Spatial Dynamic Panel Data Models with Correlated Random Effects*

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Abstract

In this paper, M-estimation and inference methods are developed for spatial dynamic panel data models with *correlated random effects*, based on *short panels*. The unobserved individual-specific effects are assumed to be correlated with the observed time-varying regressors linearly or in a *linearizable* way, giving the so-called correlated random effects model, which allows the estimation of effects of time-invariant regressors. The unbiased estimating functions are obtained by adjusting the conditional quasi-scores given the initial observations, leading to M-estimators that are consistent, asymptotically normal, and free from the initial conditions except the process starting time. By decomposing the estimating functions into sums of terms uncorrelated given idiosyncratic errors, a hybrid method is developed for consistently estimating the variance-covariance matrix of the M-estimators, which again depends only on the process starting time. Monte Carlo results demonstrate that the proposed methods perform well in finite sample.

Key Words: Adjusted quasi score; Dynamic panels; Correlated random effects; Initial-conditions; Martingale difference; Spatial effects; Short panels.

JEL classifications: C10, C13, C21, C23, C15

1. Introduction

Consider the spatial dynamic panel data (SDPD) model where the spatial effects appear in the model in the forms of spatial lag (SL), space-time lag (STL), and spatial error (SE):

$$\begin{aligned} y_t &= \rho y_{t-1} + \lambda_1 W_1 y_t + \lambda_2 W_2 y_{t-1} + X_t\beta + Z\gamma + \mu + \alpha_t 1_n + u_t, \\ & u_t = \lambda_3 W_3 u_t + v_t, \quad t = 1, 2, \ldots, T, \end{aligned} \tag{1.1}$$

where $y_t = (y_{1t}, y_{2t}, \ldots, y_{nt})'$ and $v_t = (v_{1t}, v_{2t}, \ldots, v_{nt})'$ are $n \times 1$ vectors of response values and idiosyncratic errors at time t, and $\{v_{it}\}$ are independent and identically distributed (iid) across i and t with mean zero and variance σ_v^2; the scalar parameter ρ characterizes the

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dynamic effect, λ_1 the spatial lag (SL) effect, λ_2 the space-time lag (STL) effect, and λ_3 the spatial error (SE) effect; $\{X_t\}$ are $n \times p$ matrices containing values of p time-varying exogenous variables, Z is an $n \times q$ matrix containing the values of q time-invariant exogenous variables that may include the intercept, dummy variables (e.g., individuals' gender and race), etc.; β and γ are the usual regression coefficients; $W_r, r = 1, 2, 3$, are the given $n \times n$ spatial weight matrices; and μ is an $n \times 1$ vector of unobserved individual-specific effects, $\alpha = \{\alpha_t\}_{t=1}^T$ is a $T \times 1$ vector of unobserved time-specific effects, and 1_n is an $n \times 1$ vector of ones.

According to the way (μ, α) relate to $\{X_t\}$, the model is classified as: (*i*) *fixed effects* (FE) model if (μ, α) are correlated with X_t arbitrarily; (*ii*) *random effects* (RE) model if (μ, α) are uncorrelated with X_t; and (*iii*) *correlated random effects* (CRE) if (μ, α) are correlated with X_t linearly or in a *linearizable* way (see Footnote 1). Lee, M-J (2002) called FE the *related effects*, and RE the *unrelated effects*. So, naturally the CRE can be called the *linearly related effects*. The term CRE is a tribute of Mundlak (1978), and Chamberlain (1982, 1984). In this work, we adopt the more popular terms: FE, RE and CRE, so that the SDPD models specified in (1.1) can be: FE-SDPD model, RE-SDPD model, or CRE-SDPD model.

Extensive discussions have appeared in the panel model literature, see, e.g. Cameron and Trivedi (2005), Wooldridge (2010), Baltagi (2013), and Hsiao (2014). The FE model has weaknesses (Cameron and Trivedi, 2005, p.715-716): (*i*) it does not allow the estimation of the effects of time-invariant regressors, e.g., gender, race; (*ii*) while coefficients of time-varying regressors are estimable, these estimates may be very imprecise if most of the variation in a regressor is cross sectional rather than over time; (*iii*) prediction of the conditional mean is impossible, instead only changes in conditional mean caused by the changes in time-varying regressors can be predicted; and (*iv*) even coefficients of time-varying regressors may be difficult or theoretically impossible to identify in nonlinear models. The RE model overcomes these difficulties, but causal interpretation may then be unwarranted (Cameron and Trivedi, 2005, p.715-716). The CRE model makes a compromise between the two: overcomes the weaknesses of the FE model and at the same time captures the linear or linearizable correlation between the 'effects' and the time-varying regressors.

In this paper, we consider the estimation and inference for the CRE-SDPD model, which includes the RE-SDPD model as a special case. We consider the large-n and small-T setting, i.e., the *short panels*. The literature on spatial dynamic panels is fast expanding in recent years. However, most of the research on spatial dynamic panel data models focused on the long panels (with large n and large T), see, e.g., Yang, et al. (2006), Mutl (2006), Yu, et al. (2008), Yu and Lee (2010), Lee and Yu (2010a, 2012, 2014); Bai and Li (2015), Shi and Lee (2017), with relatively fewer works on the short panels, e.g., Elhorst (2010), Su and Yang (2015), Qu, et al. (2016), Kuersteiner and Prucha (2018), and Yang (2018). Most of the works on short panels are on the FE-SDPD model, except Su and Yang (2015) who considered RE-SDPD model but with only the SE effect built in the model. The general RE-SDPD model of the form (1.1) has not been formally considered, and the more general CRE-SDPD model specification has not even appeared in the literature. See Anselin et al. (2008), and Lee and Yu (2010b, 2015) for nice surveys.

The CRE assumption renders a linear model for μ based on the observed X_t. We adopt the approach of Mundlak (1978) and specify that μ is linearly related to $\{X_t\}$ as,

$$\mu = \bar{X}\delta + \varepsilon, \tag{1.2}$$

where $\bar{X} = \frac{1}{T+1}\sum_{t=0}^{T} X_t$ and ε is an n-vector of $iid(0, \sigma_\varepsilon^2)$ errors, independent of v_t for all t. This can be extended to $\mu = X_0\delta_0 + X_1\delta_1 + \cdots + X_T\delta_T + \varepsilon$, as in Chamberlain (1982, 1984), a spatial Durbin form as in Debarsy (2012), or any *linearizable* relationship.[1]

Clearly, the advantages of the CRE-SDPD model over the FE-SPDP model are (i) it captures the typical correlation between μ and X_t and at the same time allows the effects of time-invariant variables Z, such as gender and race, be estimated, (ii) it may be more robust against possible existence of measurement errors and random coefficients, and (iii) it avoids the *incidental parameters* problem caused by the individual fixed effects, and hence may increase the estimation efficiency greatly.[2] However, the CRE-SDPD model induces another set of errors associated with the model for μ, besides the original set of idiosyncratic errors, hence it posts a much greater challenge in the estimation of model parameters and the estimation of standard errors of parameter estimates, in particular the latter, due to the fixed T nature. The key problem is that in short panels, the error components in the disturbance cannot be separately estimated, rendering the *outer-product-of-martingale-difference* (OPMD) method of Yang (2018) for the FE-SDPD model unapplicable. The full quasi maximum likelihood (QML) approach of Su and Yang (2015) is also unapplicable as a the usual way of modeling the initial observations based on a linear model may not be valid in the existence of spatial lag terms; see Yang (2018) for a detailed discussion on this.

In this paper, an M-estimation method is proposed for estimating the CRE-SDPD model based on short panels, which is **free from the initial conditions** except the process starting time $(-m)$. The method modifies the conditional quasi score function given the initial observations, to give a set of unbiased *estimation functions* or moment conditions. For statistical inferences, the vector of estimating functions is written as a sum with the n summands being martingale differences with respect to individual-specific errors given idiosyncratic errors, so that a hybrid method that combines analytical derivations and the feasible sample analogues is proposed for estimating the variance-covariance (VC) matrix of the M-estimators. The resulting VC matrix estimator is also free from the initial conditions except the process starting time. The consistency and asymptotic normality of the M-estimators are established, and the consistency of the VC matrix estimator is also proved. Extensive Monte Carlo results

[1]The intercept of Model (1.2) is absorbed into the intercept of Model (1.1), or vise versa, for parameter identifiability (see Sec. 2.1 for details on this). By 'linearizable' we mean any CRE relationship that can be written as or approximated by a model linear in a finite number of parameters. To keep our exposition simple enough, we work with the CRE form (1.2). For related issue on parameter identification under alternative spatial specifications, see, e.g., Anselin et al. (2008, p.647), Elhorst (2012), Lee and Yu (2016).

[2]Clearly, the CRE-SDPD model embeds the RE-SDPD model. The estimation of the FE-SDPD model is typically through a first difference or some orthonormal transformation to remove the fixed effects. However, it simultaneously removes all the time-invariant variables, and hence their effects cannot be estimated. Furthermore, due to the differencing or transformation, one period of the data is 'lost' which may consist of one third or one quarter of the 'usable' data if $T = 3$ or 4, making a significant difference in estimation efficiency.

show that, in finite samples, (i) proposed M-estimators perform very well, much superior to the conditional QML estimators, (ii) proposed VC matrix estimator also performs well, and (iii) in case of the simple RE-SDPD model with only `SE` effect, the proposed M-estimator performs equally well as the full QMLE of Su and Yang (2015), but is numerically much more efficient. The proposed M-method for point estimation remains valid if T goes large with n. In this case, the usual method for estimating VC matrix applies.

The CRE-SDPD model given in (1.1) is fairly general, embedding several important sub-models obtained by dropping one or two spatial effects, none of which has been formally treated in the literature except Su and Yang (2015).[3] Thus, it is highly desirable to have a unified method of inference for this general model so that the method can easily be simplified to suit each special model of interest for a particular applied problem.

The rest of the paper goes as follows. Section 2 introduces the M-estimation framework for the CRE-SDPD model, and presents asymptotic properties of the proposed M-estimator. Section 3 introduces a robust method for estimating the VC matrix of the M-estimator. Section 4 presents Monte Carlo results. Section 5 concludes the paper. All the technical proofs are relegated to the appendices.

2. Estimation of SDPD Model with CRE

2.1. Conditional QML Estimation of CRE-SDPD Model

For the SDPD model with CRE specified by (1.1)-(1.2), we focus on the case of large n and small T. **Assume** (i) data collection starts from the 0th period; the processes start from the $-m$th period, i.e., m periods before the start of data collection where $m \geq 0$, and then evolve according to the prescribed processes, i.e., one of the models described above; (ii) starting positions of the process y_{-m} are treated as exogenous; hence the exogenous variables X_t and the errors u_t start to have impact on the response from the period $-m+1$ onwards; (iii) all the exogenous quantities (y_{-m}, X_t, Z) can be fixed or random, and in the later case inferences proceed by conditioning on them.

Let $B_r \equiv B_r(\lambda_r) = I_n - \lambda_r W_r, r = 1, 3$, and $B_2 \equiv B_2(\rho, \lambda_2) = \rho I_n + \lambda_2 W_2$. The model specified by (1.1)-(1.2) has reduced form:

$$y_t = B_1^{-1} B_2 y_{t-1} + B_1^{-1}(X_t\beta + Z\gamma + \bar{X}\delta + \alpha_t 1_n) + B_1^{-1}\varepsilon + B_1^{-1}B_3^{-1}v_t. \tag{2.1}$$

Let $Y = (y_1', \ldots, y_T')'$, $Y_{-1} = (y_0', \ldots, y_{T-1}')'$, $X = (X_1', \ldots, X_T')'$, $D = (I_{T-1} \otimes 1_n',\ 0_{(T-1)}0_n')'$, and $\mathbf{X} = (1_{nT}, D, X, 1_T \otimes Z, 1_T \otimes \bar{X})$, where $\otimes$ denotes the Kronecker product, 1_k denotes a $k \times 1$ vector of ones, 0_k a $k \times 1$ vector of zeros, and I_k a $k \times k$ identity matrix. Further, let $\boldsymbol{\varepsilon} = 1_T \otimes \varepsilon$, $\mathbf{v} = (v_1', \ldots, v_T')'$, $\mathbf{W}_r = I_T \otimes W_r$, and $\mathbf{B}_r = I_T \otimes B_r, r = 1, 2, 3$. The reduced form (2.1) can be written compactly in matrix form:

[3] They considered a SDPD model with RE and spatial error (i.e., in Model (1.1) setting λ_1 and λ_2 to zero). By modeling the initial observations by a linear model based on the observed regressors, they obtained a full QML estimator (QMLE) for the model and a bootstrap estimator for the VC matrix. It would be interesting to compare the full QMLE with the M-estimators proposed in this paper.

$$Y = \mathbf{B}_1^{-1}\mathbf{B}_2 Y_{-1} + \mathbf{B}_1^{-1}\mathbf{X}\beta + \mathbf{B}_1^{-1}\boldsymbol{\varepsilon} + \mathbf{B}_1^{-1}\mathbf{B}_3^{-1}\mathbf{v}. \tag{2.2}$$

where $\boldsymbol{\beta} = (\breve{\alpha}', \beta', \gamma', \delta')'$ with $\mathtt{dim}(\boldsymbol{\beta}) = 2p + q + T$, and $\breve{\alpha} = (\alpha_T, \alpha_1 - \alpha_T, \ldots, \alpha_{T-1} - \alpha_T)'$.

Let $\mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}_3^{-1}\mathbf{v}$. As $\{\varepsilon_i\}$ are $iid(0, \sigma_\varepsilon^2)$, $\{v_{it}\}$ are $iid(0, \sigma_v^2)$, and $\boldsymbol{\varepsilon}$ and $\mathbf{v}$ are independent, the variance-covariance (VC) matrix of $\mathbf{e}$ is:

$$\text{Var}(\mathbf{e}) = \sigma_\varepsilon^2 (J_T \otimes I_n) + \sigma_v^2 (\mathbf{B}'{}_3 \mathbf{B}_3)^{-1} = \sigma_v^2[\phi(J_T \otimes I_n) + (\mathbf{B}'{}_3\mathbf{B}_3)^{-1}] \equiv \sigma_v^2 \Omega, \tag{2.3}$$

where $\phi = \sigma_\varepsilon^2/\sigma_v^2$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)'$, $\theta = (\boldsymbol{\beta}', \rho, \lambda_1, \lambda_2)$ and $\psi = (\boldsymbol{\beta}', \sigma_v^2, \phi, \rho, \lambda')'$. The **quasi** Gaussian loglikelihood, treating $\boldsymbol{\varepsilon}$ and $\mathbf{v}$ as normally distributed and y_0 as exogenously generated (or conditioning on y_0) is

$$\ell_{\mathtt{SDPD}}(\psi) = -\tfrac{nT}{2}\log(2\pi\sigma_v^2) - \tfrac{1}{2}\log|\Omega(\phi, \lambda_3)| + \log|\mathbf{B}_1(\lambda_1)| - \tfrac{1}{2\sigma_v^2} e'(\theta)\Omega^{-1}(\phi, \lambda_3) e(\theta), \tag{2.4}$$

where $e(\theta) = \mathbf{B}_1 Y - \mathbf{B}_2 Y_{-1} - \mathbf{X}\boldsymbol{\beta}$, and $|\cdot|$ denotes the determinant of a square matrix.

Maximizing $\ell_{\mathtt{SDPD}}(\psi)$ gives the conditional QML (CQML) estimator $\hat{\psi}_c$ of ψ. However, y_0 is not exogenous unless $m = 0$ (data collection starts when process starts) and ε and/or $\mathbf{v}$ may not be normal. Thus, $\ell_{\mathtt{SDPD}}(\psi)$ may not be a true loglikelihood function and maximizing it may not give a consistent estimate of ψ, in particular when $m > 0$ so that y_0 is endogenously generated. When T is also large, consistency may be achieved as ignoring the endogeneity in y_0 is asymptotically negligible. However, it may still suffer from the so-called asymptotic bias problem. To overcome these problems, Yang (2018) propose a unified M-estimation approach for the model with FE specification. In this paper, we adopt a similar approach by modifying the quasi score functions to have a set of estimating functions that are unbiased at the true parameter values so that a necessary condition for consistency is satisfied.

2.2. M-Estimation of CRE-SDPD Model

The quasi-score function $S_{\mathtt{SDPD}}(\psi) = \frac{\partial}{\partial\psi}\ell_{\mathtt{SDPD}}(\psi)$ has the form:

$$S_{\mathtt{SDPD}}(\psi) = \begin{cases} \frac{1}{\sigma_v^2}\mathbf{X}'\Omega^{-1}e(\theta), \\ \frac{1}{2\sigma_v^4}e'(\theta)\Omega^{-1}e(\theta) - \frac{nT}{2\sigma_v^2}, \\ \frac{1}{2\sigma_v^2}e'(\theta)\Omega^{-1}(J_T \otimes I_n)\Omega^{-1}e(\theta) - \frac{1}{2}\text{tr}[\Omega^{-1}(J_T \otimes I_n)], \\ \frac{1}{\sigma_v^2}e'(\theta)\Omega^{-1}Y_{-1}, \\ \frac{1}{\sigma_v^2}e'(\theta)\Omega^{-1}\mathbf{W}_1 Y - \text{tr}(\mathbf{B}_1^{-1}\mathbf{W}_1), \\ \frac{1}{\sigma_v^2}e'(\theta)\Omega^{-1}\mathbf{W}_2 Y_{-1}, \\ \frac{1}{2\sigma_v^2}e'(\theta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}e(\theta) - \frac{1}{2}\text{tr}(\Omega^{-1}\dot{\Omega}_{\lambda_3}), \end{cases} \tag{2.5}$$

where $\dot{\Omega}_{\lambda_3} = (\mathbf{B}_3'\mathbf{B}_3)^{-1}(\mathbf{B}_3'\mathbf{W}_3 + \mathbf{W}_3'\mathbf{B}_3)(\mathbf{B}_3'\mathbf{B}_3)^{-1}$, and $\text{tr}(\cdot)$ is the trace of a square matrix.

Let ψ_0 be the true value of ψ. A parametric quantity evaluated at the true parameters is denoted by adding a subscript '$_0$', e.g., B_{10}, Ω_0. The usual expectation and variance operators $\text{E}(\cdot)$ and $\text{Var}(\cdot)$ correspond to the true parameters. We derive $\text{E}[S_{\mathtt{SDPD}}(\psi_0)]$, and show that the ρ and λ-components of $\text{E}[S_{\mathtt{SDPD}}(\psi_0)]$ are generally not zero, and that the same components of $\text{plim}_{n\to\infty}\frac{1}{nT}S_{\mathtt{SDPD}}(\psi_0)$ are not zero. Thus, the CQML estimator $\hat{\psi}_c$ cannot be consistent.

Assumption A: *Assume (i) the processes started $m(\geq 0)$ periods before the start of data collection (0th period), and then evolve according to Models (1.1) and (1.2), (ii) $y_{-m}, \{X_t, t = -m+1, \ldots, T\}$, and Z are exogenous, and (iii) the individual specific effects μ are linearly related to X_t with additive errors ε independent of $v_t, t = -m+1, \ldots, v_T$.*

By using recursive substitution on (2.1), we have an important lemma:

Lemma 2.1. *Suppose Assumption A holds. Assume further that the errors $\{v_{it}\}$ in Model (1.1) are iid$(0, \sigma_{v0}^2)$ across i and t, the errors $\{\varepsilon_i\}$ in Model (1.2) are iid$(0, \sigma_{\varepsilon 0}^2)$, and $\{v_{it}\}$ and $\{\varepsilon_i\}$ are independent. If both B_{10}^{-1} and B_{30}^{-1} exist, then we have*

$$\mathrm{E}(Y_{-1}\mathbf{e}') = \phi_0 \mathbf{C}_{-10} + \mathbf{D}_{-10}, \tag{2.6}$$

$$\mathrm{E}(Y\mathbf{e}') = \phi_0 \mathbf{C}_0 + \mathbf{D}_0, \tag{2.7}$$

where $\mathbf{C} \equiv \mathbf{C}(\rho, \lambda_1, \lambda_2, m)$, $\mathbf{C}_{-1} \equiv \mathbf{C}_{-1}(\rho, \lambda_1, \lambda_2, m)$, $\mathbf{D} \equiv \mathbf{D}(\rho, \lambda_1, \lambda_2, \lambda_3)$, and $\mathbf{D}_{-1} \equiv \mathbf{D}_{-1}(\rho, \lambda_1, \lambda_2, \lambda_3)$ are $nT \times nT$ matrices, defined as follows: $\mathbf{C} = [(C_1, C_2, \ldots, C_T) \otimes 1_T]'$ and $\mathbf{C}_{-1} = [(C_0, C_1, \ldots, C_{T-1}) \otimes 1_T]'$, where $C_t = (\sum_{i=0}^{t+m-1} \mathcal{B}^i) B_1^{-1}$ and $\mathcal{B} = B_1^{-1} B_2$;

$$\mathbf{D} = \begin{pmatrix} D_0 & 0 & \ldots & 0 \\ D_1 & D_0 & \ldots & 0 \\ D_2 & D_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D_{T-1} & D_{T-2} & \ldots & D_0 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_{-1} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ D_0 & 0 & \ldots & 0 \\ D_1 & D_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D_{T-2} & D_{T-3} & \ldots & 0 \end{pmatrix},$$

where $D_t = \mathcal{B}^t B_1^{-1} B_3^{-1} (B_3^{-1})'$.

The results of Lemma 2.1 lead immediately to

$$\mathrm{E}(\mathbf{e}'\Omega_0^{-1} Y_{-1}) = \mathrm{tr}[(\phi_0 \mathbf{C}_{-10} + \mathbf{D}_{-10})\Omega_0^{-1}], \tag{2.8}$$

$$\mathrm{E}(\mathbf{e}'\Omega_0^{-1} \mathbf{W}_1 Y) = \mathrm{tr}[(\phi_0 \mathbf{C}_0 + \mathbf{D}_0)\Omega_0^{-1} \mathbf{W}_1], \tag{2.9}$$

$$\mathrm{E}(\mathbf{e}'\Omega_0^{-1} \mathbf{W}_2 Y_{-1}) = \mathrm{tr}[(\phi_0 \mathbf{C}_{-10} + \mathbf{D}_{-10})\Omega_0^{-1} \mathbf{W}_2]. \tag{2.10}$$

These results show that the ρ and λ-components of $\mathrm{E}[S_{\mathtt{SDPD}}(\psi_0)]$ are generally not zero, and that the same components of $\mathrm{plim}_{n\to\infty} \frac{1}{nT} S_{\mathtt{SDPD}}(\psi_0)$ are not zero. Therefore, the corresponding CQML estimator $\hat{\psi}_c$ cannot be consistent. Noticing that these quantities are free from the initial conditions, except the process starting time, they provide a simple way to adjust the quasi-scores so as to give a set of unbiased estimating functions free from the initial conditions except m. The adjusted quasi-score (AQS) functions are:

$$S^*_{\mathtt{SDPD}}(\psi) = \begin{cases} \frac{1}{\sigma_v^2} \mathbf{X}'\Omega^{-1} e(\theta), \\ \frac{1}{2\sigma_v^4} e'(\theta)\Omega^{-1} e(\theta) - \frac{nT}{2\sigma_v^2}, \\ \frac{1}{2\sigma_v^2} e'(\theta)\Omega^{-1}(J_T \otimes I_n)\Omega^{-1} e(\theta) - \frac{1}{2}\mathrm{tr}[\Omega^{-1}(J_T \otimes I_n)], \\ \frac{1}{\sigma_v^2} e'(\theta)\Omega^{-1} Y_{-1} - \mathrm{tr}[(\phi \mathbf{C}_{-1} + \mathbf{D}_{-1})\Omega^{-1}], \\ \frac{1}{\sigma_v^2} e'(\theta)\Omega^{-1} \mathbf{W}_1 Y - \mathrm{tr}[(\phi \mathbf{C} + \mathbf{D})\Omega^{-1} \mathbf{W}_1], \\ \frac{1}{\sigma_v^2} e'(\theta)\Omega^{-1} \mathbf{W}_2 Y_{-1} - \mathrm{tr}[(\phi \mathbf{C}_{-1} + \mathbf{D}_{-1})\Omega^{-1} \mathbf{W}_2], \\ \frac{1}{2\sigma_v^2} e'(\theta)\Omega^{-1} \dot{\Omega}_{\lambda_3} \Omega^{-1} e(\theta) - \frac{1}{2}\mathrm{tr}(\Omega^{-1} \dot{\Omega}_{\lambda_3}). \end{cases} \tag{2.11}$$

It is easy to show that $\mathrm{E}[S^*_{\mathtt{SDPD}}(\psi_0)] = 0$, and that $\mathrm{plim}_{n\to\infty}\frac{1}{nT}S^*_{\mathtt{SDPD}}(\psi_0) = 0$. Solving the estimating equations $S^*_{\mathtt{SDPD}}(\psi) = 0$ gives a potentially consistent estimator $\hat{\psi}_{\mathtt{M}}$ of ψ, termed in this paper as M-estimator similar to Yang (2018). Indeed, under some regularity conditions it is shown in Theorems 2.1 and 2.2 that $\hat{\psi}_{\mathtt{M}}$ is consistent and asymptotically normal.

The equation solving process can be simplified by first solving the equations for $\boldsymbol{\beta}$ and σ_v^2 given $\delta = (\phi, \rho, \lambda')'$ to obtain the constrained M-estimators of $\boldsymbol{\beta}$ and σ_v^2 as

$$\hat{\boldsymbol{\beta}}(\delta) = (\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}(\mathbf{B}_1 Y - \mathbf{B}_2 Y_{-1}), \tag{2.12}$$

$$\hat{\sigma}_v^2(\delta) = \tfrac{1}{nT}\hat{\mathbf{e}}'(\delta)\Omega^{-1}\hat{\mathbf{e}}(\delta), \tag{2.13}$$

where $\hat{\mathbf{e}}(\delta) = \mathbf{B}_1 Y - \mathbf{B}_2 Y_{-1} - \mathbf{X}\hat{\boldsymbol{\beta}}(\delta)$. Substituting $\hat{\boldsymbol{\beta}}(\delta)$ and $\hat{\sigma}_v^2(\delta)$ back into the last five components of the AQS functions gives the concentrated AQS functions:

$$S^{*c}_{\mathtt{SDPD}}(\delta) = \begin{cases} \frac{1}{2\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\hat{e}(\delta) - \frac{1}{2}\mathrm{tr}[(\Omega^{-1}(J_T\otimes I_n)], \\ \frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}Y_{-1} - \mathrm{tr}[(\phi\mathbf{C}_{-1} + \mathbf{D}_{-1})\Omega^{-1}], \\ \frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_1 Y - \mathrm{tr}[(\phi\mathbf{C} + \mathbf{D})\Omega^{-1}\mathbf{W}_1], \\ \frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1} - \mathrm{tr}[(\phi\mathbf{C}_{-1} + \mathbf{D}_{-1})\Omega^{-1}\mathbf{W}_2], \\ \frac{1}{2\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta) - \frac{1}{2}\mathrm{tr}(\Omega^{-1}\dot{\Omega}_{\lambda_3}). \end{cases} \tag{2.14}$$

Solving the resulted concentrated estimating equations, $S^{*c}_{\mathtt{SDPD}}(\delta) = 0$, we obtain the unconstrained M-estimators $\hat{\delta}_{\mathtt{M}}$ of δ. The unconstrained M-estimators of $\boldsymbol{\beta}$ and σ_v^2 are thus $\hat{\boldsymbol{\beta}}_{\mathtt{M}} \equiv \hat{\boldsymbol{\beta}}(\hat{\delta}_{\mathtt{M}})$ and $\hat{\sigma}^2_{v,\mathtt{M}} \equiv \hat{\sigma}_v^2(\hat{\delta}_{\mathtt{M}})$, leading to $\hat{\psi}_{\mathtt{M}} = (\hat{\boldsymbol{\beta}}'_{\mathtt{M}}, \hat{\sigma}^2_{v,\mathtt{M}}, \hat{\delta}'_{\mathtt{M}})'$.

Remark 2.1. *From the way that the AQS function is defined in* (2.11)*, we see that the M estimator* $\hat{\psi}_{\mathtt{M}}$ *for the CRE-SDPD model specified by* (1.1) *and* (1.2) *is free from the specification of the distribution of* y_0*, except the value* m*, which is different from the M-estimator of the FE-SDPD model considered in Yang (2018), but similar to the full QMLE of the RE-SDPD model with only* SE *effect considered in Su and Yang (2015).*

However, this does not pose a serious problem as (*i*) *in practice one is often able to 'tell' roughly the value of* m*, and* (*ii*) $\hat{\psi}_{\mathtt{M}}$ *is quite robust against the changes in the value of* m*. See Elhorst (2010) and Su and Yang (2015) for similar remarks.*

2.3. Asymptotic Properties of M-Estimator

To proceed with a formal study on the asymptotic properties of the proposed M-estimator, some generic notations are helpful: (*i*) `blkdiag`$(\cdots)$ places square matrices diagonally, (*ii*) $\gamma_{\min}(\cdot)$ and $\gamma_{\max}(\cdot)$ denote the smallest and largest eigenvalues of a real symmetric matrix, and (*iii*) $\|\cdot\|$ denotes the Frobenius norm of a matrix.

Assumption B: *The innovations* v_{it} *are iid for all* i *and* t *with* $E(v_{it}) = 0$*,* $Var(v_{it}) = \sigma^2_{v0}$*, and* $E|v_{it}|^{4+\epsilon_0} < \infty$ *for some* $\epsilon_0 > 0$*. The innovations* ε_i *are iid for all* i *with* $E(\varepsilon_i) = 0$*,* $Var(\varepsilon_i) = \sigma^2_{\varepsilon 0}$*, and* $E|\varepsilon_{it}|^{4+\epsilon_0} < \infty$ *for some* $\epsilon_0 > 0$*.*

Assumption C: *The space* $\boldsymbol{\Delta}$ *is compact, and the true parameter* δ_0 *lies in its interior.*

Assumption D: *The time-varying regressors* $\{X_t,\ t = 0, 1, \ldots, T\}$ *are exogenous, their values are uniformly bounded, and* $\lim_{n\to\infty}\frac{1}{nT}\mathbf{X}'\mathbf{X}$ *exists and is nonsingular.*

Assumption E: *(i) For $r = 1, 2, 3$, the elements $w_{r,ij}$ of W_r are at most of order h_n^{-1}, uniformly in all i and j, and $w_{r,ii} = 0$ for all i; (ii) $h_n/n \to 0$ as $n \to \infty$; (iii) $\{W_r, r = 1, 2, 3\}$ and $\{B_{r0}^{-1}, r = 1, 3\}$ are uniformly bounded in both row and column sums; (iv) For $r = 1, 3$, $\{B_r^{-1}\}$ are uniformly bounded in either row or column sums, uniformly in λ_r in a compact parameter space $\mathbf{\Lambda}_r$, and $0 < \underline{c}_r \leq \inf_{\lambda_r \in \mathbf{\Lambda}_r} \gamma_{\min}(B_r' B_r) \leq \sup_{\lambda_r \in \mathbf{\Lambda}_r} \gamma_{\max}(B_r' B_r) \leq \bar{c}_r < \infty$.*

Assumption F: *For an $n \times n$ matrix Φ uniformly bounded in either row or column sums, with elements of uniform order h_n^{-1}, and an $n \times 1$ vector ϕ with elements of uniform order $h_n^{-1/2}$, (i) $\frac{h_n}{n} y_0' \Phi y_0 = O_p(1)$; (ii) $\frac{h_n}{n}[y_0 - \mathrm{E}(y_0)]'\phi = o_p(1)$; (iii) $\frac{h_n}{n}[y_0' \Phi y_0 - \mathrm{E}(y_0' \Phi y_0)] = o_p(1)$.*

The consistency of the proposed M-estimators $\hat{\psi}_{\mathtt{M}}$ lies with the consistency of $\hat{\delta}_{\mathtt{M}}$, as under Assumptions D and E, the consistency of $\hat{\beta}_{\mathtt{M}}$ and $\hat{\sigma}^2_{v,\mathtt{M}}$ follows almost immediately that of $\hat{\delta}_{\mathtt{M}}$. Define $\bar{S}^*_{\mathtt{SDPD}}(\psi) = \mathrm{E}[S^*_{\mathtt{SDPD}}(\psi)]$, the population counter part of the AQS function given in (2.11). Given δ, the population AQS functions $\bar{S}^*_{\mathtt{SDPD}}(\psi) = 0$ are partially solved at

$$\bar{\beta}_{\mathtt{M}}(\delta) = (\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}(\mathbf{B}_1 \mathrm{E}Y - \mathbf{B}_2 \mathrm{E}Y_{-1}), \tag{2.15}$$

$$\bar{\sigma}^2_{v,\mathtt{M}}(\delta) = \tfrac{1}{n(T-1)} \mathrm{E}[\bar{\mathbf{e}}(\delta)'\Omega^{-1}\bar{\mathbf{e}}(\delta)], \tag{2.16}$$

where $\bar{\mathbf{e}}(\delta) = \mathbf{e}(\theta)|_{\beta = \bar{\beta}(\delta)} = \mathbf{B}_1 Y - \mathbf{B}_2 Y_{-1} - \mathbf{X}\bar{\beta}(\delta)$. Substituting $\bar{\beta}_{\mathtt{M}}(\delta)$ and $\bar{\sigma}^2_{v,\mathtt{M}}(\delta)$ into the last five equations of $\bar{S}^*_{\mathtt{SDPD}}(\psi)$ leads to the population counter part of the concentrated AQS functions given in (2.14):

$$\bar{S}^{*c}_{\mathtt{SDPD}}(\delta) = \begin{cases} \frac{1}{2\bar{\sigma}_v^2(\delta)} \mathrm{E}\big[\bar{\mathbf{e}}'(\delta)\Omega^{-1}(J_T \otimes I_n)\Omega^{-1}\bar{\mathbf{e}}(\delta)\big] - \frac{1}{2}\mathrm{tr}\big[(\Omega^{-1}(J_T \otimes I_n)\big], \\ \frac{1}{\bar{\sigma}_v^2(\delta)} \mathrm{E}\big[\bar{\mathbf{e}}'(\delta)\Omega^{-1}Y_{-1}\big] - \mathrm{tr}\big[(\phi\mathbf{C}_{-1} + \mathbf{D}_{-1})\Omega^{-1}\big], \\ \frac{1}{\bar{\sigma}_v^2(\delta)} \mathrm{E}\big[\bar{\mathbf{e}}'(\delta)\Omega^{-1}\mathbf{W}_1 Y\big] - \mathrm{tr}\big[(\phi\mathbf{C} + \mathbf{D})\Omega^{-1}\mathbf{W}_1\big], \\ \frac{1}{\bar{\sigma}_v^2(\delta)} \mathrm{E}\big[\bar{\mathbf{e}}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1}\big] - \mathrm{tr}\big[(\phi\mathbf{C}_{-1} + \mathbf{D}_{-1})\Omega^{-1}\mathbf{W}_2\big], \\ \frac{1}{2\bar{\sigma}_v^2(\delta)} \mathrm{E}\big[\bar{\mathbf{e}}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\bar{\mathbf{e}}(\delta)\big] - \frac{1}{2}\mathrm{tr}(\Omega^{-1}\dot{\Omega}_{\lambda_3}), \end{cases} \tag{2.17}$$

Note that the M-estimator $\hat{\delta}_{\mathtt{M}}$ of δ_0 is a zero of $S^{*c}_{\mathtt{SDPD}}(\delta)$. It is easy to see that $\bar{S}^{*c}_{\mathtt{SDPD}}(\delta_0) = 0$ through $\bar{\beta}(\delta_0) = \beta_0$ and $\bar{\sigma}^2_v(\delta_0) = \sigma^2_{v0}$, i.e., δ_0 is a zero of $\bar{S}^{*c}_{\mathtt{SDPD}}(\delta)$. Thus, by Theorem 5.9 of van der Vaart (1998), $\hat{\delta}_{\mathtt{M}}$ will be consistent for δ_0 if $\sup_{\delta \in \Delta} \frac{1}{nT}\big\|S^{*c}_{\mathtt{SDPD}}(\delta) - \bar{S}^{*c}_{\mathtt{SDPD}}(\delta)\big\| \xrightarrow{p} 0$, and the following identification condition holds.

Assumption G: $\inf_{\delta:\ d(\delta,\delta_0) \geq \varepsilon} \big\|\bar{S}^{*c}_{\mathtt{SDPD}}(\delta)\big\| > 0$ *for every $\varepsilon > 0$, where $d(\delta, \delta_0)$ is a measure of distance between δ_0 and δ.*

Theorem 2.1. *Suppose Assumptions A-G hold. Assume further that* (*i*) $\gamma_{\max}[\mathrm{Var}(Y)]$ *and* $\gamma_{\max}[\mathrm{Var}(Y_{-1})]$ *are bounded, and* (*ii*) $\inf_{\delta \in \Delta} \gamma_{\min}[\mathrm{Var}(\mathbf{B}_1 Y - \mathbf{B}_2 Y_{-1})] \geq \underline{c}_y > 0$. *We have, as $n \to \infty$, $\hat{\psi}_{\mathtt{M}} \xrightarrow{p} \psi_0$.*

To establish asymptotic normality of the proposed M-estimator $\hat{\psi}_{\mathtt{M}}$, the representations of Y and Y_{-1} in terms of $\mathbf{y}_0 = 1_T \otimes y_0$ and $\mathbf{e}$ given in the following lemma are very useful.

Lemma 2.2. *Under the assumptions of Lemma 2.1, we have,*

$$Y = \mathbb{Q}\mathbf{y}_0 + \boldsymbol{\eta} + \mathbb{S}\mathbf{e} \quad \textit{and} \quad Y_{-1} = \mathbb{Q}_1\mathbf{y}_0 + \boldsymbol{\eta}_{-1} + \mathbb{S}_{-1}\mathbf{e}, \tag{2.18}$$

where $\mathbb{Q} = \mathtt{blkdiag}(\mathcal{B}_0, \mathcal{B}_0^2, \ldots, \mathcal{B}_0^T)$, $\mathbb{Q}_{-1} = \mathtt{blkdiag}(I_n, \mathcal{B}_0^1, \ldots, \mathcal{B}_0^{T-1})$, $\mathbb{S} = \mathbb{R}\mathbf{B}_1^{-1}$, $\mathbb{S}_1 =$

$\mathbb{R}_1\mathbf{B}_1^{-1}$, $\boldsymbol{\eta} = \mathbb{S}\mathbf{X}\beta$, $\boldsymbol{\eta}_1 = \mathbb{S}_1\mathbf{X}\beta$,

$$\mathbb{R} = \begin{pmatrix} I_n & 0 & 0 & \dots & 0 \\ \mathcal{B}_0 & I_n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_0^{T-1} & \mathcal{B}_0^{T-2} & \mathcal{B}_0^{T-3} & \dots & I_n \end{pmatrix} \quad \textit{and} \quad \mathbb{R}_{-1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ I_n & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_0^{T-2} & \mathcal{B}_0^{T-3} & \mathcal{B}_0^{T-4} & \dots & 0 \end{pmatrix}.$$

By the representations given in (2.18) the AQS vector at ψ_0 is written as

$$S^*_{\mathtt{SDPD}}(\psi_0) = \begin{cases} \Pi_1'\mathbf{e}, \\ \mathbf{e}'\Phi_1\mathbf{e} - \frac{nT}{2\sigma^2_{v0}}, \\ \mathbf{e}'\Phi_2\mathbf{e} - \frac{1}{2}\mathrm{tr}[\Omega_0^{-1}(J_T \otimes I_n)], \\ \mathbf{e}'\Psi_1\mathbf{y}_0 + \Pi_2'\mathbf{e} + \mathbf{e}'\Phi_3\mathbf{e} - \mathrm{tr}[(\phi\mathbf{C}_{-10} + \mathbf{D}_{-10})\Omega_0^{-1}], \\ \mathbf{e}'\Psi_2\mathbf{y}_0 + \Pi_3'\mathbf{e} + \mathbf{e}'\Phi_4\mathbf{e} - \mathrm{tr}[(\phi\mathbf{C}_0 + \mathbf{D}_0)\Omega_0^{-1}\mathbf{W}_1], \\ \mathbf{e}'\Psi_3\mathbf{y}_0 + \Pi_4'\mathbf{e} + \mathbf{e}'\Phi_5\mathbf{e} - \mathrm{tr}[(\phi\mathbf{C}_{-10} + \mathbf{D}_{-10})\Omega_0^{-1}\mathbf{W}_2], \\ \mathbf{e}'\Phi_6\mathbf{e} - \frac{1}{2}\mathrm{tr}(\Omega_0^{-1}\dot{\Omega}_{\lambda_3 0}), \end{cases} \tag{2.19}$$

where $\Pi_1 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\mathbf{X}$, $\Pi_2 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\boldsymbol{\eta}_{-1}$, $\Pi_3 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\mathbf{W}_1\boldsymbol{\eta}$, and $\Pi_4 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\mathbf{W}_2\boldsymbol{\eta}_{-1}$;
$\Phi_1 = \frac{1}{2\sigma^4_{v0}}\Omega_0^{-1}$, $\Phi_2 = \frac{1}{2\sigma^2_{v0}}\Omega_0^{-1}(J_T \otimes I_n)\Omega_0^{-1}$, $\Phi_3 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\mathbb{S}_{-1}$, $\Phi_4 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\mathbf{W}_1\mathbb{S}$,
$\Phi_5 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\mathbf{W}_2\mathbb{S}_{-1}$, and $\Phi_6 = \frac{1}{2\sigma^2_{v0}}\Omega_0^{-1}\dot{\Omega}_{\lambda_3 0}\Omega_0^{-1}$;
$\Psi_1 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\mathbb{Q}_{-1}$, $\Psi_2 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\mathbf{W}_1\mathbb{Q}$, and $\Psi_3 = \frac{1}{\sigma^2_{v0}}\Omega_0^{-1}\mathbf{W}_2\mathbb{Q}_{-1}$.

Now, by Lemma 2.2 and noticing $\mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}_{30}^{-1}\mathbf{v}$, Y and Y_{-1} are further represented as

$$Y = \mathbb{Q}\mathbf{y}_0 + \boldsymbol{\eta} + \mathbb{S}\boldsymbol{\varepsilon} + \mathbb{B}\mathbf{v} \quad \text{and} \quad Y_{-1} = \mathbb{Q}_{-1}\mathbf{y}_0 + \boldsymbol{\eta}_{-1} + \mathbb{S}_{-1}\boldsymbol{\varepsilon} + \mathbb{B}_{-1}\mathbf{v}, \tag{2.20}$$

where $\mathbb{B} = \mathbb{S}\mathbf{B}_{30}^{-1}$ and $\mathbb{B}_{-1} = \mathbb{S}_{-1}\mathbf{B}_{30}^{-1}$. Thus, $S^*_{\mathtt{SDPD}}(\psi_0)$ are further expressed in terms of $\mathbf{v}$, ε and y_0. Using backward substitution on equation (2.1), we have:

$$\begin{aligned} y_0 &= \mathcal{B}^m y_{-m} + \textstyle\sum_{k=0}^{m-1}\mathcal{B}^k B_1^{-1}\mathbf{X}_{-k}\beta + \sum_{k=0}^{m-1}\mathcal{B}^k B_1^{-1}\varepsilon + \sum_{k=0}^{m-1}\mathcal{B}^k B_1^{-1}B_3^{-1}v_{-k} \\ &\equiv \eta_m + K_m\varepsilon + V_m, \end{aligned} \tag{2.21}$$

where $\eta_m = \mathcal{B}^m y_{-m} + \sum_{k=0}^{m-1}\mathcal{B}^k B_1^{-1}\mathbf{X}_{-k}\beta$, being the mean of y_0 given $\mathbf{X}_{-k}, k = 0, 1, \dots, m$ and thus exogenous; $K_m = \sum_{k=0}^{m-1}\mathcal{B}^k B_1^{-1}$; and $V_m = \sum_{k=0}^{m-1}\mathcal{B}^k B_1^{-1}B_3^{-1}v_{-k}$ which obviously is independent of ε and $v_t, t = 1, 2, \dots, T$. Therefore, the components of $S^*_{\mathtt{SDPD}}(\psi_0)$ are linear combinations of terms linear-quadratic in $\mathbf{v}$, linear-quadratic in ε, and bilinear in ε and $\mathbf{v}$, in ε and V_m, and in $\mathbf{v}$ and V_m. These lead to a simple way for establishing the asymptotic normality of the AQS vector $S^*_{\mathtt{SDPD}}(\psi_0)$, and thus the asymptotic normality of the proposed M-estimator. We have the following theorem, and see the proof in Appendix B for the details.

Theorem 2.2. *Under assumptions of Theorem 2.1, we have, as $n \to \infty$,*

$$\sqrt{nT}\big(\hat{\psi}_{\mathtt{M}} - \psi_0\big) \xrightarrow{D} N\big[0, \lim_{n\to\infty}\Sigma^{*-1}_{\mathtt{SDPD}}(\psi_0)\Gamma^*_{\mathtt{SDPD}}(\psi_0)\Sigma^{*-1}_{\mathtt{SDPD}}(\psi_0)\big],$$

where $\Sigma^*_{\mathtt{SDPD}}(\psi_0) = -\frac{1}{nT}\mathrm{E}[\frac{\partial}{\partial\psi'}S^*_{\mathtt{SDPD}}(\psi_0)]$ and $\Gamma^*_{\mathtt{SDPD}}(\psi_0) = \frac{1}{nT}\mathrm{Var}[S^*_{\mathtt{SDPD}}(\psi_0)]$, both assumed to exist and $\Sigma^*_{\mathtt{SDPD}}(\psi_0)$ to be positive definite, for sufficiently large n.

3. Robust Estimation of VC Matrix of M-Estimators

The expected negative Hessian matrix $\Sigma^*_{\texttt{SDPD}}(\psi)$ can be consistently estimated by its observed counter parts $\widehat{\Sigma}^*_{\texttt{SDPD}} = \frac{1}{nT}\frac{\partial}{\partial\psi'}S^*_{\texttt{SDPD}}(\psi)|_{\psi=\hat{\psi}_{\texttt{M}}}$. The detailed expression of $\frac{\partial}{\partial\psi'}S^*_{\texttt{SDPD}}(\psi)$ is given in Appendix B. However, the estimation of $\Gamma^*_{\texttt{SDPD}}(\psi_0) = \frac{1}{nT}\text{Var}[S^*_{\texttt{SDPD}}(\psi_0)]$ runs into difficulty. The traditional plug-in method requires the unconditional distribution of y_0 or a valid model for y_0 when T is fixed, of which neither is plausible as the the unconditional distribution involves unobservables and a valid model seems very difficult (if not impossible) to formulate, in particular when the model contains spatial lag terms (Yang, 2018). To overcome these difficulties in estimating the VC matrix for the FE-SDPD model, Yang (2018) proposed an *outer-product-of-martingale-difference* (OPMD) method, where the AQS function of the FE-SDPD model is decomposed into a sum of vector martingale difference (MD) sequences so that the average of the outer products of the MDs gives a consistent estimate of the VC matrix of that AQS function. However, this OPMD method does not apply to our CRE-SDPD model due to the existence of two error components ε and v_t.

New method of feasible and consistent VC matrix estimation is needed. The representations given in (2.19) are crucial in obtaining such an estimate. From (2.19) we see that the AQS function contains three types of elements:

$$\Pi'\mathbf{e}, \quad \mathbf{e}'\Phi\mathbf{e}, \quad \text{and } \mathbf{e}'\Psi\mathbf{y}_0,$$

where Π, Φ, and Ψ are nonstochastic matrices depending on ϕ_0 with Π being $nT\times\dim(\boldsymbol{\beta})$ or $nT\times 1$, and Φ and Ψ being $nT\times nT$. The closed form expressions for the variances of $\Pi'e$ and $e'\Phi e$ can be derived but the plug-in method cannot be applied as their analytical expressions involve the 3rd and 4th moments of both ε_i and v_{it}, which cannot be consistently estimated simultaneously with a fixed T. Furthermore, the closed-form expressions for the variance of $\mathbf{e}'\Psi\mathbf{y}_0$ and its covariances with $\Pi'e$ and $e'\Phi e$ depend on the past values of the regressors and the process starting positions, which are unobserved. Thus, the plug-in method based on the full analytical expression of $\Gamma^*_{\texttt{SDPD}}$ does not work either in this case.

As neither the traditional plug-in method nor the OPMD method works for estimating $\Gamma^*_{\texttt{SDPD}}$, an alternative method must be developed. To fix idea, we again, as in Yang (2018), endeavor to decompose $S^*_{\texttt{SDPD}}(\psi_0)$ into a sum $\sum_{i=1}^n \mathbf{g}_i$ such that $\{\mathbf{g}_i\}$ possess some 'desirable properties' and a feasible estimator for $\Gamma^*_{\texttt{SDPD}}$ can thus be developed. Difficulty lies in the fact that the composite error, $e_t = \varepsilon + B_3^{-1}v_t$, consists of two components v_t and ε, which cannot be 'consistently' estimated simultaneously due to the fixed T nature. Thus, although $\{\mathbf{g}_i\}$ can be written as MD sequences separately in terms of ε and v_t, it cannot be estimated this way as only the estimates $\hat{e}_t$ are available. However, if the decomposition $\sum_{i=1}^n \mathbf{g}_i$ is such that the covariance between $\mathbf{g}_i$ and $\mathbf{g}_j, j\neq i$, are uncorrelated with respect to ε for given $\{v_t\}$, then an *hybrid method*, i.e., combining sample analogue and the analytical expressions, can be developed for estimating $\Gamma^*_{\texttt{SDPD}}$. Note that based on $S^*_{\texttt{SDPD}}(\psi_0) = \sum_{i=1}^n \mathbf{g}_i$,

$$\Gamma^*_{\texttt{SDPD}} = \tfrac{1}{nT}\text{E}[S^*_{\texttt{SDPD}}(\psi_0)S^{*\prime}_{\texttt{SDPD}}(\psi_0)] = \tfrac{1}{nT}\textstyle\sum_{i=1}^n \text{E}(\mathbf{g}_i\mathbf{g}_i') + \tfrac{1}{nT}\sum_{i=1}^n\sum_{j=1,j\neq i}^n \text{E}(\mathbf{g}_i\mathbf{g}_j'). \quad (3.1)$$

The single sum term $\sum_{i=1}^n \text{E}(\mathbf{g}_i\mathbf{g}_i')$ may be estimated by its by its sample analogue $\sum_{i=1}^n \hat{\mathbf{g}}_i\hat{\mathbf{g}}_i'$, where $\hat{\mathbf{g}}_i$ is the plug-in estimate of $\mathbf{g}_i$ by plugging $\hat{\psi}_{\mathtt{M}}$ and $\hat{e}_{it}$ in $\mathbf{g}_i$. For the double sum term, we derive 'partial' analytical expressions in terms of ψ_0, 3rd and 4th moments of v_{it}, and the initial values y_0, so that a mixture of the plug-in method and sample analogue method can be applied. We choose $\hat{\mathbf{g}}_i$ in such a way that this method is free from the specifications of the distributions of the initial observations, and that it involves only the 3rd and 4th moments of the idiosyncratic error v_{it}, of which the estimations are readily available. The latter is achieved by transforming y_0 so that the transformed y_0 has an error structure similar to e_t:

$$y_0^* = K_m^{-1} y_0 = \varepsilon + K_m^{-1}\eta_m + K_m^{-1} V_m \equiv \varepsilon + \eta_m^* + V_m^*, \tag{3.2}$$

see (2.21). Clearly, making ε 'stand out' in the above expression as in $\mathbf{e}$ is to take a full advantage of the MD structure in ε so that the 3rd and 4th moments of ε_i do not appear and the relevant terms in the covariance part of (3.1) disappear. This is important as the 3rd and 4th moments of ε_i cannot be consistently estimated together with these of v_{it}.

To proceed, for a square matrix A, let A^u, A^l and A^d be, respectively, its upper-triangular, lower- triangular, and diagonal matrix such that $A = A^u + A^l + A^d$. Denote by Π_t, Φ_{ts} and Ψ_{ts} the submatrices of Π, Φ and Ψ partitioned according to $t, s = 2, \ldots, T$. Denote the partial sum of time-indexed quantities using the '$_+$' notation: e.g., $\Psi_{t+} = \sum_{s=1}^T \Psi_{ts}$, $\Psi_{+s} = \sum_{t=1}^T \Psi_{ts}$, $\Psi_{++} = \sum_{t=1}^T \sum_{s=1}^T \Psi_{ts}$, and similarly for Φ_{ts}, Π_t and other time-indexed quantities.

First, to estimate the variance of $\mathbf{e}'\Psi\mathbf{y}_0$, letting $\Psi_{t+}^* = \Psi_{t+}K_m$, we have:

$$\begin{aligned}
\mathbf{e}'\Psi\mathbf{y}_0 &= \textstyle\sum_{t=1}^T \sum_{s=1}^T e_t' \Psi_{ts} y_0 = \sum_{t=1}^T e_t' \Psi_{t+}^* y_0^* \\
&= \textstyle\sum_{t=1}^T e_t' \Psi_{t+}^{*d} y_0^* + \sum_{t=1}^T e_t' (\Psi_{t+}^{*l} + \Psi_{t+}^{*u}) y_0^* \\
&= \textstyle\sum_{t=1}^T e_t' \Psi_{t+}^{*d} y_0^* + \sum_{t=1}^T e_t' \xi_t \\
&= \textstyle\sum_{i=1}^n \big(\sum_{t=1}^T e_{it} \Psi_{ii,t+}^* y_{0i}^* + \sum_{t=1}^T e_{it}\xi_{it} \big),
\end{aligned}$$

where $\xi_t = (\Psi_{t+}^{*l} + \Psi_{t+}^{*u}) y_0^*$ and $\{\Psi_{ii,t+}^*, i = 1, \ldots, n\}$ are the diagonal elements of Ψ_{t+}^*.

Letting b_i' and w_{it}' be the ith row of B_3^{-1} and $(\Psi_{t+}^{*l} + \Psi_{t+}^{*u})$, and noting that $e_t = \varepsilon + B_3^{-1} v_t$, we have, $e_{it} = \varepsilon_i + b_i' v_t$, and $\xi_{it} = w_{it}' y_0^*$. It follows that $\mathrm{E}(e_{it}' \Psi_{ii,t+}^* y_{0i}^*) = \sigma_{\varepsilon_0}^2 \Psi_{ii,t+}^* \equiv d_{\Psi,it}$, and $\mathrm{E}(e_{it}'\xi_{it}) = \mathrm{E}[(\varepsilon_i + b_i' v_t) w_{it}' y_0^*] = 0$. These lead to $\mathbf{e}'\Psi\mathbf{y}_0 - \mathrm{E}(\mathbf{e}'\Psi\mathbf{y}_0) = \sum_{i=1}^n g_{\Psi,i}$, where

$$g_{\Psi,i} = \textstyle\sum_{t=1}^T \big[(e_{it} \Psi_{ii,t+}^* y_{0i}^* - d_{\Psi,it}) + e_{it}\xi_{it} \big], \tag{3.3}$$

i.e., $\mathbf{e}'\Psi\mathbf{y}_0 - \mathrm{E}(\mathbf{e}'\Psi\mathbf{y}_0)$ is decomposed into a sum of n 'gradients'.

Similarly for the terms quadratic in $\mathbf{e}$, we have

$$\begin{aligned}
\mathbf{e}'\Phi\mathbf{e} &= \textstyle\sum_{t=1}^T \sum_{s=1}^T e_t' \Phi_{ts} e_s = \sum_{t=1}^T \sum_{s=1}^T e_t' (\Phi_{ts}^d + \Phi_{ts}^u + \Phi_{ts}^l) e_s \\
&= \textstyle\sum_{t=1}^T \sum_{s=1}^T e_t' \Phi_{ts}^d e_s + \sum_{t=1}^T \sum_{s=1}^T e_t' \Phi_{ts}^l e_s + \sum_{t=1}^T \sum_{s=1}^T e_s' \Phi_{ts}^{u\prime} e_t \\
&= \textstyle\sum_{t=1}^T \sum_{s=1}^T e_t' \Phi_{ts}^d e_s + \sum_{t=1}^T \sum_{s=1}^T e_t' \Phi_{ts}^l e_s + \sum_{t=1}^T \sum_{s=1}^T e_t' \Phi_{st}^{u\prime} e_s \\
&= \textstyle\sum_{t=1}^T e_t' \sum_{s=1}^T \Phi_{ts}^d e_s + \sum_{t=1}^T e_t' \sum_{s=1}^T (\Phi_{ts}^l + \Phi_{st}^{u\prime}) e_s \\
&= \textstyle\sum_{t=1}^T e_t' e_t^* + \sum_{t=1}^T e_t' \varphi_t \\
&= \textstyle\sum_{t=1}^T \sum_{1=1}^n e_{it} e_{it}^* + \sum_{t=1}^T \sum_{i=1}^n e_{it}\varphi_{it},
\end{aligned}$$

where $e_t^* = \sum_{s=1}^T \Phi_{ts}^d e_s$ and $\varphi_t = \sum_{s=1}^T (\Phi_{ts}^l + \Phi_{st}^{u\prime}) e_s$.

Letting $a'_{i,ts}$ and $c'_{i,ts}$ be, respectively, the ith row of $(\Phi^l_{ts} + \Phi^{u\prime}_{st})$ and $(\Phi^l_{ts} + \Phi^{u\prime}_{st})B_3^{-1}$, we have $e^*_{it} = \Phi_{ii,t+}\varepsilon_i + \sum_{s=1}^T \Phi_{ii,ts} b'_i v_s$ and $\varphi_{it} = a'_{i,t+}\varepsilon + \sum_{s=1}^T c'_{i,ts} v_s$. It follows that

$$\mathrm{E}(e_{it}e^*_{it}) = \mathrm{E}[(\varepsilon_i + b'_i v_t)(\Phi_{ii,t+}\varepsilon_i + \textstyle\sum_{s=1}^T \Phi_{ii,ts} b'_i v_s)] = \sigma^2_{\varepsilon 0}\Phi_{ii,t+} + \sigma^2_{v0}\Phi_{ii,tt}(b'_i b_i) \equiv d_{1\Phi,it},$$
$$\mathrm{E}(e_{it}\varphi_{it}) = \mathrm{E}[(\varepsilon_i + b'_i v_t)(a'_{i,t+}\varepsilon + \textstyle\sum_{s=1}^T c'_{i,ts} v_s)] = \sigma^2_{v0}(b'_i c_{i,tt}) \equiv d_{2\Phi,it}.$$

These lead to $\mathbf{e}'\Phi\mathbf{e} - \mathrm{E}(\mathbf{e}'\Phi\mathbf{e}) = \sum_{i=1}^n g_{\Phi,i}$, where

$$g_{\Phi,i} = \textstyle\sum_{t=1}^T \big[(e_{it}e^*_{it} - d_{1\Phi,it}) + (e_{it}\varphi_{it} - d_{2\Phi,it})\big]. \tag{3.4}$$

Finally, for the terms linear in $\mathbf{e}$, $\mathrm{E}(\Pi'\mathbf{e}) = 0$, and, letting Π'_{it} be the ith row of Π_t,

$$\Pi'\mathbf{e} = \textstyle\sum_{i=1}^n(\sum_{t=1}^T \Pi_{it} e_{it}) \equiv \sum_{i=1}^n g_{\Pi,i}. \tag{3.5}$$

The decompositions of the three types of quantities into sums with 'gradients' given by (3.3)-(3.5) lead to a 'possible' way for a consistent estimate of the VC matrix of the AQS function.

For for each $\Psi_r, r = 1,2,3$, defined in (2.19), define $g_{\Psi_r,i}$ according to (3.3); for each $\Phi_r, r = 1,\ldots,6$, defined in (2.19), define $g_{\Phi_r,i}$ according to (3.4); and each $\Pi_r, r = 1,2,3,4$, defined in (2.19), define $g_{\Pi_r,i}$ according to (3.5). Define,

$$\mathbf{g}_i = \begin{cases} g_{\Pi_1,i}, \\ g_{\Phi_1,i}, \\ g_{\Phi_2,i}, \\ g_{\Pi_2,i} + g_{\Phi_3,i} + g_{\Psi_1,i}, \\ g_{\Pi_3,i} + g_{\Phi_4,i} + g_{\Psi_2,i}, \\ g_{\Pi_4,i} + g_{\Phi_5,i} + g_{\Psi_3,i}, \\ g_{\Phi_6,i}. \end{cases} \tag{3.6}$$

Then, the AQS vector at the true parameter value is $S^*_{\mathtt{SDPD}}(\psi_0) = \sum_{i=1}^n \mathbf{g}_i$ and its variance is given by (3.1), i.e., $\mathrm{Var}\,[S^*_{\mathtt{SDPD}}(\psi_0)] = \sum_{i=1}^n \mathrm{E}(\mathbf{g}_i\mathbf{g}'_i) + \sum_{i=1}^n\sum_{j=1,j\neq i}^n \mathrm{E}(\mathbf{g}_i\mathbf{g}'_j)$, where the single sum can be estimated by its sample counter part $\sum_{i=1}^n \hat{\mathbf{g}}_i\hat{\mathbf{g}}'_i$ with $\hat{\mathbf{g}}_i$ being obtained by replacing ψ_0 and $\mathbf{e}$ in $\mathbf{g}_i$ by their estimates $\hat{\psi}_{\mathtt{M}}$ and $\hat{\mathbf{e}}$, and the double sum is estimated using the results of the following lemma.

To simplify the representation and to facilitate the calculations, let $\boldsymbol{\pi}_r$ and $\boldsymbol{\pi}_\nu$ be the column(s) of $\mathbf{\Pi} = (\Pi_1, \Pi_2, \Pi_3, \Pi_4)$, for $r, \nu = 1, 2, \ldots, \mathtt{dim}(\boldsymbol{\beta}) + 3$, and $g_{\boldsymbol{\pi}_r}$ and $g_{\boldsymbol{\pi}_\nu}$ be the corresponding gradients vectors defined according to (3.5).

Lemma 3.1. *For* $(\boldsymbol{\pi}_r, \boldsymbol{\pi}_\nu)$ *and the corresponding gradients vectors* $(g_{\boldsymbol{\pi}_r}, g_{\boldsymbol{\pi}_\nu})$, $r, \nu = 1, 2, \ldots, k_x + 3$*;* (Φ_r, Φ_ν) *and the corresponding* $(g_{\Phi_r,i}, g_{\Phi_\nu,i}), r, \nu = 1, \ldots, 6$*; and* (Ψ_r, Ψ_ν) *and the corresponding* $(g_{\Psi_r,i}, g_{\Psi_\nu,i})$*, we have under Assumptions A and B, for* $j \neq i\ (= 1, \ldots, n)$,

$$\mathrm{E}(g_{\boldsymbol{\pi}_r i}\, g_{\boldsymbol{\pi}_\nu j}) = \sigma^2_{v0}(b'_i b_j)\textstyle\sum_{t=1}^T \boldsymbol{\pi}_{ri,t}\boldsymbol{\pi}_{\nu j,t}, \quad r, \nu = 1, 2, \ldots, k_2 + 3, \tag{3.7}$$
$$\mathrm{E}(g_{\Psi_r i}\, g_{\Psi_\nu j}) = \sigma^4_{\varepsilon_0}(w_{rij,+} w_{\nu ji,+}) + \sigma^2_{v_0}\textstyle\sum_{t=1}^T (b'_i b_j)\mathrm{E}(\xi^*_{ri,t}\xi^*_{\nu j,t}), \quad r, \nu = 1, 2, 3, \tag{3.8}$$
$$\begin{aligned}\mathrm{E}(g_{\Phi_r i}\, g_{\Phi_\nu j}) = {} & \sigma^4_{v0}\textstyle\sum_{t=1}^T\sum_{s=1}^T \big[(b'_j c^*_{ri,ts})(b'_i c^*_{\nu j,st}) + (b'_i b_j)(c^{*\prime}_{ri,ts} c^*_{\nu j,ts})\big] \\ & + \sigma^2_{v0}\sigma^2_{\varepsilon_0}\textstyle\sum_{t=1}^T \big[a_{\nu ji,t+}(b'_j c^*_{ri,+t}) + a_{rij,t+}(b'_i c^*_{\nu j,+t}) + (b'_i b_j)(a^{*\prime}_{ri,t+} a^*_{\nu j,t+})\big] \\ & + (\mu^{(4)}_{v0} - 3\sigma^4_{v0})\textstyle\sum_{t=1}^T \big[(b_i \odot c^*_{ri,tt})'(b_j \odot c^*_{\nu j,tt})\big], \quad r, \nu = 1, \ldots, 6;\end{aligned} \tag{3.9}$$

$$\mathrm{E}(g_{\Psi_r i}\, g_{\boldsymbol{\pi}_\nu j}) = \sigma_{v0}^2 \sum_{t=1}^T \boldsymbol{\pi}'_{\nu j,t}\mathrm{E}(\xi^*_{ri,t})(b'_i b_j), \tag{3.10}$$

$$\mathrm{E}(g_{\Phi_r i}\, g_{\boldsymbol{\pi}_\nu j}) = \mu_{v0}^{(3)} \sum_{t=1}^T (b_i \odot c^*_{ri,tt})' b_j \boldsymbol{\pi}_{\nu j,t}, \tag{3.11}$$

$$\begin{aligned}\mathrm{E}(g_{\Phi_r i}\, g_{\Psi_\nu j}) = \sigma_{\varepsilon_0}^2\sigma_{v0}^2 \sum_{t=1}^T \left[(b'_i b_j)(a'_{ri,t+} w^*_{\nu j,t}) + w_{\nu ji,+}(b'_j c^\circ_{ri,+t})\right] + \sigma_{\varepsilon_0}^4 (w_{ji,+} a_{ij,++}) \\ + \mu_{v0}^{(3)} \sum_{t=1}^T (b_i \odot c^*_{ri,tt})' b_j \mathrm{E}(\xi^*_{\nu j,t}),\end{aligned} \tag{3.12}$$

where $w^{*\prime}_{ri,t}$ *is the* i*th row of* $\Psi^*_{r,t+}$ *and* $\xi^*_{ri,t} = w^{*\prime}_{ri,t} y^*_0$, $a^{*\prime}_{ri,ts}$ *is the* i*th row of* $\Phi^*_{r,ts} = (\Phi^l_{r,ts} + \Phi^{u\prime}_{r,st} + \Phi^d_{r,ts})$, $c^{*\prime}_{ri,ts}$ *is the* i*th row of* $\Phi^*_{r,ts} B_3^{-1}$, $a_{rij,t+}$ *is the* (i,j)*th element of* $(\Phi^l_{r,t+} + \Phi^{u\prime}_{r,+t})$, *and* $w_{rij,+}$ *is the* (i,j)*th element of* $(\Psi^{*l}_{r,++} + \Psi^{*u}_{r,++})$.

Denote $\Upsilon_{ij} = \mathrm{E}(\mathbf{g}_i \mathbf{g}'_j)$. It is clear from (3.6) that Υ_{ij} can be obtained from the results of Lemma 3.1. Note that the (Π, Φ) terms of Υ_{ij} are analytical functions of ψ_0, $\mu^{(3)}$ and $\mu^{(4)}$, and hence can be estimated by plugging-in $\hat{\psi}$, $\hat{\mu}^{(3)}$ and $\hat{\mu}^{(4)}$ in the expressions. However, the Ψ-related terms are semi-analytical functions of ψ_0, $\mu^{(3)}$ and $\mu^{(4)}$, and $\mathrm{E}(y_0)$ and $\mathrm{E}(y_0 y'_0)$ appeared in $\mathrm{E}(\xi^*_{ri,t})$ and $\mathrm{E}(\xi^*_{ri,t}\xi^*_{\nu j,t})$. Consistent estimators of $\hat{\mu}^{(3)}$ and $\hat{\mu}^{(4)}$ are readily available as seen later, but the estimation of $\mathrm{E}(y_0)$ and $\mathrm{E}(y_0 y'_0)$ is not trivial. Their expressions involve unobservables and thus cannot be used. In this paper, we propose to estimate the terms $\mathrm{E}(\xi^*_{ri,t})$ and $\mathrm{E}(\xi^*_{ri,t}\xi^*_{\nu j,t})$ by their sample analogues and the other analytical terms by plugging-in method, i.e., removing E in the expressions and then relpacing (in all terms) ψ_0, $\mu^{(3)}$ and $\mu^{(4)}$ by $\hat{\psi}$, $\hat{\mu}^{(3)}$ and $\hat{\mu}^{(4)}$. The resulted estimator of Υ_{ij}, denoted by $\widehat{\Upsilon}_{ij}$, are thus mixtures of plug-in method and sample analogue method. The resulted estimator of the variance of the estimating functions, $\Gamma^*_{\mathtt{SDPD}}$, is given as follows,

$$\widehat{\Gamma}^*_{\mathtt{SDPD}} = \frac{1}{nT}\sum_{i=1}^n \hat{\mathbf{g}}_i \hat{\mathbf{g}}'_i + \frac{1}{nT}\sum_{i=1}^n \sum_{j=1, j\neq i}^n \widehat{\Upsilon}_{ij}. \tag{3.13}$$

Its consistency is proved in the following theorem.

Theorem 3.1. *Under the assumptions of Theorem* (2.1)*, we have, as* $n \to \infty$,

$$\widehat{\Gamma}^*_{\mathtt{SDPD}} - \Gamma^*_{\mathtt{SDPD}}(\psi_0) = \frac{1}{nT}\sum_{i=1}^n \left[\hat{\mathbf{g}}_i \hat{\mathbf{g}}'_i - \mathrm{E}(\mathbf{g}_i \mathbf{g}'_i)\right] + \frac{1}{nT}\sum_{1=1}^n \sum_{j=1, j\neq i}^n \left(\widehat{\Upsilon}_{ij} - \Upsilon_{ij}\right) \xrightarrow{p} 0,$$

and hence, $\Sigma^{*-1}_{\mathtt{SDPD}}(\hat{\psi}_{\mathtt{M}})\widehat{\Gamma}^*_{\mathtt{SDPD}}\Sigma^{*-1}_{\mathtt{SDPD}}(\hat{\psi}_{\mathtt{M}}) - \Sigma^{*-1}_{\mathtt{SDPD}}(\psi_0)\Gamma^*_{\mathtt{SDPD}}(\psi_0)\Sigma^{*-1}_{\mathtt{SDPD}}(\psi_0) \xrightarrow{p} 0.$

Finally, to estimate the third moment of v_{it}, let $\bar{e} = \frac{1}{T}\sum_{t=1}^T e_t$ and $\bar{v} = \frac{1}{T}\sum_{t=1}^T v_t$. Then, we have $v_t - \bar{v} = B_3(e_t - \bar{e})$. Letting $v^*_t = v_t - \bar{v}$, we have $\mathrm{E}(v^{*3}_{it}) = \frac{T^2-3T+2}{T^2}\mu^{(3)}_{v_0}$. Summing over i and t, an estimator of the third moment of v_{it} is naturally

$$\hat{\mu}^{(3)}_v = \tfrac{T^2}{T^2-3T+2}\tfrac{1}{nT}\sum_{t=1}^T \sum_{i=1}^n \hat{v}^{*3}_{it}.$$

To estimate the 4th moment of v_{it}, $\mu^{(4)}_{v_0} = \mathrm{E}(v^4_{it})$, we take first difference of e_{it} to get rid of the correlated random effect. After first differencing, we have $\Delta v_t = B_3 \Delta e_t, t = 2, \ldots, T$, and the following easily derived expression:

$$\mathrm{E}(\Delta v^4_{it}) = \mathrm{E}[(v_{it} - v_{i,t-1})^4] = \mathrm{E}(v^4_{it}) + \mathrm{E}(v^4_{i,t-1}) + 6\mathrm{E}(v^2_{it}v^2_{i,t-1}) = 2\mu^{(4)}_{v_0} + 6\sigma^4_{v_0}.$$

Therefore an estimator of $\mu_{v_0}^{(4)}$ can be: $\hat{\mu}_{v_0}^{(4)} = \frac{1}{2n}\sum_{i=1}^{n}\Delta\hat{v}_{it}^4 - 3\hat{\sigma}_{v_0}^4$, for any $t = 2,\ldots,T$. Obviously, one should combine these to give a pooled estimate:

$$\hat{\mu}_{v_0}^{(4)} = \tfrac{1}{2n(T-1)}\sum_{t=2}^{T}\sum_{i=1}^{n}\Delta\hat{v}_{it}^4 - 3\hat{\sigma}_{v_0}^4.$$

A computational note. The calculation of the double summation term $\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\widehat{\Upsilon}_{ij}$ in (3.13) is greatly facilitated by writing (3.7)-(3.12) in matrix forms for all i,j, using the Kronecker product $\otimes$ operator, and the Hadamard product operator $\odot$:

$$\Lambda(\pi_r,\pi_\nu) = \sigma_{v_0}^2\mathbb{B}_3\odot\big(\textstyle\sum_{t=1}^{T}\boldsymbol{\pi}_{rt}\boldsymbol{\pi}'_{\nu t}\big). \tag{3.14}$$

$$\Lambda(\Psi_r,\Psi_\nu) = \sigma_{\varepsilon_0}^4(\Psi_{r,++}^*\odot\Psi_{\nu,++}^{*\prime}) + \sigma_{v_0}^2\textstyle\sum_{t=1}^{T}\mathbb{B}_3\odot\mathrm{E}(\xi_{r,t}^*\xi_{\nu,t}^{*\prime}) \tag{3.15}$$

$$\begin{aligned}\Lambda(\Phi_r,\Phi_\nu) &= \sigma_{v0}^4\textstyle\sum_{t=1}^{T}\sum_{s=1}^{T}\big[(\Phi_{r,ts}^*\mathbb{B}_3)\odot(\mathbb{B}_3\Phi_{\nu,st}^{*\prime}) + \mathbb{B}_3\odot(\Phi_{r,ts}^*\mathbb{B}_3\Phi_{\nu,ts}^{*\prime})\big]\\ &\quad + \sigma_{v0}^2\sigma_{\varepsilon_0}^2\textstyle\sum_{t=1}^{T}\big[\Phi_{\nu,t}^{\circ\prime}\odot(\Phi_{r,+t}^*\mathbb{B}_3) + \Phi_{r,t}^{\circ}\odot(\mathbb{B}_3\Phi_{\nu,+t}^{*\prime}) + \mathbb{B}_3\odot(\Phi_{r,t+}^*\Phi_{\nu,t+}^{*\prime})\big]\\ &\quad + (\mu_{v0}^{(4)} - 3\sigma_{v0}^4)\textstyle\sum_{t=1}^{T}\big[B_3^{-1}\odot(\Phi_{r,tt}^*B_3^{-1})\big]\big[B_3^{-1}\odot(\Phi_{\nu,tt}^*B_3^{-1})\big]',\end{aligned} \tag{3.16}$$

$$\Lambda(\Psi_r,\pi_\nu) = \sigma_{v0}^2\mathbb{B}_3\odot[\textstyle\sum_{t=1}^{T}\mathrm{E}(\xi_{r,t}^*)\boldsymbol{\pi}'_{\nu t}], \tag{3.17}$$

$$\Lambda(\Phi_r,\pi_\nu) = \mu_{v0}^{(3)}\textstyle\sum_{t=1}^{T}\big[B_3^{-1}\odot(\Phi_{r,tt}^*B_3^{-1})\big]B_3^{\prime -1}\mathtt{diag}(\boldsymbol{\pi}_{\nu t}), \tag{3.18}$$

$$\begin{aligned}\Lambda(\Phi_r,\Psi_\nu) &= \sigma_{\varepsilon_0}^4(\Phi_{r,++}^{\circ}\odot\Psi_{\nu,++}^{*\prime}) + \sigma_{v0}^2\sigma_{\varepsilon_0}^2\textstyle\sum_{t=1}^{T}\big[\mathbb{B}_3\odot(\Phi_{r,t}^{\circ}\Psi_{\nu,t+}^{*\prime}) + (\Phi^{\sharp}\mathbb{B}_3)\odot\Psi_{\nu,t+}^{*\prime}\big]\\ &\quad + \mu_{v0}^{(3)}\textstyle\sum_{t=1}^{T}\big[B_3^{-1}\odot(\Phi_{r,tt}^*B_3^{-1})\big]B_3^{\prime -1}\mathtt{diag}\big[\mathrm{E}(\xi_{\nu,t}^*)\big].\end{aligned} \tag{3.19}$$

where $\mathbb{B}_3 = (B_3'B_3)^{-1}$, $\Phi_{r,t}^{\sharp} = \Phi_{r,+t}^* + \Phi_{r,t+}^d$, and $\Phi_{r,t}^{\circ} = \Phi_{r,t+}^l + \Phi_{r,+t}^{u\prime}$.

Then, it is easy to see that $\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathrm{E}(g_{\omega i}\, g_{\varpi j})$ equals the sum of the off-diagonal elements of $\Lambda(\omega,\varpi)$, for $\omega,\varpi = \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_{k_2+3}$, Ψ_1, Ψ_2, Ψ_3, and $\Phi_1,\ldots,\Phi_6$, which lead immediately to $\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\Upsilon_{ij}$ and its estimate $\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\widehat{\Upsilon}_{ij}$.

4. Monte Carlo Study

Extensive Monte Carlo experiments are run to investigate the finite sample performance of the proposed M-estimator of the CRE-SDPD model, and the finite sample performance of the proposed estimate of the VC matrix of the M-estimator. As in the special case of a RE-SDPD model with only spatial errors the full QMLE is available from Su and Yang (2015), a comparison is made between the full QMLE and the proposed M-estimator. We use the following three data generating processes (DGPs):

DGP1: $y_t = \rho y_{t-1} + \lambda_1 W_1 y_t + \lambda_2 W_2 y_{t-1} + X_t\beta_1 + Z\gamma + \mu + \alpha_t 1_n + u_t$,

DGP2: $y_t = \rho y_{t-1} + \lambda_1 W_1 y_t + \lambda_2 W_2 y_{t-1} + X_t\beta_1 + Z\gamma + \varepsilon + \alpha_t 1_n + u_t$,

DGP3: $y_t = \alpha_0 1_n + \rho y_{t-1} + X_t\beta_1 + Z\gamma + \varepsilon + u_t$,

where $u_t = \lambda_3 W_3 u_t + v_t$ for all three DGPs, and μ, ε and v_t represent, respectively, the CRE, RE, and idiosyncratic error.

The elements of X_t are generated in a similar fashion as in Hsiao et al. (2002),[4] and

[4] The detail is: $X_t = \mu_x + gt1_n + \zeta_t$, $(1-\phi_1 L)\zeta_t = \varepsilon_t + \phi_2\varepsilon_{t-1}$, $\varepsilon_t \sim N(0,\sigma_1^2 I_n)$, $\mu_x = e + \frac{1}{T+m+1}\sum_{t=-m}^{T}\varepsilon_t$,

the elements of Z are randomly generated from Bernoulli (0.5). The CRE is generated according to (1.2), and ε is generated from $N(0,1)$. The spatial weight matrices are generated according to Rook or Queen contiguity, or group interaction schemes.[5] We choose $\beta_0 = 1, \beta_1 = 1, \gamma = 1, \sigma_v^2 = 1, \alpha_0 = 1, \alpha_T = 1$, and $\alpha_t, t = 1, ..., T-1$ are generated from $N(1,1)$. We use a set of values for ρ ranging from -0.9 to 0.9, a set of values for $(\lambda_1, \lambda_2, \lambda_3)$ in the similar range, $T = 3$ or 6, and $N = 50, 100, 200, 400$. Each set of Monte Carlo results, corresponding to a combination of the values of $(n, T, m, \rho, \lambda's)$ is based on 2000 samples. The error v_t distribution can be (i) normal, (ii) normal mixture $(10\%N(0,4), 90\%N(0,1))$, or (iii) chi-squared with degree of freedom of 3. In both (ii) and (iii), the generated errors are standardized to have mean zero and variance σ_v^2.

Monte Carlo (empirical) means and standard deviations (sds) are reported for the CQML estimator (CQMLE), the M-estimator, and the full QMLE (DGP3). Empirical averages of the robust standard errors (rses) based on the VC matrix estimate $\Sigma^{*-1}_{\texttt{SDPD}}(\hat{\psi}_{\texttt{M}})\widehat{\Gamma}^{*}_{\texttt{SDPD}}\Sigma^{*-1}_{\texttt{SDPD}}(\hat{\psi}_{\texttt{M}})$ are also reported for the M-estimator, which should be compared with the corresponding empirical sds. The (standard errors) ses of the M-estimator based only on $\Sigma^{*}_{\texttt{SDPD}}$ and only on $\widehat{\Gamma}^{*}_{\texttt{SDPD}}$ are also computed but unreported to conserve space. All the Monte Carlo results that are involved in the following discussions but unreported due to space constraint can be found in the `supplement` to this paper, available from `http://www.mysmu.edu/faculty/zlyang/`.

Tables 1-3 present the results based on DGP1, the CRE-SDPD model with all three types of spatial effects. The results show an excellent performance of the proposed M-estimators of the model parameters, and the rses. The M-estimator of the dynamic parameter is nearly unbiased, whereas the CQMLE can be quite biased and as n increases it does not show a sign of convergence. The M-estimators of the spatial parameters λ_1 and λ_2 also show an excellent finite sample performance. Both CQMLE and M-estimator of the spatial parameter λ_3 show some bias. This is perhaps due to the intrinsic nature of the QML-type estimation of spatial error effects. The rses are on average very close to the corresponding Monte Carlo sds in general, showing the robustness and good finite sample performance of the proposed VC matrix estimate. The non-robust ses based on $\widehat{\Gamma}^{*-1}_{\texttt{SDPD}}$ and $\Sigma^{*-1}_{\texttt{SDPD}}(\hat{\psi}_{\texttt{M}})$ are also simulated and the results (reported in the `supplement`) show that when errors are normal, all three methods give averaged standard errors close to the corresponding Monte Carlo sds; but when the errors are not normal the non-robust ses can be quite different from the corresponding Monte Carlo sds in particular in the standard errors of σ_v^2 and ϕ.

Tables 4-6 present the results based on DGP2, the RE-SDPD with all three types of spatial effects. Similar observations hold, the proposed estimation strategy performs excellently and clearly outperforms the conditional QMLEs. The results also show that the proposed estimate of the standard deviation of M-estimator also performs very well.

Table 7-9 present the results based on DGP 3, the RE-SDPD with only spatial error effect.

and $e \sim N(0, \sigma_2^2 I_n)$. Let $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2)$.

[5]The Rook and Queen schemes are standard. For group interaction, we first generate $k = n^{\alpha}$ groups of sizes $n_g \sim U(.5\bar{n}, 1.5\bar{n})$, $g = 1, \cdots, k$, where $0 < \alpha < 1$ and $\bar{n} = n/k$, and then adjust n_g so that $\sum_{g=1}^{k} n_g = n$. The reported results correspond to $\alpha = 0.5$. See Yang (2015) for details in generating these spatial layouts.

For this model, the full QMLE (FQMLE) is available from Su and Yang (2015). As the main focus of this set of Monte Carlo experiments is to compare M-estimator with FQMLE, the rses of the M-estimator are not reported. The results show that both M-estimator and FQMLE of the dynamic parameter are nearly unbiased whereas the CQMLE is quite different from the true value and does not show a sign of convergence. Three estimators of spatial parameter λ_3 all show some bias, but the M-estimator has the smallest bias among the three. Comparing the empirical sds, we see that the M-estimator is slightly less efficient than the FQMLE, as expected. Computationally, however, the CQMLE and M-estimator are much more efficient.

Under all three DPGs, the Monte Carlo experiments are also run using a 'wrong' value of m and a larger value of $T(= 6)$. The results (unreported for brevity) show that the M-estimator is quite robust against the choice of m value, and that with a larger value of T the CQMLE perform significantly better, but is still clearly dominated by the M-estimator.

5. Conclusion and Discussion

This paper introduces M-estimation and and inference methods for the spatial dynamic panel data (SDPD) model with correlated random effects (CRE), based on the short panel set up. The estimation strategy is based on the adjusted quasi score functions following the idea of Yang (2018). For statistical inferences, a hybrid method that combines analytical derivations and the feasible sample analogues is proposed for estimating the robust standard errors of the M-estimators. The asymptotic properties of these estimators are studied in detail and Monte Carlo simulation shows that both the M-estimators and the robust standard errors perform very well in finite samples.

In this paper, we adopt the approach of Mundlak (1978) to specify the CRE for easy exposition. The results can be adapted to cover any CRE form that is linearizable in the sense that it can be written or be approximated by a linear model based on the observed time-varying regressors. The most general CRE form may be $\mu = g(X_0, X_1, \ldots, X_T) + \varepsilon$ with an unknown functional form $g(\cdot)$ and an additive error ε, giving a SDPD model with *nonlinear linear* individual-specific effects and error components. Standard semiparametric methods may be used to handle this unknown function and the model estimation may proceed in a similar way as that in this paper. This is clearly an interesting model specification, but a detailed study is beyond the scope of this paper. It would also be interesting to extend our methods to allow for heteroskedasticity in cross-section as well as in time, and serial correlation. These models and methods would be much more challenging than the already quite challenging work presented in this paper, and will be the topics of our future research.

Appendix A: Some Basic Lemmas

The following lemmas are essential to the proofs of the theorems in Sections 2 and 3.

Lemma A.1. *(Kelejian and Prucha, 1999; Lee, 2002): Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ matrices that are uniformly bounded in both row and column sums. Let C_n be a sequence of conformable matrices whose elements are uniformly $O(h_n^{-1})$. Then*

(i) the sequence $\{A_n B_n\}$ are uniformly bounded in both row and column sums,

(ii) the elements of A_n are uniformly bounded and $tr(A_n) = O(n)$, and

(iii) the elements of $A_n C_n$ and $C_n A_n$ are uniformly $O(h_n^{-1})$.

Lemma A.2. *(Lee, 2004a, p.1918): For W_1 and B_1 defined in Model* (1.1)*, if $\|W_1\|$ and $\|B_{10}^{-1}\|$ are uniformly bounded, where $\|\cdot\|$ is a matrix norm, then $\|B_1^{-1}\|$ is uniformly bounded in a neighborhood of λ_{10}.*

Lemma A.3. *(Lee, 2004a, p.1918): Let X_n be an $n \times p$ matrix. If the elements X_n are uniformly bounded and $\lim_{n\to\infty} \frac{1}{n} X_n' X_n$ exists and is nonsingular, then $P_n = X_n(X_n' X_n)^{-1} X_n'$ and $M_n = I_n - P_n$ are uniformly bounded in both row and column sums.*

Lemma A.4. *(Lemma B.4, Yang, 2015, extended): Let $\{A_n\}$ be a sequence of $n \times n$ matrices that are uniformly bounded in either row or column sums. Suppose that the elements $a_{n,ij}$ of A_n are $O(h_n^{-1})$ uniformly in all i and j. Let v_n be a random n-vector of iid elements with mean zero, variance σ^2 and finite 4th moment, and b_n a constant n-vector of elements of uniform order $O(h_n^{-1/2})$. Then*

(i) $\mathrm{E}(v_n' A_n v_n) = O(\frac{n}{h_n})$, *(ii)* $\mathrm{Var}(v_n' A_n v_n) = O(\frac{n}{h_n})$,

(iii) $\mathrm{Var}(v_n' A_n v_n + b_n' v_n) = O(\frac{n}{h_n})$, *(iv)* $v_n' A_n v_n = O_p(\frac{n}{h_n})$,

(v) $v_n' A_n v_n - \mathrm{E}(v_n' A_n v_n) = O_p((\frac{n}{h_n})^{\frac{1}{2}})$, *(vi)* $v_n' A_n b_n = O_p((\frac{n}{h_n})^{\frac{1}{2}})$,

and (vii), the results (iii) and (vi) remain valid if b_n is a random n-vector independent of v_n such that $\{\mathrm{E}(b_{ni}^2)\}$ are of uniform order $O(h_n^{-1})$.

Lemma A.5. *(Lemma A.5, Yang, 2018; Kelejian and Prucha, 2001): Let $\{\Phi_n\}$ be a sequence of $n \times n$ matrices with row and column sums uniformly bounded, and elements of uniform order $O(h_n^{-1})$. Let $v_n = (v_1, \cdots, v_n)'$ be a random vector of iid elements with mean zero, variance σ_v^2, and finite $(4+2\epsilon_0)$th moment for some $\epsilon_0 > 0$. Let $b_n = \{b_{ni}\}$ be an $n \times 1$ random vector, independent of v_n, such that (i) $\{\mathrm{E}(b_{ni}^2)\}$ are of uniform order $O(h_n^{-1})$, (ii) $sup_i E|b_{ni}|^{2+\epsilon_0} < \infty$, (iii) $\frac{h_n}{n}\sum_{i=1}^n [\phi_{n,ii}(b_{ni} - \mathrm{E}b_{ni})] = o_p(1)$ where $\{\phi_{n,ii}\}$ are the diagonal elements of Φ_n, and (iv) $\frac{h_n}{n}\sum_{i=1}^n [b_{ni}^2 - \mathrm{E}(b_{ni}^2)] = o_p(1)$. Define the bilinear-quadratic form:*

$$Q_n = b_n' v_n + v_n' \Phi_n v_n - \sigma_v^2 tr(\Phi_n),$$

and let $\sigma_{Q_n}^2$ be the variance of Q_n. If $lim_{n\to\infty} h_n^{1+2/\epsilon_0}/n = 0$ and $\{\frac{h_n}{n}\sigma_{Q_n}^2\}$ are bounded away from zero, then $Q_n/\sigma_{Q_n} \xrightarrow{d} N(0,1)$.

Lemma A.6. *(Lemma A.6, Lee 2004b): Suppose that z_{1n} and z_{2n} are sequences of n-dimensional column vectors, with elements being uniformly bounded for all n.*

(i) If $\{A_n\}$ is uniformly bounded in either row or column sums, then $|z'_{1n}A_n z_{2n}| = O(n)$.

(ii) If the sum of elements of $\{z_{1n}\}$ in absolute value is bounded, and the row sums of $\{A_n\}$ are uniformly bounded, then $|z'_{1n}A_n z_{2n}| = O(1)$.

Appendix B: Proofs for Section 2

Proof of Lemma 2.1: By (2.1), backward substitution leads to, for $t = -m+1, \ldots, T$,

$$\begin{aligned} \mathrm{E}(y_t\varepsilon') &= B_1^{-1}B_2E(y_{t-1}\varepsilon') + B_1^{-1}E(\varepsilon\varepsilon') + B_1^{-1}B_3^{-1}\mathrm{E}(v_t\varepsilon') \\ &= (B_1^{-1}B_2)^2E(y_{t-2}\varepsilon') + (B_1^{-1}B_2 + I_n)B_1^{-1}\mathrm{E}(\varepsilon\varepsilon') \\ &= \mathcal{B}^t\mathrm{E}(y_0\varepsilon') + (\textstyle\sum_{i=0}^{t-1}\mathcal{B}^i)B_1^{-1}\mathrm{E}(\varepsilon\varepsilon') \\ &= \mathcal{B}^{t+m}\mathrm{E}(y_{-m}\varepsilon') + \mathcal{B}^{t+m-1}B_1^{-1}\mathrm{E}(\varepsilon\varepsilon') + (\textstyle\sum_{i=0}^{t+m-2}\mathcal{B}^i)B_1^{-1}\mathrm{E}(\varepsilon\varepsilon') \\ &= (\textstyle\sum_{i=0}^{t+m-1}\mathcal{B}^i)B_1^{-1}\sigma_{\varepsilon 0}^2. \end{aligned}$$

Therefore, $\mathrm{E}(Y_{-1}\boldsymbol{\varepsilon}') = \sigma_{\varepsilon 0}^2\mathbf{C}_{-1}$ and $\mathrm{E}(Y\boldsymbol{\varepsilon}') = \sigma_{\varepsilon 0}^2\mathbf{C}$.

For $t, s = 1, \ldots, T$, we have $\mathrm{E}(y_t v_t') = B_1^{-1}B_2E(y_{t-1}v_t') + B_1^{-1}B_3^{-1}\mathrm{E}(v_t v_t') = \sigma_{v0}^2 B_1^{-1}B_3^{-1}$; $\mathrm{E}(y_t v_s') = 0$ when $t < s$; and

$$\begin{aligned} \mathrm{E}(y_t v_s') &= B_1^{-1}B_2E(y_{t-1}v_s') + B_1^{-1}B_3^{-1}E(v_t v_s') = \mathcal{B}^2E(y_{t-2}v_s') = \cdots \\ &= \mathcal{B}^{t-s}E(y_s v_s') = \mathcal{B}^{t-s}E(B_1^{-1}B_3^{-1}v_s v_s') = \mathcal{B}^{t-s}B_1^{-1}B_3^{-1}\sigma_{v0}^2, \end{aligned}$$

when $t > s$. Therefore, $\mathrm{E}(Y_{-1}v')(\mathbf{B}_3^{-1})' = \sigma_{v0}^2\mathbf{D}_{-1}$ and $\mathrm{E}(Yv')(\mathbf{B}_3^{-1})' = \sigma_{v0}^2\mathbf{D}$. Combining these results, we obtain the results of Lemma 2.1. ■

Proof of Lemma 2.2: Backward substitution on (2.1) gives, for $t = 1, \ldots, T$,

$$\begin{aligned} y_t &= \mathcal{B}y_{t-1} + B_1^{-1}\mathbf{X}_t\beta_0 + B_1^{-1}\varepsilon + B_1^{-1}B_3^{-1}v_t \\ &= \mathcal{B}^2y_{t-2} + \mathcal{B}B_1^{-1}\mathbf{X}_t\beta_0 + \mathcal{B}B_1^{-1}\varepsilon + \mathcal{B}B_1^{-1}B_3^{-1}v_t + B_1^{-1}\mathbf{X}_t\beta_0 + B_1^{-1}\varepsilon + B_1^{-1}B_3^{-1}v_t \\ &\;\;\vdots \\ &= \mathcal{B}^t y_0 + \textstyle\sum_{k=1}^t \mathcal{B}^{t-k}B_1^{-1}\mathbf{X}_k\beta_0 + \sum_{k=1}^t \mathcal{B}^{t-k}B_1^{-1}\varepsilon + \sum_{k=1}^t \mathcal{B}^{t-k}B_1^{-1}B_3^{-1}v_k. \end{aligned}$$

The results of Lemma 2.2 follows. ■

Following results are used in proving the theorems: (i) eigenvalues of a projection matrix are either 0 or 1; (ii) eigenvalues of a positive definite matrix are strictly positive; (iii) for symmetric matrix A and positive semidefinite (p.s.d.) matrix B, $\gamma_{\min}(A)\mathrm{tr}(B) \le \mathrm{tr}(AB) \le \gamma_{\max}(A)\mathrm{tr}(B)$; (iv) for symmetric matrices A and B, $\gamma_{\max}(A+B) \le \gamma_{\max}(A) + \gamma_{\max}(B)$; and (v) for p.s.d. matrices A and B, $\gamma_{\max}(AB) \le \gamma_{\max}(A)\gamma_{\max}(B)$. See, e.g, Bernstein (2009).

Proof of Theorem 2.1: From (2.14) and (2.17), we have

$$S^{*c}_{\mathtt{SDPD}}(\delta) - \bar{S}^{*c}_{\mathtt{SDPD}}(\delta) = \begin{cases} \frac{1}{2\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\hat{e}(\delta) - \frac{1}{2\bar{\sigma}_v^2}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\bar{e}(\delta)], \\ \frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}Y_{-1} - \frac{1}{\bar{\sigma}_v^2}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}Y_{-1}], \\ \frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_1 Y - \frac{1}{\bar{\sigma}_v^2}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}\mathbf{W}_1 Y], \\ \frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1} - \frac{1}{\bar{\sigma}_v^2}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1}], \\ \frac{1}{2\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta) - \frac{1}{2\bar{\sigma}_v^2}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta)]. \end{cases}$$

Under Assumption G, the consistency of $\hat{\delta}_{\texttt{M}}$ follows if $\sup_{\delta\in\Delta}\frac{1}{nT}[S^{*c}_{\texttt{SDPD}}(\delta)-\bar{S}^{*c}_{\texttt{SDPD}}(\delta)]\xrightarrow{p}0$ as $n\to\infty$, by Theorem 5.9 of van der Vaart (1998), boils down to the proofs of the following:

(a) $\inf_{\delta\in\Delta}\bar{\sigma}^2_{v,\texttt{M}}(\delta)$ is bounded away from zero,

(b) $\sup_{\delta\in\Delta}\big|\hat{\sigma}^2_{v,\texttt{M}}(\delta)-\bar{\sigma}^2_{v,\texttt{M}}(\delta)\big|=o_p(1)$,

(c) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{2\hat{\sigma}^2_v}\hat{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\hat{e}(\delta)-\frac{1}{2\bar{\sigma}^2_v}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\bar{e}(\delta)]\big|=o_p(1)$,

(d) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{\hat{\sigma}^2_v}\hat{e}'(\delta)\Omega^{-1}Y_{-1}-\frac{1}{\bar{\sigma}^2_v}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}Y_{-1}]\big|=o_p(1)$,

(e) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{\hat{\sigma}^2_v}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_1Y-\frac{1}{\bar{\sigma}^2_v}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}\mathbf{W}_1Y]\big|=o_p(1)$,

(f) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{\hat{\sigma}^2_v}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_2Y_{-1}-\frac{1}{\bar{\sigma}^2_v}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}\mathbf{W}_2Y_{-1}]\big|=o_p(1)$,

(g) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{\hat{\sigma}^2_v}\hat{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta)-\frac{1}{\bar{\sigma}^2_v}\mathrm{E}[\bar{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta)]\big|=o_p(1)$.

Define $\bar{\mathbf{e}}^*(\delta)=\Omega^{-\frac{1}{2}}\bar{\mathbf{e}}(\delta)$ and $\mathbf{B}^*_r=\Omega^{-\frac{1}{2}}\mathbf{B}_r, r=1,2$, where $\Omega^{\frac{1}{2}}$ is the square-root matrix of Ω. Let $Y^\circ=Y-\mathrm{E}(Y)$ and $Y^\circ_{-1}=Y_{-1}-\mathrm{E}(Y_{-1})$. We first present a useful identity:

$$\bar{\mathbf{e}}^*(\delta)=\mathbf{M}(\mathbf{B}^*_1\Delta Y-\mathbf{B}^*_2Y_{-1})+\mathbf{P}(\mathbf{B}^*_1Y^\circ-\mathbf{B}^*_2Y^\circ_{-1}), \tag{B.1}$$

where $\mathbf{M}=I_{nT}-\Omega^{-\frac{1}{2}}\mathbf{X}(\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-\frac{1}{2}}$ and $\mathbf{P}=I_{nT}-\mathbf{M}$.

Proof of (a). By (B.1) and the orthogonality between $\mathbf{M}$ and $\mathbf{P}$, we have

$$\begin{aligned}\bar{\sigma}^2_{v,\texttt{M}}(\delta)&=\tfrac{1}{nT}\mathrm{E}[\bar{\mathbf{e}}^{*\prime}(\delta)\bar{\mathbf{e}}^*(\delta)]\\&=\tfrac{1}{nT}\mathrm{E}[(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1})'\mathbf{M}(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1})]+\tfrac{1}{nT}\mathrm{E}[(\mathbf{B}^*_1Y^\circ-\mathbf{B}^*_2Y^\circ_{-1})'\mathbf{P}(\mathbf{B}^*_1Y^\circ-\mathbf{B}^*_2Y^\circ_{-1})]\\&=\tfrac{1}{nT}\mathrm{tr}[\mathrm{Var}(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1})]+\tfrac{1}{nT}(\mathbf{B}^*_1\mathrm{E}Y-\mathbf{B}^*_2\mathrm{E}Y_{-1})'\mathbf{M}(\mathbf{B}^*_1\mathrm{E}Y-\mathbf{B}^*_2\mathrm{E}Y_{-1}).\end{aligned}$$

As $\mathbf{M}$ is p.s.d, the second term is nonnegative for every in $\delta\in\Delta$. By Assumption E(iv) and the assumptions given in the theorem, the first term is such that for every $\delta\in\Delta$,

$$\begin{aligned}\tfrac{1}{nT}\mathrm{tr}[\Omega^{-1}\mathrm{Var}(\mathbf{B}_1Y-\mathbf{B}_2Y_{-1})]&\geq\tfrac{1}{nT}\gamma_{\min}(\Omega^{-1})\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y-\mathbf{B}_2Y_{-1})]\\&=\tfrac{1}{nT}\gamma^{-1}_{\max}(\Omega)\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y-\mathbf{B}_2Y_{-1})]\\&\geq\tfrac{1}{nT}[\phi\gamma_{\max}(J_T\otimes I_n)+\gamma_{\max}\big((B_3'B_3)^{-1}\big)]^{-1}\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y-\mathbf{B}_2Y_{-1})]\\&=\tfrac{1}{nT}[\phi+\gamma^{-1}_{\min}(B_3'B_3)]^{-1}\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y-\mathbf{B}_2Y_{-1})]\\&\geq\tfrac{1}{nT}\tfrac{\underline{c}_3}{1+\underline{c}_3\phi}\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y-\mathbf{B}_2Y_{-1})]\geq c>0.\end{aligned}$$

It follows that $\inf_{\delta\in\Delta}\bar{\sigma}^2_{v,\texttt{M}}(\delta)>c>0$.

Proof of (b). Let $\hat{\mathbf{e}}^*(\delta)=\Omega^{-\frac{1}{2}}\hat{\mathbf{e}}(\delta)=\mathbf{M}(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1})$, we have,

$$\hat{\sigma}^2_{v,\texttt{M}}(\delta)=\tfrac{1}{nT}\hat{\mathbf{e}}^{*\prime}(\delta)\hat{\mathbf{e}}^*(\delta)=\tfrac{1}{nT}(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1})'\mathbf{M}(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1}).$$

It follows that,

$$\begin{aligned}&\hat{\sigma}^2_{v,\texttt{M}}(\delta)-\bar{\sigma}^2_{v,\texttt{M}}(\delta)=\tfrac{1}{nT}[(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1})'\mathbf{M}(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1})]\\&\quad-\tfrac{1}{nT}\mathrm{E}[(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1})'\mathbf{M}(\mathbf{B}^*_1Y-\mathbf{B}^*_2Y_{-1})]-\tfrac{1}{nT}\mathrm{E}[(\mathbf{B}^*_1Y^\circ-\mathbf{B}^*_2Y^\circ_{-1})'\mathbf{P}(\mathbf{B}^*_1Y^\circ-\mathbf{B}^*_2Y^\circ_{-1})]\\&=\tfrac{1}{nT}[Y'\mathbf{B}^{*\prime}_1\mathbf{M}\mathbf{B}^*_1Y-\mathrm{E}(Y'\mathbf{B}^{*\prime}_1\mathbf{M}\mathbf{B}^*_1Y)]+\tfrac{1}{nT}[Y'_{-1}\mathbf{B}^{*\prime}_2\mathbf{M}\mathbf{B}^*_2Y_{-1}-\mathrm{E}(Y'_{-1}\mathbf{B}^{*\prime}_2\mathbf{M}\mathbf{B}^*_2Y_{-1})]\\&\quad-\tfrac{2}{nT}[Y'\mathbf{B}^{*\prime}_1\mathbf{M}\mathbf{B}^*_2Y_{-1}-\mathrm{E}(Y'\mathbf{B}^{*\prime}_1\mathbf{M}\mathbf{B}^*_2Y_{-1})]-\tfrac{1}{nT}\mathrm{E}[(\mathbf{B}^*_1Y^\circ-\mathbf{B}^*_2Y^\circ_{-1})'\mathbf{P}(\mathbf{B}^*_1Y^\circ-\mathbf{B}^*_2Y^\circ_{-1})]\\&\equiv(Q_1-\mathrm{E}Q_1)+(Q_2-\mathrm{E}Q_2)-2(Q_3-\mathrm{E}Q_3)-\mathrm{E}Q_4.\end{aligned}$$

The results follows if $Q_j - \mathrm{E}Q_j \xrightarrow{p} 0,\ j = 1,2,3$, and $\mathrm{E}Q_4 \longrightarrow 0$, uniformly in $\delta \in \Delta$.

By Lemma 2.2 and letting $\mathbf{M}^* = \Omega^{-\frac{1}{2}}\mathbf{M}\Omega^{-\frac{1}{2}}$, we have,

$$Q_1 = \tfrac{1}{nT}\big(\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{Q}\mathbf{y}_0 + \boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\boldsymbol{\eta} + \boldsymbol{\varepsilon}'\mathbb{S}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{S}\boldsymbol{\varepsilon} + \mathbf{v}'\mathbb{B}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{B}\mathbf{v}$$
$$+2\boldsymbol{\varepsilon}'\mathbb{S}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{B}\mathbf{v} + 2\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\boldsymbol{\eta} + 2\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{S}\boldsymbol{\varepsilon} + 2\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{B}\mathbf{v}$$
$$+2\boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{S}\boldsymbol{\varepsilon} + 2\boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{B}\mathbf{v}\big),$$

which leads to $Q_1 - \mathrm{E}Q_1 = \sum_{\ell=1}^{9}(Q_{1,\ell} - \mathrm{E}Q_{1,\ell})$, where $Q_{1,\ell}, \ell = 1,\ldots,9$, denote the nine stochastic terms of Q_1, and the expectations of the forth and the last two terms are zero;

$$Q_2 = \tfrac{1}{nT}\big(\mathbf{y}_0'\mathbb{Q}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0 + \boldsymbol{\eta}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1} + \boldsymbol{\varepsilon}'\mathbb{S}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon}$$
$$+\mathbf{v}'\mathbb{B}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + 2\boldsymbol{\varepsilon}'\mathbb{S}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + 2\mathbf{y}_0'\mathbb{Q}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1}$$
$$+2\mathbf{y}_0'\mathbb{Q}_{-1}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + 2\mathbf{y}_0'\mathbb{Q}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + 2\boldsymbol{\eta}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + 2\boldsymbol{\eta}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{B}\mathbf{v}\big),$$

which leads to $Q_2 - \mathrm{E}Q_2 = \sum_{\ell=1}^{9}(Q_{2,\ell} - \mathrm{E}Q_{2,\ell})$, where $Q_{2,\ell}, \ell = 1,\ldots,9$, denote the nine stochastic terms of Q_2, and the expectations of the forth and the last two terms are zero.

$$Q_3 = \tfrac{1}{nT}\big(\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0 + \boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1} + \boldsymbol{\varepsilon}'\mathbb{S}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + \mathbf{v}'\mathbb{B}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v}$$
$$+2\boldsymbol{\varepsilon}'\mathbb{S}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + \mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1} + \mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + \mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v}$$
$$+\boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0 + \boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + \boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + \boldsymbol{\varepsilon}'\mathbb{S}_{-1}\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0$$
$$+\mathbf{v}'\mathbb{B}_{-1}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0 + \boldsymbol{\varepsilon}'\mathbb{S}_{-1}\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1} + \mathbf{v}'\mathbb{B}_{-1}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1}\big),$$

which leads to $Q_3 - \mathrm{E}Q_3 = \sum_{\ell=1}^{14}(Q_{3,\ell} - \mathrm{E}Q_{3,\ell})$, where $Q_{3,\ell}, \ell = 1,\ldots,14$, denote the fourteen stochastic terms of Q_3. The forth,ninth, tenth and the last two terms have expectations zero.

Thus, $Q_k, k = 1,2,3$, are decomposed into sums of terms of the forms: $\frac{1}{nT}\mathbf{y}_0'\Phi\mathbf{y}_0$, $\frac{1}{nT}\mathbf{v}'\Pi\mathbf{v}$, $\frac{1}{nT}\boldsymbol{\varepsilon}'\Pi_\varepsilon\boldsymbol{\varepsilon}$, $\frac{1}{nT}\boldsymbol{\varepsilon}'\Theta\mathbf{v}$,$\frac{1}{nT}\mathbf{y}_0'\Psi\mathbf{v}$, $\frac{1}{nT}\mathbf{y}_0'\Psi_\varepsilon\boldsymbol{\varepsilon}$, $\frac{1}{nT}\mathbf{y}_0'\phi$, $\frac{1}{nT}\mathbf{v}'\psi$, and $\frac{1}{nT}\boldsymbol{\varepsilon}'\psi_\varepsilon$. The matrices Φ, Π, Π_ε, Θ, and Ψ, and the vectors ϕ, ψ and ψ_ε are defined in terms of $\mathbb{Q}$, $\mathbb{Q}_{-1}$, $\mathbb{S}$, $\mathbb{S}_{-1}$, $\mathbb{B}$, $\mathbb{B}_{-1}$, $\boldsymbol{\eta}$, and $\boldsymbol{\eta}_{-1}$, which depend on true parameter values and $\mathbf{B}_1$, which depends on λ_1, $\mathbf{B}_2$, which depends on ρ and λ_2 and $\mathbf{M}^*$, which depends on λ_3 and ϕ. By Lemma A.1, Assumption E and the expressions given under the AQS function (2.19), the $nT \times N$ matrices $\mathbb{R}$, $\mathbb{R}_{-1}$, $\mathbb{S}$, $\mathbb{S}_{-1}$, $\mathbb{B}$ and $\mathbb{B}_{-1}$ are uniformly bounded in both row and colum sums, elements of the $nT \times 1$ vectors $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_{-1}$ uniformly bounded. By Assuption E(iv), $\mathbf{B}_1$ and $\mathbf{B}_2$ are uniformly bounded in either row and colum sums. By Assumption E and the expression of Ω in eqation (2.3), we know $0 < \underline{c} \leq \inf_{\lambda_3,\phi\in\boldsymbol{\Lambda}}\gamma_{\min}(\Omega) \leq \sup_{\lambda_3,\phi\in\boldsymbol{\Lambda}}\gamma_{\max}(\Omega) \leq \bar{c} < \infty$. Therefore, $0 < \frac{1}{\bar{c}} \leq \inf_{\lambda_3,\phi\in\boldsymbol{\Lambda}}\gamma_{\min}(\Omega^{-1}) \leq \sup_{\lambda_3,\phi\in\boldsymbol{\Lambda}}\gamma_{\max}(\Omega^{-1}) \leq \frac{1}{\underline{c}} < \infty$. For $nT \times 1$ vector e_k whose kth element is one and all other elements are zeros. $\|\Omega^{-1}e_k\| \leq \|\Omega^{-1}\|\|e_k\| \leq \gamma_{\max}(\Omega^{-1}) \leq \frac{1}{\underline{c}}$. It follows that Ω^{-1} and therefore $\mathbf{M}^*$ is bounded either row and colum sums.

The quadratic terms of $\mathbf{y}_0$ can be written as $\frac{1}{nT}y_0'\Phi_{++}(\delta)y_0$ where $\Phi_{++}(\delta) = \sum_t\sum_s\Phi_{t,s}(\delta)$. Each $\delta \in \Delta$, $\Phi_{t,s}(\delta)$ are uniformly bounded in either row or column sums. The pointwise convergence of $\frac{1}{n}[y_0'\Phi_{++}(\delta)y_0 - \mathrm{E}(y_0'\Phi_{++}(\delta)y_0)]$ thus follows from Assumption F(iii). The quadratic terms of $\mathbf{v}$ can be written as $\frac{1}{nT}\sum_{t=1}^{T}\sum_{s=1}^{T}v_t'\Pi_{ts}v_s$. The quadratic terms of ε can be written as $\frac{1}{nT}\varepsilon'\Pi_{\varepsilon,++}\varepsilon$, where $\Pi_{\varepsilon,++} = \sum_t\sum_s\Pi_{\varepsilon,ts}$. The pointwise convergence of $\frac{1}{n}[v_t'\Pi_{ts}v_s - \mathrm{E}(v_t'\Pi_{ts}v_s)]$ follows from Lemma A.4 (v), for each $t,s = 1,\ldots,T$, and the pointwise convergence of $\frac{1}{n}[\varepsilon'\Pi_{\varepsilon,++}\varepsilon - \mathrm{E}(\varepsilon'\Pi_{\varepsilon,++}\varepsilon)]$ also follows from Lemma A.4 (v). The pointwise convergence of $\frac{1}{n}[\boldsymbol{\varepsilon}'\Theta\mathbf{v} - \mathrm{E}(\boldsymbol{\varepsilon}'\Theta\mathbf{v})]$ and $\frac{1}{nT}[\mathbf{y}_0'\Psi\mathbf{v} - \mathrm{E}(\mathbf{y}_0'\Psi\mathbf{v})]$ follows by writing $\boldsymbol{\varepsilon}'\Theta\mathbf{v} = \sum_s \varepsilon'\Theta_{+s}v_s$, $\mathbf{y}_0'\Psi\mathbf{v} = \sum_s y_0'\Psi_{+s}v_s$ and then applying Lemma A.4 (vii). Similarly, the pointwise convergence of $\frac{1}{nT}[\mathbf{y}_0'\Psi_\varepsilon\boldsymbol{\varepsilon} - \mathrm{E}(\mathbf{y}_0'\Psi_\varepsilon\boldsymbol{\varepsilon})]$ follows by writing, $\mathbf{y}_0'\Psi_\varepsilon\boldsymbol{\varepsilon} = y_0'\Psi_{++}\varepsilon = \varepsilon'\Psi_{++}\varepsilon + (\eta_m + V_m^*)'\Psi_{++}\varepsilon$, and then applying Lemma A.4 (i) to the quadratic term and Chebyshev inequality for the linear term. The pointwise convergence of $\frac{1}{nT}[\mathbf{y}_0'\phi - \mathrm{E}(\mathbf{y}_0'\phi)]$ follows from Assumption F(ii), and of $\frac{1}{nT}\mathbf{v}'\psi$ and $\frac{1}{nT}\boldsymbol{\varepsilon}'\psi_\varepsilon$ from Chebyshev inequality. Thus, $Q_{k,\ell}(\delta) - \mathrm{E}Q_{k,\ell}(\delta) \xrightarrow{p} 0$, for each $\delta \in \Delta$, and all k and ℓ.

Now, for all the $Q_{k,\ell}(\delta)$ terms, let δ_1 and δ_2 be in $\boldsymbol{\Delta}$. We have by the mean value theorem (MVT):

$$Q_{k,\ell}(\delta_2) - Q_{k,\ell}(\delta_1) = \tfrac{\partial}{\partial\delta'}Q_{k,\ell}(\bar{\delta})(\delta_2 - \delta_1),$$

where $\bar{\delta}$ lies between δ_1 and δ_2 elementwise. Note that $Q_{k,\ell}(\delta)$ is linear or quadratic in ρ, λ_1 and λ_2, and thus the corresponding partial derivatives takes simple form. It is easy to show that $\sup_{\delta\in\boldsymbol{\Delta}} |\frac{\partial}{\partial\omega}Q_{k,\ell}(\delta)| = O_p(1)$, for $\omega = \rho, \lambda_1, \lambda_2$. For $\frac{\partial}{\partial\lambda_3}Q_{k,\ell}(\delta)$, note that only the matrix M^* involves λ_3. Some algebra leads to the following simple expression for its derivative:

$$\tfrac{d}{d\lambda_3}M^* = \mathbf{M}^*\dot{\Omega}_{\lambda 3}\mathbf{M}^*,$$

where $\dot{\Omega}_{\lambda 3} = \frac{d}{d\lambda_3}\Omega = I_T \otimes (B_3'B_3)(B_3'W_3 + W_3'B_3)(B_3'B_3)$. Thus, it is easy to show that for all k and l $\sup_{\delta\in\boldsymbol{\Delta}} |\frac{\partial}{\partial\lambda_3}Q_{k,\ell}(\delta)| = O_p(1)$. For example, for $Q_{1,1}(\delta)$, noting that $\gamma_{\max}(\mathbf{M}) = 1$,

$$\begin{aligned}
\sup_{\delta\in\boldsymbol{\Delta}} |\tfrac{\partial}{\partial\lambda_3}Q_{1,1}(\delta)| &= \sup_{\delta\in\boldsymbol{\Delta}} |\tfrac{1}{n(T-1)}\tfrac{\partial}{\partial\lambda_3}\Delta\mathbf{y}_1'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{Q}\Delta\mathbf{y}_1| \\
&= \sup_{\delta\in\boldsymbol{\Delta}} \tfrac{1}{n(T-1)}|\Delta\mathbf{y}_1'\mathbb{R}'\mathbf{B}_1'\mathbf{M}^*\dot{\Omega}_{\lambda 3}\mathbf{M}^*\mathbf{B}_1\mathbb{R}\Delta\mathbf{y}_1| \\
&\le \sup_{\delta\in\boldsymbol{\Delta}} \tfrac{1}{n(T-1)}|\Delta\mathbf{y}_1'\mathbb{R}'\mathbf{B}_1'\dot{\Omega}_{\lambda 3}\mathbf{B}_1\mathbb{R}\Delta\mathbf{y}_1| \\
&\le \gamma_{\max}(\dot{\Omega}_{\lambda 3})\gamma_{\max}(\mathbf{B}_1'\mathbf{B}_1)\tfrac{1}{nT}|\Delta\mathbf{y}_1'\mathbb{R}'\mathbb{R}\Delta\mathbf{y}_1| \\
&= O(1) \times O(1) \times O_p(1) = O_p(1), \text{ by Assumption F(i).}
\end{aligned}$$

It follows that $Q_{k,\ell}(\delta)$ are stochastic equicontinuous, and by Theorem 1 of Andrews (1992) $Q_{k,\ell}(\delta) - \mathrm{E}Q_{k,\ell}(\delta) \xrightarrow{p} 0$, uniformly in $\delta \in \boldsymbol{\Delta}$. Thus, $Q_k(\delta) - \mathrm{E}Q_k(\delta) \xrightarrow{p} 0$, uniformly in $\delta \in \Delta$, $k = 1, 2, 3$. It left to show that $\mathrm{E}Q_4(\delta) \to 0$, uniformly in $\delta \in \Delta$. We have

$$\begin{aligned}
\mathrm{E}Q_4 &= \tfrac{1}{nT}\mathrm{tr}[\Omega^{-1}\mathbf{X}(\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})] \\
&\le \tfrac{1}{nT}\gamma_{\max}^2(\Omega^{-1})\gamma_{\max}[(\mathbf{X}'\Omega^{-1}\Delta X)^{-1}]\ \mathrm{tr}[\mathbf{X}'\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})\mathbf{X}] \\
&= \tfrac{1}{nT}\gamma_{\min}^{-2}(\Omega)\gamma_{\min}^{-1}\big(\tfrac{\mathbf{X}'\Omega^{-1}\mathbf{X}}{nT}\big)\ \tfrac{1}{nT}\mathrm{tr}[\mathbf{X}'\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})\mathbf{X}].
\end{aligned}$$

By Assumption E(iv), $\gamma_{\min}(\Omega) \ge \phi\gamma_{\min}(J_T \otimes I_n) + \gamma_{\max}^{-1}(B_3'B_3) \ge \frac{1}{\inf_{\lambda_3\in\Lambda_3}\gamma_{\max}(B_3'B_3)} \ge \frac{1}{\bar{c}_3}$. By Assumption D, $\gamma_{\min}(\frac{\mathbf{X}'\Omega^{-1}\mathbf{X}}{nT}) \ge \inf_{\lambda_3\in\Lambda_3}\gamma_{\min}(\Omega^{-1})\gamma_{\min}\big(\frac{\mathbf{X}'\mathbf{X}}{nT}\big) \ge \underline{c}_x \ge 0$. It follows that,

$$\begin{aligned}
\mathrm{E}Q_4 &\le \tfrac{1}{nT}\bar{c}_3^2\tfrac{1}{\underline{c}_x}\tfrac{1}{nT}\mathrm{tr}[\mathbf{X}'\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})\mathbf{X}] \\
&\le \tfrac{1}{n(T-1)}\bar{c}_c^2\tfrac{1}{\underline{c}_x}\bar{c}_y\ \tfrac{1}{nT}\mathrm{tr}[\mathbf{X}'\mathbf{X}], \text{ by the assumption in Theorem 2.1} \\
&= O(n^{-1}), \text{ by Assumption D.}
\end{aligned}$$

Hence, $\hat{\sigma}_{v,\mathtt{M}}^2(\delta) - \bar{\sigma}_{v,\mathtt{M}}^2(\delta) \xrightarrow{p} 0$, uniformly in $\delta \in \Delta$, completing the proof of **(b)**.

Proofs of (c)-(f). By the expressions of $\hat{\mathbf{e}}(\delta)$ and $\bar{\mathbf{e}}(\delta)$ given earlier and Lemma 2.2, all the quantities inside $|\cdot|$ in (c)-(g) can all be expressed in the forms similar to (5). Thus, the proofs of (c)-(f) follow the proof of (b). ∎

Proof of Theorem 2.2: We have by the mean value theorem,

$$0 = \tfrac{1}{\sqrt{nT}} S^*_{\texttt{SDPD}}(\hat{\psi}_{\texttt{SDPD}}) = \tfrac{1}{\sqrt{nT}} S^*_{\texttt{SDPD}}(\psi_0) + \big[\tfrac{1}{nT}\tfrac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\bar{\psi})\big]\sqrt{nT}(\hat{\psi}_{\texttt{M}} - \psi_0),$$

where $\bar{\psi}$ lies elementwise between $\hat{\psi}_{\texttt{M}}$ and ψ_0. The result of the theorem follows if

(a) $\frac{1}{\sqrt{nT}} S^*_{\texttt{SDPD}}(\psi_0) \xrightarrow{D} N\big[0, \lim_{n\to\infty} \Gamma^*_{\texttt{SDPD}}(\psi_0)\big]$,
(b) $\frac{1}{nT}\big[\frac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\bar{\psi}) - \frac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\psi_0)\big] \xrightarrow{p} 0$, and
(c) $\frac{1}{nT}\big[\frac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\psi_0) - \mathrm{E}\big(\frac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\psi_0)\big)\big] \xrightarrow{p} 0$.

Proof of (a). By $\mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}_{30}^{-1}\mathbf{v}$ and letting $\Pi_r^\circ = \mathbf{B}_{30}'^{-1}\Pi_r$, $r = 1, \ldots, 4$, $\Psi_r^\circ = \mathbf{B}_{30}'^{-1}\Psi_r$, $r = 1, 2, 3$, and $\Phi_r^\circ = \mathbf{B}_{30}'^{-1}\Phi_r\mathbf{B}_{30}^{-1}$, $r = 1, \ldots, 6$, and $\Phi_r^\circ = \mathbf{B}_{30}'^{-1}\Phi_r$ and $\Phi_r^\diamond = \mathbf{B}_{30}'^{-1}\Phi_r\mathbf{B}_{30}^{-1}$, $r = 1, \ldots, 6$, the AQS functions given by (2.19) can be further expressed as follows,

$$S^*_{\texttt{SDPD}}(\psi_0) = \begin{cases} \mathbf{\Pi}_1'\varepsilon + \Pi_1^{\circ\prime}\mathbf{v}, \\ \boldsymbol{\varepsilon}'\Phi_1\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_1^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_1^\circ\boldsymbol{\varepsilon} - \mu_{\sigma^2}, \\ \boldsymbol{\varepsilon}'\Phi_2\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_2^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_2^\circ\boldsymbol{\varepsilon} - \mu_\phi, \\ \boldsymbol{\varepsilon}'\Psi_1\mathbf{y}_0 + \mathbf{v}'\Psi_1^\circ\mathbf{y}_0 + \Pi_2'\boldsymbol{\varepsilon} + \Pi_2^{\circ\prime}\mathbf{v} + \boldsymbol{\varepsilon}'\Phi_3\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_3^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_3^\circ\boldsymbol{\varepsilon} - \mu_\rho, \\ \boldsymbol{\varepsilon}'\Psi_2\mathbf{y}_0 + \mathbf{v}'\Psi_2^\circ\mathbf{y}_0 + \Pi_3'\boldsymbol{\varepsilon} + \Pi_3^{\circ\prime}\mathbf{v} + \boldsymbol{\varepsilon}'\Phi_4\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_4^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_4^\circ\boldsymbol{\varepsilon} - \mu_{\lambda_1}, \\ \boldsymbol{\varepsilon}'\Psi_3\mathbf{y}_0 + \mathbf{v}'\Psi_3^\circ\mathbf{y}_0 + \Pi_4'\boldsymbol{\varepsilon} + \Pi_4^{\circ\prime}\mathbf{v} + \boldsymbol{\varepsilon}'\Phi_5\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_5^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_5^\circ\boldsymbol{\varepsilon} - \mu_{\lambda_{12}}, \\ \boldsymbol{\varepsilon}'\Phi_6\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_6^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_6^\circ\boldsymbol{\varepsilon} - \mu_{\lambda_3}, \end{cases} \tag{B.2}$$

where $\mu_{\sigma^2} = \frac{nT}{2\sigma_{v0}^2}$, $\mu_\phi = \frac{1}{2}\mathrm{tr}[\Omega_0^{-1}(J_T \otimes I_n)]$, $\mu_\rho = \mathrm{tr}[(\phi_0\mathbf{C}_{-10} + \mathbf{D}_{-10})\Omega_0^{-1}]$, $\mu_{\lambda_1} = \mathrm{tr}[(\phi_0\mathbf{C}_0 + \mathbf{D}_0)\Omega_0^{-1}\mathbf{W}_1]$, $\mu_{\lambda_2} = \mathrm{tr}[(\phi_0\mathbf{C}_{-10} + \mathbf{D}_{-10})\Omega_0^{-1}\mathbf{W}_2]$, and $\mu_{\lambda_3} = \mathrm{tr}(\Omega_0^{-1}\dot{\Omega}_{\lambda_3 0})$.

Partition the vectors or matrices Π_r and Π_r° according to $t = 1, \ldots, T$, and denote the partitioned vectors or matrices, respectively, by $\{\Pi_{rt}\}$ and $\{\Pi_{rt}^\circ\}$; partition the matrices Φ_r, Φ_r°, $\Phi_r^\diamond$, Ψ_r, and Ψ_r° according to $t, s = 1, \ldots, T$, and denote the partitioned matrices, respectively, by $\{\Phi_{rts}\}$, $\{\Phi_{rts}^\circ\}$, $\{\Phi_{rts}^\diamond\}$, $\{\Psi_{rts}\}$, and $\{\Phi_{rts}^\circ\}$. As $\boldsymbol{\varepsilon} = 1_T \otimes \varepsilon$ and $\mathbf{y}_0 = 1_T \otimes y_0$, denoting $\Pi_{r+} = \sum_{t=1}^T \Pi_{rt}$, $\Phi_{rt+}^\circ = \sum_{s=1}^T \Phi_{rts}^\circ$, $\Phi_{r++} = \sum_{s=1}^T\sum_{s=1}^T \Phi_{rts}$, we have

$$\begin{aligned} &\Pi_r'\boldsymbol{\varepsilon} = \Pi_{r+}\varepsilon, \quad \boldsymbol{\varepsilon}'\Phi\boldsymbol{\varepsilon} = \varepsilon'\Phi_{r++}\varepsilon, \quad \boldsymbol{\varepsilon}'\Psi\mathbf{y}_0 = \varepsilon'\Psi_{r++}y_0, \\ &\mathbf{v}'\Psi_r^\circ\mathbf{y}_0 = \mathbf{v}'\Psi_{r+}^\circ y_0, \quad \mathbf{v}'\Phi_r^\circ\boldsymbol{\varepsilon} = \mathbf{v}'\Phi_{r+}^\circ\varepsilon. \end{aligned}$$

where $\Psi_{r+}^\circ = \Psi_r^\circ(1_T \otimes I_n)$ and $\Phi_{r+}^\circ = \Phi_r^\circ(1_T \otimes I_n)$. Now, by (3.2), the terms bilinear in ε and y_0, and the terms bilinear in $\mathbf{v}$ and y_0 can be expressed as

$$\begin{aligned} &\varepsilon'\Psi_{r++}y_0 = \varepsilon'\Psi_{r++}K_m\varepsilon + \varepsilon'\Psi_{r++}K_m(\eta_m^* + V_m^*), \quad \text{and} \\ &\mathbf{v}'\Psi_{r+}^\circ y_0 = \mathbf{v}'\Psi_{r+}^\circ K_m\varepsilon + \mathbf{v}'\Psi_{r+}^\circ K_m(\eta_m^* + V_m^*). \end{aligned}$$

Therefore, the AQS vector at the true parameters consists of terms linear-quadratic in $\mathbf{v}$, linear-quadratic in ε, and bilinear in ε and $\mathbf{v}$. Thus, for every non-zero $\dim(\psi) \times 1$ vector of constants c, $c'S^*_{\texttt{SDPD}}(\psi_0)$ can be expressed as

$$c'S^*_{\texttt{SDPD}}(\psi_0) = \mathbf{v}'A\mathbf{v} + \mathbf{v}'\pi + \varepsilon'B\varepsilon + \varepsilon'\varphi + \mathbf{v}D\varepsilon - c'\mu_\psi,$$

for suitably defined non-stochastic matrices A, B and D, and (random) vectors π and φ, where $\mu_\psi = \{0_p', \mu_{\sigma^2}, \mu_\rho, \mu_{\lambda_1}, \mu_{\lambda_2}, \mu_{\lambda_3}, \}'$. Both π and φ are measurable functions of V_m, and hence are independent of ε and $\mathbf{v}$. Putting $c'S^*_{\mathtt{SDPD}}(\psi_0)$ in a more compact form: $\mathbb{V}'\mathbb{A}\mathbb{V}+\mathbb{V}'\varpi-c'\mu_\psi$, where $\mathbb{V}=(\mathbf{v}',\varepsilon')'$, $\mathbb{A}=\{A,D;\mathbf{0},B\}$, $\varpi=(\pi',\varphi)'$, and $\mathbf{0}$ denotes a matrix of zeros, the asymptotic normality of $\frac{1}{\sqrt{nT}}c'S^*_{\mathtt{SDPD}}(\psi_0)$ follows from Lemma A.5. Finally, the Cramér-Wold devise leads to the joint asymptotic normality of $\frac{1}{\sqrt{nT}}S^*_{\mathtt{SDPD}}(\psi_0)$.

Proof of (b). The Hessian matrix,$H^*_{\mathtt{SDPD}}(\psi)=\frac{\partial}{\partial\psi'}S^*_{\mathtt{SDPD}}(\psi)$, has the elements:

$$H^*_{\beta\beta}=-\tfrac{1}{\sigma_v^2}X'\Omega^{-1}X,\quad H^*_{\beta\sigma_v^2}=-\tfrac{1}{\sigma_v^4}X'\Omega^{-1}\mathbf{e}(\theta),\quad H^*_{\beta\phi}=\tfrac{1}{\sigma_v^2}X'\dot{\Omega}^-_{\phi}\mathbf{e}(\theta),$$

$$H^*_{\beta\rho}=-\tfrac{1}{\sigma_v^2}X'\Omega^{-1}Y_{-1},\quad H^*_{\beta\lambda_1}=-\tfrac{1}{\sigma_v^2}X'\Omega^{-1}W_1Y,\quad H^*_{\beta\lambda_2}=-\tfrac{1}{\sigma_v^2}X'\Omega^{-1}W_2Y_{-1}$$

$$H^*_{\beta\lambda_3}=\tfrac{1}{\sigma_v^2}X'\dot{\Omega}^-_{\lambda_3}\mathbf{e}(\theta),\quad H^*_{\sigma_v^2\sigma_v^2}=-\tfrac{1}{\sigma_v^6}\mathbf{e}'(\theta)\Omega^{-1}\mathbf{e}(\theta)+\tfrac{nT}{2\sigma_v^4},\quad H^*_{\sigma_v^2\phi}=\tfrac{1}{2\sigma_v^4}\mathbf{e}'(\theta)\dot{\Omega}^-_{\phi}\mathbf{e}(\theta),$$

$$H^*_{\sigma_v^2\rho}=-\tfrac{1}{\sigma_v^4}\mathbf{e}'(\theta)\Omega^{-1}Y_{-1},\quad H^*_{\sigma_v^2\lambda_1}=-\tfrac{1}{\sigma_v^4}\mathbf{e}'(\theta)\Omega^{-1}W_1Y,\quad H^*_{\sigma_v^2\lambda_2}=-\tfrac{1}{\sigma_v^4}\mathbf{e}'(\theta)\Omega^{-1}W_2Y_{-1},$$

$$H^*_{\sigma_v^2\lambda_3}=\tfrac{1}{2\sigma_v^4}\mathbf{e}'(\theta)\dot{\Omega}^-_{\lambda_3}\mathbf{e}(\theta),\quad H^*_{\phi\rho}=\tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_{\phi}Y_{-1},\quad H^*_{\phi\lambda_1}=\tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_{\phi}W_1Y,$$

$$H^*_{\phi\lambda_2}=\tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_{\phi}W_2Y_{-1},\quad H^*_{\phi\phi}=-\tfrac{1}{2\sigma_v^2}\mathbf{e}'(\theta)\ddot{\Omega}^-_{\phi}\mathbf{e}(\theta)-\tfrac{1}{2}\mathrm{tr}[\dot{\Omega}^-_{\phi}(J_T\otimes I_n)],$$

$$H^*_{\phi\lambda_3}=-\tfrac{1}{2\sigma_v^2}\mathbf{e}'(\theta)\ddot{\Omega}^-_{\phi,\lambda_3}\mathbf{e}(\theta)-\tfrac{1}{2}\mathrm{tr}[\dot{\Omega}^-_{\lambda_3}(J_T\otimes I_n)],$$

$$H^*_{\rho\rho}=-\tfrac{1}{\sigma_v^2}Y'_{-1}\Omega^{-1}Y_{-1}-\mathrm{tr}[(\phi\dot{\mathbf{C}}_{-1,\rho}+\dot{\mathbf{D}}_{-1,\rho})\Omega^{-1}],$$

$$H^*_{\rho\lambda_1}=-\tfrac{1}{\sigma_v^2}Y'W_1'\Omega^{-1}Y_{-1}-\mathrm{tr}[(\phi\dot{\mathbf{C}}_{-1,\lambda_1}+\dot{\mathbf{D}}_{-1,\lambda_1})\Omega^{-1}],$$

$$H^*_{\rho\lambda_2}=-\tfrac{1}{\sigma_v^2}Y_1'W_2'\Omega^{-1}Y_{-1}-\mathrm{tr}[(\phi\dot{\mathbf{C}}_{-1,\lambda_2}+\dot{\mathbf{D}}_{-1,\lambda_2})\Omega^{-1}],$$

$$H^*_{\rho\lambda_3}=\tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_{\lambda_3}Y_{-1}-\mathrm{tr}[(\dot{D}_{-1,\lambda_3}\Omega^{-1})+(\phi\mathbf{C}_1+\mathbf{D}_1)\dot{\Omega}^-_{\lambda_3}],$$

$$H^*_{\lambda_1\lambda_1}=-\tfrac{1}{\sigma_v^2}Y'W_1'\Omega^{-1}W_1Y-\mathrm{tr}[(\phi\dot{\mathbf{C}}_{\lambda_1}+\dot{\mathbf{D}}_{\lambda_1})\Omega^{-1}W_1],$$

$$H^*_{\lambda_1\lambda_2}=-\tfrac{1}{\sigma_v^2}Y'_{-1}W_2'\Omega^{-1}W_1Y-\mathrm{tr}[(\phi\dot{\mathbf{C}}_{\lambda_2}+\dot{\mathbf{D}}_{\lambda_2})\Omega^{-1}W_2],$$

$$H^*_{\lambda_1\lambda_3}=\tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_{\lambda_3}W_1Y-\mathrm{tr}\{[\dot{D}_{\lambda_3}\Omega^{-1}+(\phi\mathbf{C}+\mathbf{D})\dot{\Omega}^-_{\lambda_3}]W_1\},$$

$$H^*_{\lambda_2\lambda_2}=-\tfrac{1}{\sigma_v^2}Y'_{-1}W_2'\Omega^{-1}W_2Y_1-\mathrm{tr}[(\phi\dot{\mathbf{C}}_{-1,\lambda_2}+\dot{\mathbf{D}}_{-1,\lambda_2})\Omega^{-1}W_2],$$

$$H^*_{\lambda_2\lambda_3}=\tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_{\lambda_3}W_2Y_{-1}-\mathrm{tr}\{[\dot{D}_{-1\lambda_3}\Omega^{-1}+(\phi\mathbf{C}_1+\mathbf{D}_1)\dot{\Omega}^-_{\lambda_3}]W_2\},$$

$$H^*_{\lambda_3\lambda_3}=-\tfrac{1}{2\sigma_v^2}\mathbf{e}'(\theta)\ddot{\Omega}^-_{\lambda_3}\mathbf{e}(\theta)-\tfrac{1}{2}\mathrm{tr}(\dot{\Omega}^-_{\lambda_3}\dot{\Omega}_{\lambda_3}+\Omega^{-1}\ddot{\Omega}_{\lambda_3}),$$

where $\dot{\mathbf{C}}_\omega=\frac{\partial\mathbf{C}}{\partial\omega}$, $\dot{\mathbf{D}}_\omega=\frac{\partial\mathbf{D}}{\partial\omega}$, $\dot{\mathbf{C}}_{-1,\omega}=\frac{\partial\mathbf{C}_{-1}}{\partial\omega}$, $\dot{\mathbf{D}}_{-1,\omega}=\frac{\partial\mathbf{D}_{-1}}{\partial\omega}$, for $\omega=\rho,\lambda_1,\lambda_2,\lambda_3$, and these expressions can easily be obtained from the expressions of $\mathbf{C}$, $\mathbf{C}_{-1}$, $\mathbf{D}$, and $\mathbf{D}_{-1}$ given in Lemma 2.1; and further,

$$\dot{\Omega}_{\lambda_3}=\tfrac{\partial\Omega_{\lambda_3}}{\partial\lambda_3}=(\mathbf{B}_3'\mathbf{B}_3)^{-1}(\mathbf{B}_3'\mathbf{W}_3+\mathbf{W}_3'\mathbf{B}_3)(\mathbf{B}_3'\mathbf{B}_3)^{-1},$$

$$\ddot{\Omega}_{\lambda_3}=\tfrac{\partial\dot{\Omega}_{\lambda_3}}{\partial\lambda_3}=2[\dot{\Omega}_{\lambda_3}(\mathbf{B}_3'\mathbf{W}_3+\mathbf{W}_3'\mathbf{B}_3)(\mathbf{B}_3'\mathbf{B}_3)^{-1}-(\mathbf{B}_3'\mathbf{B}_3)^{-1}(\mathbf{W}_3'\mathbf{W}_3)(\mathbf{B}_3'\mathbf{B}_3)^{-1}],$$

$$\dot{\Omega}^-_{\lambda_3}=\tfrac{\partial\Omega^{-1}}{\partial\lambda_3}=-\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1},\qquad \ddot{\Omega}^-_{\lambda_3}=\tfrac{\partial\dot{\Omega}^-_{\lambda_3}}{\partial\lambda_3}=-2\Omega^{-1}\dot{\Omega}_{\lambda_3}\dot{\Omega}^-_{\lambda_3}-\Omega^{-1}\ddot{\Omega}_{\lambda_3}\Omega^{-1},$$

$$\dot{\Omega}^-_{\phi}=\tfrac{\partial\Omega^{-1}}{\partial\phi}=\Omega^{-1}(J_T\otimes I_n)\Omega^{-1},\quad \ddot{\Omega}^-_{\phi}=\tfrac{\partial\dot{\Omega}^{-1}_{\phi}}{\partial\phi}=2\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1},$$

$$\ddot{\Omega}^-_{\phi,\lambda_3}=\tfrac{\partial\dot{\Omega}^-_{\phi}}{\partial\lambda_3}=2\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}.$$

It is easy to show that $\frac{1}{nT}H^*_{\mathtt{SDPD}}(\psi_0)=O_p(1)$ by Lemma A.1 and the model assumptions. Thus, $\frac{1}{nT}H^*_{\mathtt{SDPD}}(\bar{\psi})=O_p(1)$ because $\bar{\psi}-\psi_0=o_p(1)$ which is implied by $\hat{\psi}_{\mathtt{M}}\xrightarrow{p}\psi_0$. As $\bar{\sigma}^2\xrightarrow{p}\sigma^2_{v0}$, $\bar{\sigma}^{-r}=\sigma^{-r}_{v0}+o_p(1), r=2,4,6$. As σ^r appears in $H^*_{\mathtt{SDPD}}(\psi)$ multiplicatively,

$$\tfrac{1}{n(T-1)}H^*_{\mathtt{SDPD}}(\bar{\psi})=\tfrac{1}{n(T-1)}H^*_{\mathtt{SDPD}}(\bar{\beta},\sigma^2_{v0},\bar{\rho},\bar{\lambda},\phi)+o_p(1).$$

The proof of (**b**) is thus equivalent to the proof of

$$\tfrac{1}{n(T-1)}\left[H^*_{\mathtt{SDPD}}(\bar{\beta},\sigma^2_{v0},\bar{\rho},\bar{\lambda},\bar{\phi})-H^*_{\mathtt{SDPD}}(\psi_0)\right]\xrightarrow{p}0.$$

Writing $\mathbf{e}(\theta) = \mathbf{e} - (\lambda_1 - \lambda_{10})\mathbf{W}_1 Y - (\rho - \rho_0)Y_{-1} - (\lambda_2 - \lambda_{20})\mathbf{W}_2 Y_{-1} - X(\beta - \beta_0)$, and by the expressions for Y and Y_{-1} given in Lemma 2.2, we see that all the random elements of $H^*_{\mathtt{SDPD}}(\psi)$ can be written as linear combinations of terms:

quadratic in $\mathbf{e}$:	$(\varpi - \varpi_0)^j(\omega - \omega_0)^k \mathbf{e}'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbf{e}$,
quadratic in $\mathbf{y}_0$:	$(\varpi - \varpi_0)^j(\omega - \omega_0)^k \mathbf{y}_0'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbf{y}_0$,
linear in $\mathbf{e}$:	$(\varpi - \varpi_0)^j \mathbf{e}'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbb{Z}$,
linear in $\mathbf{y}_0$:	$(\varpi - \varpi_0)^j \mathbf{y}_0'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbb{Z}$,
bylinear in $\mathbf{e}$ *and* $\mathbf{y}_0$:	$(\varpi - \varpi_0)^j(\omega - \omega_0)^k \mathbf{e}'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbf{y}_0$,

for $j, k = 0, 1,\ \varpi, \omega = \rho, \lambda_1, \lambda_2$, where $\mathbb{A}$ and $\mathbb{B}$ denote generically $nT \times nT$ nonstochastic matrices, and $\mathbb{Z}$ generically $nT \times d$ nonstochastic vector or matrices, free from parameters; and $\mathbf{G}(\phi, \lambda_3)$ can be Ω^{-1}, $\dot{\Omega}^-_{\lambda_3}$, $\ddot{\Omega}^-_{\lambda_3}$, $\dot{\Omega}^-_{\phi}$, $\ddot{\Omega}^-_{\phi}$, and $\ddot{\Omega}^-_{\phi,\lambda_3}$.

Take a typical quadratic term of $\mathbf{e}$, $\mathbf{e}'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbf{e}$, for example. Letting (λ_3^*, ϕ^*) be between $(\bar{\lambda}_3, \bar{\phi})$ and (λ_{30}, ϕ_0), we have by MVT,

$$\tfrac{1}{nT}[\mathbf{e}'\mathbb{A}\mathbf{G}(\bar{\lambda}_3, \bar{\phi})\mathbb{B}\mathbf{e} - \mathbf{e}'\mathbb{A}\mathbf{G}(\lambda_{30}, \phi_0)\mathbb{B}\mathbf{e}] = \tfrac{\bar{\phi}-\phi_0}{nT}\mathbf{e}'\mathbb{A}\dot{\mathbf{G}}_{\phi^*}\mathbb{B}\mathbf{e} + \tfrac{\bar{\lambda}_3-\lambda_0}{nT}\mathbf{e}'\mathbb{A}\dot{\mathbf{G}}_{\lambda_3^*}\mathbb{B}\mathbf{e},$$

where $\dot{\mathbf{G}}_\phi$ and $\dot{\mathbf{G}}_{\lambda_3}$ are the partial derivatives of $\mathbf{G}$ evaluated at (λ_3^*, ϕ^*). From the expression of the Hessian matrix given earlier, we see that $\mathbf{G}$ depends on λ_3 and ϕ, and is the multiplications and linear combinations of matrices Ω^{-1}, $\mathbf{B}_3^{-1}$ and $\mathbf{W}_3$. Therefore, its partial derivatives evaluated at (λ_3, ϕ) are the multiplications and linear combinations of $\Omega^{-1}(\lambda_3, \phi)$, $\mathbf{B}_3^{-1}(\lambda_3)$ and $\mathbf{W}_3$, and hence are uniformly bounded in both row and column sums for (λ_3, ϕ) in a neighborhood of (λ_{30}, ϕ_0). Recall that $\mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}_0^{-1}\mathbf{v}$. By applying Lemma A.4 (i) and using the consistency of $\hat{\psi}_{\mathtt{M}}$, we have $\frac{1}{nT}[\mathbf{e}'\mathbb{A}\mathbf{G}(\bar{\lambda}_3, \bar{\phi})\mathbb{B}\mathbf{e} - \mathbf{e}'\mathbb{A}\mathbf{G}(\lambda_{30}, \phi_0)\mathbb{B}\mathbf{e}] \xrightarrow{p} 0$. The convergence of all other terms can be shown similarly by using Lemma A.4, Assumption F, and the consistency of the M-estimator.

It left to show that all the 'trace' terms in $\frac{1}{nT}\left[H^*_{\mathtt{SDPD}}(\bar{\beta}, \sigma^2_{v0}, \bar{\rho}, \bar{\lambda}, \bar{\phi}) - H^*_{\mathtt{SDPD}}(\psi_0)\right]$ are $o_p(1)$. For example, let $(\phi^*, \rho^*, \lambda^*)$ be between $(\bar{\phi}, \bar{\rho}, \bar{\lambda})$ and $(\phi_0, \rho_0, \lambda_0)$. By MVT,

$$\begin{aligned}
&\tfrac{1}{nT}\{\mathrm{tr}[\dot{\mathbf{E}}_{-1,\rho}(\bar{\phi}, \bar{\rho}, \bar{\lambda})\Omega^{-1}(\bar{\phi}, \bar{\lambda}_3)] - \mathrm{tr}[\dot{\mathbf{E}}_{-1,\rho}(\phi_0, \rho_0, \lambda_0)\Omega^{-1}(\phi_0, \lambda_{30})]\} \\
&= \tfrac{\bar{\phi}-\phi_0}{nT}\mathrm{tr}[\phi^*\Omega^{-1}(\phi^*, \lambda_3^*) + \dot{\mathbf{E}}_{-1,\rho^*}\dot{\Omega}^{-1}_{\phi^*}] + \tfrac{\bar{\rho}-\rho_0}{nT}\mathrm{tr}[\ddot{\mathbf{E}}^{\rho^*}_{-1,\rho}\Omega^{-1}(\phi^*, \lambda_3^*)] \\
&+ \tfrac{\bar{\lambda}_1-\lambda_{10}}{nT}\mathrm{tr}[\ddot{\mathbf{E}}^{\lambda_1^*}_{-1,\rho}\Omega^{-1}(\phi^*, \lambda_3^*)] + \tfrac{\bar{\lambda}_2-\lambda_{20}}{nT}\mathrm{tr}[\ddot{\mathbf{E}}^{\lambda_2^*}_{-1,\rho}\Omega^{-1}(\phi^*, \lambda_3^*)] \\
&+ \tfrac{\bar{\lambda}_3-\lambda_{30}}{nT}\mathrm{tr}[\ddot{\mathbf{E}}^{\lambda_3^*}_{-1,\rho}\Omega^{-1}(\phi^*, \lambda_3^*) + \dot{\mathbf{E}}_{-1,\rho}\dot{\Omega}^{-1}(\lambda_3^*)],
\end{aligned}$$

where $\ddot{\mathbf{E}}^{r^*}_{-1,\rho}, r = \phi, \rho, \lambda_1, \lambda_2, \lambda_3$,are the partial derivatives of $\dot{\mathbf{E}}_\rho$ evaluated at $(\phi^*, \rho^*, \lambda^*)$. Consider w.l.o.g. $T = 2$. Recall the expression of $\mathbf{E}$ and the definitions of C and D, we have,

$$\mathbf{D}(\rho, \lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} B_1^{-1}(B_3'B_3)^{-1}, & B_1^{-1}(B_3'B_3)^{-1} \\ \mathcal{B}B_1^{-1}(B_3'B_3)^{-1}, & \mathcal{B}B_1^{-1}(B_3'B_3)^{-1} \end{pmatrix},$$

$$\mathbf{C}(\rho, \lambda_1, \lambda_2) = \begin{pmatrix} (\sum_{i=0}^m \mathcal{B}^i)B_1^{-1}, & (\sum_{i=0}^m \mathcal{B}^i)B_1^{-1} \\ (\sum_{i=0}^{m+1} \mathcal{B}^i)B_1^{-1}, & (\sum_{i=0}^{m+1} \mathcal{B}^i)B_1^{-1} \end{pmatrix}.$$

This shows that elements of $\mathbf{E}$ and $\mathbf{E}_\rho$ are linear combinations of multiplications of the matricies W_1, B_1^{-1}, B_2 and B_3^{-1}. Therefore, $\ddot{\mathbf{E}}^r_{-1,\rho}$ have elements being the linear combinations of W_1, W_2,W_3, $B_1^{-1}(\lambda_1)$, $B_2(\rho,\lambda_2)$,and $B_3^{-1}(\lambda_3)$ and hence are uniformly bounded in both row and column sums for (ρ,λ) in the neighborhood of (ρ_0,λ_0) by Lemmas A.1 and A.2. Therefore, each trace term in the equation above divided by nT such as $\frac{1}{nT}\mathrm{tr}[\phi^*\Omega^{-1}(\phi^*,\lambda_3^*)+\dot{\mathbf{E}}_{-1,\rho^*}\dot{\Omega}_{\phi^*}^{-1}]$ is $O_p(1)$. So, (b) is proved.

Proof of (c). By Lemma 2.2 and the definition of $\mathbf{e}$, elements of Hessian matrix can be written as linear combinations of quadratic and linear terms of $\mathbf{v}$ and $\boldsymbol{\varepsilon}$, quadratic and linear terms of $\mathbf{y}_0$, bilinear terms of $\mathbf{v}$ and $\mathbf{y}_0$, $\boldsymbol{\varepsilon}$ and $\mathbf{y}_0$, $\mathbf{v}$ and $\boldsymbol{\varepsilon}$. Thus, the results follow by repeatedly applying Lemma A.1, Lemma A.4, and Assumption F. ■

Appendix C: Proofs for Section 3

Proof of Lemma 3.1. The result (3.7) is obvious. To show (3.8), write $g_{\Psi i}$ defined in (3.3) as $g_{\Psi i}=Q_{1i}+Q_{2i}$, where $Q_{1i}=\sum_{t=1}^T(e_{it}\Psi_{ii,t+}^*y_{0i}^*-d_{\Psi it})$ and $Q_{2i}=\sum_{t=1}^T e_{it}'\xi_{it}$. Then, $\mathrm{E}(g_{\Psi i}\ g_{\Psi j})=\mathrm{E}[(Q_{1i}+Q_{2i})(Q_{1j}+Q_{2j})]$. Defined above (3.3), b_i' and w_{it}' are the ith row of B_3^{-1} and $(\Psi_{t+}^{*l}+\Psi_{t+}^{*u})$. Thus, $\xi_{it}=w_{it}'y_0^*$ and $e_{it}=\varepsilon_i+b_i'v_t$. It is easy to show that

$$
\begin{aligned}
\mathrm{E}(Q_{1i}Q_{1j}) &= \textstyle\sum_{t=1}^T \mathrm{Cov}(e_{it}'\Psi_{ii,t+}^*y_{0i}^*, e_{jt}'\Psi_{jj,t+}^*y_{0j}^*)+\sum_{t=1}^T\sum_{s(\neq t)}\mathrm{Cov}(e_{it}'\Psi_{ii,t+}^*y_{0i}^*, e_{js}'\Psi_{jj,s+}^*y_{0j}^*)\\
&= \sigma_{v_0}^2(b_i'b_j)\textstyle\sum_{t=1}^T(\Psi_{ii,t+}^*\Psi_{jj,t+}^*)\mathrm{E}(y_{0i}^*y_{0j}^*),
\end{aligned}
$$

where the double summation part vanishes, as for $i\neq j$ and $t\neq s$, $e_{it}'\Psi_{ii,t+}^*y_{0i}^*$ and $e_{js}'\Psi_{jj,s+}^*y_{0j}^*$ (respectively measurable-(ε_i,v_t,V_m) and (ε_j,v_s,V_m)) are conditionally independent given V_m. Similarly, $\mathrm{E}(Q_{ri}Q_{\nu j}), r,\nu=1,2$, are

$$
\begin{aligned}
\mathrm{E}(Q_{1i}Q_{2j}) &= \sigma_{v_0}^2(b_i'b_j)\textstyle\sum_{t=1}^T\Psi_{ii,t+}^*\mathrm{E}(y_{0i}^*\xi_{jt}),\\
\mathrm{E}(Q_{2i}Q_{1j}) &= \sigma_{v_0}^2(b_i'b_j)\textstyle\sum_{t=1}^T\Psi_{jj,t+}^*\mathrm{E}(y_{0j}^*\xi_{it}),\\
\mathrm{E}(Q_{2i}Q_{2j}) &= \sigma_{v_0}^2(b_i'b_j)\textstyle\sum_{t=1}^T\mathrm{E}(\xi_{it}\xi_{jt})+\sigma_{\epsilon_0}^4(1_j'w_{i+})(1_i'w_{j+}),
\end{aligned}
$$

where 1_i denotes an $n\times 1$ vector of element 1 at the ith position and zero elsewhere. Summarizing and simplifying, we have $\mathrm{E}(g_{\Psi i}\ g_{\Psi j})=\sigma_{\varepsilon_0}^4(w_{ij,+}w_{ji,+})+\sigma_{v_0}^2\sum_{t=1}^T(b_i'b_j)\mathrm{E}[(w_{ti}^{*\prime}y_0^*)(w_{tj}^{*\prime}y_0^*)]$.

More generally, for Ψ_r and Ψ_ν, $r,\nu=1,2,3$, we have,

$$
\mathrm{E}(g_{\Psi_r i}\ g_{\Psi_\nu j})=\sigma_{\varepsilon_0}^4(w_{rij,+}w_{\nu ji,+})+\sigma_{v_0}^2\textstyle\sum_{t=1}^T(b_i'b_j)\mathrm{E}[(w_{ri,t}^{*\prime}y_0^*)(w_{\nu j,t}^{*\prime}y_0^*)], \qquad \text{(C.1)}
$$

which is the result (3.8) in Lemma 3.1.

To show (3.9), first note that, for $n\times 1$ vectors a,b,c, and d:

$$
\mathrm{E}[(a'v_t)(b'v_t)(c'v_t)(d'v_t)]=(\mu_{v_0}^{(4)}-3\sigma_{v0}^4)(a\odot b)'(c\odot d)+\sigma_{v0}^4[(a'b)(c'd)+(a'c)(b'd)+(a'd)(b'c)],
$$

where $\odot$ denotes the Hadamard product, and $\mu_{v_0}^{(4)}$ is the 4th moment of v_{it}. Immediately following this result we have:

$$
\mathrm{E}[(a'v_t)^2(b'v_t)^2]=(\mu_{v_0}^{(4)}-3\sigma_{v0}^4)(a\odot a)'(b\odot b)+\sigma_{v0}^4[(a'a)(b'b)+2(a'b)^2].
$$

Writing $g_{\Phi i}=Q_{1i}+Q_{2i}$, where $Q_{1i}=\sum_{t=1}^T(e_{it}e_{it}^*-d_{1\Phi it})$ and $Q_{2i}=\sum_{t=1}^T(e_{it}\varphi_{it}-d_{2\Phi it})$ by

(3.4), we have $\text{E}(g_{\Phi i}g_{\Phi j}) = \text{E}[(Q_{1i}+Q_{2i})(Q_{1j}+Q_{2j})]$. Some tedious algebra leads to

$$\begin{aligned}
\text{E}(Q_{1i}Q_{1j}) &= \textstyle\sum_{t=1}^T \text{Cov}(e_{it}e_{it}^*, e_{jt}e_{jt}^*) + \sum_{t=1}^T \sum_{s=1,s\neq t}^T \text{Cov}(e_{it}e_{it}^*, e_{js}e_{js}^*) \\
&= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T \left[\Phi_{ii,ts}\Phi_{jj,st}(b_i'b_j)^2 + \Phi_{ii,ts}\Phi_{jj,ts}(b_i'b_j)^2\right] \\
&\quad +(\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T \Phi_{ii,tt}\Phi_{jj,tt}(b_i \odot b_i)'(b_j \odot b_j), \\
\text{E}(Q_{1i}Q_{2j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [\Phi_{ii,ts}(b_i'c_{j,st})(b_i'b_j) + \Phi_{ii,ts}(b_i'c_{j,ts})(b_i'b_j)] \\
&\quad +\sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T (\Phi_{ii,t+} + \Phi_{ii,+t})(1_i'a_{j,t+})(b_i'b_j) \\
&\quad +(\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T \Phi_{ii,tt}(b_i \odot b_i)'(b_j \odot c_{j,tt}), \\
\text{E}(Q_{2i}Q_{1j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [\Phi_{jj,ts}(b_j'c_{i,st})(b_i'b_j) + \Phi_{jj,ts}(b_j'c_{i,ts})(b_i'b_j)] \\
&\quad +\sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T (\Phi_{jj,t+} + \Phi_{jj,+t})(1_j'a_{i,t+})(b_i'b_j) \\
&\quad +(\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T \Phi_{jj,tt}(b_j \odot b_j)'(b_i \odot c_{i,tt}), \\
\text{E}(Q_{2i}Q_{2j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [(b_i'c_{j,st})(b_j'c_{i,ts}) + (b_i'b_j)(c_{i,ts}'c_{j,ts})] \\
&\quad +\sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T [(1_j'a_{i,t+})(b_i'c_{j,+t}) + (1_i'a_{j,t+})(b_j'c_{i,+t}) + (a_{i,t+}'a_{j,t+})(b_i'b_j)] \\
&\quad +(\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T (b_i \odot c_{i,tt})'(b_j \odot c_{j,tt}),
\end{aligned}$$

Denote $\Phi_{ts}^* = \Phi_{ts}^l + \Phi_{st}^{w\prime} + \Phi_{ts}^d$, and let $c_{i,ts}^*$ be ith row of $\Phi_{ts}^* B_3^{-1}$. Further let $a_{ji,ts}$ be the ith element of $a_{j,ts}$. Then we have,

$$\begin{aligned}
\text{E}(g_{\Phi i}g_{\Phi j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [(b_j'c_{i,ts}^*)(b_i'c_{j,st}^*) + (b_i'b_j)(c_{i,ts}^{*\prime}c_{j,ts}^*)] \\
&\quad +\sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T [a_{ji,t+}(b_j'c_{i,+t}^*) + a_{ij,t+}(b_i'c_{j,+t}^*) + (b_i'b_j)(a_{i,t+}^{*\prime}a_{j,t+}^*)] \\
&\quad +(\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T [(b_i \odot c_{i,tt}^*)'(b_j \odot c_{j,tt}^*)].
\end{aligned}$$

More generally, for Φ_r and Φ_ν, $r, \nu = 1, \ldots, 6$, we have

$$\begin{aligned}
\text{E}(g_{\Phi_r i}g_{\Phi_\nu j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [(b_j'c_{ri,ts}^*)(b_i'c_{\nu j,st}^*) + (b_i'b_j)(c_{ri,ts}^{*\prime}c_{\nu j,ts}^*)] \\
&\quad +\sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T a_{rji,t+}(b_j'c_{\nu i,+t}^*) + a_{\nu ij,t+}(b_i'c_{rj,+t}^*) + (b_i'b_j)(a_{ri,t+}^{*\prime}a_{\nu j,t+}^*)] \\
&\quad +(\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T [(b_i \odot c_{ri,tt}^*)'(b_j \odot c_{\nu j,tt}^*)].
\end{aligned}$$

To show (3.11), write $g_{\Pi i} = \sum_{t=1}^T \Pi_{jt}'e_{jt} = P_i$ and write $g_{\Phi i} = Q_{1i} + Q_{2i}$, where $Q_r, r = 1, 2$, are given above. Then, $\text{E}(g_{\Phi i}g_{\Pi j}) = \text{E}[(Q_{1i}+Q_{2i})P_j]$. Some algebra leads to

$$\text{E}(Q_{1i}P_j) = \mu_{v0}^{(3)} \textstyle\sum_{t=1}^T \Phi_{ii,tt}(b_i \odot b_i)'b_j\Pi_{jt}', \quad \text{E}(Q_{2i}P_j) = \mu_{v0}^{(3)} \sum_{t=1}^T (b_i \odot c_{i,tt})'b_j\Pi_{jt}'.$$

Combining the two terms we have, $\text{E}(g_{\Phi i}g_{\Pi j}) = \mu_{v0}^{(3)} \sum_{t=1}^T (b_i \odot c_{i,tt}^*)'b_j\Pi_{jt}'$.

Proof of (3.10) is similar. Write $g_{\Psi i} = Q_{1i} + Q_{2i}$ and $g_{\Pi i} = P_i$. Then $\text{E}(g_{\Psi i}g_{\Pi j}') = \text{E}[(Q_{1i}+Q_{2i})P_j]$. After some algebra, we obtain

$$\text{E}(Q_{1i}P_j) = \sigma_{v0}^2 \textstyle\sum_{t=1}^T (b_i'b_j)\Pi_{jt}'\Psi_{ii,t}^*\text{E}(y_{0i}^*) \text{ and } \text{E}(Q_{2i}P_j) = \sigma_{v0}^2 \sum_{t=1}^T (b_i'b_j)\Pi_{jt}'\text{E}(\xi_{it}),$$

leading to $\text{E}(g_{\Psi i}g_{\Pi j}') = \sigma_{v0}^2 \sum_{t=1}^T \Pi_{jt}'\text{E}(w_{it}^{*\prime}y_0^*)(b_i'b_j)$.

To show the result (3.12) in Lemma 3.1, Write $g_{\Phi i} = Q_{1i} + Q_{2i}$ and $g_{\Psi i} = P_{1i} + P_{2i}$ where $P_{1i} = \sum_{t=1}^T (e_{it}\Psi_{ii,t}^*y_{0i}^* - d_{\Psi it})$ and $P_{2i} = \sum_{t=1}^T e_{it}'\xi_{it}$. Then $\text{E}(g_{\Phi i}g_{\Psi j}) = \text{E}[(Q_{1i} +$

$Q_{2i})(P_{1j}+P_{2j})]$. Some tedious algebra leads to

$$\begin{aligned}
\mathrm{E}(Q_{1i}P_{1j}) &= \mu_{v0}^{(3)} \textstyle\sum_{t=1}^T \Phi_{ii,tt}\Psi^*_{jj,t}(b_i \odot b_i)' b_j \mathrm{E}(y^*_{0j}),\\
\mathrm{E}(Q_{1i}P_{2j}) &= \sigma^2_{\varepsilon 0}\sigma^2_{v0} \textstyle\sum_{t=1}^T \left[\Phi_{ii,+t}(1_i' w_{jt})(b_i' b_j) + \Phi_{ii,t+}(1_i' w_{jt})(b_i' b_j)\right]\\
&\quad + \mu_{v0}^{(3)} \textstyle\sum_{t=1}^T \Phi_{ii,tt}(b_i \odot b_i)' b_j \mathrm{E}(\xi_{jt}),\\
\mathrm{E}(Q_{2i}P_{1j}) &= \sigma^2_{\varepsilon 0}\sigma^2_{v0} \textstyle\sum_{t=1}^T \Psi^*_{jj,t}(1_j' a_{i,t+})(b_i' b_j) + \mu_{v0}^{(3)} \sum_{t=1}^T \Psi^*_{jj,t}(b_i \odot b_j)' c_{i,tt}\mathrm{E}(y^*_{0j}),\\
\mathrm{E}(Q_{2i}P_{2j}) &= \sigma^2_{\varepsilon 0}\sigma^2_{v0} \textstyle\sum_{t=1}^T \left[(1_i' w_{jt})(b_j' c_{i,+t}) + (b_i' b_j)(a'_{i,t+} w_{jt})\right] + \sigma^4_{\varepsilon 0}(1_i' w_{j+})(1_j' a_{i,++})\\
&\quad + \mu_{v0}^{(3)} \textstyle\sum_{t=1}^T (b_i \odot b_j)' c_{i,tt}\mathrm{E}(\xi_{jt}).
\end{aligned}$$

Summarizing the above and simplifying give the result (3.12). ■

Proof of Theorem 3.1. First, the result $\Sigma^*_{\mathtt{SDPD}}(\hat\psi_{\mathtt{M}}) - \Sigma^*_{\mathtt{SDPD}}(\psi_0) \xrightarrow{p} 0$ is implied by the result **(b)** in the proof of Theorem 2.2. To show $\widehat\Gamma^*_{\mathtt{SDPD}} - \Gamma^*_{\mathtt{SDPD}}(\psi_0) \xrightarrow{p} 0$, for the single summation part, the result $\frac{1}{nT}\sum_{i=1}^n[\hat{\mathbf{g}}_i\hat{\mathbf{g}}_i' - \mathrm{E}(\mathbf{g}_i\mathbf{g}_i')] \xrightarrow{p} 0$ follows from $\frac{1}{nT}\sum_{i=1}^n[\hat{\mathbf{g}}_i\hat{\mathbf{g}}_i' - \mathbf{g}_i\mathbf{g}_i'] \xrightarrow{p} 0$ and $\frac{1}{nT}\sum_{i=1}^n[\mathbf{g}_i\mathbf{g}_i' - \mathrm{E}(\mathbf{g}_i\mathbf{g}_i')] \xrightarrow{p} 0$. The proof of the former is straightforward by MVT. We focus on the proof of the latter result. As the elements of $S^*_{\mathtt{SDPD}}(\psi_0)$ are mixtures of terms of the forms $\Pi'\mathbf{e} = \sum_{i=1}^n g_{\Pi i}$, $\mathbf{e}'\Psi\mathbf{y}_0 - \mathrm{E}(\mathbf{e}'\Psi\mathbf{y}_0) = \sum_{i=1}^n g_{\Psi i}$, and $\mathbf{e}'\Phi\mathbf{e} - \mathrm{E}(\mathbf{e}'\Phi\mathbf{e}) = \sum_{i=1}^n g_{\Phi i}$, it suffices to show that

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^n[g_{ki}g'_{ri} - \mathrm{E}(g_{ki}g'_{ri})] = o_p(1), \quad \text{for } g_{ki}, g_{ri} = g_{\Pi i}, g_{\Psi i}, g_{\Phi i}. \tag{C.2}$$

First, assuming, W.L.O.G, $\{\Pi_{it}\}$ are scalars, we have

$$g_{\Pi i} = \textstyle\sum_{t=1}^T \Pi_{it}\mathbf{e}_{it} = \sum_{t=1}^T \Pi_{it}(\varepsilon_i + b_i' v_t) = \Pi_{i+}\varepsilon_i + b_i'\mathbf{v}_i, \tag{C.3}$$

where $\Pi_{i+} = \sum_{t=1}^T \Pi_{it}$ and $\mathbf{v}_i = \sum_{t=1}^T \Pi_{it}v_t$. It follows that

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^n \left[g^2_{\Pi i} - \mathrm{E}(g^2_{\Pi i})\right] \equiv U_1 + U_2 + U_3,$$

where $U_1 = \frac{1}{nT}\sum_{i=1}^n \Pi^2_{i+}(\varepsilon_i^2 - \sigma^2_{\varepsilon 0})$, $U_2 = \frac{2}{nT}\sum_{i=1}^n(\Pi_{i+}\varepsilon_i)(b_i'\mathbf{v}_i)$ and $U_3 = \frac{1}{nT}\sum_{i=1}^n \left[(b_i'\mathbf{v}_i)^2 - \sigma^2_{v0}(\sum_{t=1}^T \Pi^2_{it})(b_i' b_i)\right]$. It is obvious that $\mathrm{E}(U_1) = 0$ and $\mathrm{Var}(U_1) = \frac{1}{n^2T^2}\sum_{i=1}^n \Pi^4_{i+}\mathrm{E}[(\varepsilon_i^2 - \sigma^2_{\varepsilon 0})^2]$ $= \frac{1}{n^2T^2}(\mu^{(4)}_\varepsilon - \sigma^4_{\varepsilon 0})\sum_{i=1}^n \Pi^4_{i+}$. Given Assumption B and that the elements of Π_{it} are uniformly bounded for each t, $\mathrm{Var}(U_1) = o(1)$. Hence $U_1 \xrightarrow{p} 0$ by Chebyshev's inequality.

Write $U_2 = \frac{2}{nT}\sum_{i=1}^n(\Pi_{i+}\varepsilon_i)(b_i'\mathbf{v}_i) = \frac{2}{nT}\sum_{t=1}^T \varepsilon'\Theta_t v_t$, where $\Theta_t = \mathtt{diag}(\Pi_+)\mathtt{diag}(\Pi_t)B_3^{-1}$, and is uniformly bounded in either row or column sum. Then $U_2 \xrightarrow{p} 0$, by Lemma A.4 (vii) and independence of ε and v_t. For U_3, we have

$$\begin{aligned}
\tfrac{1}{nT}\textstyle\sum_{i=1}^n(b_i'\mathbf{v}_i)^2 &= \tfrac{1}{nT}\textstyle\sum_{i=1}^n \left[(\sum_{t=1}^T \Pi_{it}v_t')(b_i b_i')(\sum_{t=1}^T \Pi_{it}v_t)\right]\\
&= \tfrac{1}{nT}\textstyle\sum_{t=1}^T\sum_{s=1}^T v_t'\big(\sum_{i=1}^n \Pi_{it}b_i b_i'\Pi_{is}\big)v_s\\
&= \tfrac{1}{nT}\textstyle\sum_{t=1}^T\sum_{s=1}^T v_t'\big[B_3^{-1}\mathtt{diag}(\Pi_t)\mathtt{diag}(\Pi_s)B_3'^{-1}\big]v_s.
\end{aligned}$$

Denote $A_{ts} = B_3^{-1}\mathtt{diag}(\Pi_t)\mathtt{diag}(\Pi_s)B_3'^{-1}$. As t is fixed and the elements of Π_t are uniformly bounded, one can easily show that $\frac{1}{n}[v_s'A_{ts}v_s - \mathrm{E}(v_t'A_{ts}v_s)] \xrightarrow{p} 0$ by applying Lemma A.4 (v) for the case of $t = s$, and Lemma A.4 (vii) for the case of $t \neq s$. Thus, $U_3 \xrightarrow{p} 0$. Therefore,

$\frac{1}{nT}\sum_{i=1}^{n}\left[g_{\Pi i}^2 - \mathrm{E}(g_{\Pi i}^2)\right] = o_p(1)$.

Second, for $g_{\Phi i} = \sum_{t=1}^{T} e_{it}e_{it}^{*} + \sum_{t=1}^{T} e_{it}\varphi_{it} - d_{\Phi i}$, recall $e_{it} = \varepsilon_i + b_i' v_t$, $e_{it}^{*} = \Phi_{ii,t+}\varepsilon_i + b_i'\mathbf{v}_{it}^{*}$ where $\mathbf{v}_{it}^{*} = \sum_{s=1}^{T}\Phi_{ii,ts}v_s$, and $\varphi_{it} = a_{i,t+}'\varepsilon + \sum_{s=1}^{T} c_{i,ts}' v_s$. Some algebra leads to

$$g_{\Phi i} = k_i(\varepsilon_i^2 - \sigma_\varepsilon^2) + \varepsilon_i z_{1i} + \varepsilon_i(r_i'\varepsilon) + (u_i - \mu_{u_i}) + \sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i' v_t), \tag{C.4}$$

where $z_{1i} = \sum_{t=1}^{T} p_{it}' v_t$ with p_{it}' being the ith row of some non-stochastic matrix uniformly bounded in row or column sums; $u_i = \sum_{t=1}^{T}\sum_{s=1}^{T} v_t' A_{i,ts} v_s$ with mean $\mu_{u_i} = \sigma_v^2 \sum_{t=1}^{T} \mathtt{tr}(A_{i,tt})$, where $A_{i,ts} = \Phi_{ii,ts}(b_i b_i') + (b_i c_{i,ts}')$; k_i are uniformly bounded scalar constants, b_i is defined as before, and r_i' and q_{it}' represent ith row of some non-stochastic lower triangular matrices which are uniformly bounded in either row or column sums. Noticing that the terms in (C.4) are uncorrelated, it follows that

$$\frac{1}{nT}\sum_{i=1}^{n}\left(g_{\Phi i}^2 - \mathrm{E}(g_{\Phi i}^2)\right) = \sum_{r=1}^{15} U_r, \quad \text{where} \tag{C.5}$$

$$\begin{aligned}
&U_1 = \tfrac{1}{nT}\sum_{i=1}^{n} k_i^2\{(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)^2 - \mathrm{E}[(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)^2]\}, && U_2 = \tfrac{1}{nT}\sum_{i=1}^{n}[\varepsilon_i^2 z_{1i}^2 - (\textstyle\sum_{t=1}^{T} p_{it}'p_{it})\sigma_{v0}^2\sigma_{\varepsilon 0}^2],\\
&U_3 = \tfrac{1}{nT}\sum_{i=1}^{n}[\varepsilon_i^2(r_i'\varepsilon)^2 - \sigma_{\varepsilon 0}^4\textstyle\sum_{j=1}^{n} r_{ij}^2], && U_4 = \tfrac{1}{nT}\sum_{i=1}^{n}\{(u_i - \mu_{u_i})^2 - \mathrm{E}[(u_i - \mu_{u_i})^2]\},\\
&U_5 = \tfrac{1}{nT}\sum_{i=1}^{n}\{[\textstyle\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i'v_t)]^2 - \sigma_{v0}^2\sigma_{\varepsilon 0}^2(\sum_{t=1}^{T} q_{it}'q_{it})(b_i'b_i)\}, &&\\
&U_6 = \tfrac{2}{nT}\sum_{i=1}^{n} k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)\varepsilon_i z_{1i}, && U_7 = \tfrac{2}{nT}\sum_{i=1}^{n} k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)\varepsilon_i(r_i'\varepsilon),\\
&U_8 = \tfrac{2}{nT}\sum_{i=1}^{n} k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)(u_i - \mu_{u_i}), && U_9 = \tfrac{2}{nT}\sum_{i=1}^{n}\varepsilon_i^2(r_i'\varepsilon)z_{1i},\\
&U_{10} = \tfrac{2}{nT}\sum_{i=1}^{n} k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)\textstyle\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i'v_t), && U_{11} = \tfrac{1}{nT}\sum_{i=1}^{n}\varepsilon_i z_{1i}(u_i - \mu_{u_i}),\\
&U_{12} = \tfrac{2}{nT}\sum_{i=1}^{n}[\varepsilon_i z_{1i}\textstyle\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i'v_t)], && U_{13} = \tfrac{2}{nT}\sum_{i=1}^{n}\varepsilon_i(r_i'\varepsilon)(u_i - \mu_{u_i}),\\
&U_{14} = \tfrac{2}{nT}\sum_{i=1}^{n}\varepsilon_i(r_i'\varepsilon)\textstyle\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i'v_t), && U_{15} = \tfrac{2}{nT}\sum_{i=1}^{n}(u_i - \mu_{u_i})\textstyle\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i'v_t).
\end{aligned}$$

Each of the fifteen terms above is or can be written as the sum of a MD array, and thus the weak law of large numbers (WLLN) for a MD array, i.e., Theorem 19.7 of Davidson (1994, p.299), can be applied to prove its convergence in probability to zero. Details are as follows.

For the first term, writing $U_1 = \frac{1}{nT}\sum_{i=1}^{n} k_i^2[(\varepsilon_i^4 - \mu_{\varepsilon 0}^{(4)}) + 2\sigma_\varepsilon^2(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)] = \sum_{i=1}^{n} H_{ni}$. As H_{ni} are independent across i and $\{k_i\}$ are uniformly bounded, the conditions for WLLN for a MD array of Davidson can easily be verified and thus $U_1 \xrightarrow{p} 0$.

For the 14th term, denote $U_{14} = \frac{2}{nT}\sum_{i=1}^{n}\varepsilon_i(r_i'\varepsilon)\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i'v_t) = \sum_{i=1}^{n} H_{ni}$. Let $\mathcal{G}_{ni}$ be the increasing σ-field generated by $(\mathbf{v}, \varepsilon_1, \ldots, \varepsilon_i)$. Notice that $\mathrm{E}(H_{ni}|\mathcal{G}_{n,i-1}) = 0$. Thus $\{H_{ni}, \mathcal{G}_{ni}\}$ form a MD array. It is easy to show that $\mathrm{E}|H_{n,i}^{1+\epsilon}| \le K_h < \infty$, for some $\epsilon > 0$. In particular, $\mathrm{E}(H_{ni}^2) = \sum_{t=1}^{T}(b_i'b_i)\sigma_{\varepsilon 0}^2\sigma_{v0}^2\{(\mu_{\varepsilon 0}^{(4)} - 3\sigma_{\varepsilon 0}^4)(r_i \odot r_i)'(q_{it} \odot q_{it}) + \sigma_{\varepsilon 0}^4[(r_i'r_i)(q_{it}'q_{it}) + 2(r_i'q_{it})^2]\}$, which is bounded by Lemma A.1. Thus, $\{H_{ni}\}$ is uniformly integrable. The other two conditions of the WLLN for MD arrays of Davidson are satisfied. So we have $U_{14} = \sum_{i=1}^{n} H_{ni} \xrightarrow{p} 0$. The proofs of the terms U_{11}, U_{12}, and U_{13} proceed similarly.

The 8th term can be written as $U_8 = \frac{2}{nT}\sum_{i=1}^{n} k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)(u_i - \mu_{u_i}) = \frac{2}{nT}\sum_{i=1}^{n} k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)u_i - \frac{2}{nT}\sum_{i=1}^{n} k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)\mu_{ui} = \frac{2}{nT}\sum_{i=1}^{n} V_{1n,i} + \frac{2}{nT}\sum_{i=1}^{n} V_{2n,i}$. As k_i and μ_{ui} are uniformly bounded, we immediately have $\frac{2}{nT}\sum_{i=1}^{n} V_{2n,i} \xrightarrow{p} 0$ by invoking Kolmogorov's law of large numbers (LLN). For $V_{1n,i}$, first we notice that u_i is independent of ε_i for all i. Let $\mathcal{F}_{ni}$ be sigma field generated by $(u_i, \varepsilon_1, \ldots, \varepsilon_i)$, then $\mathrm{E}(V_{1n,i}|\mathcal{F}_{n,i-1}) = 0$. So, $\{V_{1n,i}, \mathcal{F}_{n,i}\}$ form a MD array. Now, $\mathrm{E}(V_{1n,i}^2) = \mathrm{E}(\varepsilon_i^2 - \sigma_{\varepsilon_0}^2)^2\,\mathrm{E}(u_i^2)$, and

$$\begin{aligned}\mathrm{E}(u_i^2) &= \sum_{t=1}^T \mathrm{E}(v_t' A_{i,tt} v_t v_t' A_{i,tt} v_t) + \sum_{t=1}^T \sum_{s\neq t} [\mathrm{E}(v_t' A_{i,tt} v_t)\mathrm{E}(v_s' A_{i,ss} v_s) \\ &\quad + \mathrm{E}(v_t' A_{i,ts} v_s v_t' A_{i,ts} v_s) + \mathrm{E}(v_t' A_{i,ts} v_s v_s' A_{i,st} v_t)] \\ &= \sigma_{v_0}^4 \sum_{t=1}^T \sum_{s\neq t} [\mathrm{tr}(A_{i,tt})\mathrm{tr}(A_{i,ss}) + \mathrm{tr}(A_{i,ts}A_{i,ts}') + \mathrm{tr}(A_{i,ts}A_{i,st})] \\ &\quad + (\mu_{v_0}^{(4)} - 3\sigma_{v_0}^4) \sum_{t=1}^T \sum_{j=1}^n a_{itt,jj}^2,\end{aligned}$$

where $a_{its,ij}$ is the (i,j) element of $A_{i,ts}$. $\sum_{j=1}^n a_{itt,jj}^2 \leq \mathrm{tr}(A_{i,tt}A_{i,tt}')$. Recall that $A_{i,ts} = \Phi_{ii,ts}(b_i b_i') + (b_i c_{i,ts}')$. So we have $\mathrm{tr}(A_{i,ts}) = \Phi_{ii,ts}\sum_{j=1}^n (b_{ij}c_{ij}^*) = O(1)$ as $(B_3' B_3')^{-1}$ and Φ_{ts}^* are uniformly bounded in both row and column sums at true parameter values. Similarly we have $\mathrm{tr}(A_{i,ts}A_{i,ts}') = O(1)$ and $\mathrm{tr}(A_{i,ts}A_{i,st}) = O(1)$. Therefore, the condition, $\mathrm{E}(|V_{1n,i}|^{1+\epsilon}) < K_v < \infty$ for some $\epsilon > 0$, is satisfied. With constant coefficients $\frac{1}{nT}$, the other two conditions of WLLN for MD array of Davidson are satisfied. So we have $\frac{2}{nT}\sum_{i=1}^n V_{1n,i} \xrightarrow{p} 0$ and thus, $U_8 \xrightarrow{p} 0$. The proofs of convergence of U_2, U_5 and U_{15} are similar as that of U_8.

The third term can be written as: $U_3 = \frac{1}{nT}\sum_{i=1}^n (\varepsilon_i^2 - \sigma_{v_0}^2)(r_i'\varepsilon)^2 + \frac{1}{nT}\sum_{i=1}^n \sigma_{v_0}^2[(r_i'\varepsilon)^2 - \sigma_{v_0}^2\sum_{j=1}^n r_{ij}^2]$. Similarly, the first term is the average of a MD array and its convergence follows from Davidson's WLLN for MD array. Letting $n\times n$ matrix $\mathbf{r} = (r_1,\ldots,r_n)(r_1,\ldots,r_n)'$, the second term becomes $\frac{1}{nT}\sigma_{v_0}^2[\varepsilon'\mathbf{r}\varepsilon - \mathrm{E}(\varepsilon'\mathbf{r}\varepsilon)]$. Then, by Lemma A.1 and Lemma A.4 (v), we have $\frac{1}{nT}\sigma_{v_0}^2[\varepsilon'\mathbf{r}\varepsilon - \mathrm{E}(\varepsilon'\mathbf{r}\varepsilon)] \xrightarrow{p} 0$. So, we have $U_3 \xrightarrow{p} 0$.

Next, define $n\times 1$ vectors $p_{it}^l = (p_{1t},\ldots,p_{i-1,t},0,\ldots,0)'$, and $p_{it}^u = (0,\ldots,0,p_{it},\ldots,p_{nt},)'$. Then, U_9 can be written as $\frac{1}{nT}\sum_{i=1}^n(\varepsilon_i^2 - \sigma_{\varepsilon_0}^2)(r_i'\varepsilon)(\sum_{t=1}^T p_{it}^{l\prime} v_t) + \frac{1}{nT}\sum_{i=1}^n \varepsilon_i(r_i'\varepsilon)(\sum_{t=1}^T p_{it}^{u\prime} v_t)$. It can be easily seen that the first term is the average of a MD array as $(r_i'\varepsilon)(\sum_{t=1}^T p_{it}^{u\prime} v_t)$ is $\mathcal{G}_{n,i-1}$ measurable, and the second term is the average of n independent terms. Conditions of Theorem 19.7 of Davidson (1994) are easily verified and hence $U_9 \xrightarrow{p} 0$. Convergence of U_6 and U_{10} can be proved similarly as the first term of U_9.

For the 7th term, we have $U_7 = \frac{2}{nT}\sum_{i=1}^n k_i\varepsilon_i^3(r_i'\varepsilon) - \frac{2}{nT}\sigma_{\varepsilon 0}^2\sum_{i=1}^n k_i\varepsilon_i(r_i'\varepsilon) = \frac{2}{nT}\sum_{i=1}^n V_{1n,i} - \frac{2}{nT}\sigma_{\varepsilon 0}^2\sum_{i=1}^n V_{2n,i}$. For $V_{1n,i}$, we can write $\frac{2}{nT}\sum_{i=1}^n V_{1n,i} = \frac{2}{nT}\sum_{i=1}^n k_i\varepsilon_i^3(r_i'\varepsilon) = \frac{2}{nT}\sum_{i=1}^n k_i(\varepsilon_i^3 - \mu_{\varepsilon_0}^3)(r_i'\varepsilon) + \frac{2}{nT}\sum_{i=1}^n k_i\mu_{\varepsilon_0}^3(r_i'\varepsilon)$. The convergence of the second term follows immediately from Lemma A.4 (vi). For the first term, let $H_{n,i} = k_i(\varepsilon_i^3 - \mu_{\varepsilon_0}^3)(r_i'\varepsilon)$. As $(r_i'\varepsilon)$ is $\mathcal{G}_{n,i-1}$ measurable, $\mathrm{E}(H_{n,i}|\mathcal{G}_{n,i-1}) = 0$. Therefore $\{H_{n,i},\mathcal{G}_{n,i}\}$ form a MD array and $\mathrm{E}|H_{n,i}^{1+\epsilon}| \leq K_v < \infty$, for some $\epsilon > 0$. The other two conditions of Davidson's WLLN for MD arrays are satisfied. Thus, $\frac{1}{nT}\sum_{i=1}^n H_{n,i} \xrightarrow{p} 0$, leading to $\frac{2}{nT}\sum_{i=1}^n V_{1n,i} \xrightarrow{p} 0$. It is easy to see that $\frac{2}{nT}\sigma_\varepsilon^2\sum_{i=1}^n V_{2n,i}$ is the average of a MD array and its convergence follows from Davidson's WLLN for MD arrays, and therefore we have $U_7 \xrightarrow{p} 0$.

Lastly, for the 4th term, we have $U_4 = \frac{1}{nT}\sum_{i=1}^n\{(u_i - \mu_{u_i})^2 - \mathrm{E}[(u_i - \mu_{u_i})^2]\} = \frac{1}{nT}\sum_{i=1}^n[u_i^2 - \mathrm{E}(u_i^2)] - \frac{1}{nT}\sum_{i=1}^n \mu_{u_i}(u_i - \mu_{u_i})$. The convergence of the second term follows from Lemma A.4. For the first term, note that $(\sum_{t=1}^T\sum_{s=1}^T v_t' A_{i,ts} v_s)^2$ can be written as a sum of four types of terms: $H_{r,ni}, r = 1,2,3,4$. The first type is $H_{1,ni} = \sum_t\sum_s\sum_k\sum_{l\neq t,s,k} v_t' A_{i,ts} v_s v_k' A_{i,kl} v_l = \sum_l v_l'\varphi_{il}$, where $\varphi_{il} = \sum_{t\neq l}\sum_{s\neq l}\sum_{k\neq l} A_{i,kl}' v_k v_t' A_{i,ts} v_s$. By the independence between v_l and φ_{il}, we have $\mathrm{E}(v_l'\varphi_{il}) = 0$. As T is fixed, we ignore the sum over t and we have $\frac{1}{nT}\sum_{i=1}^n v_l'\varphi_{il} = \frac{1}{nT}v_l'\sum_{i=1}^n \varphi_{il} = \frac{1}{nT}v_l'\varphi_l = \frac{1}{nT}\sum_{j=1}^n v_{lj}\varphi_{lj}$. Therefore we have average of n uncorrelated terms. It is easy to verify that the conditions of WLLN for MD array of

Davidson are satisfied, and thus $\frac{1}{nT}\sum_{i=1}^n H_{1,ni} \xrightarrow{p} 0$.

The second type of terms is $H_{2,ni} = \sum_t \sum_{s\neq t} v_t' A_{i,tt} v_t v_s' A_{i,ss} v_s = \sum_t \sum_{s\neq t} u_{it}u_{is}$. For each t and s, we can write $\frac{1}{nT}\sum_{i=1}^n [u_{it}u_{is} - \mathrm{E}(u_{it})\mathrm{E}(u_{is})] = \frac{1}{nT}\sum_{i=1}^n [u_{it} - \mathrm{E}(u_{it})]\mathrm{E}(u_{is}) + \frac{1}{nT}\sum_{i=1}^n [u_{is} - \mathrm{E}(u_{is})]u_{it} \equiv \frac{1}{nT}\sum_{i=1}^n V_{1n,i} + \frac{1}{nT}\sum_{i=1}^n V_{2n,i}$. We have,

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^n V_{1n,i} = \frac{1}{nT}\sum_{i=1}^n \left[\left(v_t' A_{i,tt}^d v_t - \sigma_{v0}^2 \mathrm{tr}(A_{i,tt})\right) + v_t'(A_{i,tt}^l + A_{i,tt}^{u\prime})v_t\right]\mathrm{E}(u_{is}) \\
&= \frac{1}{nT} v_t' [\textstyle\sum_{i=1}^n A_{i,tt}^d \sigma_{v0}^2 \mathrm{tr}(A_{i,ss})] v_t - \frac{1}{nT}\sum_{i=1}^n \sigma_{v0}^4 \mathrm{tr}(A_{i,tt})\mathrm{tr}(A_{i,ss}) + \frac{1}{nT} v_t' [\sum_{i=1}^n (A_{i,tt}^l + A_{i,tt}^{u\prime})] v_t \\
&= \frac{1}{nT}[v_t' v_t^* - \mathrm{E}(v_t' v_t^*)] + \frac{1}{nT} v_t' \xi_t = \frac{1}{nT}\textstyle\sum_{j=1}^n (v_{jt}v_{jt}^* - \mathrm{E}(v_{jt}v_{jt}^*)) + \frac{1}{nT}\sum_{j=1}^n v_{jt}\xi_{jt},
\end{aligned}$$

where $v_t^* = [\sum_{i=1}^n A_{i,tt}^d \sigma_{v0}^2 \mathrm{tr}(A_{i,ss})] v_t$ and $\xi_t = [\sum_{i=1}^n (A_{i,tt}^l + A_{i,tt}^{u\prime})] v_t$. Clearly, the first term is the average of n independent terms, and the second term is the average of an MD array as ξ_{jt} is $\mathcal{G}_{n,j-1}$-measurable and $\{v_{jt}\xi_{jt}, \mathcal{G}_{n,j}\}$ form an MD array. Conditions of Theorem 19.7 of Davidson(1994) are easily verified and hence $\frac{1}{nT}\sum_{i=1}^n V_{1n,i} \xrightarrow{p} 0$. Similarly, we have $\frac{1}{nT}\sum_{i=1}^n V_{2n,i} \xrightarrow{p} 0$, and therefore $\frac{1}{nT}\sum_{i=1}^n (H_{2,ni} - \mathrm{E}H_{2,ni}) \xrightarrow{p} 0$.

The third type of terms is $H_{3,ni} = \sum_t \sum_{s\neq t} v_t' A_{i,ts} v_s v_t' A_{i,ts} v_s = \sum_t \sum_{s\neq t} (v_t' \xi_{its})^2 = \sum_t \sum_{s\neq t} v_t' \xi_{its} \xi_{its}' v_t = \sum_t \sum_{s\neq t} v_t' \mathbb{A}_{its} v_t$. For each t and s, we have $\frac{1}{nT}\sum_{i=1}^n v_t' \mathbb{A}_{its} v_t = \frac{1}{nT} v_t' \mathbb{A}_{+ts} v_t = \frac{1}{nT} v_t' v_t^* + \frac{1}{nT} v_t' \xi_t$. Therefore, similar to the proof of the second type of terms, we have $\frac{1}{nT}\sum_{i=1}^n [H_{3,ni} - \mathrm{E}H_(3,ni)] \xrightarrow{p} 0$.

The forth type of terms: $H_{4,ni} = \sum_t v_t' A_{i,tt} v_t v_t' A_{i,tt} v_t = \sum_t (v_t' v_{it}^* + v_t' \xi_{it})^2$, where $v_{it}^* = A_{i,tt}^d v_t$, and $\xi_{it} = (A_{i,tt}^l + A_{i,tt}^{u\prime})v_t$. For each t, we have $\frac{1}{nT}\sum_{i=1}^n (v_t' v_{it}^* + v_t' \xi_{it})^2 = \frac{1}{nT}\sum_{i=1}^n (v_t' v_{it}^*)^2 + \frac{1}{nT}\sum_{i=1}^n (v_t' \xi_{it})^2 + \frac{2}{nT}\sum_{i=1}^n v_t' v_{it}^* v_t' \xi_{it} \equiv \frac{1}{nT}\sum_{i=1}^n V_{1n,i} + \frac{1}{nT}\sum_{i=1}^n V_{2n,i} + \frac{1}{nT}\sum_{i=1}^n V_{3n,i}$. First, $\frac{1}{nT}\sum_{i=1}^n [V_{1n,i} - \mathrm{E}(V_{1n,i})] = \frac{1}{nT}\sum_{j=1}^n (v_j^4 - \mu_v^{(4)})a_{jj} + \frac{1}{nT}\sum_{j=1}^n \sum_{k\neq j} (v_j^2 v_k^2 - \sigma_{v0}^4) a_{kj} \xrightarrow{p} 0$. Similarly, $\frac{1}{nT}\sum_{i=1}^n [V_{3n,i} - \mathrm{E}(V_{3n,i})] \xrightarrow{p} 0$. Now, $\frac{1}{nT}\sum_{i=1}^n [V_{2n,i} - \mathrm{E}(V_{2,ni})]$ van be written as

$$\begin{aligned}
&\frac{1}{nT}\sum_{j=1}^n [v_j^2 (\textstyle\sum_{i=1}^n \xi_{i,j}^2) - \sigma_{v0}^2 \mathrm{E}(\sum_{i=1}^n \xi_{i,j}^2)] + \frac{1}{nT}\sum_{j=1}^n \sum_{k\neq j} v_j v_k (\sum_{i=1}^n \xi_{i,j}\xi_{i,k}) \\
&= \frac{1}{nT}\textstyle\sum_{j=1}^n (v_j^2 - \sigma_{v0}^2)(\sum_{i=1}^n \xi_{i,j}^2) + \frac{1}{nT}\sum_{j=1}^n [(\sum_{i=1}^n \xi_{i,j}^2) - \mathrm{E}(\sum_{i=1}^n \xi_{i,j}^2)] \\
&\quad + \frac{\sigma_{v0}^2}{nT}\textstyle\sum_{j=1}^n v_j [\sum_{k\neq j} v_k (\sum_{i=1}^n \xi_{i,j}\xi_{i,k})].
\end{aligned}$$

The first term and third term can be proved by WLLN for MD arrays as $\xi_{i,j}$ is $\mathcal{G}_{n,j-1}$ measurable and the third term is average of n uncorrelated terms. Let $a_{i,j}'$ be the j th row of $A_{i,tt}^l + A_{i,tt}^{u\prime}$. Then, $\xi_{i,j} = a_{i,j}' v_t$ and the second term becomes $\frac{1}{nT}\sum_{j=1}^n \left[v_t'(\sum_{i=1}^n a_{i,j}a_{i,j}')v_t - \sigma_{v0}^2 \mathrm{tr}(\sum_{i=1}^n a_{i,j}a_{i,j}')\right] = o_p(1)$ by Lemma A.1, A.2 and A.4. So we have $\frac{1}{nT}\sum_{i=1}^n H_{4,ni} \xrightarrow{p} 0$. Combining these results, we have $U_4 = o_p(1)$. We have proved that each of $U_r \xrightarrow{p} 0$ for $r = 1, \ldots, 15$. Therefore, $\frac{1}{nT}\sum_{i=1}^n \left[g_{\Phi i}^2 - \mathrm{E}(g_{\Phi i}^2)\right] \xrightarrow{p} 0$.

By (C.3) and (C.4), we have for the cross-product term,

$$\frac{1}{nT}\sum_{i=1}^n \left[g_{\Pi i} g_{\Phi i} - \mathrm{E}(g_{\Pi i} g_{\Phi i})\right] = \sum_{r=1}^{10} U_r, \quad \text{where} \tag{C.6}$$

$$\begin{aligned}
&U_1 = \frac{1}{nT}\sum_{i=1}^n k_i \Pi_{i+}[(\varepsilon_i^2 - \sigma_\varepsilon^2)\varepsilon_i - \mu_{\varepsilon_0}^{(3)}], && U_2 = \frac{1}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2 - \sigma_\varepsilon^2) b_i' \mathbf{v}_i, \\
&U_3 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}\varepsilon_i^2 z_{1i}, && U_4 = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i z_{1i} b_i' \mathbf{v}_i, \\
&U_5 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}[\varepsilon_i^2 (r_i'\varepsilon) + r_{ii}\mu_{\varepsilon_0}^{(3)}], && U_6 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}\varepsilon_i^2 z_{1i}, \\
&U_7 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}\varepsilon_i (u_i - \mu_{ui}), && U_8 = \frac{1}{nT}\sum_{i=1}^n b_i' \mathbf{v}_i (u_i - \mu_{ui}), \\
&U_9 = \frac{1}{nT}\sum_{i=1}^n b_i' \mathbf{v}_i \sum_{t=1}^T (q_{it}'\varepsilon)(b_i' v_t), && U_{10} = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}\varepsilon_i \sum_{t=1}^T (q_{it}'\varepsilon)
\end{aligned}$$

As the terms contained in (C.6) are similar to the terms contained (C.5), we skip the proofs.

Third, similarly, for $g_{\Psi i} = \sum_{t=1}^{T} e_{it}\Psi_{ii,t}^{*} y_{0i}^{*} + \sum_{t=1}^{T} e_{it}\xi_{it} - d_{\Psi i}$, recall $\xi_{it} = w_{it}' y_0^*$ and $y_0^* = \eta_m^* + \varepsilon + V_m^*$. Some algebra lead to

$$g_{\Psi i} = \varepsilon_i h_i + \Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2) + \varepsilon_i(w_{i+}'\varepsilon) + z_{2i} + \textstyle\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon), \tag{C.7}$$

where $h_i = a_i' V_m^* + \sum_{t=1}^{T} c_{it}' v_t$, $z_{2i} = \sum_{t=1}^{T} s_{it}' v_t$, and a_i', s_i', and c_{it}' are non-stochastic vectors. It follows that

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\left(g_{\Psi i}^2 - \mathrm{E}(g_{\Psi i}^2)\right) = \sum_{r=1}^{15} U_r, \quad \text{where} \tag{C.8}$$

$U_1 = \frac{1}{nT}\sum_{i=1}^{n}(\varepsilon_i^2 h_i^2 - \sigma_{\varepsilon 0}^2 \mathrm{E}(h_i^2))$, $U_2 = \frac{1}{nT}\sum_{i=1}^{n}\Psi_{ii+}^{*2}\{(\varepsilon_i^2 - \sigma_\varepsilon^2)^2 - \mathrm{E}[(\varepsilon_i^2 - \sigma_\varepsilon^2)^2]\}$,
$U_3 = \frac{1}{nT}\sum_{i=1}^{n}[\varepsilon_i^2 (w_{i+}'\varepsilon)^2 - \sigma_{\varepsilon 0}^4 \sum_{j=1}^{n} w_{ij}^2]$, $U_4 = \frac{1}{nT}\sum_{i=1}^{n}[z_{2i}^2 - \mathrm{E}(z_{2i}^2)]$,
$U_6 = \frac{2}{nT}\sum_{i=1}^{n}\Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2)\varepsilon_i h_i$, $U_7 = \frac{2}{nT}\sum_{i=1}^{n}\varepsilon_i^2 (w_{i+}'\varepsilon) h_i$,
$U_8 = \frac{2}{nT}\sum_{i=1}^{n}\varepsilon_i h_i z_{2i}$, $U_9 = \frac{2}{nT}\sum_{i=1}^{n}\varepsilon_i h_i \sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$,
$U_{10} = \frac{2}{nT}\sum_{i=1}^{n}\Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2)\varepsilon_i (w_{i+}'\varepsilon)$, $U_{11} = \frac{2}{nT}\sum_{i=1}^{n}\Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2)\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$,
$U_{12} = \frac{2}{nT}\sum_{i=1}^{n}\Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2) z_{2i}$, $U_{13} = \frac{2}{nT}\sum_{i=1}^{n}\varepsilon_i (w_{i+}'\varepsilon)\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$,
$U_{14} = \frac{2}{nT}\sum_{i=1}^{n}\varepsilon_i (w_{i+}'\varepsilon) z_{2i}$, $U_{15} = \frac{2}{nT}\sum_{i=1}^{n} z_{2i}\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$.
$U_5 = \frac{1}{nT}\sum_{i=1}^{n}[(\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon))^2 - \sigma_{v_0}^2 \sigma_{\varepsilon 0}^2 (b_i' b_i)\sum_{t=1}^{T}(w_{it}' w_{it})]$.

By (C.3) and (C.7), we have,

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\left(g_{\Pi i} g_{\Psi i} - \mathrm{E}(g_{\Pi i} g_{\Psi i})\right) = \sum_{r=1}^{10} U_r, \quad \text{where} \tag{C.9}$$

$U_1 = \frac{1}{nT}\sum_{i=1}^{n}\Pi_{i+}[\varepsilon_i^2 h_i - \sigma_\varepsilon^2 \mathrm{E}(h_i)]$, $U_2 = \frac{1}{nT}\sum_{i=1}^{n}\Pi_{i+}\Psi_{ii+}^*[(\varepsilon_i^2 - \sigma_\varepsilon^2)\varepsilon_i - \mu_{\varepsilon 0}^{(3)}]$,
$U_3 = \frac{1}{nT}\sum_{i=1}^{n}\Pi_{i+}\varepsilon_i^2 (w_{i+}'\varepsilon)$, $U_4 = \frac{1}{nT}\sum_{i=1}^{n}\Pi_{i+}\varepsilon_i z_{2i}$,
$U_5 = \frac{1}{nT}\sum_{i=1}^{n}\Pi_{i+}\varepsilon_i \sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$, $U_6 = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i h_i b_i' \mathbf{v}_i$,
$U_7 = \frac{1}{nT}\sum_{i=1}^{n} b_i' \mathbf{v}_i \Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2)$, $U_8 = \frac{1}{nT}\sum_{i=1}^{n} b_i' \mathbf{v}_i \varepsilon_i (w_{i+}'\varepsilon)$,
$U_9 = \frac{1}{nT}\sum_{i=1}^{n}[b_i' \mathbf{v}_i z_{2i} - \mathrm{E}(b_i' \mathbf{v}_i z_{2i})]$, $U_{10} = \frac{1}{nT}\sum_{i=1}^{n} b_i' \mathbf{v}_i \sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$.

By (C.4) and (C.7), we have,

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\left(g_{\Phi i} g_{\Psi i} - \mathrm{E}(g_{\Phi i} g_{\Psi i})\right) = \sum_{r=1}^{25} U_r, \quad \text{where} \tag{C.10}$$

$U_1 = \frac{1}{nT}\sum_{i=1}^{n} k_i (\varepsilon_i^2 - \sigma_\varepsilon^2)\varepsilon_i h_i$, $U_2 = \frac{1}{nT}\sum_{i=1}^{n} k_i \Psi_{ii+}^*\{(\varepsilon_i^2 - \sigma_\varepsilon^2)^2 - \mathrm{E}[(\varepsilon_i^2 - \sigma_\varepsilon^2)^2]\}$,
$U_3 = \frac{1}{nT}\sum_{i=1}^{n} k_i (\varepsilon_i^2 - \sigma_\varepsilon^2)\varepsilon_i (w_{i+}'\varepsilon)$, $U_4 = \frac{1}{nT}\sum_{i=1}^{n} k_i (\varepsilon_i^2 - \sigma_\varepsilon^2) z_{2i}$,
$U_5 = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i^2 h_i z_{1i}$, $U_6 = \frac{1}{nT}\sum_{i=1}^{n} k_i (\varepsilon_i^2 - \sigma_\varepsilon^2)\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$,
$U_7 = \frac{1}{nT}\sum_{i=1}^{n}\Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2)\varepsilon_i z_{1i}$, $U_8 = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i^2 (w_{i+}'\varepsilon) z_{1i}$,
$U_9 = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i z_{1i} z_{2i}$, $U_{10} = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i z_{1i}\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$,
$U_{11} = \frac{1}{nT}\sum_{i=1}^{n} h_i \varepsilon_i^2 (r_i'\varepsilon)$, $U_{12} = \frac{1}{nT}\sum_{i=1}^{n}\Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2)\varepsilon_i (r_i'\varepsilon)$,
$U_{13} = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i (r_i'\varepsilon) z_{2i}$, $U_{14} = \frac{1}{nT}\sum_{i=1}^{n}[\varepsilon_i^2 (w_{i+}'\varepsilon)(r_i'\varepsilon) - \sigma_{\varepsilon 0}^4 (w_{i+}' r_i)]$,
$U_{15} = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i (r_i'\varepsilon)\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$, $U_{16} = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i h_i (u_i - \mu_{ui})$,
$U_{17} = \frac{1}{nT}\sum_{i=1}^{n}\Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2)(u_i - \mu_{ui})$, $U_{18} = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i (w_{i+}'\varepsilon)(u_i - \mu_{ui})$,
$U_{19} = \frac{1}{nT}\sum_{i=1}^{n}[(u_i - \mu_{ui}) z_{2i} - \mathrm{E}(u_i z_{2i})]$, $U_{20} = \frac{1}{nT}\sum_{i=1}^{n}(u_i - \mu_{ui})\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)$,
$U_{21} = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i h_i \sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i' v_t)$, $U_{22} = \frac{1}{nT}\sum_{i=1}^{n}\Psi_{ii+}^*(\varepsilon_i^2 - \sigma_\varepsilon^2)\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i' v_t)$,
$U_{23} = \frac{1}{nT}\sum_{i=1}^{n}\varepsilon_i (w_{i+}'\varepsilon)\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i' v_t)$, $U_{25} = \frac{1}{nT}\sum_{i=1}^{n} z_{2i}\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i' v_t)$.
$U_{24} = \frac{1}{nT}\sum_{i=1}^{n}[\sum_{t=1}^{T}(b_i' v_t)(w_{it}'\varepsilon)\sum_{t=1}^{T}(q_{it}'\varepsilon)(b_i' v_t) - \sigma_{\varepsilon 0}^2 \sigma_{v_0}\sum_{t=1}^{T}(b_i' b_i)(w_{it}' q_{it})]$,

As V_m^* is independent of ε and v_t, and η_m^* is exogenous. The terms in (C.8)-(C.10) are similar to those in (C.5), and therefore their convergence is proved similarly. These complete the prove of convergence in the single summation part of Theorem 3.1.

To prove the convergence of the **double summation** part in Theorem 3.1, the result $\frac{1}{nT}\sum_{1=1}^n \sum_{j=1,j\neq i}^n (\widehat{\Upsilon}_{ij} - \Upsilon_{ij}) \xrightarrow{p} 0$ follows if

(a) $\frac{1}{nT}\sum_{i=1}^n \sum_{j=1,j\neq i}^n (\widehat{\Upsilon}_{ij} - \widetilde{\Upsilon}_{ij}) \xrightarrow{p} 0$, and

(b) $\frac{1}{nT}\sum_{i=1}^n \sum_{j=1,j\neq i}^n [\widetilde{\Upsilon}_{ij} - \Upsilon_{ij}] \xrightarrow{p} 0$,

where $\widetilde{\Upsilon}_{ij}$ is Υ_{ij} with $\mathrm{E}(\cdot)$ corresponding to y_0^* being removed. As each element of Υ_{ij} is a linear combination of the terms specified in Lemma 3.1. So, we only need to prove the consistency of those terms. Also, as T is fixed, the proof can be done with a fixed t and s.

Proof of (a): (i) By lemma 3.1 we have, $\mathrm{E}(g_{\boldsymbol{\pi}_r} g'_{\boldsymbol{\pi}_\nu}) = \sigma_{v0}^2 \sum_{t=1}^T (b_i' b_j) \boldsymbol{\pi}'_{r,it} \boldsymbol{\pi}_{\nu,jt}$. As σ_{v0}^2 enters linearly and $\hat{\sigma}_v^2$ is consistent, and as this term does not involve y_0^*, it suffices to prove

$$Q_0^t = \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j\neq i} [(\hat{b}_i'\hat{b}_j)\hat{\boldsymbol{\pi}}_{it}\hat{\boldsymbol{\pi}}_{jt} - (b_i'b_j)\boldsymbol{\pi}_{it}\boldsymbol{\pi}_{jt}] \xrightarrow{p} 0, \quad \text{for each } t = 1, \ldots, T.$$

Denote $\eta_t = \Pi_t \odot \Pi_t$. Let $\mathbb{B} = (B_3' B_3)^{-1}$ and $\mathbb{B}_{ij}$ be its $(i,j)th$ element. Rewrite,

$$Q_0^t = \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j\neq i} (\hat{b}_i'\hat{b}_j - b_i'b_j)\hat{\Pi}_{it}\hat{\Pi}_{jt} + \tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i} (\hat{\Pi}_{it}\hat{\Pi}_{jt} - \Pi_{it}\Pi_{jt})(b_i'b_j) \equiv Q_{0,1}^t + Q_{0,2}^t.$$

By Holder's inequality, $Q_{0,1}^t \leq \big[\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} (\hat{b}_i'\hat{b}_j - b_i'b_j)^2\big]^{\frac{1}{2}} \big[\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} (\hat{\Pi}_{it}\hat{\Pi}_{jt})^2\big]^{\frac{1}{2}}$. By Lemma A.6, $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} (\hat{\Pi}_{it}\hat{\Pi}_{jt})^2 \leq \frac{1}{n}\eta_t'\eta_t = O(1)$. By MVT, $\hat{\mathbb{B}} - \mathbb{B} = (\hat{\lambda}_3 - \lambda_{30})\dot{\mathbb{B}}_{\lambda_3^*}$, where $\dot{\mathbb{B}}_{\lambda_3^*} = \frac{d}{d\lambda_3}\mathbb{B}(\lambda_3^*) = \mathbb{B}_{\lambda_3^*}(W_3'B_3 + B_3'W_3)\mathbb{B}_{\lambda_3^*}$, with λ_3^* lying between $\hat{\lambda}_3$ and λ_{30}. Then, by Lemmas A.1 and A.2, and the consistency of $\hat{\lambda}_3$, $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} (\hat{b}_i'\hat{b}_j - b_i'b_j)^2 = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} \big[(\hat{\lambda}_3 - \lambda_{30})\dot{\mathbb{B}}_{\lambda_3^*,ij}\big]^2 \leq \frac{1}{n}(\hat{\lambda}_3 - \lambda_{30})^2 \mathrm{tr}(\dot{\mathbb{B}}_{\lambda_3^*}\dot{\mathbb{B}}'_{\lambda_3^*}) = o_p(1)$. Therefore, $Q_{0,1}^t \xrightarrow{p} 0$.

By Holder's inequality, $Q_{0,2}^t \leq \big[\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} (\hat{\Pi}_{it}\hat{\Pi}_{jt} - \Pi_{it}\Pi_{jt})^2\big]^{\frac{1}{2}} \big[\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} (b_i'b_j)^2\big]^{\frac{1}{2}}$. Applying MVT on $\hat{\Pi}_t\hat{\Pi}_t' - \Pi_t\Pi_t'$, we have $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} (\hat{\Pi}_{it}\hat{\Pi}_{jt} - \Pi_{it}\Pi_{jt})^2 = o_p(1)$ by the consistency of the estimator, and Lemmas A.1 and A.2. Next, by Assumption E(iii) and Lemma A.1, we have $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} (b_i'b_j)^2 = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} \mathbb{B}_{ij}^2 \leq \frac{1}{n}\mathrm{tr}(\mathbb{B}\mathbb{B}') = O(1)$, and thus $Q_{0,2}^t \xrightarrow{p} 0$. Therefore, $Q_0^t = Q_{0,1}^t + Q_{0,2}^t \xrightarrow{p} 0$.

(ii) By lemma 3.1, $\mathrm{E}(g_{\Phi_r i}\, g_{\Phi_\nu j}) = \sigma_{v0}^4 \sum_{t=1}^T \sum_{s=1}^T [(b_j' c^*_{ri,ts})(b_i' c^*_{\nu j,st}) + (b_i'b_j)(c^{*\prime}_{ri,ts} c^*_{\nu j,ts})] + \sigma_{v0}^2\sigma_{\varepsilon_0}^2 \sum_{t=1}^T [a_{\nu ji,t+}(b_j' c^*_{ri,+t}) + a_{rij,t+}(b_i' c^*_{\nu j,+t}) + (b_i'b_j)(a^{*\prime}_{ri,t+} a^*_{\nu j,t+})] + (\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \sum_{t=1}^T [(b_i \odot c^*_{ri,tt})'(b_j \odot c^*_{\nu j,tt})]$, $r, \nu = 1, \ldots, 6$. Therefore, we need to prove:

$$\begin{aligned}
Q_1^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j\neq i} [(\hat{b}_j'\hat{c}^*_{i,ts})(\hat{b}_i'\hat{c}^*_{j,st}) - (b_j'c^*_{i,ts})(b_i'c^*_{j,st})] \xrightarrow{p} 0,\\
Q_2^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j\neq i} [(\hat{b}_i'\hat{b}_j)(\hat{c}^{*\prime}_{i,ts}\hat{c}^*_{j,ts}) - (b_i'b_j)(c^{*\prime}_{i,ts}c^*_{j,ts})] \xrightarrow{p} 0,\\
Q_3^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j\neq i} [\hat{a}_{ji,t+}(\hat{b}_j'\hat{c}^*_{i,+t}) - a_{ji,t+}(b_j'c^*_{i,+t})] \xrightarrow{p} 0,\\
Q_4^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j\neq i} [\hat{a}_{ij,t+}(b_i'\hat{c}^*_{j,+t}) - a_{ij,t+}(b_i'c^*_{j,+t})] \xrightarrow{p} 0,\\
Q_5^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j\neq i} [(\hat{a}^{*\prime}_{i,t+}\hat{a}^*_{j,t+})(\hat{b}_i'\hat{b}_j) - (a^{*\prime}_{i,t+}a^*_{j,t+})(b_i'b_j)] \xrightarrow{p} 0,\\
Q_6^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j\neq i} [(\hat{b}_i \odot \hat{c}^*_{i,tt})'(\hat{b}_j \odot \hat{c}^*_{j,tt}) - (b_i \odot c^*_{i,tt})'(b_j \odot c^*_{j,tt})] \xrightarrow{p} 0.
\end{aligned}$$

To prove $Q_1^t \xrightarrow{p} 0$, rewrite

$$Q_1^t = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^* - b_i'c_{j,st}^*)(b_j'c_{i,ts}^*) + \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_j'\hat{c}_{i,ts}^* - b_j'c_{i,ts}^*)(\hat{b}_i'\hat{c}_{j,st}^*) \equiv Q_{1,1}^t + Q_{1,2}^t.$$

By Holder's inequality, $Q_{1,1}^t \leq (\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^* - b_i'c_{j,st}^*)^2)^{\frac{1}{2}}(\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(b_j'c_{i,ts}^*)^2)^{\frac{1}{2}}$. Let $Q_{ts} = \mathbb{B}\Phi_{ts}^*$ and $Q_{ts,ij}$ be its (i,j)th element. By MVT, we have $\hat{Q}_{st} - Q_{st} = \dot{Q}_{st}(\delta^*)(\hat{\delta} - \delta)$, where $\dot{Q}_{st}(\delta^*) = \frac{\partial}{\partial \delta'}Q_{st}(\delta^*)$, and δ^* lies between $\hat{\delta}$ and δ elementwise. Then, by Lemmas A.1 and A.2, and the consistency of $\hat{\delta}$, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^* - b_i'c_{j,st}^*)^2 &= \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{Q}_{st,ij} - Q_{st,ij})^2 \\ &= \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{\delta} - \delta)^2\dot{Q}_{st,ij}^2(\delta^*) \leq \frac{1}{n}(\hat{\delta} - \delta)^2\mathrm{tr}(\dot{Q}_{st}(\delta^*)\dot{Q}_{st}'(\delta^*)) = o_p(1).\end{aligned}$$

Next, $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(b_j'c_{i,ts}^*)^2 = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} Q_{ts,ij}^2 \leq \frac{1}{n}\mathrm{tr}(Q_{ts}Q_{ts}') = O(1)$ by Assumption E(iii) and Lemma A(?). Thus, $Q_{1,1}^t \xrightarrow{p} 0$. Similarly for the 2nd term, by Holder's inequality, $Q_{1,2}^t \leq \big(\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_j'\hat{c}_{i,ts}^* - b_j'c_{i,ts}^*)^2\big)^{\frac{1}{2}}\big(\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^*)^2\big)^{\frac{1}{2}}$. By MVT, Assumption E$(iv)$, Lemmas A.1 and A.2, and the consistency of the estimator, we have the 1st part $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_j'\hat{c}_{i,ts}^* - b_j'c_{i,ts}^*)^2 \xrightarrow{p} 0$. For the 2nd part, we have $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^*)^2 \leq \frac{1}{n}\mathrm{tr}(\hat{Q}_{st}\hat{Q}_{st}') = O(1)$ by Assumption E(iv) and Lemmas A.1 and A.2, leading to $Q_{1,2}^t \xrightarrow{p} 0$. Therefore, $Q_1^t \xrightarrow{p} 0$. The results $Q_r^t \xrightarrow{p} 0$, $r = 2, \ldots, 5$, can be proved in a similar manner.

To prove $Q_6^t \xrightarrow{p} 0$, let $Q_{tt} = B_3^{-1} \odot \Phi_{tt}^* B_3^{-1}$, q_i' be its ith row, and q_{ij} be its (i,j)th element, then we can rewrite Q_6^t as:

$$Q_6^t = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{q}_j' - q_j')q_i + \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{q}_i' - q_i')\hat{q}_j \equiv Q_{6,1}^t + Q_{6,2}^t.$$

By MVT, $\hat{Q}_{tt} - Q_{tt} = \dot{Q}_{tt}(\delta^*)(\hat{\delta} - \delta)$, where $\dot{Q}_{tt}(\delta^*) = \frac{\partial}{\partial \delta'}Q_{tt}(\delta^*)$, and δ^* lies between $\hat{\delta}$ and δ elementwise. By Lemmas A.1 and A.2, it can be easily seen that $\dot{Q}_{tt}(\delta^*)$ is uniformly bounded in either row or column sum. Let $\dot{q}_i'$ be the i th row of $\dot{Q}_{tt}(\delta^*)$. We have, $Q_{6,1}^t = \frac{1}{n}(\sum_{i=1}^n q_i')(\sum_{j\neq i}\hat{q}_j - q_j) = \frac{1}{n}(\hat{\delta} - \delta)(\sum_{i=1}^n q_i')(\sum_{j\neq i}\dot{q}_j) \leq \frac{1}{n}(\hat{\delta} - \delta)\sum_{m=1}^n \sum_{i=1}^n |q_{im}| \sum_{j=1}^n |\dot{q}_{jm}|$ $\leq \frac{1}{n}(\hat{\delta} - \delta)c_1 \sum_{m=1}^n \sum_{j=1}^n |\dot{q}_{jm}| = \frac{1}{n}(\hat{\delta} - \delta)c_1 n c_2 = o_p(1)$. The convergence of $Q_{6,2}^t$ proceeds similarly as the first term. Therefore, $Q_6^t \xrightarrow{p} 0$.

(iii) $\mathrm{E}(g_{\Psi_r,i}g_{\Psi_\nu,j}) = \sigma_{\varepsilon 0}^4(w_{rij,+}w_{\nu ji,+}) + \sigma_{v0}^2\sum_{t=1}^T(b_i'b_j)\mathrm{E}(\xi_{ri,t}^*\xi_{\nu j,t}^*)$ from Lemma 3.1, and thus we need to show:

$$\begin{aligned}Q_7^t &= \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{w}_{ij,+}\hat{w}_{ji,+} - w_{ij,+}w_{ji,+}) \xrightarrow{p} 0, \quad \text{and} \\ Q_8^t &= \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}[(\hat{b}_i'\hat{b}_j)(\hat{\xi}_{i,t}^*\hat{\xi}_{j,t}^*) - (b_i'b_j)(\xi_{i,t}^*\xi_{j,t}^*)] \xrightarrow{p} 0.\end{aligned}$$

(iv) $\mathrm{E}(g_{\Phi i}g_{\Pi' j}) = \mu_{v0}^{(3)}\sum_{t=1}^T(b_i \odot c_{i,tt}^*)'b_j\boldsymbol{\pi}_{j,t}$ from Lemma 3.1, and thus we need to show:

$$Q_9^t = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}[(\hat{b}_i \odot \hat{c}_{i,tt}^*)'\hat{b}_j\hat{\boldsymbol{\pi}}_{j,t} - (b_i \odot c_{i,tt}^*)'b_j\boldsymbol{\pi}_{j,t}] \xrightarrow{p} 0.$$

(v) $\mathrm{E}(g_{\Psi i}g_{\Pi' j}) = \sigma_{v0}^2\sum_{t=1}^T \boldsymbol{\pi}_{j,t}\mathrm{E}(\xi_{ri,t}^*)(b_i'b_j)$ from Lemma 3.1, and thus we need to show:

$$Q_{10}^t = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}[(\hat{\boldsymbol{\pi}}_{j,t}\hat{\xi}_{i,t}^*)(\hat{b}_i'\hat{b}_j) - (\boldsymbol{\pi}_{j,t}\xi_{ri,t}^*)(b_i'b_j)] \xrightarrow{p} 0.$$

(vi) Finally by Lemma 3.1, $\mathrm{E}(g_{\Phi i}g_{\Psi j}) = \sigma_{\varepsilon_0}^2\sigma_{v0}^2\sum_{t=1}^T\big[(b_i'b_j)(a_{ri,t+}'w_{\nu j,t}^*) + w_{\nu ji,+}(b_j'c_{ri,+t}^\circ)\big] +$

$\sigma^4_{\varepsilon_0}(w_{ji,+}a_{ij,++}) + \mu^{(3)}_{v0}\sum_{t=1}^T(b_i \odot c^*_{ri,tt})'b_j\mathrm{E}(\xi^*_{\nu j,t})$, and thus we need to show:

$$\begin{aligned}
Q^t_{11} &= \tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j\neq i}[(\hat{b}_i'\hat{b}_j)(\hat{a}'_{i,t+}\hat{w}^*_{j,t}) - (b_i'b_j)(a'_{i,t+}w^*_{j,t})] \xrightarrow{p} 0,\\
Q^t_{12} &= \tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j\neq i}[\hat{w}_{ji,+}(\hat{b}_j'\hat{c}^\circ_{i,+t}) - w_{ji,+}(b_j'c^\circ_{i,+t})] \xrightarrow{p} 0,\\
Q^t_{13} &= \tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j\neq i}[(\hat{w}_{ji,+}\hat{a}_{ij,++}) - (w_{ji,+}a_{ij,++})] \xrightarrow{p} 0,\\
Q^t_{14} &= \tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j\neq i}[(\hat{b}_i \odot \hat{c}^*_{i,tt})'\hat{b}_j(\hat{\xi}^*_{j,t}) - (b_i \odot c^*_{i,tt})'b_j(\xi^*_{j,t})] \xrightarrow{p} 0.
\end{aligned}$$

All the terms in (iii)-(vi) are similar to the terms in (i) and (ii), and therefore their convergence in probability to zero is proved similarly to that of the terms in (i) and (ii).

Proof of (b): The proofs for the terms not involving y^*_0 are trivial. We focus on the terms which involve y^*_0. Therefore, we need to prove:

$$\begin{aligned}
R^t_1 &= \tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j\neq i}(b_i'b_j)[\xi^*_{i,t}\xi^*_{j,t} - \mathrm{E}(\xi^*_{i,t}\xi^*_{j,t})] \xrightarrow{p} 0,\\
R^t_2 &= \tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j\neq i}\boldsymbol{\pi}_{j,t}[\xi^*_{i,t} - \mathrm{E}(\xi^*_{i,t})] \xrightarrow{p} 0,\\
R^t_3 &= \tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j\neq i}[(b_i \odot c^*_{i,tt})'b_j][\xi^*_{j,t} - \mathrm{E}(\xi^*_{j,t})] \xrightarrow{p} 0.
\end{aligned}$$

Recall: $\xi^*_{i,t} = w^{*\prime}_{it}y^*_0$ where $w^{*\prime}_{it}$ is the ith row of Ψ^*_t, and $y^*_0 = V^*_m + \eta^*_m + \varepsilon$. We have $\xi^*_{j,t} - \mathrm{E}(\xi^*_{j,t}) = w^{*\prime}_{it}(V^*_m + \varepsilon)$. The convergence of R^t_2 and R^t_3 thus follow by Assumption F(ii), Lemma A.4(vi) and Lemma A.6(ii). To show $R^t_1 \xrightarrow{p} 0$, note that $\Psi^*_t = \Psi_t K_m$, and $y^*_0 = K_m^{-1}y_0$, so we can write, $\xi^*_{i,t} = a'_{it}y_0$, where a'_{it} is the ith row of Ψ_t. Then we have, $\sum_{i=1}^n\sum_{j\neq i}(b_i'b_j)\xi^*_{i,t}\xi^*_{j,t} = y_0'[\sum_{i=1}^n(a_{it}b_i')\sum_{j\neq i}(b_ja'_{jt})]y_0 = y_0'A_ty_0$, where $A_t = \Psi_t'\mathbb{B}\Psi_t - \Psi_t'\mathtt{diag}(\mathbb{B})\Psi_t$, and $\mathbb{B} = (B_3'B_3)^{-1}$. Clearly, A_t is bounded in both row and column sums by Assuption E(iii) and Lemma A.1(i). Therefore, $R^t_1 = \frac{1}{n}[y_0'A_ty_0 - \mathrm{E}(y_0'A_ty_0)] = o_p(1)$, by Assumption F$(iii)$. These complete the prove of convergence in the double summation part of Theorem 3.1.

■

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Table 1. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP1**, $T = 3$, $m = 10$
$W_1 = W_3$**: Queen Contiguity;** W_2**: Group Interaction**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.8634(.417)	1.0514(.428)[.372]	.8819(.398)	1.0673(.411)[.376]	.8615(.394)	1.0472(.412)[.371]
	.26	.3213(.270)	.2599(.263)[.232]	.3083(.272)	.2475(.266)[.232]	.3240(.260)	.2623(.254)[.228]
	.23	.2577(.260)	.2317(.251)[.228]	.2498(.261)	.2242(.253)[.227]	.2604(.260)	.2345(.251)[.227]
	1	.9993(.053)	1.0028(.052)[.050]	.9979(.054)	1.0012(.053)[.050]	.9994(.053)	1.0028(.052)[.050]
	1	.9146(.319)	.9964(.339)[.322]	.9250(.311)	1.0070(.331)[.320]	.9113(.313)	.9922(.332)[.320]
	1	.8564(.163)	.9947(.153)[.136]	.8538(.166)	.9912(.156)[.137]	.8501(.167)	.9866(.157)[.136]
	1	.9948(.163)	.9397(.147)[.135]	1.0081(.261)	.9524(.242)[.205]	1.0038(.235)	.9489(.216)[.188]
	1	.7614(.374)	.9923(.432)[.374]	.7745(.424)	1.0027(.485)[.397]	.7740(.422)	1.0007(.486)[.390]
	.3	.3530(.055)	.2995(.050)[.043]	.3547(.055)	.3016(.050)[.043]	.3543(.055)	.3015(.051)[.043]
	.2	.1887(.052)	.1933(.053)[.049]	.1859(.050)	.1904(.051)[.049]	.1884(.053)	.1931(.054)[.049]
	.2	.1927(.039)	.1970(.040)[.037]	.1911(.039)	.1955(.040)[.037]	.1906(.039)	.1947(.039)[.036]
	.2	.1014(.182)	.0957(.184)[.169]	.0983(.183)	.0928(.188)[.165]	.1059(.174)	.0994(.177)[.165]
100	1	.8798(.269)	1.0172(.279)[.270]	.8851(.274)	1.0227(.284)[.271]	.8902(.274)	1.0275(.285)[.267]
	-.44	-.3924(.192)	-.4432(.186)[.176]	-.3972(.187)	-.4479(.182)[.176]	-.4017(.194)	-.4522(.188)[.173]
	.33	.3991(.197)	.3308(.189)[.179]	.4033(.195)	.3346(.187)[.179]	.3986(.197)	.3303(.190)[.175]
	1	1.0006(.036)	1.0001(.035)[.035]	.9996(.036)	.9991(.036)[.035]	1.0011(.036)	1.0004(.036)[.035]
	1	.9189(.224)	.9966(.237)[.230]	.9160(.217)	.9929(.229)[.231]	.9213(.220)	.9985(.232)[.230]
	1	.8425(.117)	.9940(.106)[.099]	.8489(.119)	1.0000(.109)[.100]	.8465(.118)	.9974(.107)[.100]
	1	1.0350(.117)	.9792(.105)[.100]	1.0389(.188)	.9825(.170)[.155]	1.0336(.171)	.9776(.156)[.143]
	1	.7552(.254)	.9813(.295)[.259]	.7701(.284)	.9958(.322)[.284]	.7660(.278)	.9913(.315)[.280]
	.3	.3547(.037)	.3014(.033)[.030]	.3532(.037)	.3001(.033)[.030]	.3535(.037)	.3004(.033)[.030]
	.2	.1851(.025)	.1979(.026)[.025]	.1849(.026)	.1977(.027)[.025]	.1846(.026)	.1974(.027)[.025]
	.2	.1908(.028)	.1980(.029)[.028]	.1909(.027)	.1980(.028)[.028]	.1892(.028)	.1963(.029)[.028]
	.2	.1648(.118)	.1521(.121)[.116]	.1650(.118)	.1511(.120)[.114]	.1621(.117)	.1498(.120)[.114]
200	1	.9030(.207)	1.0226(.217)[.213]	.9003(.206)	1.0193(.216)[.213]	.9007(.206)	1.0201(.218)[.213]
	-.25	-.2330(.136)	-.2578(.132)[.131]	-.2248(.139)	-.2498(.135)[.130]	-.2240(.138)	-.2493(.135)[.130]
	.30	.3411(.135)	.2996(.130)[.126]	.3400(.133)	.2989(.129)[.125]	.3419(.134)	.3005(.129)[.125]
	1	.9994(.025)	1.0007(.025)[.025]	.9989(.026)	1.0003(.026)[.025]	.9996(.025)	1.0010(.025)[.025]
	1	.9250(.160)	.9969(.171)[.167]	.9290(.156)	1.0005(.167)[.167]	.9265(.160)	.9983(.172)[.167]
	1	.8292(.086)	.9963(.078)[.072]	.8326(.086)	.9979(.077)[.073]	.8328(.087)	.9993(.078)[.072]
	1	1.0502(.085)	.9872(.075)[.072]	1.0458(.129)	.9838(.116)[.113]	1.0523(.122)	.9893(.109)[.105]
	1	.7383(.179)	.9919(.208)[.190]	.7528(.196)	1.0044(.227)[.207]	.7438(.184)	.9961(.210)[.203]
	.3	.3608(.028)	.3010(.024)[.022]	.3595(.028)	.3003(.024)[.023]	.3594(.028)	.2998(.024)[.023]
	.2	.1859(.022)	.1979(.023)[.023]	.1866(.022)	.1984(.023)[.023]	.1862(.023)	.1981(.023)[.023]
	.2	.1846(.023)	.1974(.023)[.022]	.1847(.022)	.1973(.023)[.022]	.1854(.022)	.1982(.023)[.022]
	.2	.1861(.083)	.1761(.085)[.084]	.1854(.084)	.1754(.086)[.083]	.1873(.085)	.1775(.087)[.083]
400	1	.8942(.155)	1.0108(.160)[.158]	.8917(.151)	1.0085(.157)[.158]	.8946(.153)	1.0114(.160)[.157]
	-.21	-.1986(.093)	-.2091(.090)[.088]	-.1938(.092)	-.2045(.089)[.088]	-.1969(.090)	-.2076(.087)[.088]
	.42	.4432(.095)	.4170(.091)[.088]	.4485(.094)	.4222(.090)[.088]	.4428(.091)	.4167(.087)[.088]
	1	.9995(.019)	.9995(.019)[.019]	1.0002(.019)	1.0001(.019)[.019]	1.0007(.019)	1.0006(.018)[.019]
	1	.9231(.110)	1.0016(.118)[.117]	.9166(.112)	.9947(.119)[.117]	.9204(.113)	.9987(.120)[.117]
	1	.8415(.061)	1.0001(.054)[.051]	.8394(.061)	.9982(.055)[.052]	.8388(.061)	.9976(.055)[.052]
	1	1.0532(.060)	.9946(.054)[.051]	1.0538(.094)	.9948(.085)[.082]	1.0538(.088)	.9950(.080)[.075]
	1	.7564(.122)	.9954(.140)[.133]	.7607(.143)	1.0013(.163)[.148]	.7593(.141)	.9989(.160)[.144]
	.3	.3561(.019)	.3000(.017)[.015]	.3565(.019)	.3004(.017)[.016]	.3566(.019)	.3005(.017)[.016]
	.2	.1837(.014)	.1995(.015)[.014]	.1836(.014)	.1995(.014)[.014]	.1831(.014)	.1989(.014)[.014]
	.2	.1889(.021)	.1986(.021)[.021]	.1885(.020)	.1983(.021)[.021]	.1890(.021)	.1988(.021)[.021]
	.2	.2031(.059)	.1870(.060)[.059]	.2050(.059)	.1886(.061)[.059]	.2028(.059)	.1870(.061)[.059]

Note: $\psi = (\alpha', \beta', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 2. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP1**, $T = 3$, $m = 10$
$W_1 = W_2 = W_3$: **Group Interaction**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.8959(.340)	1.0452(.355)[.310]	.9057(.335)	1.0545(.350)[.315]	.8792(.329)	1.0277(.344)[.309]
	.30	.3776(.294)	.2918(.282)[.248]	.3677(.295)	.2820(.284)[.255]	.3885(.290)	.3022(.277)[.247]
	.24	.2781(.269)	.2393(.255)[.234]	.2648(.272)	.2264(.259)[.237]	.2807(.273)	.2418(.258)[.235]
	1	.9920(.058)	.9982(.057)[.054]	.9943(.057)	1.0003(.057)[.056]	.9937(.059)	.9999(.058)[.054]
	1	.8928(.323)	.9849(.343)[.324]	.9059(.315)	1.0008(.333)[.326]	.8992(.327)	.9931(.349)[.322]
	1	.8579(.161)	1.0003(.155)[.139]	.8465(.162)	.9906(.154)[.142]	.8442(.168)	.9866(.160)[.136]
	1	.9859(.162)	.9295(.150)[.139]	1.0016(.264)	.9422(.240)[.208]	.9974(.241)	.9396(.222)[.186]
	1	.7696(.378)	1.0209(.449)[.388]	.7750(.422)	1.0248(.491)[.410]	.7778(.433)	1.0268(.503)[.407]
	.3	.3580(.056)	.3006(.053)[.045]	.3609(.056)	.3031(.051)[.047]	.3611(.057)	.3037(.053)[.044]
	.2	.1754(.101)	.1957(.096)[.090]	.1721(.104)	.1920(.098)[.099]	.1722(.107)	.1922(.102)[.092]
	.2	.2003(.111)	.1964(.100)[.096]	.2007(.113)	.1974(.102)[.104]	.2020(.116)	.1985(.106)[.097]
	.2	.1124(.206)	.0965(.201)[.180]	.1141(.209)	.0985(.204)[.180]	.1164(.204)	.1009(.199)[.176]
100	1	1.0010(.236)	1.0155(.232)[.218]	.9966(.231)	1.0079(.225)[.230]	.9981(.233)	1.0120(.229)[.232]
	-.47	-.5815(.282)	-.4764(.254)[.243]	-.5891(.285)	-.4809(.257)[.262]	-.5913(.276)	-.4857(.252)[.281]
	-.42	-.4808(.236)	-.4301(.213)[.198]	-.4890(.236)	-.4362(.213)[.211]	-.4823(.236)	-.4318(.214)[.216]
	1	.9933(.040)	.9987(.039)[.038]	.9929(.040)	.9986(.039)[.040]	.9926(.039)	.9981(.038)[.041]
	1	.9001(.227)	.9898(.244)[.236]	.9153(.224)	1.0070(.242)[.238]	.9068(.222)	.9971(.238)[.237]
	1	.8294(.121)	.9959(.111)[.102]	.8335(.120)	.9990(.111)[.103]	.8318(.124)	.9978(.113)[.105]
	1	1.0292(.118)	.9678(.104)[.100]	1.0280(.185)	.9674(.167)[.155]	1.0259(.171)	.9646(.153)[.146]
	1	.7447(.257)	1.0020(.303)[.270]	.7651(.287)	1.0231(.337)[.294]	.7612(.286)	1.0204(.333)[.290]
	.3	.3633(.039)	.3017(.035)[.032]	.3623(.040)	.3009(.036)[.033]	.3631(.041)	.3017(.037)[.035]
	.2	.1433(.121)	.1957(.108)[.109]	.1369(.127)	.1914(.113)[.122]	.1365(.122)	.1893(.111)[.131]
	.2	.2254(.120)	.1996(.105)[.108]	.2322(.125)	.2040(.109)[.119]	.2328(.122)	.2065(.108)[.128]
	.2	.1725(.178)	.1228(.176)[.178]	.1791(.186)	.1274(.184)[.180]	.1868(.171)	.1374(.172)[.183]
200	1	.8902(.157)	1.0022(.161)[.157]	.8967(.157)	1.0087(.162)[.157]	.8924(.152)	1.0048(.159)[.156]
	.14	.1524(.131)	.1426(.127)[.121]	.1493(.132)	.1394(.128)[.120]	.1535(.130)	.1435(.126)[.121]
	.35	.3631(.137)	.3517(.132)[.123]	.3586(.135)	.3472(.131)[.123]	.3576(.135)	.3464(.130)[.123]
	1	.9959(.028)	.9999(.028)[.027]	.9962(.028)	1.0002(.027)[.027]	.9951(.028)	.9990(.028)[.027]
	1	.9104(.156)	.9972(.165)[.165]	.9095(.160)	.9958(.170)[.166]	.9103(.158)	.9977(.167)[.165]
	1	.8464(.081)	.9990(.073)[.070]	.8493(.082)	1.0016(.074)[.070]	.8450(.082)	.9982(.074)[.070]
	1	1.0399(.081)	.9829(.073)[.071]	1.0414(.131)	.9838(.118)[.113]	1.0462(.119)	.9881(.107)[.105]
	1	.7680(.179)	1.0045(.207)[.189]	.7779(.196)	1.0157(.221)[.209]	.7609(.188)	.9969(.214)[.202]
	.3	.3561(.027)	.3004(.024)[.022]	.3549(.027)	.2993(.024)[.022]	.3566(.027)	.3007(.024)[.022]
	.2	.1854(.066)	.1996(.064)[.065]	.1854(.068)	.2000(.066)[.065]	.1841(.069)	.1983(.067)[.065]
	.2	.1847(.069)	.1993(.065)[.066]	.1853(.070)	.1995(.066)[.066]	.1856(.072)	.2003(.068)[.066]
	.2	.1537(.131)	.1464(.129)[.126]	.1516(.137)	.1431(.135)[.127]	.1507(.135)	.1430(.133)[.126]
400	1	.9001(.124)	1.0059(.128)[.123]	.8973(.125)	1.0030(.128)[.122]	.9042(.120)	1.0098(.125)[.121]
	.32	.3528(.101)	.3250(.096)[.096]	.3526(.102)	.3248(.097)[.096]	.3513(.104)	.3237(.099)[.095]
	-.09	-.0897(.094)	-.0849(.089)[.087]	-.0887(.093)	-.0839(.089)[.086]	-.0906(.094)	-.0857(.090)[.086]
	1	.9974(.019)	1.0001(.018)[.018]	.9970(.019)	.9995(.019)[.018]	.9978(.019)	1.0003(.018)[.018]
	1	.9265(.110)	1.0015(.117)[.118]	.9231(.113)	.9980(.120)[.117]	.9215(.112)	.9960(.119)[.117]
	1	.8621(.055)	1.0013(.050)[.049]	.8609(.056)	1.0002(.052)[.049]	.8600(.056)	.9989(.051)[.049]
	1	1.0431(.057)	.9917(.051)[.051]	1.0466(.089)	.9950(.081)[.082]	1.0439(.086)	.9927(.079)[.075]
	1	.7907(.118)	1.0066(.134)[.133]	.7887(.136)	1.0039(.153)[.146]	.7901(.137)	1.0039(.155)[.142]
	.3	.3494(.017)	.2994(.015)[.015]	.3500(.018)	.3000(.016)[.015]	.3500(.017)	.3002(.016)[.015]
	.2	.1744(.061)	.1971(.058)[.059]	.1742(.061)	.1969(.057)[.059]	.1762(.062)	.1985(.059)[.058]
	.2	.2019(.061)	.2020(.057)[.058]	.2019(.061)	.2020(.057)[.059]	.1991(.063)	.1996(.059)[.058]
	.2	.1857(.108)	.1672(.107)[.107]	.1845(.109)	.1660(.108)[.107]	.1787(.108)	.1603(.107)[.107]

Note: $\psi = (\alpha', \beta', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 3. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP1**, $T = 3$, $m = 10$
$W_1 = W_3$**: Group Interaction;** W_2**: Queen Contiguity**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.9775(.299)	1.0164(.309)[.288]	.9880(.294)	1.0257(.303)[.287]	.9727(.304)	1.0120(.314)[.287]
	-.19	-.2419(.272)	-.1893(.266)[.243]	-.2520(.272)	-.2007(.265)[.241]	-.2375(.278)	-.1860(.273)[.245]
	-.10	-.1026(.256)	-.0957(.249)[.231]	-.1100(.257)	-.1037(.250)[.230]	-.0975(.262)	-.0910(.255)[.232]
	1	.9943(.048)	.9997(.048)[.046]	.9980(.049)	1.0027(.048)[.046]	.9953(.051)	1.0010(.050)[.046]
	1	.9061(.330)	.9908(.345)[.327]	.9110(.329)	.9941(.343)[.326]	.8988(.331)	.9830(.346)[.326]
	1	.8827(.141)	.9956(.135)[.123]	.8840(.147)	.9953(.141)[.121]	.8808(.143)	.9930(.136)[.121]
	1	.9896(.158)	.9460(.146)[.136]	.9960(.256)	.9527(.240)[.205]	.9971(.230)	.9530(.214)[.189]
	1	.7841(.351)	.9654(.400)[.357]	.8110(.416)	.9896(.465)[.384]	.7989(.411)	.9800(.462)[.377]
	.3	.3438(.046)	.3017(.043)[.038]	.3430(.048)	.3014(.045)[.038]	.3444(.047)	.3030(.044)[.038]
	.2	.1874(.026)	.1970(.027)[.024]	.1870(.027)	.1966(.027)[.024]	.1866(.027)	.1960(.028)[.024]
	.2	.1862(.037)	.1962(.037)[.035]	.1880(.037)	.1980(.037)[.035]	.1871(.038)	.1970(.038)[.035]
	.2	.1272(.152)	.1226(.152)[.132]	.1220(.157)	.1170(.159)[.129]	.1256(.151)	.1210(.152)[.129]
100	1	.9101(.300)	1.0463(.311)[.289]	.9060(.290)	1.0413(.300)[.288]	.8845(.284)	1.0200(.296)[.288]
	.44	.4569(.188)	.4403(.183)[.169]	.4560(.179)	.4396(.175)[.170]	.4598(.183)	.4430(.178)[.169]
	-.08	-.1073(.187)	-.0813(.181)[.169]	-.1010(.187)	-.0757(.180)[.170]	-.0953(.189)	-.0700(.183)[.171]
	1	.9971(.034)	1.0012(.034)[.033]	.9970(.033)	1.0007(.033)[.033]	.9966(.035)	1.0010(.035)[.033]
	1	.9078(.225)	.9974(.237)[.234]	.9060(.232)	.9950(.243)[.234]	.9082(.228)	.9970(.238)[.235]
	1	.8677(.106)	.9973(.099)[.092]	.8690(.107)	.9981(.098)[.092]	.8688(.108)	.9980(.099)[.092]
	1	1.0233(.114)	.9738(.104)[.099]	1.0300(.184)	.9800(.169)[.154]	1.0236(.169)	.9740(.156)[.143]
	1	.7861(.257)	.9922(.298)[.260]	.7890(.282)	.9916(.318)[.280]	.7982(.277)	1.0050(.313)[.280]
	.3	.3476(.035)	.2999(.032)[.029]	.3480(.034)	.3008(.031)[.029]	.3476(.036)	.3000(.032)[.029]
	.2	.1869(.025)	.1970(.026)[.024]	.1860(.024)	.1957(.025)[.024]	.1874(.024)	.1970(.025)[.024]
	.2	.1911(.030)	.1968(.031)[.029]	.1910(.030)	.1972(.031)[.029]	.1934(.031)	.1990(.031)[.029]
	.2	.1484(.117)	.1454(.118)[.109]	.1540(.117)	.1509(.118)[.108]	.1573(.116)	.1540(.117)[.107]
200	1	.9001(.217)	1.0286(.224)[.213]	.8980(.211)	1.0246(.218)[.213]	.8933(.210)	1.0220(.218)[.212]
	-.40	-.3976(.133)	-.4101(.131)[.125]	-.3980(.136)	-.4100(.134)[.125]	-.3927(.132)	-.4050(.130)[.125]
	.43	.4684(.133)	.4271(.129)[.123]	.4610(.133)	.4204(.129)[.123]	.4672(.128)	.4260(.125)[.122]
	1	.9969(.025)	.9990(.025)[.025]	.9980(.026)	1.0005(.026)[.025]	.9981(.025)	1.0000(.025)[.025]
	1	.9397(.167)	.9963(.177)[.165]	.9490(.157)	1.0061(.165)[.166]	.9411(.165)	.9980(.174)[.165]
	1	.8763(.074)	1.0006(.068)[.066]	.8770(.074)	1.0007(.069)[.067]	.8731(.076)	.9980(.071)[.067]
	1	1.0339(.083)	.9871(.075)[.071]	1.0280(.125)	.9818(.116)[.112]	1.0342(.118)	.9870(.108)[.104]
	1	.7997(.178)	.9956(.201)[.184]	.8150(.197)	1.0109(.220)[.204]	.8011(.185)	.9980(.207)[.198]
	.3	.3447(.023)	.2997(.021)[.020]	.3440(.023)	.2991(.021)[.020]	.3453(.024)	.3000(.021)[.020]
	.2	.1893(.028)	.1950(.029)[.028]	.1910(.029)	.1962(.030)[.028]	.1899(.028)	.1960(.030)[.028]
	.2	.1918(.020)	.1987(.021)[.020]	.1920(.021)	.1986(.021)[.020]	.1919(.021)	.1990(.021)[.020]
	.2	.1649(.095)	.1626(.097)[.094]	.1620(.100)	.1600(.102)[.094]	.1646(.095)	.1620(.097)[.094]
400	1	.9113(.166)	1.0230(.174)[.169]	.9070(.164)	1.0182(.172)[.169]	.9083(.164)	1.0200(.173)[.168]
	.18	.1762(.092)	.1777(.089)[.086]	.1740(.091)	.1758(.088)[.086]	.1725(.092)	.1740(.089)[.086]
	-.23	-.2411(.093)	-.2256(.090)[.086]	-.2420(.091)	-.2261(.087)[.086]	-.2430(.091)	-.2280(.088)[.087]
	1	.9961(.019)	1.0001(.019)[.018]	.9970(.018)	1.0006(.018)[.018]	.9961(.019)	1.0000(.019)[.018]
	1	.9158(.108)	.9956(.116)[.117]	.9240(.110)	1.0036(.116)[.117]	.9172(.109)	.9970(.117)[.117]
	1	.8380(.059)	.9988(.053)[.051]	.8380(.059)	.9987(.053)[.051]	.8396(.062)	1.0000(.056)[.051]
	1	1.0550(.059)	.9945(.053)[.051]	1.0530(.092)	.9922(.083)[.082]	1.0535(.087)	.9930(.078)[.075]
	1	.7505(.121)	.9963(.141)[.134]	.7570(.138)	1.0027(.158)[.148]	.7544(.135)	1.0000(.155)[.145]
	.3	.3582(.019)	.3003(.016)[.016]	.3580(.019)	.3004(.016)[.016]	.3580(.020)	.3000(.017)[.016]
	.2	.1839(.021)	.1970(.022)[.021]	.1840(.021)	.1968(.022)[.021]	.1841(.021)	.1970(.022)[.021]
	.2	.1865(.013)	.1991(.014)[.014]	.1870(.013)	.1991(.014)[.014]	.1870(.013)	.2000(.014)[.014]
	.2	.1816(.079)	.1736(.080)[.077]	.1810(.079)	.1727(.081)[.076]	.1839(.080)	.1760(.082)[.076]

Note: $\psi = (\alpha', \beta', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 4. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP2**, $T = 3$, $m = 10$

$W_1 = W_3$: **Queen Contiguity;** W_2: **Group Interaction**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.8245(.541)	1.0988(.529)[.481]	.8562(.531)	1.1262(.522)[.486]	.8320(.541)	1.1044(.534)[.483]
	.26	.3172(.272)	.2533(.269)[.238]	.3027(.276)	.2393(.274)[.239]	.3194(.266)	.2550(.262)[.236]
	.23	.2532(.256)	.2302(.252)[.230]	.2453(.259)	.2221(.255)[.229]	.2568(.258)	.2336(.253)[.229]
	1	.9788(.047)	1.0008(.045)[.042]	.9776(.047)	.9995(.045)[.042]	.9770(.047)	.9989(.046)[.042]
	1	.9360(.321)	.9998(.335)[.325]	.9474(.315)	1.0108(.330)[.323]	.9301(.314)	.9931(.329)[.322]
	1	.9706(.153)	.9371(.144)[.134]	.9825(.248)	.9487(.236)[.203]	.9801(.223)	.9470(.213)[.187]
	1	.8608(.373)	1.0237(.424)[.371]	.8748(.419)	1.0366(.474)[.398]	.8732(.416)	1.0323(.469)[.392]
	.3	.3331(.042)	.2970(.040)[.035]	.3342(.041)	.2984(.039)[.036]	.3335(.041)	.2977(.040)[.035]
	.2	.2034(.076)	.1886(.075)[.067]	.1991(.072)	.1844(.072)[.067]	.2028(.076)	.1882(.075)[.068]
	.2	.2043(.061)	.1943(.059)[.055]	.2020(.061)	.1925(.059)[.055]	.2017(.061)	.1916(.058)[.055]
	.2	.0817(.194)	.0949(.192)[.175]	.0820(.191)	.0949(.191)[.171]	.0875(.183)	.0997(.182)[.170]
100	1	.9179(.371)	1.0566(.367)[.362]	.9060(.380)	1.0455(.378)[.362]	.9363(.389)	1.0747(.386)[.361]
	-.44	-.4017(.202)	-.4535(.197)[.187]	-.4042(.202)	-.4563(.197)[.187]	-.4109(.207)	-.4626(.201)[.184]
	.33	.3945(.203)	.3244(.197)[.187]	.4008(.202)	.3299(.197)[.186]	.3983(.199)	.3276(.194)[.184]
	1	.9794(.032)	.9989(.031)[.030]	.9794(.032)	.9990(.031)[.030]	.9794(.032)	.9989(.031)[.029]
	1	.9559(.226)	.9977(.234)[.229]	.9667(.221)	1.0091(.229)[.230]	.9576(.229)	.9994(.237)[.229]
	1	1.0017(.107)	.9739(.103)[.098]	1.0127(.175)	.9841(.168)[.155]	1.0059(.161)	.9777(.154)[.143]
	1	.8693(.244)	.9963(.276)[.252]	.8771(.279)	1.0056(.310)[.278]	.8736(.267)	1.0011(.298)[.272]
	.3	.3289(.026)	.2995(.025)[.023]	.3287(.026)	.2991(.026)[.024]	.3287(.026)	.2992(.025)[.024]
	.2	.1861(.044)	.1951(.044)[.042]	.1864(.044)	.1954(.045)[.042]	.1858(.045)	.1949(.046)[.042]
	.2	.2042(.046)	.1955(.045)[.044]	.2040(.046)	.1951(.045)[.044]	.2000(.047)	.1913(.046)[.044]
	.2	.1630(.126)	.1538(.128)[.120]	.1606(.125)	.1514(.126)[.119]	.1599(.126)	.1509(.128)[.119]
200	1	.9363(.293)	1.0573(.296)[.285]	.9108(.288)	1.0315(.291)[.282]	.9168(.280)	1.0375(.282)[.283]
	-.25	-.2454(.150)	-.2671(.149)[.143]	-.2357(.145)	-.2571(.144)[.141]	-.2357(.145)	-.2571(.144)[.142]
	.30	.3259(.138)	.2972(.136)[.131]	.3237(.139)	.2954(.137)[.129]	.3270(.132)	.2986(.129)[.129]
	1	.9770(.023)	.9989(.022)[.022]	.9781(.024)	.9999(.023)[.022]	.9781(.023)	.9999(.022)[.022]
	1	.9379(.162)	1.0022(.169)[.168]	.9406(.166)	1.0048(.172)[.168]	.9358(.162)	.9996(.169)[.168]
	1	1.0207(.076)	.9905(.073)[.071]	1.0138(.121)	.9837(.116)[.113]	1.0194(.113)	.9893(.108)[.105]
	1	.8613(.169)	.9963(.191)[.180]	.8788(.195)	1.0154(.217)[.202]	.8672(.185)	1.0021(.206)[.195]
	.3	.3314(.019)	.2996(.018)[.017]	.3313(.019)	.2997(.019)[.017]	.3313(.019)	.2996(.019)[.017]
	.2	.1953(.036)	.1940(.037)[.035]	.1990(.036)	.1977(.036)[.035]	.1977(.035)	.1964(.035)[.035]
	.2	.1967(.037)	.1957(.037)[.035]	.1976(.037)	.1966(.036)[.035]	.1979(.036)	.1969(.036)[.035]
	.2	.1783(.089)	.1797(.089)[.088]	.1728(.089)	.1744(.089)[.087]	.1766(.088)	.1780(.088)[.087]
400	1	.9025(.220)	1.0133(.219)[.215]	.9248(.229)	1.0352(.227)[.215]	.9128(.221)	1.0239(.218)[.215]
	-.21	-.1996(.089)	-.2097(.087)[.089]	-.2006(.090)	-.2108(.088)[.088]	-.1978(.092)	-.2082(.090)[.088]
	.42	.4336(.096)	.4152(.093)[.089]	.4352(.092)	.4167(.089)[.089]	.4380(.093)	.4194(.090)[.089]
	1	.9786(.017)	.9998(.016)[.016]	.9794(.017)	1.0006(.016)[.016]	.9786(.017)	.9998(.016)[.016]
	1	.9529(.112)	.9965(.116)[.117]	.9546(.113)	.9980(.117)[.117]	.9568(.114)	1.0005(.118)[.117]
	1	1.0225(.052)	.9921(.050)[.051]	1.0251(.090)	.9947(.086)[.082]	1.0231(.080)	.9928(.076)[.075]
	1	.8665(.116)	1.0025(.132)[.128]	.8683(.136)	1.0035(.152)[.142]	.8703(.131)	1.0056(.146)[.139]
	.3	.3315(.013)	.2997(.013)[.012]	.3309(.013)	.2992(.013)[.012]	.3316(.013)	.2999(.013)[.012]
	.2	.1897(.023)	.1993(.023)[.023]	.1886(.023)	.1981(.024)[.023]	.1887(.023)	.1981(.023)[.023]
	.2	.2027(.033)	.1988(.032)[.031]	.1998(.033)	.1959(.032)[.031]	.2009(.033)	.1970(.032)[.031]
	.2	.1992(.062)	.1882(.063)[.061]	.1957(.062)	.1852(.063)[.061]	.1978(.062)	.1874(.063)[.061]

Note: $\psi = (\alpha', \beta', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 5. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP2**, $T = 3$, $m = 10$
$W_1 = W_2 = W_3$**: Group Interaction**

n	ψ	Normal Error		Normal Mixture		Chi-Square	
		CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.8801(.415)	1.0891(.424)[.366]	.8940(.414)	1.1002(.423)[.385]	.8672(.407)	1.0750(.416)[.370]
	.30	.3669(.295)	.2779(.285)[.251]	.3571(.299)	.2685(.290)[.260]	.3779(.292)	.2886(.282)[.249]
	.24	.2724(.266)	.2346(.255)[.235]	.2602(.271)	.2227(.261)[.239]	.2767(.269)	.2390(.258)[.234]
	1	.9737(.051)	.9986(.049)[.045]	.9725(.050)	.9973(.049)[.049]	.9721(.052)	.9969(.050)[.045]
	1	.9110(.321)	.9898(.338)[.326]	.9296(.316)	1.0091(.331)[.328]	.9192(.329)	.9984(.347)[.325]
	1	.9698(.151)	.9297(.142)[.135]	.9836(.250)	.9435(.237)[.207]	.9781(.228)	.9378(.214)[.186]
	1	.8418(.371)	1.0420(.440)[.385]	.8510(.421)	1.0471(.492)[.411]	.8580(.427)	1.0573(.498)[.408]
	.3	.3417(.046)	.2987(.045)[.039]	.3435(.045)	.3009(.044)[.041]	.3426(.045)	.2999(.044)[.039]
	.2	.1875(.102)	.1919(.100)[.093]	.1805(.112)	.1850(.108)[.110]	.1844(.109)	.1887(.106)[.094]
	.2	.2065(.109)	.1928(.099)[.096]	.2098(.114)	.1960(.105)[.109]	.2086(.113)	.1948(.105)[.096]
	.2	.1008(.209)	.0964(.204)[.178]	.1058(.211)	.1014(.205)[.182]	.1049(.206)	.1001(.201)[.175]
100	1	1.0003(.259)	1.0352(.258)[.247]	1.0010(.260)	1.0330(.258)[.545]	1.0095(.262)	1.0425(.260)[.255]
	-.47	-.5729(.280)	-.4722(.255)[.245]	-.5938(.299)	-.4893(.269)[.798]	-.5972(.278)	-.4935(.253)[.262]
	-.42	-.4792(.236)	-.4321(.216)[.202]	-.4876(.237)	-.4387(.215)[.492]	-.4896(.234)	-.4412(.214)[.210]
	1	.9705(.035)	.9977(.034)[.033]	.9707(.036)	.9979(.034)[.077]	.9701(.036)	.9974(.034)[.034]
	1	.9625(.223)	.9978(.232)[.232]	.9672(.225)	1.0034(.235)[.266]	.9587(.223)	.9940(.233)[.232]
	1	1.0020(.106)	.9698(.100)[.099]	1.0045(.173)	.9723(.166)[.223]	1.0043(.157)	.9721(.150)[.143]
	1	.8488(.243)	1.0045(.282)[.258]	.8674(.270)	1.0263(.312)[.305]	.8531(.263)	1.0088(.301)[.280]
	.3	.3377(.028)	.3011(.027)[.026]	.3363(.029)	.2996(.028)[.056]	.3386(.029)	.3021(.028)[.027]
	.2	.1482(.125)	.1898(.113)[.113]	.1394(.134)	.1828(.120)[.415]	.1430(.128)	.1856(.115)[.120]
	.2	.2415(.121)	.1996(.107)[.109]	.2519(.129)	.2082(.114)[.378]	.2462(.124)	.2032(.109)[.116]
	.2	.1727(.185)	.1264(.182)[.178]	.1762(.192)	.1284(.189)[.294]	.1800(.184)	.1332(.180)[.181]
200	1	.9434(.184)	1.0206(.187)[.182]	.9371(.184)	1.0147(.187)[.182]	.9360(.180)	1.0136(.183)[.182]
	.14	.1501(.134)	.1413(.130)[.121]	.1501(.130)	.1414(.127)[.121]	.1555(.132)	.1468(.129)[.121]
	.35	.3582(.135)	.3494(.131)[.123]	.3598(.133)	.3509(.128)[.123]	.3608(.136)	.3519(.131)[.124]
	1	.9734(.024)	.9991(.023)[.023]	.9726(.024)	.9982(.023)[.023]	.9728(.024)	.9984(.023)[.023]
	1	.9387(.159)	.9956(.165)[.165]	.9393(.161)	.9961(.168)[.165]	.9482(.158)	1.0054(.164)[.165]
	1	1.0191(.078)	.9881(.074)[.071]	1.0163(.128)	.9856(.122)[.113]	1.0196(.113)	.9887(.107)[.104]
	1	.8562(.171)	.9940(.193)[.180]	.8719(.192)	1.0098(.214)[.200]	.8632(.184)	1.0011(.205)[.195]
	.3	.3335(.019)	.3009(.018)[.018]	.3338(.019)	.3013(.019)[.018]	.3333(.019)	.3007(.018)[.018]
	.2	.1831(.071)	.1970(.068)[.066]	.1817(.072)	.1960(.069)[.066]	.1791(.070)	.1933(.066)[.067]
	.2	.1982(.071)	.1987(.067)[.066]	.2002(.072)	.2004(.068)[.066]	.2024(.071)	.2026(.066)[.067]
	.2	.1574(.136)	.1445(.132)[.127]	.1584(.134)	.1452(.131)[.126]	.1605(.133)	.1477(.129)[.126]
400	1	.9062(.153)	1.0183(.155)[.152]	.9064(.157)	1.0186(.159)[.152]	.9048(.154)	1.0172(.157)[.152]
	.32	.3478(.100)	.3233(.097)[.095]	.3485(.104)	.3237(.100)[.096]	.3510(.101)	.3261(.097)[.095]
	-.09	-.0873(.094)	-.0850(.090)[.086]	-.0887(.096)	-.0863(.092)[.087]	-.0888(.094)	-.0864(.090)[.086]
	1	.9781(.017)	.9999(.016)[.016]	.9772(.016)	.9991(.015)[.016]	.9782(.016)	1.0000(.016)[.016]
	1	.9628(.113)	1.0026(.117)[.116]	.9626(.114)	1.0023(.118)[.116]	.9613(.110)	1.0010(.115)[.116]
	1	1.0209(.053)	.9929(.051)[.050]	1.0209(.088)	.9930(.085)[.082]	1.0219(.081)	.9939(.078)[.075]
	1	.8742(.117)	1.0004(.131)[.127]	.8767(.137)	1.0029(.152)[.141]	.8754(.129)	1.0013(.142)[.138]
	.3	.3294(.013)	.2998(.012)[.012]	.3299(.013)	.3004(.012)[.012]	.3296(.013)	.3001(.012)[.012]
	.2	.1790(.063)	.1954(.060)[.061]	.1774(.065)	.1944(.062)[.062]	.1788(.063)	.1957(.059)[.061]
	.2	.2124(.061)	.2008(.057)[.058]	.2136(.063)	.2013(.058)[.059]	.2126(.061)	.2004(.057)[.058]
	.2	.1796(.111)	.1619(.109)[.108]	.1858(.110)	.1674(.108)[.108]	.1810(.112)	.1625(.110)[.108]

Note: $\psi = (\alpha', \beta', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 6. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP2**, $T = 3$, $m = 10$
$W_1 = W_3$: **Group Interaction;** W_2: **Queen Contiguity**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.8718(.507)	1.0873(.499)[.459]	.8997(.513)	1.1112(.508)[.459]	.8749(.500)	1.0893(.495)[.461]
	-.16	-.1143(.272)	-.1670(.265)[.240]	-.1295(.274)	-.1817(.268)[.241]	-.1102(.269)	-.1631(.262)[.238]
	.22	.2634(.267)	.2236(.258)[.239]	.2506(.269)	.2112(.262)[.240]	.2648(.268)	.2253(.260)[.238]
	1	.9778(.046)	1.0000(.045)[.042]	.9763(.047)	.9983(.046)[.041]	.9769(.046)	.9990(.045)[.042]
	1	.9538(.328)	1.0188(.341)[.324]	.9375(.338)	1.0010(.351)[.323]	.9450(.326)	1.0094(.339)[.325]
	1	.9710(.150)	.9391(.143)[.134]	.9822(.247)	.9499(.237)[.204]	.9773(.223)	.9452(.213)[.187]
	1	.8687(.373)	1.0224(.426)[.371]	.8772(.420)	1.0294(.470)[.399]	.8882(.417)	1.0426(.473)[.397]
	.3	.3322(.040)	.2981(.039)[.035]	.3333(.041)	.2996(.040)[.035]	.3313(.040)	.2974(.039)[.035]
	.2	.1860(.051)	.1865(.051)[.047]	.1867(.053)	.1873(.053)[.047]	.1872(.053)	.1875(.053)[.047]
	.2	.2089(.075)	.1955(.070)[.067]	.2054(.076)	.1924(.071)[.067]	.2073(.073)	.1941(.069)[.068]
	.2	.1296(.156)	.1274(.158)[.134]	.1325(.154)	.1304(.155)[.130]	.1307(.153)	.1297(.154)[.129]
100	1	.9334(.377)	1.0749(.376)[.357]	.9368(.374)	1.0787(.373)[.356]	.9242(.363)	1.0663(.363)[.356]
	.49	.5288(.203)	.4753(.201)[.189]	.5280(.205)	.4742(.203)[.188]	.5320(.202)	.4785(.199)[.187]
	.17	.1845(.186)	.1698(.183)[.170]	.1886(.186)	.1737(.183)[.170]	.1832(.182)	.1685(.179)[.170]
	1	.9795(.032)	.9985(.031)[.030]	.9806(.032)	.9996(.031)[.030]	.9806(.033)	.9995(.032)[.030]
	1	.9880(.223)	1.0038(.231)[.228]	.9957(.226)	1.0119(.234)[.228]	.9815(.232)	.9971(.240)[.228]
	1	1.0032(.108)	.9772(.104)[.099]	1.0041(.172)	.9777(.165)[.154]	1.0017(.156)	.9757(.150)[.142]
	1	.8754(.246)	.9942(.275)[.251]	.8921(.266)	1.0124(.293)[.278]	.8903(.267)	1.0091(.293)[.273]
	.3	.3270(.025)	.2996(.025)[.023]	.3268(.025)	.2995(.025)[.023]	.3267(.025)	.2994(.024)[.023]
	.2	.1933(.041)	.1914(.041)[.039]	.1910(.040)	.1891(.041)[.039]	.1933(.040)	.1914(.041)[.039]
	.2	.1921(.047)	.1937(.045)[.044]	.1927(.046)	.1942(.045)[.044]	.1955(.046)	.1970(.045)[.043]
	.2	.1449(.120)	.1466(.121)[.111]	.1481(.120)	.1497(.120)[.110]	.1526(.117)	.1546(.117)[.107]
200	1	.9393(.265)	1.0328(.265)[.260]	.9416(.266)	1.0347(.268)[.260]	.9530(.261)	1.0461(.262)[.259]
	.22	.2086(.130)	.2144(.127)[.123]	.2064(.132)	.2124(.129)[.122]	.2086(.130)	.2144(.127)[.123]
	-.11	-.1147(.134)	-.1122(.130)[.121]	-.1158(.132)	-.1132(.128)[.121]	-.1186(.130)	-.1160(.126)[.121]
	1	.9769(.022)	.9996(.021)[.021]	.9769(.023)	.9995(.022)[.022]	.9779(.023)	1.0004(.022)[.021]
	1	.9327(.161)	1.0011(.168)[.168]	.9300(.162)	.9978(.169)[.168]	.9307(.166)	.9981(.173)[.167]
	1	1.0176(.077)	.9876(.074)[.071]	1.0141(.122)	.9841(.117)[.113]	1.0192(.114)	.9892(.109)[.104]
	1	.8672(.171)	1.0012(.194)[.180]	.8765(.193)	1.0112(.216)[.201]	.8646(.187)	.9970(.209)[.194]
	.3	.3315(.018)	.3002(.018)[.017]	.3312(.019)	.3000(.018)[.017]	.3308(.019)	.2997(.019)[.017]
	.2	.1859(.031)	.1952(.031)[.030]	.1866(.030)	.1960(.030)[.030]	.1855(.031)	.1947(.031)[.029]
	.2	.2035(.033)	.1984(.033)[.032]	.2027(.033)	.1976(.033)[.032]	.2023(.033)	.1972(.032)[.032]
	.2	.1665(.101)	.1608(.103)[.095]	.1707(.096)	.1648(.097)[.094]	.1648(.097)	.1592(.099)[.094]
400	1	.8668(.232)	1.0319(.234)[.234]	.8790(.243)	1.0437(.246)[.233]	.8694(.239)	1.0344(.243)[.233]
	.29	.3115(.089)	.2897(.088)[.088]	.3093(.095)	.2876(.094)[.088]	.3127(.088)	.2908(.087)[.087]
	.42	.4322(.090)	.4225(.089)[.087]	.4330(.093)	.4232(.092)[.088]	.4331(.091)	.4234(.089)[.087]
	1	.9770(.017)	.9999(.016)[.016]	.9774(.017)	1.0003(.016)[.016]	.9765(.017)	.9995(.016)[.016]
	1	.9583(.113)	1.0005(.117)[.116]	.9555(.109)	.9975(.114)[.116]	.9576(.112)	.9998(.117)[.116]
	1	1.0260(.056)	.9948(.053)[.051]	1.0246(.087)	.9935(.083)[.081]	1.0248(.081)	.9935(.078)[.075]
	1	.8609(.118)	.9989(.134)[.127]	.8665(.135)	1.0047(.151)[.142]	.8623(.133)	1.0009(.148)[.139]
	.3	.3319(.013)	.2996(.013)[.012]	.3316(.013)	.2993(.013)[.012]	.3322(.013)	.2999(.013)[.012]
	.2	.1970(.032)	.1958(.033)[.032]	.1957(.033)	.1945(.034)[.032]	.1963(.033)	.1951(.033)[.032]
	.2	.1978(.023)	.1988(.023)[.023]	.1979(.023)	.1988(.023)[.023]	.1975(.024)	.1985(.024)[.023]
	.2	.1721(.082)	.1736(.083)[.078]	.1760(.081)	.1772(.081)[.078]	.1700(.081)	.1716(.082)[.078]

Note: $\psi = (\alpha', \beta', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 7. Empirical Mean(sd) of CQMLE, M-Estimator, and FQMLE, **DGP3**, $T = 3$, $m = 10$; W_3**: Rook Contiguity**

		Normal Error			Normal Mixture			Chi-Square		
n	ψ	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE
50	1	.9539(.224)	1.0005(.238)	1.0101(.239)	.9525(.227)	.9980(.241)	1.0078(.241)	.9546(.225)	1.0017(.241)	1.0116(.242)
	1	.9715(.045)	.9979(.045)	.9993(.044)	.9726(.044)	.9988(.043)	1.0003(.043)	.9701(.045)	.9965(.044)	.9983(.044)
	1	.8896(.318)	1.0008(.339)	.9926(.339)	.8853(.333)	.9951(.354)	.9873(.355)	.8813(.321)	.9918(.342)	.9851(.341)
	1	1.0053(.076)	.9837(.074)	.9820(.071)	1.0068(.124)	.9855(.120)	.9832(.119)	1.0052(.113)	.9835(.109)	.9806(.107)
	1	.7896(.332)	.9931(.407)	1.0009(.373)	.8093(.388)	1.0091(.459)	1.0240(.436)	.8108(.376)	1.0168(.451)	1.0364(.426)
	.5	.5384(.031)	.5004(.032)	.4978(.031)	.5390(.032)	.5015(.032)	.4987(.032)	.5394(.032)	.5015(.033)	.4985(.032)
	.3	.2819(.114)	.2865(.114)	.2528(.114)	.2864(.115)	.2917(.115)	.2561(.115)	.2829(.116)	.2874(.116)	.2516(.118)
100	1	.9521(.169)	.9971(.177)	.9954(.177)	.9517(.179)	.9960(.188)	.9945(.188)	.9571(.177)	1.0019(.187)	1.0001(.186)
	1	.9760(.032)	1.0001(.031)	1.0004(.031)	.9757(.031)	.9995(.031)	.9998(.030)	.9751(.031)	.9989(.030)	.9993(.030)
	1	.9218(.217)	1.0012(.229)	1.0071(.229)	.9197(.227)	.9988(.239)	1.0047(.239)	.9203(.233)	.9997(.245)	1.0058(.245)
	1	1.0100(.054)	.9933(.052)	.9923(.051)	1.0055(.084)	.9889(.082)	.9881(.081)	1.0074(.078)	.9907(.076)	.9901(.076)
	1	.8334(.232)	.9940(.275)	.9977(.252)	.8586(.266)	1.0218(.313)	1.0245(.292)	.8485(.251)	1.0094(.293)	1.0121(.274)
	.5	.5311(.019)	.5004(.020)	.4997(.019)	.5305(.019)	.5000(.019)	.4993(.019)	.5310(.019)	.5004(.020)	.4997(.019)
	.3	.2879(.085)	.2902(.086)	.2836(.081)	.2904(.084)	.2935(.084)	.2858(.079)	.2940(.084)	.2969(.085)	.2889(.081)
200	1	.9518(.118)	1.0042(.124)	1.0096(.124)	.9504(.119)	1.0027(.125)	1.0081(.125)	.9413(.120)	.9934(.128)	.9986(.128)
	1	.9718(.023)	.9995(.022)	1.0007(.022)	.9726(.023)	1.0002(.022)	1.0014(.022)	.9722(.023)	.9999(.022)	1.0010(.022)
	1	.9439(.160)	1.0029(.169)	.9944(.169)	.9421(.158)	1.0013(.167)	.9927(.166)	.9419(.159)	1.0010(.169)	.9924(.168)
	1	1.0140(.039)	.9962(.037)	.9954(.036)	1.0130(.061)	.9951(.059)	.9943(.058)	1.0134(.056)	.9955(.055)	.9950(.054)
	1	.8264(.164)	.9947(.197)	.9982(.181)	.8380(.181)	1.0087(.213)	1.0118(.199)	.8326(.177)	1.0027(.211)	1.0034(.199)
	.5	.5320(.014)	.4996(.014)	.4990(.014)	.5320(.014)	.4996(.014)	.4991(.014)	.5330(.014)	.5006(.014)	.5001(.014)
	.3	.2947(.060)	.2981(.060)	.2833(.057)	.2929(.062)	.2962(.062)	.2815(.059)	.2921(.061)	.2958(.061)	.2814(.058)
400	1	.9300(.088)	1.0000(.093)	1.0103(.096)	.9305(.085)	1.0003(.090)	1.0092(.093)	.9295(.088)	.9997(.094)	1.0093(.094)
	1	.9727(.015)	1.0003(.014)	1.0062(.017)	.9729(.016)	1.0004(.015)	1.0058(.017)	.9719(.015)	.9995(.015)	1.0051(.016)
	1	.9345(.113)	1.0013(.120)	1.0189(.123)	.9374(.110)	1.0043(.117)	1.0206(.120)	.9372(.115)	1.0041(.122)	1.0209(.125)
	1	1.0178(.026)	.9982(.025)	1.0220(.030)	1.0161(.044)	.9966(.042)	1.0222(.040)	1.0178(.041)	.9982(.040)	1.0226(.038)
	1	.8173(.112)	1.0005(.136)	1.0560(.252)	.8268(.132)	1.0107(.156)	1.0489(.250)	.8211(.129)	1.0050(.154)	1.0517(.249)
	.5	.5349(.010)	.4998(.010)	.4931(.014)	.5343(.011)	.4992(.011)	.4932(.014)	.5351(.010)	.5000(.010)	.4936(.013)
	.3	.2956(.042)	.2991(.042)	.2858(.040)	.2943(.042)	.2978(.042)	.2842(.040)	.2936(.042)	.2970(.043)	.2834(.040)

Note: $\psi = (\alpha_0, \beta', \sigma_v^2, \phi, \rho, \lambda_3)'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 8. Empirical Mean(sd) of CQMLE, M-Estimator, and FQMLE, **DGP3**, $T = 3$, $m = 10$; W_3**: Queen Contiguity**

		Normal Error			Normal Mixture			Chi-Square		
n	ψ	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE
50	1	.9522(.223)	1.0007(.237)	1.0068(.238)	.9565(.230)	1.0050(.244)	1.0113(.244)	.9542(.226)	1.0024(.241)	1.0089(.242)
	1	.9712(.045)	.9983(.044)	.9992(.044)	.9736(.046)	1.0005(.045)	1.0015(.044)	.9730(.045)	.9998(.043)	1.0010(.043)
	1	.8877(.317)	.9999(.338)	.9910(.337)	.8888(.335)	1.0011(.359)	.9926(.357)	.8786(.316)	.9897(.338)	.9822(.338)
	1	1.0063(.076)	.9842(.073)	.9834(.071)	1.0033(.124)	.9815(.120)	.9800(.117)	1.0064(.117)	.9847(.113)	.9827(.111)
	1	.7847(.325)	.9922(.400)	.9942(.366)	.8392(.400)	1.0520(.474)	1.0579(.446)	.8146(.391)	1.0195(.458)	1.0331(.435)
	.5	.5394(.032)	.5004(.032)	.4989(.031)	.5383(.032)	.4994(.032)	.4977(.030)	.5399(.032)	.5012(.032)	.4993(.031)
	.3	.2654(.142)	.2693(.142)	.2353(.139)	.2660(.151)	.2712(.151)	.2356(.146)	.2624(.146)	.2668(.147)	.2327(.141)
100	1	.9549(.173)	1.0014(.181)	1.0002(.181)	.9500(.175)	.9963(.184)	.9952(.184)	.9511(.172)	.9978(.181)	.9967(.182)
	1	.9747(.032)	.9993(.031)	.9999(.031)	.9742(.032)	.9987(.032)	.9994(.031)	.9747(.032)	.9994(.032)	1.0001(.031)
	1	.9183(.224)	.9961(.237)	.9999(.236)	.9237(.227)	1.0018(.240)	1.0051(.239)	.9223(.228)	1.0004(.241)	1.0045(.240)
	1	1.0099(.052)	.9931(.051)	.9930(.050)	1.0091(.086)	.9923(.084)	.9925(.083)	1.0110(.080)	.9940(.078)	.9937(.075)
	1	.8304(.229)	.9919(.274)	.9963(.247)	.8490(.259)	1.0116(.302)	1.0141(.277)	.8411(.252)	1.0044(.295)	1.0101(.273)
	.5	.5309(.020)	.5000(.021)	.4993(.020)	.5309(.020)	.4999(.020)	.4993(.019)	.5308(.020)	.4998(.020)	.4990(.020)
	.3	.2828(.106)	.2851(.107)	.2890(.098)	.2834(.104)	.2854(.104)	.2897(.096)	.2814(.106)	.2840(.106)	.2875(.096)
200	1	.9428(.117)	1.0044(.123)	1.0064(.123)	.9435(.118)	1.0051(.124)	1.0071(.124)	.9376(.118)	.9992(.125)	1.0013(.125)
	1	.9679(.023)	.9992(.023)	1.0002(.022)	.9688(.023)	1.0000(.023)	1.0010(.022)	.9688(.023)	1.0001(.023)	1.0011(.023)
	1	.9541(.157)	.9977(.166)	.9962(.166)	.9497(.157)	.9928(.166)	.9914(.166)	.9507(.158)	.9937(.167)	.9923(.167)
	1	1.0135(.037)	.9956(.036)	.9948(.035)	1.0118(.061)	.9940(.060)	.9934(.059)	1.0122(.056)	.9943(.054)	.9934(.053)
	1	.8207(.156)	.9903(.189)	.9929(.176)	.8368(.181)	1.0073(.213)	1.0092(.204)	.8334(.175)	1.0042(.208)	1.0069(.193)
	.5	.5331(.014)	.5004(.014)	.4998(.014)	.5325(.014)	.4998(.014)	.4992(.014)	.5331(.014)	.5003(.014)	.4998(.014)
	.3	.2861(.073)	.2887(.074)	.2663(.072)	.2918(.075)	.2950(.076)	.2710(.074)	.2884(.073)	.2911(.074)	.2682(.071)
400	1	.9293(.086)	.9998(.091)	1.0001(.092)	.9284(.084)	.9988(.089)	.9999(.089)	.9289(.085)	.9992(.090)	1.0002(.092)
	1	.9723(.015)	1.0000(.015)	1.0018(.016)	.9725(.016)	1.0001(.015)	1.0023(.016)	.9725(.015)	1.0001(.015)	1.0022(.016)
	1	.9333(.112)	1.0001(.120)	1.0096(.120)	.9353(.110)	1.0022(.117)	1.0126(.118)	.9318(.114)	.9982(.121)	1.0080(.122)
	1	1.0179(.027)	.9983(.026)	1.0288(.027)	1.0164(.043)	.9969(.042)	1.0271(.036)	1.0165(.041)	.9971(.040)	1.0279(.035)
	1	.8129(.113)	.9955(.137)	.9548(.211)	.8230(.127)	1.0069(.151)	.9708(.212)	.8188(.126)	1.0011(.150)	.9612(.213)
	.5	.5353(.010)	.5000(.010)	.4978(.012)	.5350(.011)	.4998(.011)	.4971(.012)	.5352(.011)	.5000(.011)	.4975(.012)
	.3	.2927(.051)	.2965(.051)	.2853(.053)	.2912(.053)	.2948(.053)	.2832(.053)	.2925(.052)	.2961(.053)	.2839(.053)

Note: $\psi = (\alpha_0, \beta', \sigma_v^2, \phi, \rho, \lambda_3)'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 9. Empirical Mean(sd) of CQMLE, M-Estimator, and FQMLE, **DGP3**, $T = 3$, $m = 10$; W_3**: Group Interaction**

		Normal Error			Normal Mixture			Chi-Square		
n	ψ	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE
50	1	.9349(.226)	1.0001(.243)	1.0009(.242)	.9316(.223)	.9961(.238)	.9979(.239)	.9406(.222)	1.0064(.241)	1.0083(.242)
	1	.9658(.046)	.9982(.045)	.9994(.045)	.9665(.049)	.9986(.048)	1.0004(.047)	.9645(.049)	.9969(.047)	.9988(.046)
	1	.8877(.311)	.9950(.337)	.9997(.336)	.8887(.312)	.9953(.338)	1.0016(.338)	.8783(.321)	.9854(.347)	.9922(.348)
	1	1.0108(.078)	.9843(.074)	.9839(.072)	1.0103(.127)	.9837(.122)	.9818(.120)	1.0142(.119)	.9873(.113)	.9850(.112)
	1	.7506(.332)	.9934(.419)	.9940(.392)	.7697(.385)	1.0151(.473)	1.0252(.447)	.7697(.386)	1.0153(.469)	1.0305(.448)
	.5	.5473(.035)	.5011(.035)	.4999(.034)	.5474(.036)	.5015(.036)	.4996(.035)	.5480(.036)	.5016(.036)	.4996(.035)
	.3	.2611(.139)	.2675(.139)	.2432(.136)	.2607(.141)	.2659(.143)	.2415(.139)	.2631(.132)	.2687(.132)	.2455(.125)
100	1	.9566(.162)	.9985(.170)	.9962(.170)	.9622(.162)	1.0046(.171)	1.0022(.171)	.9659(.164)	1.0080(.174)	1.0058(.174)
	1	.9712(.032)	.9990(.031)	.9990(.030)	.9710(.032)	.9989(.031)	.9990(.031)	.9725(.031)	1.0002(.030)	1.0003(.030)
	1	.9104(.230)	.9945(.245)	1.0020(.245)	.9095(.224)	.9940(.237)	1.0014(.238)	.9105(.221)	.9940(.234)	1.0017(.234)
	1	1.0127(.055)	.9940(.053)	.9935(.052)	1.0132(.088)	.9940(.085)	.9934(.084)	1.0135(.080)	.9947(.078)	.9940(.077)
	1	.8191(.233)	.9969(.280)	.9998(.261)	.8323(.258)	1.0146(.304)	1.0173(.286)	.8183(.250)	.9952(.297)	1.0008(.283)
	.5	.5349(.020)	.5008(.021)	.5004(.020)	.5340(.020)	.4997(.021)	.4993(.020)	.5338(.021)	.5000(.021)	.4994(.021)
	.3	.2669(.113)	.2684(.115)	.2632(.109)	.2703(.113)	.2721(.114)	.2674(.108)	.2690(.111)	.2706(.113)	.2650(.107)
200	1	.9344(.130)	.9984(.136)	1.0006(.136)	.9372(.127)	1.0014(.134)	1.0035(.134)	.9342(.126)	.9984(.134)	1.0009(.134)
	1	.9708(.022)	.9997(.021)	1.0000(.021)	.9710(.023)	1.0000(.022)	1.0001(.022)	.9705(.023)	.9994(.022)	.9997(.022)
	1	.9596(.160)	1.0016(.169)	.9971(.169)	.9590(.157)	1.0011(.166)	.9964(.166)	.9588(.160)	1.0007(.169)	.9958(.169)
	1	1.0145(.037)	.9966(.036)	.9965(.035)	1.0144(.061)	.9963(.059)	.9966(.058)	1.0140(.057)	.9960(.055)	.9957(.054)
	1	.8267(.159)	.9958(.189)	.9971(.174)	.8360(.181)	1.0079(.213)	1.0068(.201)	.8333(.180)	1.0042(.211)	1.0062(.195)
	.5	.5328(.014)	.5003(.014)	.5005(.013)	.5323(.014)	.4997(.014)	.5000(.014)	.5324(.014)	.4999(.014)	.5000(.014)
	.3	.2803(.092)	.2831(.093)	.2964(.086)	.2793(.094)	.2818(.094)	.2964(.086)	.2796(.096)	.2823(.096)	.2961(.089)
400	1	.9347(.085)	.9961(.089)	1.0061(.090)	.9634(.084)	.9961(.089)	1.0011(.090)	.9534(.085)	.9991(.086)	1.0010(.084)
	1	.9720(.016)	1.0003(.016)	1.0045(.016)	.9750(.017)	1.0005(.016)	1.0005(.015)	.9753(.016)	1.0001(.016)	1.0005(.015)
	1	.9150(.111)	1.0025(.117)	1.0170(.119)	.9380(.111)	1.0015(.112)	1.0017(.111)	.9273(.110)	1.0005(.112)	.9971(.111)
	1	1.0162(.026)	.9975(.025)	1.0088(.023)	1.0169(.043)	.9974(.042)	1.0005(.041)	1.0259(.043)	.9984(.042)	.9965(.041)
	1	.8288(.114)	1.0052(.137)	1.0679(.131)	.8389(.115)	1.0032(.128)	1.0022(.109)	.8385(.115)	1.0002(.123)	1.0023(.112)
	.5	.5339(.010)	.5001(.010)	.4945(.012)	.5333(.011)	.5001(.010)	.4975(.010)	.5342(.010)	.5000(.011)	.4995(.011)
	.3	.2811(.076)	.2966(.077)	.2902(.075)	.2801(.075)	.2919(.076)	.2899(.074)	.2798(.076)	.2917(.077)	.2901(.074)

Note: $\psi = (\alpha_0, \beta', \sigma_v^2, \phi, \rho, \lambda_3)'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.