	.088	.123	.126
.3	.038	.036	.064	.035	.037	.036	.076	.035	.038	.036	.068	.035
.2	.077	.074	.130	.073	.074	.073	.146	.071	.072	.073	.136	.071
.2	.064	.062	.105	.061	.063	.061	.118	.061	.063	.061	.110	.061
.2	.121	.110	.161	.113	.115	.110	.176	.113	.112	.110	.167	.113
	$n = 100$											
1	.044	.043	.055	.042	.045	.043	.061	.043	.046	.043	.057	.043
1	.046	.048	.059	.047	.048	.048	.066	.046	.049	.048	.061	.047
1	.066	.064	.080	.064	.151	.065	.047	.139	.109	.065	.061	.101
.3	.026	.026	.033	.026	.027	.026	.037	.027	.026	.026	.035	.026
.2	.060	.059	.073	.059	.060	.059	.081	.059	.059	.059	.077	.059
.2	.042	.041	.051	.041	.044	.041	.056	.042	.042	.041	.054	.042
.2	.088	.083	.097	.084	.087	.083	.105	.084	.084	.083	.101	.084
	$n = 200$											
1	.029	.030	.033	.030	.032	.030	.036	.030	.031	.030	.034	.030
1	.033	.033	.037	.033	.034	.033	.039	.033	.033	.033	.037	.033
1	.046	.047	.052	.046	.108	.047	.027	.105	.079	.047	.037	.075
.3	.018	.018	.020	.018	.019	.018	.021	.018	.018	.018	.020	.018
.2	.038	.037	.041	.037	.038	.037	.044	.037	.038	.037	.043	.037
.2	.029	.029	.032	.029	.029	.029	.034	.029	.029	.029	.033	.029
.2	.056	.057	.061	.057	.058	.057	.063	.057	.058	.057	.062	.057
	$n = 400$											
1	.023	.023	.024	.023	.024	.023	.025	.023	.023	.023	.024	.023
1	.022	.023	.024	.023	.023	.023	.025	.023	.023	.023	.025	.023
1	.033	.033	.035	.033	.078	.033	.017	.076	.057	.033	.024	.054
.3	.013	.013	.014	.014	.014	.013	.014	.014	.014	.013	.014	.014
.2	.028	.029	.030	.029	.029	.029	.031	.029	.029	.029	.031	.029
.2	.021	.021	.022	.021	.021	.021	.023	.021	.021	.021	.023	.021
.2	.042	.042	.043	.042	.043	.042	.045	.042	.043	.042	.044	.042

Table V. Empirical Mean(sd)[$\widehat{\text{rse}}$] of BC-CQMLE and M-Estimator: DGP1, $T = 10$, $m = 10$
$W_1 = W_3$: Rook Contiguity; W_2: Group Interaction, $r_0 = 1$, $r = 1$

	Normal Error		Normal Mixture		Chi-Square	
ψ	BC-CQMLE	M-Est	BC-CQMLE	M-Est	BC-CQMLE	M-Est
	$n = 25$					
1	1.0018(.071)[.062]	1.0017(.071)[.065]	1.0015(.070)[.062]	1.0013(.070)[.065]	1.0000(.070)[.062]	.9999(.070)[.066]
1	.9966(.069)[.062]	.9965(.069)[.064]	1.0007(.067)[.061]	1.0005(.067)[.063]	1.0017(.066)[.062]	1.0015(.066)[.065]
1	.8284(.078)[−]	.9208(.087)[.085]	.8161(.180)[−]	.9071(.200)[.169]	.8347(.130)[−]	.9278(.145)[.129]
.3	.2970(.037)[.033]	.2982(.037)[.035]	.2970(.037)[.032]	.2981(.037)[.036]	.2937(.039)[.033]	.2949(.039)[.036]
.2	.1950(.081)[.071]	.1949(.082)[.075]	.1956(.079)[.070]	.1952(.078)[.074]	.1975(.078)[.071]	.1971(.078)[.075]
.2	.1995(.051)[.046]	.1992(.051)[.049]	.1960(.053)[.046]	.1957(.053)[.048]	.1951(.052)[.046]	.1947(.052)[.048]
.2	.1888(.140)[.117]	.1888(.145)[.146]	.1798(.141)[.117]	.1795(.145)[.150]	.1851(.136)[.118]	.1844(.141)[.146]
	$n = 50$					
1	.9956(.046)[.041]	.9960(.046)[.043]	1.0012(.046)[.041]	1.0010(.046)[.044]	1.0014(.045)[.041]	1.0009(.045)[.042]
1	.9999(.046)[.045]	1.0001(.046)[.047]	.9995(.047)[.045]	.9997(.047)[.047]	1.0020(.047)[.045]	1.0022(.047)[.047]
1	.8663(.059)[−]	.9628(.066)[.063]	.8676(.137)[−]	.9643(.152)[.140]	.8673(.100)[−]	.9640(.111)[.102]
.3	.3017(.023)[.021]	.3004(.023)[.022]	.3001(.025)[.021]	.2988(.025)[.023]	.2990(.022)[.021]	.2977(.023)[.022]
.2	.1999(.043)[.042]	.2003(.043)[.044]	.1989(.045)[.042]	.1992(.045)[.043]	.1956(.046)[.042]	.1959(.046)[.044]
.2	.1986(.028)[.026]	.1993(.028)[.026]	.1986(.028)[.026]	.1987(.028)[.027]	.2004(.027)[.026]	.2005(.027)[.026]
.2	.1869(.090)[.081]	.1942(.092)[.091]	.1896(.090)[.081]	.1890(.092)[.090]	.1945(.091)[.081]	.1948(.092)[.090]
	$n = 100$					
1	1.0006(.031)[.028]	1.0005(.031)[.030]	1.0004(.031)[.028]	1.0004(.031)[.030]	1.0001(.030)[.028]	1.0001(.030)[.030]
1	1.0002(.034)[.031]	1.0002(.034)[.033]	1.0000(.033)[.031]	1.0000(.033)[.033]	.9986(.032)[.031]	.9986(.032)[.033]
1	.8836(.042)[−]	.9928(.047)[.046]	.8795(.095)[−]	.9773(.105)[.102]	.8807(.070)[−]	.9786(.078)[.075]
.3	.2993(.018)[.016]	.2998(.018)[.018]	.3012(.018)[.016]	.3006(.018)[.018]	.2992(.018)[.016]	.2987(.018)[.018]
.2	.1964(.041)[.038]	.1986(.041)[.040]	.1976(.040)[.038]	.1997(.040)[.040]	.1957(.040)[.038]	.1959(.040)[.040]
.2	.1997(.033)[.031]	.1999(.033)[.033]	.1976(.033)[.031]	.1997(.033)[.033]	.1984(.033)[.031]	.2003(.033)[.033]
.2	.2005(.069)[.063]	.2001(.070)[.069]	.1962(.067)[.063]	.1978(.068)[.068]	.1992(.067)[.063]	.2007(.067)[.068]
	$n = 200$					
1	.9994(.021)[.020]	.9996(.021)[.021]	.9990(.021)[.020]	.9993(.021)[.021]	.9993(.021)[.020]	.9995(.021)[.021]
1	.9999(.023)[.022]	.9999(.023)[.023]	.9986(.023)[.022]	.9996(.023)[.023]	.9994(.024)[.022]	.9995(.024)[.023]
1	.8909(.029)[−]	.9902(.032)[.033]	.8927(.071)[−]	.9923(.079)[.075]	.8944(.049)[−]	.9941(.055)[.055]
.3	.3002(.012)[.012]	.2999(.012)[.012]	.3008(.013)[.012]	.2999(.013)[.013]	.3001(.013)[.012]	.2999(.013)[.012]
.2	.1986(.028)[.026]	.1996(.028)[.028]	.1994(.026)[.026]	.1996(.026)[.026]	.1998(.029)[.026]	.2000(.029)[.028]
.2	.1992(.027)[.025]	.1998(.027)[.026]	.1984(.026)[.025]	.1988(.026)[.026]	.1979(.028)[.025]	.1983(.028)[.026]
.2	.1986(.050)[.045]	.1998(.050)[.049]	.1963(.047)[.045]	.1995(.048)[.048]	.1953(.048)[.045]	.2001(.048)[.048]

Note: 1. $\psi = (\beta', \sigma_v^2, \rho, \lambda')'$; 2. r_0 = true number of factor, r = assumed number of factor.

Table V-a. Empirical sd and average of estimated standard errors of M-Estimator
Parameter configurations as in Table V.

	Normal Error				Normal Mixture				Chi-Square			
ψ	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$	sd	$\widehat{\text{rse}}$	$\widehat{\text{se}}$	$\widetilde{\text{se}}$
	$n = 25$											
1	.071	.067	.120	.065	.070	.066	.136	.065	.070	.067	.127	.066
1	.069	.065	.113	.064	.067	.065	.131	.063	.066	.066	.121	.065
1	.087	.088	.155	.085	.200	.086	.099	.169	.145	.088	.126	.129
.3	.037	.035	.063	.035	.037	.035	.072	.036	.039	.035	.067	.036
.2	.080	.076	.132	.075	.078	.074	.150	.074	.078	.075	.142	.075
.2	.051	.049	.088	.049	.053	.049	.103	.048	.052	.049	.095	.048
.2	.145	.132	.179	.146	.145	.132	.196	.153	.141	.132	.189	.147
	$n = 50$											
1	.046	.043	.056	.043	.046	.043	.061	.043	.045	.043	.059	.042
1	.046	.048	.060	.047	.047	.048	.066	.047	.047	.048	.062	.047
1	.066	.065	.081	.063	.152	.065	.045	.140	.111	.065	.061	.102
.3	.023	.023	.029	.022	.025	.022	.033	.022	.023	.022	.031	.022
.2	.043	.044	.055	.044	.045	.044	.061	.043	.046	.045	.059	.044
.2	.028	.027	.036	.026	.028	.027	.040	.026	.027	.027	.038	.026
.2	.092	.089	.100	.091	.092	.089	.106	.090	.092	.089	.103	.090
	$n = 100$											
1	.031	.030	.034	.030	.031	.030	.036	.030	.030	.030	.035	.030
1	.034	.033	.037	.033	.033	.033	.039	.033	.032	.033	.037	.033
1	.047	.047	.052	.046	.105	.046	.026	.102	.078	.046	.037	.075
.3	.018	.018	.020	.018	.018	.018	.020	.018	.018	.018	.020	.018
.2	.041	.041	.045	.040	.040	.040	.047	.040	.040	.041	.048	.040
.2	.033	.033	.036	.033	.033	.033	.038	.033	.033	.033	.038	.033
.2	.070	.067	.072	.068	.068	.068	.075	.068	.067	.067	.074	.068
	$n = 200$											
1	.021	.021	.022	.021	.021	.021	.023	.021	.021	.021	.023	.021
1	.023	.023	.024	.023	.023	.023	.025	.023	.024	.023	.025	.023
1	.032	.033	.035	.033	.079	.033	.017	.075	.055	.033	.024	.055
.3	.012	.012	.013	.012	.013	.012	.013	.012	.013	.012	.013	.012
.2	.028	.028	.029	.028	.026	.028	.030	.028	.029	.028	.030	.028
.2	.027	.026	.028	.026	.026	.026	.029	.026	.028	.026	.029	.026
.2	.050	.048	.049	.048	.048	.048	.050	.048	.048	.048	.050	.048

Table VI. Empirical Mean(sd)[$\widehat{\text{rse}}$] of BC-CQMLE and M-Estimator: DGP1, $T = 3$, $m = 10$
$W_1 = W_2 = W_3$: `Rook Contiguity`, $r_0 = 1$, $r = 2$

	Normal Error		Normal Mixture		Chi-Square	
ψ	BC-CQMLE	M-Est	BC-CQMLE	M-Est	BC-CQMLE	M-Est
	$n = 50$					
1	.7243(.174)[.063]	.9988(.154)[.151]	.7857(.196)[.061]	.9895(.142)[.137]	.7507(.185)[.062]	.9899(.155)[.151]
1	.7370(.181)[.076]	.9838(.172)[.160]	.8110(.199)[.071]	.9965(.154)[.148]	.7728(.190)[.074]	.9877(.162)[.157]
1	.1701(.039)[−]	.6797(.144)[.147]	.1607(.048)[−]	.6254(.195)[.155]	.1653(.043)[−]	.6560(.180)[.155]
.3	-.1715(.210)[.054]	.2939(.107)[.102]	-.0564(.262)[.049]	.2932(.096)[.090]	-.1249(.234)[.052]	.2885(.100)[.102]
.2	.0957(.282)[.130]	.1870(.190)[.191]	.1202(.246)[.114]	.1806(.171)[.167]	.1032(.264)[.124]	.1622(.183)[.191]
.2	.1705(.246)[.111]	.2053(.143)[.150]	.1852(.209)[.093]	.2024(.121)[.126]	.1716(.234)[.103]	.1899(.135)[.147]
.2	.1402(.356)[.151]	.1876(.303)[.301]	.1377(.329)[.142]	.1767(.290)[.284]	.1523(.345)[.147]	.1980(.307)[.315]
	$n = 100$					
1	.8124(.195)[.055]	.9979(.111)[.119]	.8778(.179)[.052]	.9943(.109)[.110]	.8396(.192)[.053]	.9967(.106)[.118]
1	.8458(.149)[.055]	.9950(.107)[.115]	.8929(.150)[.052]	.9886(.105)[.109]	.8674(.161)[.053]	.9924(.113)[.122]
1	.2444(.033)[−]	.7933(.103)[.119]	.2243(.052)[−]	.7402(.178)[.158]	.2360(.042)[−]	.7570(.144)[.139]
.3	.1514(.258)[.040]	.2972(.076)[.077]	.2215(.228)[.035]	.2999(.073)[.073]	.1780(.253)[.038]	.2965(.073)[.083]
.2	.1662(.177)[.091]	.1997(.147)[.149]	.1676(.162)[.079]	.1961(.135)[.140]	.1666(.176)[.085]	.2028(.136)[.160]
.2	.1957(.149)[.067]	.1976(.124)[.135]	.1806(.142)[.060]	.1973(.120)[.123]	.1877(.142)[.065]	.1985(.119)[.140]
.2	.1591(.243)[.111]	.1921(.206)[.224]	.1900(.214)[.104]	.1930(.197)[.212]	.1819(.248)[.106]	.1960(.207)[.233]
	$n = 200$					
1	.8223(.069)[.037]	.9987(.081)[.087]	.8431(.084)[.036]	.9986(.076)[.087]	.8371(.078)[.037]	1.0006(.079)[.085]
1	.7641(.076)[.038]	.9978(.078)[.086]	.7966(.102)[.037]	.9985(.073)[.085]	.7788(.088)[.038]	.9971(.078)[.085]
1	.2247(.025)[−]	.8775(.088)[.100]	.2199(.035)[−]	.8154(.140)[.145]	.2229(.028)[−]	.8644(.115)[.123]
.3	-.0425(.074)[.029]	.2982(.060)[.060]	-.0077(.120)[.028]	.2998(.057)[.063]	-.0255(.098)[.029]	.2987(.059)[.065]
.2	.1351(.143)[.068]	.1993(.101)[.112]	.1421(.133)[.065]	.1976(.099)[.109]	.1467(.133)[.066]	.2017(.101)[.111]
.2	.1101(.101)[.055]	.1999(.094)[.095]	.1249(.104)[.051]	.1996(.082)[.093]	.1195(.100)[.053]	.1996(.089)[.096]
.2	.1984(.185)[.082]	.2003(.131)[.152]	.1959(.170)[.080]	.1966(.130)[.151]	.1922(.168)[.081]	.1963(.138)[.151]
	$n = 400$					
1	.9381(.056)[.029]	.9991(.055)[.060]	.9462(.057)[.028]	.9994(.053)[.060]	.9397(.058)[.028]	.9995(.053)[.065]
1	.9412(.058)[.028]	.9980(.055)[.059]	.9548(.056)[.028]	.9986(.052)[.059]	.9449(.058)[.028]	.98941(.053)[.063]
1	.2859(.022)[−]	.9591(.059)[.070]	.2745(.035)[−]	.9230(.108)[.118]	.2811(.028)[−]	.9025(.085)[.088]
.3	.2165(.082)[.017]	.2993(.036)[.039]	.2236(.078)[.017]	.2990(.038)[.046]	.2217(.083)[.017]	.2973(.039)[.036]
.2	.2168(.072)[.040]	.2002(.069)[.078]	.2034(.071)[.039]	.1992(.070)[.074]	.2047(.076)[.039]	.1977(.071)[.084]
.2	.2156(.049)[.025]	.2001(.048)[.054]	.2166(.046)[.024]	.2005(.047)[.053]	.2150(.047)[.025]	.2008(.047)[.058]
.2	.1888(.096)[.054]	.1998(.097)[.108]	.1989(.099)[.053]	.1998(.098)[.106]	.1960(.105)[.054]	.2008(.101)[.117]

Note: 1. $\psi = (\beta', \sigma_v^2, \rho, \lambda')'$; 2. r_0 = true number of factor, r = assumed number of factor.