Supplementary Material for “Diagnostic Tests for Homoskedasticity in Spatial Cross-Sectional or Panel Models”

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Appendix A: Some Basic Lemmas

The proofs of the main results depend on the following lemmas. The results state explicitly that the *degree of spatial dependence* may grow with the sample size. A way to formulate this is to consider that the elements of $W_{rn}, r = 1, 2$, are of uniform order $O(h_n^{-1})$ where h_n is such that $\lim_{n\to\infty}(h_n/n) = 0$, see Lee (2004).

Lemma A.1. *(Kelejian and Prucha, 1999; Lee, 2002): Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ matrices that are uniformly bounded in both row and column sums in absolute value. Let C_n be a sequence of conformable matrices whose elements are uniformly $O(h_n^{-1})$, where $\{h_n\}$ is a sequence of constants bounded or divergent with n. Then,*

(i) the sequence $\{A_nB_n\}$ are uniformly bounded in both row and column sums,

(ii) the elements of A_n are uniformly bounded and $tr(A_n) = O(n)$, and

(iii) the elements of A_nC_n and C_nA_n are uniformly $O(h_n^{-1})$.

Lemma A.2. *(Lee, 2004, Appendix A): For W_{rn} and $B_{rn}(\lambda_r)$, $r = 1, 2$, defined for the SLR model, if $\|W_{rn}\|$ and $\|B_{rn}^{-1}\|$ at true λ_{r0} are uniformly bounded, where $\|\cdot\|$ is a matrix norm, then $\|B_{rn}^{-1}(\lambda_r)\|$ is uniformly bounded for λ_r in a neighborhood of λ_{r0}.*

Lemma A.3. *(Lee, 2004, Appendix A): Let X_n be an $n \times p$ matrix. If the elements X_n are uniformly bounded and $\lim_{n\to\infty}\frac{1}{n}X_n'X_n$ exists and is nonsingular, then $P_n = X_n(X_n'X_n)^{-1}X_n'$ and $M_n = I_n - P_n$ are uniformly bounded in both row and column sums.*

Lemma A.4. *(Lemma A.4, Yang, 2018a): Let $\{A_n\}$ be a sequence of $n \times n$ matrices that are uniformly bounded in either row or column sums. Suppose that the elements $a_{n,ij}$ of A_n are $O(h_n^{-1})$ uniformly in all i and j. Let v_n be a random n-vector of iid elements with mean zero, variance σ^2 and finite 4th moment, and b_n a constant n-vector of elements of uniform order $O(h_n^{-1/2})$. Then*

(i) $\mathrm{E}(v_n'A_nv_n) = O(\frac{n}{h_n})$, *(ii)* $\mathrm{Var}(v_n'A_nv_n) = O(\frac{n}{h_n})$,

(iii) $\mathrm{Var}(v_n'A_nv_n + b_n'v_n) = O(\frac{n}{h_n})$, *(iv)* $v_n'A_nv_n = O_p(\frac{n}{h_n})$,

(v) $v_n'A_nv_n - \mathrm{E}(v_n'A_nv_n) = O_p((\frac{n}{h_n})^{\frac{1}{2}})$, *(vi)* $v_n'A_nb_n = O_p((\frac{n}{h_n})^{\frac{1}{2}})$,

and (vii), the results (iii) and (vi) remain valid if b_n is a random n-vector independent of v_n such that $\{\mathrm{E}(b_{ni}^2)\}$ are of uniform order $O(h_n^{-1})$.

Lemma A.5. *(Lemma A.5, Yang, 2018a): Let $\{\Phi_n\}$ be a sequence of $n \times n$ matrices with row and column sums uniformly bounded, and elements of uniform order $O(h_n^{-1})$. Let*

$v_n = (v_1, \cdots, v_n)'$ be a random vector of iid elements with mean zero, variance σ_v^2, and finite $(4+2\epsilon_0)$th moment for some $\epsilon_0 > 0$. Let $b_n = \{b_{ni}\}$ be an $n \times 1$ random vector, independent of v_n, such that (i) $\{\mathrm{E}(b_{ni}^2)\}$ are of uniform order $O(h_n^{-1})$, (ii) $sup_i E|b_{ni}|^{2+\epsilon_0} < \infty$, (iii) $\frac{h_n}{n}\sum_{i=1}^n[\phi_{n,ii}(b_{ni} - \mathrm{E}b_{ni})] = o_p(1)$ where $\{\phi_{n,ii}\}$ are the diagonal elements of Φ_n, and (iv) $\frac{h_n}{n}\sum_{i=1}^n[b_{ni}^2 - \mathrm{E}(b_{ni}^2)] = o_p(1)$. Define the bilinear-quadratic form:

$$Q_n = b_n'v_n + v_n'\Phi_n v_n - \sigma_v^2 tr(\Phi_n),$$

and let $\sigma_{Q_n}^2$ be the variance of Q_n. If $lim_{n\to\infty} h_n^{1+2/\epsilon_0}/n = 0$ and $\{\frac{h_n}{n}\sigma_{Q_n}^2\}$ are bounded away from zero, then $Q_n/\sigma_{Q_n} \xrightarrow{d} N(0,1)$.[15]

Lemma A.6. *Let $\mathbf{Q}_n = (Q_{rn}, r = 1, \ldots, m)'$, where $Q_{rn} = b_{rn}'V_n + V_n'\Phi_{rn}V_n$ with V_n, b_{rn} and Φ_{rn} satisfying the conditions of Lemma A.5. Write $\Phi_{rn} = \Phi_{rn}^u + \Phi_{rn}^l + \Phi_{rn}^d$, the sum of the upper triangular, lower triangular, and diagonal matrices of Φ_{rn}. Define*

$$g_{r,ni} = v_{ni}\xi_{r,ni} + b_{r,ni}v_{ni} + (v_{ni}^2 - \sigma^2)\phi_{rn,ii}, \; r = 1, \ldots, m,$$

where $\{\xi_{r,ni}\} = \xi_{rn} = (\Phi_{rn}^{u\prime} + \Phi_{rn}^l)V_n$. Let $\mathbf{g}_{ni} = (g_{r,ni}, r = 1, \ldots, m)'$. Then, $\{\mathbf{g}_{ni}, \mathcal{F}_{n,i}\}$ form a vector martingale difference sequence with respect to the increasing σ-fields $\mathcal{F}_{n,i}$ generated by $\{b_{1n} \ldots, b_{mn}; v_{n1}, \cdots, v_{ni}\}$, such that (i) $\mathbf{Q}_n - \mathrm{E}(\mathbf{Q}_n) = \sum_{i=1}^n \mathbf{g}_{ni}$,
(ii) $\mathrm{Var}(\mathbf{Q}_n) = \sum_{i=1}^n \mathrm{E}(\mathbf{g}_{ni}\mathbf{g}_{ni}')$, and (iii) $\frac{h_n}{n}[\sum_{i=1}^n \mathbf{g}_{ni}\mathbf{g}_{ni}' - \mathrm{Var}(\mathbf{Q}_n)] = o_p(1)$.

Proof of Lemma A.6: We have for each $Q_{jn}, j = 1, \cdots, m$,

$$\begin{aligned}
Q_{rn} - \mathrm{E}(Q_{rn}) &= b_{rn}'V_n + V_n'\Phi_{rn}V_n - \sigma^2\mathrm{tr}(\Phi_{rn}) \\
&= b_{rn}'V_n + V_n'(\Phi_{rn}^u + \Phi_{rn}^l + \Phi_{rn}^d)V_n - \sigma^2\mathrm{tr}(\Phi_{rn}) \\
&= V_n'(\Phi_{rn}^{u\prime} + \Phi_{rn}^l)V_n + b_{rn}'V_n + V_n'\Phi_{rn}^dV_n - \sigma^2\mathrm{tr}(\Phi_{rn}) \\
&= V_n'\xi_n + b_{rn}'V_n + V_n'\Phi_{rn}^dV_n - \sigma^2\mathrm{tr}(\Phi_{rn}) \\
&= \textstyle\sum_{i=1}^n[v_{ni}\xi_{r,ni} + b_{r,ni}v_{ni} + (v_{ni}^2 - \sigma^2)\phi_{rn,ii}] = \sum_{i=1}^n g_{r,ni}.
\end{aligned}$$

Thus, $\mathbf{Q}_n - \mathrm{E}(\mathbf{Q}_n) = \sum_{i=1}^n \mathbf{g}_{ni}$. As $\xi_{r,ni}$ is $\mathcal{F}_{n,i-1}$ measurable, $\mathrm{E}(g_{j,ni}|\mathcal{F}_{n,i-1}) = 0$ for $j = 1, \cdots, m$. It follows that $\{\mathbf{g}_{ni}, \mathcal{F}_{n,i}\}$ form a vector MD sequence with respect to $\mathcal{F}_{n,i}$, and that $\mathrm{Var}(\mathbf{Q}_n) = \sum_{i=1}^n \mathrm{E}(\mathbf{g}_{ni}\mathbf{g}_{ni}')$, since martingale differences $\{\mathbf{g}_{ni}\}$ are uncorrelated.

It left to show *(iii)*. It is easy to show that, for $r, s = 1, \cdots, m$, **conditional** on (b_{rn}, b_{sn}),

$$\begin{aligned}
\mathrm{Cov}[(Q_{rn}, Q_{sn})|_{(b_{rn}, b_{sn})}] &= 2\sigma^4 \textstyle\sum_{i=1}^n\sum_{j=1}^n \phi_{rn,ij}\phi_{sn,ij} + \sigma^2\sum_{i=1}^n b_{r,ni}b_{s,ni} \\
&\quad + (\mu^{(4)} - 3)\textstyle\sum_{i=1}^n \phi_{rn,ii}\phi_{sn,ii} + \mu^{(3)}\sum_{i=1}^n(b_{r,ni}\phi_{sn,ii} + b_{s,ni}\phi_{rn,ii}),
\end{aligned}$$

where $\mu^{(3)} = \mathrm{E}(v_{ni}^3)$ and $\mu^{(4)} = \mathrm{E}(v_{ni}^4)$. This gives, for $r, s = 1, \cdots, m$,

[15] In a special case where $\{b_n\}$ is a sequence of constant vectors, it is assumed that the elements of b_n are uniformly bounded and are of uniform order $h_n^{-1/2}$. See Lee (2004, Appendix A).

$$\begin{aligned}
&\textstyle\sum_{i=1}^n g_{r,ni}g'_{s,ni} - \mathrm{Cov}[(Q_{rn}, Q_{sn})|_{(b_{rn},b_{sn})}] \\
&= \textstyle\sum_{i=1}^n (v_{ni}^2\xi_{r,ni}\xi_{s,ni} - 2\sigma^2 \sum_{j=1,j\neq i}^n \phi_{rn,ij}\phi_{sn,ij}) + \sum_{i=1}^n [v_{ni}^2(\xi_{r,ni}b_{s,ni} + \xi_{s,ni}b_{r,ni})] \\
&\quad + \textstyle\sum_{i=1}^n [(v_{ni}^3 - \sigma^2 v_{ni})(\xi_{rn,i}\phi_{sn,ii} + \xi_{sn,i}\phi_{rn,ii})] + \sum_{i=1}^n (v_{ni}^2 - \sigma^2) b_{r,ni}b_{s,ni} \\
&\quad + \textstyle\sum_{i=1}^n (v_{ni}^3 - \mu^{(3)})(b_{r,ni}\phi_{sn,ii} + b_{sn,i}\phi_{rn,ii}) + \sum_{i=1}^n [(v_{ni}^3 - \mu^{(3)} - 2\sigma^2(v_{ni}^2 - \sigma^2))\phi_{rn,ii}\phi_{sn,ii},
\end{aligned}$$

where each of the six terms can be shown to be the sum of one or several MD sequences. Under Assumptions 2.1-2.4 and using Lemmas A.1-A.5, the conditions for the weak law of large numbers (WLLN) for martingale difference arrays in Davidson (1994, p. 299) can be verified, leading to $\frac{h_n}{n}\{\sum_{i=1}^n g_{rn,i}g'_{sn,i} - \mathrm{Cov}[(Q_{rn}, Q_{sn})|_{(b_{rn},b_{sn})}]\} = o_p(1)$, for $r, s = 1, \cdots, m$. It follows that $\frac{h_n}{n}[\sum_{i=1}^n \mathbf{g}_{ni}\mathbf{g}'_{ni} - \mathrm{Var}(\mathbf{Q}_n|_{(b_{1n},\ldots,b_{mn})})] = o_p(1)$. The unconditional version follows from the conditions on b_{rn} given in Lemma A.5.[16] ∎

Appendix B: Proofs for the Cross-Sectional SLR Model

Proof of Theorem 2.1: To show $T^{\mathtt{r}}_{\mathtt{SLR}}|_{H_0} \xrightarrow{D} \chi_k^2$, we first prove the following results:

(a) $\frac{1}{\sqrt{n}} S^{\circ}_{\mathtt{SLR}}(\theta_0) \xrightarrow{D} N(0_k, \lim_{n\to\infty} \frac{1}{n}\Omega_n)$, where $\Omega_n = \mathrm{Var}[S^{\circ}_{\mathtt{SLR}}(\theta_0)]$;

(b) $\frac{1}{n}\sum_{i=1}^n \mathbf{g}_{ni}\mathbf{g}'_{ni} - \frac{1}{n}\Omega_n = o_p(1)$;

(c) $\frac{1}{n}\sum_{i=1}^n (\tilde{\mathbf{g}}_{ni}\tilde{\mathbf{g}}'_{ni} - \mathbf{g}_{ni}\mathbf{g}'_{ni}) = o_p(1)$;

(d) $\frac{1}{n}(\widetilde{\Sigma}_{n,\alpha\theta} - \Sigma_{n,\alpha\theta}) = o_p(1)$ and $\frac{1}{n}(\widetilde{\Sigma}_{n,\theta\theta} - \Sigma_{n,\theta\theta}) = o_p(1)$.

To prove (a), we have the score function at the null $S^{\circ}_{\mathtt{SLR}}(\theta)$, obtained from $S_{\mathtt{SLR}}(\psi)$ given in (2.4) by setting $\alpha = 0$, $h(0) = 1$, and dropping $\dot{h}(0)$ as it is a constant being canceled out in the final expression of the test statistic:

$$S^{\circ}_{\mathtt{SLR}}(\theta) = \begin{cases} \frac{1}{\sigma^2} X'_n(\lambda_2) V_n(\beta, \lambda), \\ \frac{1}{2\sigma^4} V'_n(\beta,\lambda)V_n(\beta,\lambda) - \frac{n}{2\sigma^2}, \\ \frac{1}{\sigma^2} V'_n(\beta,\lambda) B_{2n}(\lambda_2) W_{1n} Y_n - \mathrm{tr}[G_{1n}(\lambda_1)], \\ \frac{1}{\sigma^2} V'_n(\beta,\lambda) G_{2n}(\lambda_2) V_n(\beta,\lambda) - \mathrm{tr}[G_{2n}(\lambda_2)], \\ \frac{1}{2\sigma^2}\dot{h}(0)\sum_{i=1}^n \left[\left(v_{ni}^2(\beta,\lambda) - \sigma^2\right) z_{ni}\right]. \end{cases} \tag{B.1}$$

At the true θ_0, $S^{\circ}_{\mathtt{SLR}}(\theta_0)$ reduces to that given in (2.5). The first component of (2.5) is a vector of components linear in V_n, the middle three are either quadratic or linear-quadratic

[16]Details are lengthy and are available from the authors upon request. Under an additional condition that the smallest eigenvalue of $\mathrm{Var}(\mathbf{Q}_n)$ is strictly positive, the joint asymptotic normality of the LQ vector, $\mathbf{Q}_n$, can be established using Lemma A.5 and the Cramér-Wold device.

(LQ) in V_n, and the last component can easily be written as a vector of quadratic forms in V_n, i.e., $V_n'\Phi_{3+r}V_n - \mathrm{E}(V_n'\Phi_{3+r}V_n), r = 1, \ldots, k$, where $\Phi_{3+r} = \mathtt{diag}(\frac{1}{2\sigma_0^2}z_{n1,r}, \ldots, \frac{1}{2\sigma_0^2}z_{nn,r})$ and $z_{ni,r}$ is the rth heteroskedasticity variable. Under Assumptions 2.1-2.4, it is easy to show by Lemma A.1 that the elements of Π_1 and Π_2 defined below (2.5) are uniformly bounded and are of uniform order $O(h_n^{-1/2})$; and that $\Phi_r, r = 1, 2, 3$, defined below (2.5) are uniformly bounded in both row and column sums in absolute value. Obviously, the latter is also true for $\Phi_{3+r}, r = 1 \ldots, k$, just defined. Thus, the CLT for LQ form of Kelejian and Prucha (2001) or its alternative version (under homoskedastic errors) given in Lemma A.5 is applicable to give asymptotic normality for each of the components in (2.5), where the 'b_n' vector is non-stochastic (see Footnote 15). Clearly, $c'S^{\circ}_{\mathtt{SLR}}(\theta)$ is also an LQ form in V_n for any non-zero vector c, which can be shown to be asymptotically normal by Lemma A.5 and under Assumptions 2.1-2.4. Therefore, Cramér-Wold device leads to the joint asymptotic normality of $S^{\circ}_{\mathtt{SLR}}(\theta_0)$, i.e., $\frac{1}{\sqrt{n}}S^{\circ}_{\mathtt{SLR}}(\theta_0) \xrightarrow{D} N(0_k, \lim_{n\to\infty}\frac{1}{n}\Omega_n)$.

To prove (b), consider a special case of Lemma A.6 where b_{rn} are nonstochastic. The conditions of Lemma A.6 are easily verified under Assumptions 2.1-2.4, and the result follows.

To prove (c), note that the elements of $S^{\circ}_{\mathtt{SLR}}(\theta_0)$ are mixtures of linear and quadratic forms, $\Pi_n V_n$ and $V_n'\Phi_n V_n - \mathrm{tr}(\Phi_n)$, all having an MD form $\sum_{i=1}^n g_{ni}$. It suffices to show that

$$\frac{1}{n}\sum_{i=1}^n(\tilde{g}_{k,ni}\tilde{g}_{r,ni} - g_{k,ni}g_{r,ni}) = o_p(1), \quad k, r = 1, 2, \ldots, 5+k.$$

As they are similar, we pick a typical and complicate element, corresponding to λ_1 in $S^{\circ}_{\mathtt{SLR}}(\theta_0)$, $V_n'\Phi_2V_n - \mathrm{tr}(\Phi_2) = \sum_{i=1}^n g_{2,ni}$, to prove the result. Let $\mathbf{g}_{2n} = (g_{2,n1}, \ldots, g_{2,nn})'$. Write $\tilde{\mathbf{g}}_{2n} = \mathbf{g}_{2n}(\tilde{\theta}_n)$ and denote $\dot{\mathbf{g}}_{2n}(\theta) = \frac{\partial}{\partial\theta'}\mathbf{g}_{2n}(\theta)$. By the mean value theorem (MVT), we have

$$\frac{1}{n}\sum_{i=1}^n(\tilde{g}_{2,ni}^2 - g_{k,ni}^2) = \frac{1}{n}[\mathbf{g}_{2n}'(\tilde{\theta}_n)\mathbf{g}_{2n}(\tilde{\theta}_n) - \mathbf{g}_{2n}'\mathbf{g}_{2n}] = \frac{2}{n}\mathbf{g}_{2n}'(\bar{\theta}_n)\dot{\mathbf{g}}_{2n}(\bar{\theta}_n)(\tilde{\theta}_n - \theta_0),$$

where $\bar{\theta}_n$ lies elementwise between $\tilde{\theta}_n$ and θ_0. Referring to (2.6), we have

$$\mathbf{g}_{2n}(\theta) = V_n(\theta_1) \odot \xi_{2n}(\theta_1) + [V_n(\theta_1) \odot V_n(\theta_1) - \sigma^2 1_n] \odot \phi_2(\lambda),$$

where $\odot$ denotes the Hadamard product, $\theta_1 = (\beta', \lambda')'$, and $\phi_2(\lambda)$ is a vector of diagonal elements of Φ_2. Thus, $\dot{\mathbf{g}}_{2n}(\theta) = \dot{V}_n(\theta_1)\odot\xi_{2n}(\theta_1) + V_n(\theta_1)\odot\dot{\xi}_{2n}(\theta_1) + [2V_n(\theta_1)\odot\dot{V}_n(\theta_1) - J_{n,\sigma^2}]\odot\phi_2(\lambda) + [V_n(\theta_1)\odot V_n(\theta_1) - \sigma^2 1_n]\odot\dot{\phi}_2(\lambda)$, where $\dot{V}_n(\theta_1) = \frac{\partial}{\partial\theta'}V_n(\theta_1)$, $\dot{\xi}_{2n}(\theta_1) = \frac{\partial}{\partial\theta'}\xi_{2n}(\theta_1)$, and J_{n,σ^2} is an $n \times \dim(\theta)$ matrix with its σ^2 column being 1_n and the other columns being zero. As $V_n(\theta_1) = B_{2n}(\lambda_2)[B_{1n}(\lambda_1)Y_n - X_n\beta]$ from Model (2.1), we obtain

$$\dot{V}_n(\theta_1) = [-B_{2n}(\lambda_2)X_n,\ 0_n,\ -B_{2n}(\lambda_2)W_{1n}Y_n,\ -W_{2n}(B_{1n}(\lambda_1)Y_n - X_n\beta)],$$

and by $\xi_{2n}(\theta_1) = (\Phi_2^{u\prime}(\lambda) + \Phi_2^l(\lambda))V_n(\theta_1)$, we obtain $\dot{\xi}_{2n}(\theta_1)$. It is easy to see that

$$V_n(\theta_1) = V_n - \mathbb{Z}_n(\theta_1 - \theta_{10}) + (\lambda_1 - \lambda_{10})(\lambda_2 - \lambda_{20})W_{2n}W_{1n}Y_n + (\lambda_2 - \lambda_{20})W_{2n}X_n(\beta - \beta_0),$$

where $\mathbb{Z}_n = [B_{2n}X_n,\ B_{2n}W_{1n}Y_n,\ G_{2n}V_n]$. Further, $Y_n = B_{1n}^{-1}X_n\beta_0 + B_{1n}^{-1}B_{2n}^{-1}V_n$. Therefore, $\frac{2}{n}\mathbf{g}'_{2n}(\bar{\theta}_n)\dot{\mathbf{g}}_{2n}(\bar{\theta}_n)$ can be written as sums of weighted averages of $v_{ni}^r, r = 1, 2, 3, 4$, with weights depending on the elements of matrices W_{1n}, W_{2n} and $\Phi_2(\bar{\lambda}_n)$ and the elements of $(\bar{\theta}_n - \theta_0)$ appearing in the 'weights' either multiplicatively except $(\bar{\lambda}_n - \lambda_0)$ which also appears in $\Phi_2(\bar{\lambda}_n)$. Clearly, $\Phi_2(\bar{\lambda}_n) \sim \Phi_2(\lambda_0)$, where ' $\sim$' denotes asymptotic equivalence. This implies that, e.g., $\frac{1}{n}\mathrm{tr}(\Phi_2(\bar{\lambda}_n) = \frac{1}{n}\mathrm{tr}(\Phi_2(\lambda_0) + o_p(1)$, because $\bar{\theta}_n - \theta_0 = o_p(1)$ due to $\tilde{\theta}_n - \theta_0 = o_p(1)$. With these, Assumptions 2.1-2.4 and Lemmas A.1, A.2 and A.4, straightforward but very tedious process leads to $\frac{2}{n}\mathbf{g}'_{2n}(\bar{\theta}_n)\dot{\mathbf{g}}_{2n}(\bar{\theta}_n) = O_p(1)$. Hence, the result (c) follows.

To prove (d), we choose the negative Hessian matrices evaluated at the null estimate, $\tilde{\theta}_n$, to be the estimators of $\Sigma_{n,\alpha\theta}$ and $\Sigma_{n,\theta\theta}$, i.e., $\tilde{\Sigma}_{n,\alpha\theta} = \mathbb{H}^\circ_{n,\alpha\theta}(\tilde{\theta}_n)$ and $\tilde{\Sigma}_{n,\theta\theta} = \mathbb{H}^\circ_{n,\theta\theta}(\tilde{\theta}_n)$. The expressions of $\mathbb{H}^\circ_{n,\alpha\theta}(\theta)$ and $\mathbb{H}^\circ_{n,\theta\theta}(\theta)$ are given at the end of the proof of Theorem 2.1, from which $\Sigma_{n,\alpha\theta}$ and $\Sigma_{n,\theta\theta}$ are obtained. The result $\frac{1}{n}(\tilde{\Sigma}_{n,\alpha\theta} - \Sigma_{n,\alpha\theta}) = o_p(1)$ follows if:

$$(i)\ \tfrac{1}{n}\big(\mathbb{H}^\circ_{n,\alpha\theta}(\tilde{\theta}_n) - \mathbb{H}^\circ_{n,\alpha\theta}(\theta_0)\big) = o_p(1),\ \text{ and } (ii)\ \tfrac{1}{n}\big(\mathbb{H}^\circ_{n,\alpha\theta}(\theta_0) - \Sigma_{n,\alpha\theta}\big) = o_p(1);$$

and similarly, the result $\frac{1}{n}(\tilde{\Sigma}_{n,\theta\theta} - \Sigma_{n,\theta\theta}) = o_p(1)$ follows if

$$(iii)\ \tfrac{1}{n}\big(\mathbb{H}^\circ_{n,\theta\theta}(\tilde{\theta}_n) - \mathbb{H}^\circ_{n,\theta\theta}(\theta_0)\big) = o_p(1), \text{ and } (iv)\ \tfrac{1}{n}\big(\mathbb{H}^\circ_{n,\theta\theta}(\theta_0) - \Sigma_{n,\theta\theta}\big) = o_p(1).$$

The proofs of (i) and (iii) are straightforward applications of the MVT. We thus focus on the proofs (ii) and (iv) by picking up a key term, $\frac{1}{n}(\mathbb{H}^\circ_{n,\lambda_1\lambda_1} - \Sigma_{n,\lambda_1\lambda_1})$, to show the details. From the $\mathbb{H}^\circ_{n,\theta\theta}(\theta)$, we have $\mathbb{H}^\circ_{n,\lambda_1\lambda_1} = \frac{1}{\sigma^2}\|B_{2n}W_{1n}Y_n\|^2 + \mathrm{tr}(G_{1n}^2)$. It follows that

$$\tfrac{1}{n}(\mathbb{H}^\circ_{n,\lambda_1\lambda_1} - \Sigma_{n,\lambda_1\lambda_1}) = \tfrac{1}{n}[V'_nB'^{-1}_{2n}G'_{1n}G_{1n}B_{2n}^{-1}V_n - \sigma_0^2\mathrm{tr}(B'^{-1}_{2n}G'_{1n}G_{1n}B_{2n}^{-1})] + \tfrac{1}{n}\eta'_nG_{1n}B_{2n}^{-1}V_n,$$

where $\eta_n = G_{1n}X_n\beta_0$. Hence, by Lemmas A.1 and A.4, $\frac{1}{n}(\mathbb{H}^\circ_{n,\lambda_1\lambda_1} - \Sigma_{n,\lambda_1\lambda_1}) = o_p(1)$.

Now, consider the θ- and α-components of $S^\circ_{\mathtt{SLR}}(\theta)$ evaluated at the null estimate $\tilde{\theta}_n$ of θ, $S^\circ_{\mathtt{SLR},\theta}(\tilde{\theta}_n) = \sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\theta}$ and $S^\circ_{\mathtt{SLR},\alpha}(\tilde{\theta}_n) = \sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}$. We have by the mean value theorem:

$$0 = S^\circ_{\mathtt{SLR},\theta}(\tilde{\theta}_n) = S^\circ_{\mathtt{SLR},\theta}(\theta_0) + \tfrac{\partial}{\partial\theta'}S^\circ_{\mathtt{SLR},\theta}(\bar{\theta}_n)(\tilde{\theta}_n - \theta_0), \tag{B.2}$$

$$S^\circ_{\mathtt{SLR},\alpha}(\tilde{\theta}_n) = S^\circ_{\mathtt{SLR},\alpha}(\theta_0) + \tfrac{\partial}{\partial\theta'}S^\circ_{\mathtt{SLR},\alpha}(\bar{\theta}_n)(\tilde{\theta}_n - \theta_0), \tag{B.3}$$

where $\bar{\theta}_n$ lies elementwise between $\tilde{\theta}_n$ and θ_0. As $\tilde{\theta}_n \xrightarrow{p} \theta_0$, $\bar{\theta}_n \xrightarrow{p} \theta_0$. Hence, by the results in (d), $-\frac{\partial}{\partial\theta'}S^\circ_{\mathtt{SLR},\theta}(\bar{\theta}_n) = \Sigma_{n,\theta\theta} + o_p(n)$ and $-\frac{\partial}{\partial\theta'}S^\circ_{\mathtt{SLR},\alpha}(\bar{\theta}_n) = \Sigma_{\alpha\theta} + o_p(n)$. It follows from (B.2) that, $\sqrt{n}[\tilde{\theta}_n - \theta_0] = [n\Sigma^{-1}_{n,\theta\theta}]\frac{1}{\sqrt{n}}S^{\circ}_{\mathtt{SLR},\theta}(\theta_0) + o_p(1)$, and by substituting this into (B.3),

$$\tfrac{1}{\sqrt{n}}S^{\circ}_{\mathtt{SLR},\alpha}(\tilde{\theta}_n) = \tfrac{1}{\sqrt{n}}S^{\circ}_{\mathtt{SLR},\alpha}(\theta_0) - \tfrac{1}{\sqrt{n}}\Sigma_{\alpha\theta}\Sigma^{-1}_{n,\theta\theta}S^{\circ}_{\mathtt{SLR},\theta}(\theta_0) + o_p(1), \tag{B.4}$$

which is the asymptotic representation given in (2.10). Clearly, if $S^{\circ}_{\mathtt{SLR}}(\theta_0)$ has the MD decomposition (2.7), then (B.4) or (2.10) reduces to the asymptotic MD decomposition:

$$\tfrac{1}{\sqrt{n}}S^{\circ}_{\mathtt{SLR},\alpha}(\tilde{\theta}_n) = \tfrac{1}{\sqrt{n}}\textstyle\sum_{i=1}^n(\mathbf{g}_{ni,\alpha} - \Gamma_n\mathbf{g}_{ni,\theta}) + o_p(1)$$

given in (2.10). Therefore, the joint asymptotic normality of $S^{\circ}_{\mathtt{SLR}}(\theta_0)$ given in (a) and the asymptotic representation (B.4) or (2.10) show that

$$\tfrac{1}{\sqrt{n}}S^{\circ}_{\mathtt{SLR},\alpha}(\tilde{\theta}_n) \sim N(0_k, \lim_{n\to\infty}\tfrac{1}{n}\Upsilon_n), \tag{B.5}$$

where $\Upsilon_n \equiv \mathrm{Var}[S^{\circ}_{\mathtt{SLR},\alpha}(\tilde{\theta}_n)]$. Furthermore, (B.4) immediately leads to

$$\Upsilon_n = \Omega_{n,\alpha\alpha} - \Omega_{n,\alpha\theta}\Gamma'_n - \Gamma_n\Omega_{n,\theta\alpha} + \Gamma_n\Omega_{n,\theta\theta}\Gamma'_n + o(n), \tag{B.6}$$

where $\Gamma_n = \Sigma_{n,\alpha\theta}\Sigma^{-1}_{n,\theta\theta}$, and $(\Omega_{n,\theta\theta}, \Omega_{n,\theta\alpha};\ \Omega_{n,\alpha\theta}, \Omega_{n,\alpha\alpha}) = \Omega_n$. When $S^{\circ}_{\mathtt{SLR}}(\theta_0)$ has the MD decomposition (2.7) so that Ω_n has an OPMD form, then (B.6) can be written as

$$\Upsilon_n = \textstyle\sum_{i=1}^n(\mathbf{g}_{ni,\alpha} - \Gamma_n\mathbf{g}_{ni,\theta})(\mathbf{g}_{ni,\alpha} - \Gamma_n\mathbf{g}_{ni,\theta})' + o(n). \tag{B.7}$$

Based on the approximation to Υ_n in (B.6), a consistent estimator would naturally be,

$$\tilde{\Upsilon}_n = \widetilde{\Omega}_{n,\alpha\alpha} - \widetilde{\Omega}_{n,\alpha\theta}\Gamma'_n - \widetilde{\Gamma}_n\widetilde{\Omega}_{n,\theta\alpha} + \widetilde{\Gamma}_n\widetilde{\Omega}_{n,\theta\theta}\widetilde{\Gamma}'_n, \tag{B.8}$$

which reduces to a consistent OPMD estimator based on the approximation to Υ_n in (B.7),

$$\tilde{\Upsilon}_n = \textstyle\sum_{i=1}^n(\tilde{\mathbf{g}}_{ni,\alpha} - \tilde{\Gamma}_n\tilde{\mathbf{g}}_{ni,\theta})(\tilde{\mathbf{g}}_{ni,\alpha} - \tilde{\Gamma}_n\tilde{\mathbf{g}}_{ni,\theta})'. \tag{B.9}$$

With the results (b)-(d), and using (B.5) and (B.7), it is easy to show that $\frac{1}{n}(\tilde{\Upsilon}_n - \Upsilon_n) = o_p(1)$. Positive definiteness of $\frac{1}{n}\Upsilon_n$ (for large enough n) follows from the positive definiteness of $\frac{1}{n}\Sigma_{n,\theta\theta}$ and $\frac{1}{n}\Omega_n$ stated in the theorem, which can be seen by the simpler form of (B.6): $\Upsilon_n = (-\Gamma_n, I_k)\Omega_n(-\Gamma_n, I_k)' + o(n)$, completing the proof of the result for the robust test.

If V_n is normally distributed, $\Sigma_{n,,\alpha\theta} = \Omega_{n,\alpha\theta}$ and $\Sigma_{n,\theta\theta} = \Omega_{n,\theta\theta}$. Hence, Γ_n can be consistently estimated by $(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}\tilde{\mathbf{g}}'_{ni,\theta})(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\theta}\tilde{\mathbf{g}}'_{ni,\theta})^{-1}$, leading to the test $T_{\mathtt{SLR}}$ and the second part of the results in Theorem 2.1. ■

Hessian Matrices. The negative Hessian matrix, $\mathbb{H}^{\circ}_{n,\alpha\theta}(\theta) = -\frac{\partial}{\partial\theta'}S^{\circ}_{\mathtt{SLR},\alpha}(\theta)$ required for the estimation of $\Sigma_{n,\alpha\theta}$ and the proof of (d), has elements: $\frac{1}{\sigma^2}[V_n(\beta,\lambda)\odot Z_n]'X_n(\lambda_2)$,

$\frac{1}{2\sigma^4}Z_n'\texttt{diag}(V_n(\beta,\lambda)V_n'(\beta,\lambda))$, $\frac{1}{\sigma^2}Z_n'[(B_{2n}(\lambda_2)W_{1n}Y_n)\odot V_n(\beta,\lambda)]$, and $\frac{1}{\sigma^2}Z_n'[(W_{1n}B_{1n}(\lambda_1)Y_n - W_{1n}X_n\beta)\odot V_n(\beta,\lambda)]$; and the negative Hessian matrix, $\mathbb{H}^{\circ}_{n,\theta\theta}(\theta) = -\frac{\partial}{\partial\theta'}S^{\circ}_{\texttt{SLR},\theta}(\theta)$, required for the estimation of $\Sigma_{n,\theta\theta}$ and the proof of (d) above equals:

$$\begin{pmatrix} \mathbb{H}_{n,\beta\beta}, & \frac{1}{\sigma^4}X_n'(\lambda_2)V_n(\beta,\lambda), & \frac{1}{\sigma^2}X_n'(\lambda_2)B_{2n}(\lambda_2)W_{1n}Y_n, & \frac{1}{\sigma^2}X_n'A_n(\lambda_2)U_n(\beta,\lambda_1) \\ \sim, & \frac{1}{\sigma^6}\|V_n(\beta,\lambda)\|^2 - \frac{n}{2\sigma^4}, & \frac{1}{\sigma^4}V_n'(\beta,\lambda)B_{2n}(\lambda_2)W_{1n}Y_n, & \frac{1}{\sigma^4}V_n'(\beta,\lambda)G_{2n}(\lambda_2)V_n(\beta,\lambda) \\ \sim, & \sim, & \frac{1}{\sigma^2}\|B_{2n}(\lambda_2)W_{1n}Y_n\|^2 + \mathrm{tr}[G_{1n}^2(\lambda_1)], & \frac{1}{\sigma^2}Y_n'W_{1n}'A_n(\lambda_2)U_n(\beta,\lambda_1) \\ \sim, & \sim, & \sim, & \frac{1}{\sigma^2}\|G_{2n}(\lambda_2)V_n(\beta,\lambda)\|^2 \end{pmatrix}$$

where $\mathbb{H}_{n,\beta\beta} = \frac{1}{\sigma^2}X_n'(\lambda_2)X_n(\lambda_2)$, $A_n(\lambda_2) = W_{2n}'B_{2n}(\lambda_2) + B_{2n}'(\lambda_2)W_{2n}$ and $U_n(\beta,\lambda_1) = B_{1n}(\lambda_1)Y_n - X_n\beta$; $\|\cdot\|$ denotes the Euclidean norm, and recall that $\odot$ denotes the Hadamard product and that $\texttt{diag}(\cdot)$ forms a vector by the diagonal elements of a square matrix.

Proof of Theorem 2.2: Similar to the proof of Theorem 2.1. ■

Hessian Matrices. To facilitate the derivations of the Hessian matrices required for estimating $\Sigma^*_{n,\alpha\lambda}$ and $\Sigma^*_{n,\lambda\lambda}$, write the first two components of (2.17) as

$$\begin{aligned} Y_n'(\lambda)\Phi_1(\lambda)Y_n(\lambda) - \tfrac{n}{n-p}\tilde{\sigma}_n^2(\lambda)\mathrm{tr}[\Phi_1(\lambda)] &= Y_n'B_{1n}'(\lambda_1)M_n^*(\lambda_2)W_{1n}Y_n \\ &\quad -\tfrac{1}{n-p}\mathrm{tr}[G_{1n}(\lambda_1) - X_nD_n^{-1}(\lambda_2)X_n'C_n(\lambda_2)G_{1n}(\lambda_1)]Y_n'B_{1n}'(\lambda_1)M_n^*(\lambda_2)B_{1n}(\lambda_1)Y_n, \\ Y_n'(\lambda)\Phi_2(\lambda)Y_n(\lambda) - \tfrac{n}{n-p}\tilde{\sigma}_n^2(\lambda)\mathrm{tr}[\Phi_2(\lambda)] &= Y_n'B_{1n}'(\lambda_1)M_n^{**}(\lambda_2)B_{1n}(\lambda_1)Y_n \\ &\quad -\tfrac{1}{n-p}\mathrm{tr}[G_{2n}(\lambda_2) - B_{2n}'(\lambda_2)W_{2n}X_nD_n^{-1}(\lambda_2)X_n']Y_n'B_{1n}'(\lambda_1)M_n^*(\lambda_2)B_{1n}(\lambda_1)Y_n, \end{aligned}$$

where $C_n(\lambda_2) = B_{2n}'(\lambda_2)B_{2n}(\lambda_2)$, $D_n = X_n'C_n(\lambda_2)X_n$, $M_n^*(\lambda_2) = B_{2n}'(\lambda_2)M_n(\lambda_2)B_{2n}(\lambda_2)$, and $M_n^{**}(\lambda_2) = B_{2n}'(\lambda_2)M_n(\lambda_2)G_{2n}(\lambda_2)M_n(\lambda_2)B_{2n}(\lambda_2)$.

To simplify the presentation, we write $B_{rn} \equiv B_n(\lambda_r), r = 1,2$, $G_{rn} \equiv B_n(\lambda_r), r = 1,2$, $C_n \equiv C_n(\lambda_2)$, $D_n \equiv D_n(\lambda_2)$, $M_n = M_n(\lambda_2)$, $M_n^* = M_n^*(\lambda_2)$, and $M_n^{**} = M_n^{**}(\lambda_2)$. Let $\dot{C}_n$, $\dot{M}_n^*$ and $\dot{M}_n^{**}$ be, respectively, the derivatives of C_n, M_n^* and M_n^{**}, and $\check{D}_n$ the derivative of D_n^{-1}, with respect to λ_2.[17] The negative Hessian matrix, $\mathbb{H}^*_{n,\alpha\lambda}(\lambda) = -\frac{\partial}{\partial\lambda'}S^*_{\texttt{SLR},\alpha}(\lambda)$, takes the form

$$\mathbb{H}^*_{n,\alpha\lambda}(\lambda) = \begin{cases} \frac{1}{2}Z_n'\big[\frac{q_{1i}^*}{m_i(\lambda_2)} + \frac{1}{n-p}\big(Y_n'(B_{1n}M_n^*W_{1n}' + W_{1n}M_n^*B_{1n}')Y_n\big)\big]_{(n\times1)}, \\ \frac{1}{2}Z_n'\big[q_{2i}^* - \frac{1}{n-p}\big(Y_n'B_{1n}'\dot{M}_n^*B_{1n}Y_n\big)\big]_{(n\times1)}, \end{cases}$$

where $q_1^* = -2\widetilde{V}_n(\lambda)\odot(B_{2n}W_{1n}Y_n)$, $q_2^* = \big[-\frac{2}{m_i(\lambda_2)^2}w_{1i}\tilde{v}_{ni}^2(\lambda) + \frac{1}{m_i(\lambda_2)}w_{2i}\tilde{v}_{ni}(\lambda)\big]_{(n\times1)}$, $w_1 = (M_n\odot\varphi)\iota_n$, $\varphi = W_{2n}X_nD_n^{-1}X_n'B_{2n}' + B_{2n}X_nD_n^{-1}X_n'W_{2n} - B_{2n}X_n\check{D}_nX_n'B_{2n}'$, $w_2 = -2\widetilde{V}_n(\lambda)\odot(W_{2n}B_{1n}Y_n - W_{2n}X_n\beta)$, and $\{\tilde{v}_{ni}(\lambda)\} = \widetilde{V}_n(\lambda) = V_n(\tilde{\beta}_n(\lambda),\lambda)$.

[17] We have $\dot{C}_n = -(B_{2n}'W_{2n} + W_{2n}'B_{2n})$, $\check{D}_n = -D_n^{-1}X_n\dot{C}_nX_n'D_n^{-1}$, $\dot{M}_n^* = \dot{C}_n - \dot{C}_nX_nD_n^{-1}X_n'C_n + C_nX_nD_n^{-1}X_n'\dot{C}_n + C_nX_n\check{D}_nX_n'C_n$, and $\dot{M}_n^{**}$ can easily be expressed in terms of $\dot{C}_n$, $\check{D}_n$, and $\dot{M}_n^*$.

The negative Hessian matrix, $\mathbb{H}^*_{n\lambda\lambda}(\lambda) = -\frac{\partial}{\partial\lambda'} S^*_{\texttt{SLR},\lambda}(\lambda)$ has the elements:

$$
\begin{aligned}
\mathbb{H}^*_{n\lambda_1\lambda_1} &= Y_n' W_{1n}' M_n^* W_{1n} Y_n + \tfrac{1}{n-p}\mathrm{tr}(G_{1n}^2 - X_n D_n^{-1} X_n' C_n G_{1n}^2) Y_n' B_{1n}' M_n^* B_{1n} Y_n \\
&\quad -\tfrac{1}{n-p}\mathrm{tr}(G_{1n} - X_n D_n^{-1} X_n' C_n G_{1n}) Y_n'(B_{1n}' M_n^* W_{1n} + W_{1n}' M_n^* B_{1n}) Y_n, \\
\mathbb{H}^*_{n\lambda_1\lambda_2} &= -Y_n' B_{1n}' \dot{M}_n^* W_{1n} Y_n + \tfrac{1}{n-p}\mathrm{tr}(G_{1n} - X_n D_n^{-1} X_n' C_n G_{1n}) Y_n' B_{1n}' \dot{M}_n^* B_{1n} Y_n \\
&\quad -\tfrac{1}{n-p}\mathrm{tr}(X_n \check{D}_n X_n' C_n G_{1n} + X_n D_n^{-1} X_n' \dot{C}_n G_{1n}) Y_n' B_{1n}' M_n^* B_{1n} Y_n, \\
\mathbb{H}^*_{n\lambda_2\lambda_1} &= -Y_n'(B_{1n}' M_n^{**} W_{1n} + W_{1n}' M_n^{**} B_{1n}) Y_n \\
&\quad +\tfrac{1}{n-p}[\mathrm{tr}(G_{2n} - B_{2n}' W_{2n} X_n D_n^{-1} X_n') Y_n'(B_{1n}' M_n^* W_{1n} + W_{1n}' M_n^* B_{1n}) Y_n, \\
\mathbb{H}^*_{n\lambda_2\lambda_2} &= Y_n' B_{1n}' \dot{M}_n^{**} B_{1n} Y_n - \tfrac{1}{n-p}\mathrm{tr}(G_{2n} - W_{2n}' B_{2n} X_n D_n^{-1} X_n') Y_n' B_{1n}' \dot{M}_n^* B_{1n} Y_n \\
&\quad -\tfrac{1}{n-p}\mathrm{tr}(G_{2n}^2 + W_{2n}' W_{2n} X_n D_n^{-1} X_n' - W_{2n}' B_{2n} X_n \check{D}_n X_n') Y_n' B_{1n}' M_n^* B_{1n} Y_n.
\end{aligned}
$$

Appendix C: Proofs for the Panel FE-SPD Model

Proof of Theorem 3.1: To show $T_{\texttt{SPD}}|_{H_0} \xrightarrow{D} \chi^2_k$ when the original errors $\{v_{it}\}$ are iid normal, with the help of Lemmas A.1-A.6, using the fact that the elements $\{v_j^*\}$ of $\mathbf{V}_N$ are totally independent (iid normal), and referring to the increasing σ -fields $\mathcal{F}_{N,j}$ generated by $(v_1^*, \cdots, v_j^*)$, one can easily show, in the same way as the proof of Theorem 2.1, the following:

(a) $\frac{1}{\sqrt{N}} S^\circ_{\texttt{SPD},\alpha}(\tilde{\theta}_N) \xrightarrow{D} N(0_k, \lim_{N\to\infty} \frac{1}{N}\Upsilon_N)$, where $\Upsilon_N = \mathrm{Var}[S^\circ_{\texttt{SPD},\alpha}(\tilde{\theta}_N)]$.

(b) $\frac{1}{N}\sum_{j=1}^N \mathbf{g}_{Nj}(\theta_0)\mathbf{g}'_{Nj}(\theta_0) - \frac{1}{N}\mathrm{Var}[S^\circ_{\texttt{SPD}}(\theta_0)] = o_p(1)$;

(c) $\frac{1}{N}\sum_{j=1}^N [\tilde{\mathbf{g}}_{Nj}\tilde{\mathbf{g}}'_{Nj} - \mathbf{g}_{Nj}(\theta_0)\mathbf{g}'_{Nj}(\theta_0)] = o_p(1)$.

The result, $T_{\texttt{SPD}}|_{H_0} \xrightarrow{D} \chi^2_k$, thus follows when $\{v_{it}\}$ are iid normal.

The proof of $T^{\mathrm{r}}_{\texttt{SPD}}|_{H_0} \xrightarrow{D} \chi^2_k$ is much trickier when the original errors $\{v_{it}\}$ are allowed to be nonnormal (though still iid), since in this case it is not guaranteed that $\{v_j^*\}$ will be again totally independent. It amounts to show

(a) $\frac{1}{\sqrt{N}} S^\circ_{\texttt{SPD},\alpha}(\tilde{\theta}_N) \xrightarrow{D} N(0_k, \lim_{N\to\infty} \frac{1}{N}\Upsilon_N)$, where $\Upsilon_N = \mathrm{Var}[S^\circ_{\texttt{SPD},\alpha}(\tilde{\theta}_N)]$.

(b) $\frac{1}{N}\tilde{\Omega}^r_N - \frac{1}{N}\mathrm{Var}[S^\circ_{\texttt{SPD}}(\theta_0)] = o_p(1)$;

(c) $\frac{1}{N}\sum_{j=1}^N (\tilde{\mathbf{g}}_{Nj}\tilde{\mathbf{g}}'_{Nj} - \mathbf{g}_{Nj}\mathbf{g}'_{Nj}) = o_p(1)$ and $\frac{1}{N}\sum_{j=1}^N (\tilde{\mathbf{d}}_{Nj}\tilde{\mathbf{d}}^{\circ\prime}_{Nj} - \mathbf{d}_{Nj}\mathbf{d}^{\circ\prime}_{Nj}) = o_p(1)$;

(d) $\frac{1}{N}(\tilde{\mathbf{\Sigma}}_{N,\alpha\theta} - \mathbf{\Sigma}_{N,\alpha\theta}) = o_p(1)$ and $\frac{1}{N}(\tilde{\mathbf{\Sigma}}_{N,\theta\theta} - \mathbf{\Sigma}_{N,\theta\theta}) = o_p(1)$.

To show (a), noting that $\mathbf{V}_N = (F'_{T,T-1} \otimes I_n)\mathbb{V}_{nT}$, the components of the score function $S^\circ_{\texttt{SPD}}(\theta_0)$ given in (3.7) can all be written as linear, or quadratic, or linear-quadratic forms of $\mathbb{V}_{nT}$, a vector of iid elements. Lemma A.5 and Cramér-Wold device lead to the asymptotic normality of $\frac{1}{\sqrt{N}} S^{\circ}_{\texttt{SPD}}(\theta_0)$, and hence the asymptotic normality of $\frac{1}{\sqrt{N}} S^{\circ}_{\texttt{FE-SPD},\alpha}(\tilde{\theta}_N)$.

To prove (b), note that $S^{\circ}_{\texttt{SPD}}(\theta_0) = \sum_{j=1}^N \mathbf{g}_{Nj}(\theta_0) \equiv \sum_{j=1}^N \mathbf{g}_{Nj}$, where

$$\mathbf{g}_{Nj} = \begin{cases} \Pi_{1j} v_j^*, \\ v_j^* \xi_{1j} + (v_j^{*2} - \sigma_0^2)\phi_{1j}, \\ v_j^* \xi_{2j} + (v_j^{*2} - \sigma_0^2)\phi_{2j} + \Pi_{2j} v_j^*, \\ v_j^* \xi_{3j} + (v_j^{*2} - \sigma_0^2)\phi_{3j}, \\ \frac{1}{2\sigma_0^2} z_j (v_j^{*2} - \sigma_0^2), \end{cases} \tag{C.1}$$

where $\{\xi_{rj}\} = \xi_r = (\mathbf{\Phi}_r^{u\prime} + \mathbf{\Phi}_r^l)\mathbf{V}_N$, and ϕ_{rj} are the diagonal elements of $\mathbf{\Phi}_r, r = 1, 2, 3$. All quantities are defined in (3.7), and $\dot{h}(0)$ in the last element of $\mathbf{g}_{Nj}$ is dropped as it is canceled out in the final expression of the test statistic. We have,

$$\mathrm{Var}[S^{\circ}_{\texttt{SPD}}(\theta_0)] = \sum_{j=1}^N \mathrm{Var}(\mathbf{g}_{Nj}) + \sum_{j=1}^N \sum_{\ell \neq j}^N \mathrm{Cov}(\mathbf{g}_{Nj},\ \mathbf{g}_{N\ell}). \tag{C.2}$$

Recall that $\odot$ denotes the Hadamard product. A vector raised to rth power is operated elementwise. Let $\mathbf{f}_j$ be the jth column of $F_{T,T-1} \otimes I_n$ and $\mathbf{q}_{rj}$ be the jth column of $(F_{T,T-1} \otimes I_n)(\mathbf{\Phi}_r^u + \mathbf{\Phi}_r^{l\prime})$, for $j = 1, \ldots, N$. We have $v_j^* = \mathbf{f}_j' \mathbb{V}_{nT}$ and $\xi_{rj} = \mathbf{q}_{rj}' \mathbb{V}_{nT}$; $v_j^* \xi_{rj} = \mathbb{V}_{nT}' (\mathbf{f}_j \mathbf{q}_{rj}') \mathbb{V}_{nT}$; and $v_j^{*2} = \mathbb{V}_{nT}' (\mathbf{f}_j \mathbf{f}_j') \mathbb{V}_{nT}$. Using the following easily proved results:

$$\begin{aligned} &\mathrm{Cov}(c_N' \mathbb{V}_{nT}, \mathbb{V}_{nT}' A \mathbb{V}_{nT}) = \mu_0^{(3)} c_N' a_N, \quad \text{and} \\ &\mathrm{Cov}(\mathbb{V}_{nT}' A_N \mathbb{V}_{nT}, \mathbb{V}_{nT}' B_N \mathbb{V}_{nT}) = (\mu_0^{(4)} - 3\sigma_0^4) a_N' b_N + \sigma_0^4 \mathrm{tr}[A_N(B_N + B_N')], \end{aligned}$$

for conformable matrices A_N and B_N and vector c_N, with a_N and b_N being the vectors formed by the diagonal elements of A_N and B_N, respectively, and $\mu_0^{(3)}$ and $\mu_0^{(4)}$ being, respectively, the 3rd and 4th moments of v_{it}, we have the key elements in $\mathrm{Cov}(\mathbf{g}_{Nj},\ \mathbf{g}_{N\ell})$:

$$\begin{aligned} \mathrm{Cov}(v_j^*, v_\ell^* \xi_{r\ell}) &= \mu_0^{(3)} \mathbf{f}_j' (\mathbf{f}_\ell \odot \mathbf{q}_{r\ell}), \\ \mathrm{Cov}(v_j^*, v_\ell^{*2}) &= \mu_0^{(3)} \mathbf{f}_j' (\mathbf{f}_\ell \odot \mathbf{f}_\ell), \\ \mathrm{Cov}(v_j^* \xi_{rj}, v_\ell^* \xi_{r\ell}) &= (\mu_0^{(4)} - 3\sigma_0^4)(\mathbf{f}_j \odot \mathbf{q}_{rj})'(\mathbf{f}_\ell \odot \mathbf{q}_{r\ell}) + \sigma_0^4 \mathrm{tr}[(\mathbf{f}_j \mathbf{q}_{rj}')(\mathbf{f}_\ell \mathbf{q}_{r\ell}' + \mathbf{q}_{r\ell} \mathbf{f}_\ell')], \\ \mathrm{Cov}(v_j^{*2}, v_\ell^* \xi_{r\ell}) &= (\mu_0^{(4)} - 3\sigma_0^4)(\mathbf{f}_j \odot \mathbf{f}_j)'(\mathbf{f}_\ell \odot \mathbf{q}_{r\ell}) + \sigma_0^4 \mathrm{tr}[(\mathbf{f}_j \mathbf{f}_j')(\mathbf{f}_\ell \mathbf{q}_{r\ell}' + \mathbf{q}_{r\ell} \mathbf{f}_\ell')], \\ \mathrm{Cov}(v_j^{*2}, v_\ell^{*2}) &= (\mu_0^{(4)} - 3\sigma_0^4)(\mathbf{f}_j \odot \mathbf{f}_j)'(\mathbf{f}_\ell \odot \mathbf{f}_\ell) + \sigma_0^4 \mathrm{tr}[(\mathbf{f}_j \mathbf{f}_j')(\mathbf{f}_\ell \mathbf{f}_\ell' + \mathbf{f}_\ell \mathbf{f}_\ell')], \end{aligned}$$

$r = 1, 2, 3$. It is easy to see that (i) $\mathbf{f}_j' \mathbf{f}_\ell = 0$ for all $j \neq \ell$, (ii) $\mathbf{f}_j' \mathbf{q}_{r\ell} = 0$ for $\ell \leq j$, and (iii) $\mathbf{f}_j \odot \mathbf{q}_{rj} = 0$.[18] Thus, all terms vanish except $\mathbf{f}_j'(\mathbf{f}_\ell \odot \mathbf{f}_\ell)$ and $(\mathbf{f}_j \odot \mathbf{f}_j)'(\mathbf{f}_\ell \odot \mathbf{f}_\ell)$, and

[18] The result (ii) is due to the fact that v_j^* is uncorrelated with ξ_ℓ for $\ell \leq j$, and (iii) follows from $(F_{T,T-1} \otimes I_n)(\mathbf{\Phi}_r^u + \mathbf{\Phi}_r^{l\prime}) = F_{T,T-1} \otimes (\Phi_r^u + \Phi_r^{l\prime})$ and hence $(F_{T,T-1} \otimes I_n) \odot [(F_{T,T-1} \otimes (\Phi_r^u + \Phi_r^{l\prime})] = 0$, where $\mathbf{\Phi}_r = I_{T-1} \otimes \Phi_r$.

subsequently all covariances vanish except,

$$\mathrm{Cov}(v_j^*, v_\ell^{*2}) = \mu_0^{(3)}\mathbf{f}_j'(\mathbf{f}_\ell \odot \mathbf{f}_\ell) \text{ and } \mathrm{Cov}(v_j^{*2}, v_\ell^{*2}) = (\mu_0^{(4)} - 3\sigma_0^4)(\mathbf{f}_j \odot \mathbf{f}_j)'(\mathbf{f}_\ell \odot \mathbf{f}_\ell). \tag{C.3}$$

Note that (i) the vector $\mathbf{f}_j$ has only $(T-1)$ nonzero elements, and (ii) for integers $k \geq 1$ and $m \geq 1$, $\mathbf{f}_j^k \odot \mathbf{f}_\ell^m \neq 0_n$ only when the indices $j = (i,t)$ and $\ell = (i,s)$, $t \neq s$. These show that,

$$\sum_{j=1}^{N}\sum_{\ell\neq j}^{N} \mathrm{Cov}(\mathbf{g}_{Nj},\ \mathbf{g}_{N\ell}) = \sum_{i=1}^{n}\sum_{t=1}^{T-1}\Big(\sum_{s(\neq t)=1}^{T-1} \mathrm{E}(\mathbf{d}_{N,it}\mathbf{d}_{N,is}')\Big) = \sum_{i=1}^{n}\sum_{t=1}^{T-1} \mathrm{E}(\mathbf{d}_{N,it}\mathbf{d}_{N,it}^{\circ\prime}), \tag{C.4}$$

where $\mathbf{d}_{N,it} = \big\{\Pi_{1,it}' v_{it}^*, (v_{it}^{*2}-\sigma_0^2)\phi_{1,it}, (v_{it}^{*2}-\sigma_0^2)\phi_{2,it}+\Pi_{2,it}v_{it}^*, (v_{it}^{*2}-\sigma_0^2)\phi_{3,it}, \frac{1}{2\sigma_0^2}z_{ni}'(v_{it}^{*2}-\sigma_0^2)\big\}'$, and $\mathbf{d}_{N,it}^\circ = \sum_{s(\neq t)=1}^{T-1}\mathbf{d}_{N,is}$. Letting $\tilde{\mathbf{d}}_{N,it}$ and $\tilde{\mathbf{d}}_{N,it}^\circ$ be theestimates of $\mathbf{d}_{N,it}$ and $\mathbf{d}_{N,it}^\circ$ at the null, one can show (details are available upon request from the authors) that

$$\frac{1}{N}\sum_{j=1}^{N}\sum_{\ell\neq j}^{N} \mathrm{Cov}(\mathbf{g}_{Nj},\ \mathbf{g}_{N\ell}) - \frac{1}{N}\sum_{i=1}^{n}\sum_{t=1}^{T-1}(\tilde{\mathbf{d}}_{N,it}\tilde{\mathbf{d}}_{N,it}^{\circ\prime}) = o_p(1). \tag{C.5}$$

It left to prove $\frac{1}{N}\big\{\sum_{j=1}^N \mathbf{g}_{Nj}\mathbf{g}_{Nj}' - \sum_{j=1}^N \mathrm{E}(\mathbf{g}_{Nj}\mathbf{g}_{Nj}')\big\} = o_p(1)$, which can be done by referring to the proof of Lemma A.6.

The proofs of (c) and (d) can be carried out by referencing to the proofs of (c) and (d) of Theorem 2.1, with details being available upon request from the authors. ■

Estimation of $\mathbf{\Sigma}_{N,\alpha\theta}$ and $\mathbf{\Sigma}_{N,\theta\theta}$. A pair of consistent estimators of $\mathbf{\Sigma}_{N,\alpha\theta}$ and $\mathbf{\Sigma}_{N,\theta\theta}$ are the negative Hessian matrices, $\mathbb{H}_{N,\alpha\theta}^\circ(\lambda) = -\frac{\partial}{\partial\theta'}S_{\mathtt{SPD},\alpha}^\circ(\theta)$ and $\mathbb{H}_{N,\theta\theta}^\circ(\theta) = -\frac{\partial}{\partial\theta'}S_{\mathtt{SPD},\lambda}^\circ(\theta)$, evaluated at the null estimate $\tilde{\theta}_n$, which take identical forms as these for the SLR model given in the proof of Theorem 2.1, Appendix B, except that n is replaced by N and the relevant quantities are replaced by the corresponding bold-faced quantities for the SPD model, and hence are not repeated here. These matrices are also required in proving (a) and (d) above.

Proof of Theorem 3.2: Similar to the proof of Theorem 3.1. ■

Estimation of $\mathbf{\Sigma}_{N,\alpha\lambda}^*$ and $\mathbf{\Sigma}_{N,\lambda\lambda}^*$. A pair of consistent estimators of $\mathbf{\Sigma}_{N,\alpha\lambda}^*$ and $\mathbf{\Sigma}_{N,\lambda\lambda}^*$ are the negative Hessian matrices, $\mathbb{H}_{N,\alpha\lambda}^*(\lambda) = -\frac{\partial}{\partial\lambda'}S_{\mathtt{SPD},\alpha}^*(\lambda)$ and $\mathbb{H}_{N,\lambda\lambda}^*(\lambda) = -\frac{\partial}{\partial\lambda'}S_{\mathtt{SPD},\lambda}^*(\lambda)$, evaluated at the null estimate $\tilde{\lambda}_N$, which take identical forms as these for the SLR model given in the proof of Theorem 2.2, Appendix B, except that n is replaced by N and the relevant quantities are replaced by the corresponding bold-faced quantities for the SPD model, and hence are not repeated here. These matrices are required in the proof of Theorem 3.2.

Appendix D: Proofs for the Panel FE-DSPD Model

Proof of Theorem 4.1: To show $T^{\mathrm{r}}_{\mathtt{DSPD}}|_{H_0} \xrightarrow{D} \chi^2_k$, it suffices to show

(a) $\frac{1}{\sqrt{N}} S^{\circ}_{\mathtt{DSPD},\alpha}(\tilde{\theta}_N) \xrightarrow{D} N(0_k, \lim_{N\to\infty} \frac{1}{N}\Upsilon_N)$, where $\Upsilon_N = \mathrm{Var}[S^{\circ}_{\mathtt{DSPD},\alpha}(\tilde{\theta}_N)]$;

(b) $\frac{1}{N}\sum_{i=1}^{n} \mathbf{g}_{ni}(\theta_0)\mathbf{g}'_{ni}(\theta_0) - \frac{1}{N}\mathrm{Var}[S^{\circ}_{\mathtt{DSPD}}(\theta_0)] = o_p(1)$;

(c) $\frac{1}{N}\sum_{i=1}^{n}[\tilde{\mathbf{g}}_{ni}\tilde{\mathbf{g}}'_{ni} - \mathbf{g}_{ni}(\theta_0)\mathbf{g}'_{ni}(\theta_0)] = o_p(1)$;

(d) $\frac{1}{N}(\tilde{\mathbf{\Sigma}}_{N,\alpha\theta} - \mathbf{\Sigma}_{N,\alpha\theta}) = o_p(1)$ and $\frac{1}{N}(\tilde{\mathbf{\Sigma}}_{N,\theta\theta} - \mathbf{\Sigma}_{N,\theta\theta}) = o_p(1)$.

To show (a), we first establish the joint asymptotic normality of $S^{\circ}_{\mathtt{DSPD}}(\theta_0)$ given in (4.4). While this can be done along the same line as that for $S^{\circ}_{\mathtt{DSPD},\theta}(\theta_0)$ of the null model given in Yang (2018a), it is useful to give some technical details in order for a better understanding of our methodology in constructing the tests for homoskedasticity for FE-DSPD model. Note that $S^{\circ}_{\mathtt{DSPD}}(\theta_0)$ contains three types of terms: $\Pi\Delta\mathbf{V}_N$, $\Delta\mathbf{V}'_N\mathbf{\Phi}\Delta\mathbf{V}_N$ and $\Delta\mathbf{V}'_N\mathbf{\Psi}\Delta\mathbf{Y}_{N1}$. Let $\mathbb{V}_{nT} = (V_{n1},\ldots,V_{nT})'$, the $nT\times 1$ vector or the original iid errors. Then, $\Delta\mathbf{V}_N = \mathbf{F}_{\mathtt{D}}\mathbb{V}_{nT}$, where $\mathbf{F}_{\mathtt{D}}$ is the first-differencing transformation matrix. Therefore, we have,

$$\Pi\Delta\mathbf{V}_N = \Pi^*\mathbb{V}_{nT} = \textstyle\sum_{t=1}^{T}\Pi^*_t V_{nt},$$
$$\Delta\mathbf{V}'_N\mathbf{\Phi}\Delta\mathbf{V}_N = \mathbb{V}'_{nT}\mathbf{\Phi}^*\mathbb{V}_{nT} = \textstyle\sum_{t=1}^{T}\sum_{s=1}^{T} V'_{nt}\Phi^*_{ts}V_{ns},$$
$$\Delta\mathbf{V}'_N\mathbf{\Psi}\Delta\mathbf{Y}_{N1} = \mathbb{V}'_{nT}\mathbf{\Psi}^*\Delta\mathbf{Y}_{N1} = \textstyle\sum_{t=1}^{T} V'_{nt}\Psi^*_{t\cdot}\Delta Y_{n1},$$

for suitably defined *-quantities, where $\Psi^*_{t\cdot} = \sum_{s=1}^{T}\Psi^*_{ts}$, and Π^*_t, Φ^*_{ts} and Ψ^*_{ts} are, respectively, the sub-vectors or sub-matrices of Π^*, $\mathbf{\Phi}^*$ and $\mathbf{\Psi}$, partitioned according to $t,s = 1,\ldots,T$. Now, based on the original model from which (2.3) is obtained, we have $Y_{n1} = B_{1n}^{-1}B_{2n}Y_{n0} + \eta_{n1} + B_{1n}^{-1}B_{3n}^{-1}V_{n1}$, where η_{n1} collects all the other terms in the model. Thus,

$$\textstyle\sum_{t=1}^{T} V'_{nt}\Psi^*_{t\cdot}\Delta Y_{n1} = \sum_{t=1}^{T} V'_{nt}\Psi^{\circ}_{t\cdot}Y_{n0} + \sum_{t=1}^{T} V'_{nt}\Psi^{\dagger}_{t\cdot}V_{n1} + \sum_{t=1}^{T} V'_{nt}\Psi^*_{t\cdot}\eta_{n1}$$

as in Yang (2018a). Therefore, for every non-zero $(p+k+5)\times 1$ vector c, $c'S^{\circ}_{\mathtt{DSPD}}(\theta_0)$ is a sum of liner, bilinear and quadratic forms in Y_{n0}, V_t and V_s, $t,s = 1,\ldots,T$, and the asymptotic normality of $c'S^{\circ}_{\mathtt{DSPD}}(\theta_0)$ can be proved under the assumptions stated in the theorem and using Lemma A.5. Finally, Cramér-Wold devise leads to the joint asymptotic normality of $S^{\circ}_{\mathtt{DSPD}}(\theta_0)$.

Next, similar to (2.11), an asymptotic expansion can be developed for $S^{\circ}_{\mathtt{DSPD},\alpha}(\tilde{\theta}_N)$:

$$S^{\circ}_{\mathtt{DSPD},\alpha}(\tilde{\theta}_N) = S^{\circ}_{\mathtt{DSPD},\alpha}(\theta_0) - \mathbf{\Gamma}_N S^{\circ}_{\mathtt{DSPD},\theta}(\theta_0) + o_p(\sqrt{N}),$$

by applying MVT and the results in (d), where $\mathbf{\Gamma}_N = \mathbf{\Sigma}_{N,\alpha\theta}\mathbf{\Sigma}^{-1}_{N,\theta\theta}$, $\mathbf{\Sigma}_{N,\alpha\theta} = -\mathrm{E}[\frac{\partial}{\partial\theta'}S^{\circ}_{\mathtt{DSPD},\alpha}(\theta_0)]$, and $\boldsymbol{\Sigma}_{N,\theta\theta} = -\mathrm{E}[\frac{\partial}{\partial\theta'} S^{\circ}_{\mathtt{DSPD},\theta}(\theta_0)]$. This and the joint asymptotic normality of $S^{\circ}_{\mathtt{DSPD}}(\theta_0)$ lead to

$$\frac{1}{\sqrt{N}} S^{\circ}_{\mathtt{DSPD},\alpha}(\tilde{\theta}_N) \xrightarrow{D} N(0, \lim_{n\to\infty} \frac{1}{N}\Upsilon_N),$$

where $\Upsilon_N = \boldsymbol{\Omega}_{N,\alpha\alpha} - \boldsymbol{\Gamma}_N \boldsymbol{\Omega}_{N,\theta\alpha} - \boldsymbol{\Omega}_{N,\alpha\theta}\boldsymbol{\Gamma}'_N + \boldsymbol{\Gamma}_N \boldsymbol{\Omega}_{N,\theta\theta}\boldsymbol{\Gamma}'_N$. Using the MD decompositions for $S^{\circ}_{\mathtt{DSPD},\alpha}(\theta_0)$ and $S^{\circ}_{\mathtt{DSPD},\theta}(\theta_0)$, we have $\Upsilon_N = \sum_{i=1}^{n} \mathrm{E}[(\mathbf{g}_{ni,\alpha} - \boldsymbol{\Gamma}_N \mathbf{g}_{ni,\theta})(\mathbf{g}_{ni,\alpha} - \boldsymbol{\Gamma}_N \mathbf{g}_{ni,\theta})']$.

The proof of (b) follows closely to the proof of Theorem 3.3 of Yang (2018a) using the Hessian matrix given below. Proof of (c) can be carried out along the same line as that of the proof of Theorem 2.1. The proof of the second part of (d) is given in the proof of Theorem 3.3 of Yang (2018a), and that of the first part can be done in a similar manner. ∎

Hessian Matrices. We now give the negative Hessian matrices $\mathbb{H}^{\circ}_{N,\alpha\theta}(\theta) = -\frac{\partial}{\partial\theta'} S^{\circ}_{\mathtt{DSPD},\alpha}(\theta)$ and $\mathbb{H}^{\circ}_{N,\theta\theta}(\theta) = -\frac{\partial}{\partial\theta'} S^{\circ}_{\mathtt{DSPD},\theta}(\theta)$ required for estimating $\boldsymbol{\Sigma}_{N,\alpha\theta}$ and $\boldsymbol{\Sigma}_{N,\theta\theta}$, and for proving (a) and (d) above. Recall $\Delta\mathbf{V}_N(\beta,\delta) = \mathbf{B}_{3N}(\lambda_3)[\mathbf{B}_{1N}(\lambda_1)\Delta\mathbf{Y}_N - \mathbf{B}_{2N}(\rho,\lambda_2)\Delta\mathbf{Y}_{N,-1} - \Delta\mathbf{X}_N\beta]$. Denote $\Delta\mathbf{U}(\theta_1) = \mathbf{B}^{-1}_{3N}(\lambda_3)\Delta\mathbf{V}_N(\beta,\delta) = \mathbf{B}_{1N}(\lambda_1)\Delta\mathbf{Y}_N - \mathbf{B}_{2N}(\rho,\lambda_2)\Delta\mathbf{Y}_{N,-1} - \Delta\mathbf{X}_N\beta$, where $\theta_1 = (\beta', \delta'_1)'$ and $\delta_1 = (\rho, \lambda_1, \lambda_2)'$. First, $\mathbb{H}^{\circ}_{N,\alpha\theta}(\theta)$ has its jth row, $j = 1, \ldots, k$:

$$-\tfrac{1}{\sigma^2}\Delta\mathbf{V}'_N(\beta,\delta)(C^{-1}_{T-1}\otimes\mathcal{Z}_{nj})\big[\mathbf{B}_{3N}(\lambda_3)\Delta\mathbf{X}_N,\ \tfrac{1}{2\sigma^2}\Delta\mathbf{V}_N(\beta,\delta),\ \mathbf{B}_{3N}(\lambda_3)\Delta\mathbf{Z}_N,\ \mathbf{W}_{3N}\Delta\mathbf{U}_N(\beta,\delta_1)\big],$$

where $\Delta\mathbf{Z}_N = [\Delta\mathbf{Y}_{N,-1}, \mathbf{W}_{1N}\Delta\mathbf{Y}_N, \mathbf{W}_{2N}\Delta\mathbf{Y}_{N,-1}]$. The expression for $\mathbb{H}_{N,\theta\theta}(\theta)$ is available from Yang (2018a, Appendix C). Here we give a simpler form to facilitate the numerical implementation of our testing methods. Denote $\Omega_{\mathtt{u}} \equiv \Omega_{\mathtt{u}}(\lambda_3) = \frac{1}{\sigma^2}\mathrm{Var}(\Delta\mathbf{U}) = C_{T-1} \otimes (B'_{3n}B_{3n})^{-1}$. The AQS subvector $S^{\circ}_{\mathtt{DSPD},\theta}(\theta)$ defined in (4.2) can be rewritten as

$$S^{\circ}_{\mathtt{DSPD},\theta}(\theta) = \begin{cases} \frac{1}{\sigma^2}\Delta\mathbf{X}_N\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{U}(\theta_1), \\ \frac{1}{2\sigma^4}\Delta\mathbf{U}'(\theta_1)\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{U}(\theta_1) - \frac{N}{2\sigma^2}, \\ \frac{1}{\sigma^2}\Delta\mathbf{Z}_N\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{U}(\theta_1) + \mu(\delta_1), \\ \frac{1}{2\sigma^2}\Delta\mathbf{U}'(\theta_1)(C^{-1}_{T-1}\otimes A(\lambda_3))\Delta\mathbf{U}(\theta_1), \end{cases}$$

where $\mu(\delta_1) = (\mathrm{tr}(\mathbf{C}^{-1}_N\mathbf{D}_{N,-1}), \mathrm{tr}(\mathbf{C}^{-1}_N\mathbf{D}_N\mathbf{W}_{1N}), \mathrm{tr}(\mathbf{C}^{-1}_N\mathbf{D}_{N,-1}\mathbf{W}_{2N}))'$ and $A(\lambda_3) = W'_{3n}B_{3n}(\lambda_3) + B'_{3n}(\lambda_3)W_{3n}$. We have the rows of $\mathbb{H}^{\circ}_{N,\theta\theta}(\theta)$:

$$\begin{aligned}
\mathbb{H}^{\circ}_{\beta\theta} &= \tfrac{1}{\sigma^2}\Delta\mathbf{X}'_N\big[\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{X}_N,\ \tfrac{1}{\sigma^2_v}\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{U}(\theta_1),\ \Omega^{-1}_{\mathtt{u}}\Delta\mathbf{Z}_N,\ -\dot{\Omega}^{-}_{\mathtt{u}}\Delta\mathbf{U}(\theta_1)\big],\\
\mathbb{H}^{\circ}_{\sigma^2\theta} &= \tfrac{1}{\sigma^4}\big[\Delta\mathbf{U}'(\theta_1)\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{X}_N, \tfrac{1}{\sigma^2_v}\Delta\mathbf{U}'(\theta_1)\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{U}(\theta_1) - \tfrac{N}{2}, \Delta\mathbf{Z}'\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{U}(\theta_1),\ -\tfrac{1}{2}\Delta\mathbf{U}'(\theta_1)\dot{\Omega}^{-}_{\mathtt{u}}\Delta\mathbf{U}(\theta)\big],\\
\mathbb{H}^{\circ}_{\delta_1\theta} &= \tfrac{1}{\sigma^2}\big[\Delta\mathbf{Z}'_N\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{X}_N,\ \Delta\mathbf{Z}'_N\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{U}'(\theta_1),\ \Delta\mathbf{Z}'_N\Omega^{-1}_{\mathtt{u}}\Delta\mathbf{Z}_N - \dot{\mu}(\delta_1),\ -\Delta\mathbf{Z}'_N\dot{\Omega}^{-}_{\mathtt{u}}\Delta\mathbf{U}'(\theta_1)\big],\\
\mathbb{H}^{\circ}_{\lambda_3\theta} &= \tfrac{1}{\sigma^2}\big[\Delta\mathbf{U}'(\theta_1)\dot{\Omega}^{-}_{\mathtt{u}}[\Delta\mathbf{X}_N,\ \Delta\mathbf{U}(\theta_1),\ \Delta\mathbf{Z}_N],\ \Delta\mathbf{U}'(\theta_1)\mathbf{W}'_{3N}\mathbf{W}_{3N}\Delta\mathbf{U}(\theta_1) - (T-1)\mathrm{tr}(G^2_{3n})\big],
\end{aligned}$$

where $\dot{\Omega}^{-}_{\mathtt{u}} = \frac{\partial}{\partial\lambda_3}\Omega^{-1}_{\mathtt{u}}$, and $\dot{\mu}(\delta_1) = \frac{\partial}{\partial\delta'_1}\mu(\delta_1)$.

Proof of Theorem 4.2: Similar to the proof of Theorem 4.1. ■

Hessian Matrices. The negative Hessian matrices, $\mathbb{H}^*_{N,\alpha\delta}(\delta) = -\frac{\partial}{\partial\delta'}S^*_{\mathtt{DSPD},\alpha}(\delta)$ and $\mathbb{H}^*_{N,\delta\delta}(\delta) = -\frac{\partial}{\partial\delta'}S^*_{\mathtt{DSPD},\delta}(\delta)$, for the estimation of $\mathbf{\Sigma}^*_{N,\alpha\delta}$ and $\mathbf{\Sigma}^*_{N,\delta\delta}$ and for the proof of Theorem 4.2 are given below. As in the regular AQS test, $\mathbb{H}^*_{N,\alpha\delta}(\delta)$ has its jth row:

$$\Delta\widetilde{\mathbf{V}}'_N(\delta)\big(C^{-1}_{T-1}\otimes\mathcal{Z}_{nj}\big)\Delta\widetilde{\mathbf{V}}_{N,\delta}(\delta) - \mu^*_{\alpha_j,\delta}(\lambda_3)\tilde{\sigma}^{*2}_N(\delta) + \mu^*_{\alpha_j}(\lambda_3)\tilde{\sigma}^{*2}_{N,\delta}(\delta),\ \ j=1,\dots,k,$$

where $\Delta\widetilde{\mathbf{V}}_{N,\delta}(\delta) = \frac{\partial}{\partial\delta'}\Delta\widetilde{\mathbf{V}}_N(\delta)$, $\mu^*_{\alpha_j,\delta}(\lambda_3) = \frac{\partial}{\partial\delta'}\mu^*_{\alpha_j}(\lambda_3)$ and $\tilde{\sigma}^{*2}_{N,\delta}(\delta) = \frac{\partial}{\partial\delta'}\tilde{\sigma}^{*2}_N(\delta)$. Using the relation $\Delta\widetilde{\mathbf{V}}_N(\delta) = \mathbf{C}^{1/2}_N\mathbf{M}_N(\lambda_3)\Delta\mathbb{Y}_N(\delta)$, and the quantities $\Delta\mathbb{Y}_N(\delta)$ and $\tilde{\sigma}^{*2}_N(\delta)$ defined in Section 4.3, the δ_1-components of these derivatives can easily be obtained. The derivatives w.r.t. λ_3 are more tedious as they involve the projection matrix $\mathbf{M}_N(\lambda_3)$ but are straightforward. Alternatively, numerical derivatives can be used.

Now, for $\mathbb{H}^*_{N,\delta\delta}(\delta)$, using the $\Delta\mathbf{Z}_n$ notation introduced above and denoting $\mu^*_{\delta_1}(\delta) = (\mu^*_\rho(\delta),\mu^*_{\lambda_1}(\delta),\mu^*_{\lambda_2}(\delta))'$, $S^*_{\mathtt{DSPD},\delta}(\delta)$ can be written more compactly as:

$$S^*_{\mathtt{DSPD}_\delta}(\delta) = \begin{cases} \Delta\mathbf{Z}'_N\mathbf{B}'_{3N}(\lambda_3)\mathbf{C}^{-1}_N\Delta\widetilde{\mathbf{V}}_N(\delta) - \mu^*_{\delta_1}(\delta)\tilde{\sigma}^{*2}_N(\delta), \\ \Delta\widetilde{\mathbf{V}}'_N(\delta)\big[C^{-1}_{T-1}\otimes G_{3n}(\lambda_3)\big]\Delta\widetilde{\mathbf{V}}_N(\delta) - \mu^*_{\lambda_3}(\lambda_3)\tilde{\sigma}^{*2}_N(\delta). \end{cases}$$

Therefore, we obtain the components of $\mathbb{H}^*_{N,\delta\delta}(\delta)$:

$$\begin{aligned}
\mathbb{H}^*_{\delta_1\delta_1} &= \Delta\mathbf{Z}'_N\mathbf{B}'_{3N}(\lambda_3)\mathbf{C}^{-1}_N\Delta\widetilde{\mathbf{V}}_{N,\delta_1}(\delta) - \mu^*_{\delta_1\delta_1}(\delta)\tilde{\sigma}^{*2}_N(\delta) - \mu^*_{\delta_1}(\delta)\tilde{\sigma}^{*2}_{N,\delta_1}(\delta),\\
\mathbb{H}^*_{\delta_1\lambda_3} &= \Delta\mathbf{Z}'_N\mathbf{B}'_{3N}(\lambda_3)\mathbf{C}^{-1}_N\Delta\widetilde{\mathbf{V}}_{N,\lambda_3}(\delta) - \Delta\mathbf{Z}'_N\mathbf{W}'_{3N}\mathbf{C}^{-1}_N\Delta\widetilde{\mathbf{V}}_N(\delta) - \mu^*_{\delta_1\lambda_3}(\delta)\tilde{\sigma}^{*2}_N(\delta) - \mu^*_{\delta_1}(\delta)\tilde{\sigma}^{*2}_{N,\lambda_3}(\delta),\\
\mathbb{H}^*_{\lambda_3\delta_1} &= 2\Delta\widetilde{\mathbf{V}}'_N(\delta)\big[C^{-1}_{T-1}\otimes G_{3n}(\lambda_3)\big]\Delta\widetilde{\mathbf{V}}_{N,\delta_1}(\delta) - \mu^*_{\lambda_3}(\lambda_3)\tilde{\sigma}^{*2}_{N,\delta_1}(\delta),\\
\mathbb{H}^*_{\lambda_3\lambda_3} &= 2\Delta\widetilde{\mathbf{V}}'_N(\delta)\big[C^{-1}_{T-1}\otimes G_{3n}(\lambda_3)\big]\Delta\widetilde{\mathbf{V}}_{N,\lambda_3}(\delta) + \Delta\widetilde{\mathbf{V}}'_N(\delta)\big[C^{-1}_{T-1}\otimes G^2_{3n}(\lambda_3)\big]\Delta\widetilde{\mathbf{V}}_N(\delta)\\
&\quad -\mu^*_{\lambda_3\lambda_3}(\lambda_3)\tilde{\sigma}^{*2}_N(\delta) - \mu^*_{\lambda_3}(\lambda_3)\tilde{\sigma}^{*2}_{N,\lambda_3}(\delta),
\end{aligned}$$

where an quantity with an extra subscript indicates the partial derivative in row direction, e.g., $\Delta\widetilde{\mathbf{V}}_{N,\delta_1}(\delta) = \frac{\partial}{\partial\delta'_1}\Delta\widetilde{\mathbf{V}}_N(\delta)$, and $\mu^*_{\delta_1\delta_1}(\delta) = \frac{\partial}{\partial\delta'_1}\mu^*_{\delta_1}(\delta)$. The rest are straightforward, although the derivatives w.r.t. λ_3 are tedious. Again, numerical derivatives can be used in these cases.