LM Tests of Spatial Dependence Based on Bootstrap Critical Values

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Abstract

To test the existence of spatial dependence in an econometric model, a convenient test is the Lagrange Multiplier (LM) test. However, evidence shows that, in finite samples, the LM test referring to asymptotic critical values may suffer from the problems of size distortion and low power, which become worse with a denser spatial weight matrix. In this paper, residual-based bootstrap methods are introduced for approximating the finite sample critical values of the LM statistics. Conditions for their validity are clearly laid out and justified. The methods are demonstrated using several popular spatial LM tests. Monte Carlo results show that when the conditions are not fully met, bootstrap may lead to unstable critical values that change significantly with the alternative, whereas when all conditions are met, bootstrap critical values are very stable, approximate much better the finite sample critical values than those based on asymptotics, and lead to significantly improved size and power.

Key Words: LM Tests; Bootstrapped critical values; Power; Size; Spatial dependence; Heteroscedasticity.

JEL Classification: C12, C13, C21

1 Introduction.

To test the existence of spatial dependence in an econometric model, a convenient test is the Lagrange Multiplier (LM) test as it requires model estimation only under the null hypothesis. However, evidence shows that, in finite samples, the true sizes of the LM test referring to the asymptotic critical values can be quite different from their nominal sizes, and more so with a denser spatial weight matrix and with one-sided tests. As a result, the LM tests in such circumstances may have low power in detecting a negative or 'positive' spatial dependence. Also, LM tests may not be robust against the misspecification in error

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distribution. Standardization (Koenker, 1981; Robinson, 2008; Yang, 2010; Baltagi and Yang, 2010; Yang and Shen, 2011)[2] robustifies the LM tests. It also helps alleviate the problem of size distortion for two-sided tests, but not for one-sided tests. Furthermore, standardization does not solve the problem of low power in detecting a negative or positive spatial dependence. The reason is that a denser spatial weight matrix makes the shape of the finite-sample distribution of the LM statistic deviate further away from the shape of its limiting distribution (thus the asymptotic critical values become less accurate in approximating the exact finite sample critical values). In the special case where the LM test is univariate and asymptotic standard normal under the null, a denser spatial weight makes its finite sample distribution more skewed to the left or right depending on the design of the regressors (thus more size distortion and less power for certain one-sided tests). Standardization only changes the location and scale, but not the shape of the distribution of the LM test. This is why it cannot solve the problem of size distortion and low power for one-sided tests. However, we demonstrate in this paper that standardization coupled with bootstrap provide a satisfactory solution to these problems.

It is well documented in the econometrics literature that bootstrap method is able to provide asymptotic refinements on the critical values of a test statistic if this statistic is asymptotically pivotal under the null hypothesis. See, among others, Beran (1988), Hall (1992), Horowitz (1994, 1997), Hall and Horowitz (1996), Davidson and MacKinnon (1999, 2006), van Giersbergen and Kiviet (2002), MacKinnon (2002), Cameron and Trivedi (2005, Ch. 11), and Godfrey (2009), for theoretical analyses and numerical evidence for many different type of econometric models. However, as pointed out by Davidson and reiterated in Godfrey (2009, p. 82), it is not always the case that the asymptotic analysis seems to provide a good explanation of what is observed in finite samples. For the residual-based bootstrap method which is followed in this paper, Godfrey (2009, Ch. 3), based on the work of van Giersbergen and Kiviet (2002) and MacKinnon (2002), give a detailed discussion on the type of residuals (restricted under the null hypothesis or unrestricted) to be resampled and the type of estimates (restricted or unrestricted) of the nuisance parameters to be used as parameters in the bootstrap world. However, the debate on the choices of parameter estimates and residuals does no seem to have been settled. Furthermore, in contrast to the vast literature on the bootstrap tests in general econometrics, such a literature in spatial econometrics is rather thin and to the best of our knowledge Lin et al. (2010) and Burridge and Fingleton (2010) seem to be the only formal publications although there are many ongoing research activities in this area. This research represents one such an activity.

In this paper, residual-based bootstrap methods are introduced in LM test framework for the purpose of better approximating the finite sample critical values of an LM statistic, which are applicable to a wide class of testing situations including tests for spatial depen-

[2]See Moulton and Randolph (1989), and Honda (1991) for standardizing LM tests in a different context.

dence. We demonstrate that for these methods to work well, it is important that (i) the bootstrap DGP resembles the null model, (ii) the LM statistic is asymptotically pivotal under the null or its robustified/standardized version must be used, (iii) the estimates of the nuisance parameters, to be used as parameters in the bootstrap world, are consistent whether or not the null hypothesis is true, (iv) the empirical distribution function (EDF) of the residuals to be resampled consistently estimates the error distribution whether or not the null hypothesis is true, and (v) calculation of the bootstrapped values of the LM statistic is done by imposing the null hypothesis.

Among these points, (i) and (ii) are well understood and agreed among the researchers, (v) follows the nature of LM or score tests, and (iii) and (iv) lead in general to the use of unrestricted parameter estimates and unrestricted residuals. Points (iii) and (iv), related to the major subjects of debate, make sense because in reality one does not know whether or not the null hypothesis is true. In order for the bootstrap world to be able to mimic the real world at the null, it must be set up such that the 'parameters' in the bootstrap world mimic (converge to) the nuisance parameters in the real world, and the errors in the bootstrap world mimic the true errors in the real world whether the null is true or false. These can only be guaranteed in general if the unrestricted estimates and unrestricted residuals are used. Clearly, (ii) is typically true (under regularity conditions) when error distributions are correctly specified, but may not be so when they are misspecified. In this case, the suggested bootstrap method may not be able to provide the desired level of improvement on the critical values, and a robust version of the LM statistic needs to be in place.

However, there are important special cases where it doesn't matter whether to use the unrestricted estimates/residuals, or the restricted versions, e.g., tests of linear constrains on regression coefficients in linear regression, Moran's I test of spatial error dependence in linear regression, etc. (see the following sections for detailed discussions on this), which is perhaps why some authors advocate the use of the restricted estimates and residuals as they are often much simpler than their unrestricted counterparts. The arguments given in the following section show that these special cases occur when (i) the LM statistic does not depend on the nuisance parameters, or the estimates of the nuisance parameters are consistent under both the null and alternative, and (ii) the LM tests are robust against the distributional misspecifications.

The suggested methods are demonstrated using several popular spatial LM tests in linear regressions, namely, the tests for spatial error dependence (SED), the LM tests for spatial lag dependence (SLD), and the LM tests for spatial error components (SEC). Monte Carlo results show that with the unrestricted estimates and residuals, bootstrap is able to provide critical values that are stable (with respect to the true value of the parameter that the test concerns) and provide very accurate approximations to the finite sample critical values of the test statistic, leading to more reliable size and power. Section 2 outlines the general bootstrap methods and provide simple arguments for their validity. Section 3 demonstrates these methods and validity arguments using, respectively, the LM tests for SED, the LM tests for SLD, and the LM tests for SEC. Each of the three cases is supplemented with a set of Monte Carlo results. Section 4 concludes the paper. Appendix A contains some theoretical results, and Appendix B describes the general setting of the Monte Carlo experiments.

2 Bootstrap Critical Values for LM Tests

Consider an LM test statistic $T_n(\lambda) \equiv T_n(Y_n, X_n, W_n; \lambda)$ for testing the spatial dependence represented by the parameter (vector) λ, in a model with dependent variable Y_n conditional on a set of independent variables X_n, a spatial weight matrix W_n, and parameters θ and λ, where the parameter vector θ may contain the regression coefficients, error standard deviations, etc., depending on the model considered. Typically, $T_n(\lambda)$ is not a pivotal quantity as its finite sample distribution, denoted as $\mathcal{G}_n(\cdot, \theta, \lambda)$, depends on the parameters θ and λ, but is asymptotically pivotal in the sense that its limiting distribution, denoted as $\mathcal{G}(\cdot)$, is free of parameters, such as standard normal or chi-square, depending on whether λ is a scalar or a vector. However, the latter result depends on strong assumptions on the error distribution. If the error distribution is misspecified, $T_n(\lambda)$ may not be an asymptotically pivotal quantity, and this will have important implications on the performance of the bootstrap procedures.

The most interesting inference in a spatial model is perhaps to test $H_0 : \lambda = 0$, i.e., non-existence of spatial dependence, versus $H_a : \lambda \neq 0$ ($< 0, > 0$), i.e., existence of spatial dependence (negative spatial dependence or positive spatial dependence). To test this hypothesis using the test statistic $T_n(0)$, one often refers to the asymptotic critical values of $T_n(0)$. However, as argued in the introduction, these asymptotic critical values may give poor approximations in cases of heavy spatial dependence. It is thus desirable to find better approximations to the finite sample critical values of $T_n(0)$.[3] As $T_n(\lambda)$ is not a pivotal quantity, it is not possible to find the exact finite sample critical values. However, if $T_n(\lambda)$ is asymptotically pivotal, the bootstrap approach can be used to obtain critical values that are more accurate than the asymptotic critical values, according to Beran (1988), Horowitz (1994) and Hall and Horowitz (1996). See also Cameron and Trivedi (2005, Ch. 11) and Godfrey (2009, Ch. 2 & 3) for detailed descriptions on bootstrap tests.

Our discussions above and below are for the LM tests of spatial regressions models. However, they can be applied to the LM tests of other types of models as well. It is the

[3] A unique feature for the tests of spatial dependence is that the finite sample distribution can be heavily affected by the spatial weight matrix W_n, a known matrix that specifies the relationship (neighborhood or distance) among the spatial units. The denser the matrix W_n is, the more skewed is the finite sample distribution of $T_n(\lambda)$. This makes it more meaningful to use the finite sample critical values of the test statistic instead of the asymptotic ones.

unique feature of LM tests (requiring the estimation of the null model only) and the unique feature of the spatial models (see footnote 3) that make it more appealing to study bootstrap methods in approximating the finite sample critical values of spatial LM statistics.

2.1 The methods

To facilitate our discussions, suppose that the model can be written as,

$$q(Y_n, X_n, W_n; \theta, \lambda) = e_n, \tag{1}$$

where e_n is an n-vector of model errors, with iid elements of zero mean, unit variance, and cumulative distribution function (CDF) $\mathcal{F}$. The error standard deviation σ is absorbed into θ.[4] Suppose that the model can be inverted to give

$$Y_n = h(X_n, W_n; \theta, \lambda; e_n). \tag{2}$$

Consider a general hypothesis

$$H_0 : \lambda = \lambda_0 \text{ versus } H_a : \lambda \neq \lambda_0 \ (< \lambda_0, \ > \lambda_0).$$

The test statistic to be used is the LM $T_n(\lambda_0)$, derived under $\mathcal{F}$, typically $N(0,1)$. We are interested in the finite sample null distribution of $T_n(\lambda_0)$, in particular the finite sample critical values of $T_n(\lambda_0)$ at the null, and investigate how bootstrap can provide a valid method for approximating these critical values.

In what follows, $\tilde{\theta}_n$ denotes the restricted estimate of θ under H_0, and $(\hat{\theta}_n, \hat{\lambda}_n)$ the unrestricted estimates of (θ, λ). The observable counterpart of e_n is referred to as *residuals*. If the residuals are obtained from the null model, i.e., $\tilde{e}_n = q(Y_n, X_n, W_n; \tilde{\theta}_n, \lambda_0)$, they are called the *restricted residuals*; if they are obtained from the full model, i.e., $\hat{e}_n = q(Y_n, X_n, W_n; \hat{\theta}_n, \hat{\lambda}_n)$, they are called the *unrestricted residuals*. The corresponding empirical distribution function (EDF) of the restricted residuals is denoted as $\tilde{\mathcal{F}}_n$, and that of the unrestricted residuals as $\hat{\mathcal{F}}_n$.

Note that the null model is determined by the pair $\{\theta, \mathcal{F}\}$, and that under the LM framework only the estimation of the null model is required. In order to approximate the finite sample null distribution (in particular the critical values) of $T_n(\lambda_0)$, the bootstrap world must be set up so that it is able to mimic the real world at the null. Thus, the bootstrap DGP should take the following form

$$Y_n^* = h(X_n, W_n; \ddot{\theta}_n, \lambda_0; e_n^*), \quad e_n^* \overset{iid}{\sim} \ddot{\mathcal{F}}_n. \tag{3}$$

[4]Model (1) encompasses many popular spatial models, linear or nonlinear, such as SAR, SARAR, SEC, spatial probit, spatial Tobit, etc.; see Kelejian and Prucha (2001). It can be extended to include more than one spatial weight matrix and to have non-spherical disturbances of the form $u_n \sim (0, \sigma^2\Omega(\rho))$, where $\Omega(\rho)$ is an $n \times n$ positive definite matrix, known up to a finite number of parameters ρ. In this case, writing $u_n = \sigma\Omega^{\frac{1}{2}}(\rho)e_n$ and merging σ and ρ into θ give the form of Model (1).

where $\ddot{\theta}_n$ is the bootstrap parameter vector (an estimate of the nuisance parameter vector based on the original data) which mimics (consistently estimates) θ, and $\ddot{\mathcal{F}}_n$ is the bootstrap error distribution (the EDF of some type of residuals) mimicking (consistently estimating) $\mathcal{F}$. The steps for finding the bootstrap critical values for $T_n(\lambda_0)$ is summarized as follows:

(a) Draw a bootstrap sample e_n^* from $\ddot{\mathcal{F}}_n$;

(b) Compute $Y_n^* = h(X_n, W_n; \ddot{\theta}_n, \lambda_0;\ e_n^*)$ to obtain the bootstrap data $\{Y_n^*, X_n, W_n\}$;

(c) Estimate the **null model** based on $\{Y_n^*, X_n, W_n\}$, and then compute a bootstrapped value $T_n^b(\lambda_0)$ of $T_n(\lambda_0)$;

(d) Repeat (a)-(c) B times to obtain the EDF $\ddot{\mathcal{G}}_n$ of $\{T_n^b(\lambda_0)\}_{b=1}^B$. The quantiles of $\ddot{\mathcal{G}}_n$ give the bootstrap critical values of $T_n(\lambda_0)$.

In reality, one does not know whether or not H_0 is true, thus it incurs an important issue: the choice of the pair $\{\ddot{\theta}_n, \ddot{\mathcal{F}}_n\}$. We argue in this paper that for the bootstrap DGP $Y_n^* = h(X_n, W_n; \ddot{\theta}_n, \lambda_0; e_n^*)$ to be able to mimic the real world null DGP $Y_n = h(X_n, W_n; \theta, \lambda_0; e_n)$ in general, $\{\ddot{\theta}_n, \ddot{\mathcal{F}}_n\}$ must be consistent for $\{\theta, \mathcal{F}\}$ whether or not H_0 is true. In this spirit, the only choice for $\{\ddot{\theta}_n, \ddot{\mathcal{F}}_n\}$ that can be correct in general is $\{\hat{\theta}_n, \hat{\mathcal{F}}_n\}$. As this resampling scheme is based on the unrestricted estimates of the nuisance parameters and the unrestricted residuals, it is termed as the *unrestricted resampling scheme*, or the **r**esampleing **s**cheme with **u**nrestricted estimates and **u**nrestricted residuals ($\texttt{RS}_{uu}$).

There are many special cases where $\tilde{\theta}_n$ and/or $\tilde{\mathcal{F}}_n$ are consistent whether or not H_0 is true. This leads to other choices for the pair $\{\ddot{\theta}_n, \ddot{\mathcal{F}}_n\}$: $\{\tilde{\theta}_n, \tilde{\mathcal{F}}_n\}$, $\{\hat{\theta}_n, \tilde{\mathcal{F}}_n\}$, or $\{\tilde{\theta}_n, \hat{\mathcal{F}}_n\}$, giving the so-called the *restricted resampling scheme* ($\texttt{RS}_{rr}$), and the *hybrid resampling schemes* 1 ($\texttt{RS}_{ur}$) and the *hybrid resampling schemes* 2 ($\texttt{RS}_{ru}$), to adopt the similar terms as in Godfrey (2009).

Alternative to the bootstrap method based on $\texttt{RS}_{uu}$, one may consider the bootstrap analog of H_0, $H_0^* : \lambda = \hat{\lambda}_n$. To test H_0^*, one generates the response values through the estimated full model, and performs bootstrap estimation conditional on estimated spatial parameter $\hat{\lambda}_n$. Thus, the bootstrap critical values of $T_n(\lambda_0)$ are simply the empirical quantiles of the bootstrap distribution of $T_n(\hat{\lambda}_n)$ conditional on $\hat{\lambda}_n$. This resampling scheme is denoted as $\texttt{RS}_{uf}$, and the corresponding bootstrap procedure is as follows:

(a) Draw a bootstrap sample $\hat{e}_n^*$ from the EDF $\hat{\mathcal{F}}_n$ of $\hat{e}_n$,

(b) Compute $Y_n^* = h(X_n, W_n; \hat{\theta}_n, \hat{\lambda}_n; \hat{e}_n^*)$ to obtain the bootstrap data $\{Y_n^*, X_n, W_n\}$,

(c) Conditional on $\hat{\lambda}_n$, estimate the model based on $\{Y_n^*, X_n, W_n\}$, and then compute $T_n(\hat{\lambda}_n)$ and denote its value as $\hat{T}_n^b(\hat{\lambda}_n)$,

(d) Repeat (a)-(c) B times to obtain the EDF $\hat{\mathcal{G}}_n$ of $\{\hat{T}_n^b(\hat{\lambda}_n), b = 1, \cdots, B\}$. The quantiles of $\hat{\mathcal{G}}_n$ give the bootstrap critical values of $T_n(\lambda_0)$.

Among the five resampling schemes ($\texttt{RS}_{uu}$, $\texttt{RS}_{rr}$, $\texttt{RS}_{ur}$, $\texttt{RS}_{ru}$, $\texttt{RS}_{uf}$) described above, $\texttt{RS}_{rr}$ is the simplest as the estimation of λ is not required in either the model estimation based on the original data or the model estimation based on the bootstrap data. This method is attractive, but it is valid only under special scenarios. Other schemes all require the estimation of λ based on the original data, but not based on the bootstrapped data, to be in line with the LM principle. The proposed bootstrap methods preserve the feature of LM tests in the process of bootstrapping the values of the test statistic, thus greatly alleviate the computational burden as compared with bootstrapping, e.g., a Wald type test, or a likelihood ratio type test where the full model is estimated in every bootstrap sample. This point is particularly relevant to the tests of spatial dependence as spatial parameters often enter the model in a nonlinear fashion, and hence the estimation of them must be through a numerical optimization, which is avoided by the LM tests.

2.2 Validity of the methods and simple justifications

Assume as in the standard likelihood-based inferences that (i) $T_n(\lambda_0)$ is asymptotically pivotal under H_0 when $\mathcal{F}$ is correctly specified; (ii) under H_0, $(\tilde{\theta}_n, \tilde{\mathcal{F}}_n)$ is consistent for $(\theta, \mathcal{F})$; (iii) whether or not H_0 is true, $(\hat{\theta}_n, \hat{\mathcal{F}}_n)$ is consistent for $(\theta, \mathcal{F})$. The validity of the bootstrap methods given above is summarized below. Simple justifications are given.

Proposition A. *Under assumptions (i)-(iii), the bootstrap methods under* $\texttt{RS}_{uu}$ *and* $\texttt{RS}_{uf}$ *are generally valid in that they are both able to provide asymptotic refinements on the critical values of the LM tests.*

Proposition B. *If either* $T_n(\lambda_0)$ *is robust against distributional misspecification or a robust version of it is used, then* $\tilde{\mathcal{F}}_n$ *can be used in place of* $\hat{\mathcal{F}}_n$*, and thus the bootstrap method with* $\texttt{RS}_{ur}$ *is also valid.*

Proposition C. *Under assumptions (i)-(iii), if either* $\tilde{\theta}_n$ *is also consistent when* H_0 *is false or* $T_n(\lambda_0)$ *is invariant of* θ*, then* $\tilde{\theta}_n$ *can be used in place of* $\hat{\theta}_n$ *and thus the bootstrap method with* $\texttt{RS}_{ru}$ *is also valid.*

Proposition D. *Under assumptions (i)-(iii), if the conditions for both Propositions B and C hold, then the all the five bootstrap methods are valid.*

The above four propositions clearly spell out the framework under which the use of unrestricted/restricted estimates of nuisance parameters, and unrestricted/restricted residuals leads to refined approximations to the finite sample critical values of an LM test.

To help gain insights into these results, consider a classical linear regression model: $Y_n = X_{1n}\beta_1 + X_{2n}\beta_2 + \sigma e_n$ and the hypothesis $H_0 : \beta_2 = 0$.[5] In this case $\theta = (\beta_1', \sigma)'$,

[5] The test of linear constrains in regression coefficients considered in van Giersbergen and Kiviet (2002) can be written in this form through reparameterization.

$\lambda = \beta_2$, and $T_n(\lambda)$ is the LM test which is nR_a^2, n times the R-squared from an auxiliary regression of $Y_n - X_1\tilde{\beta}_{1n}$ on X_{1n} and X_{2n}. Some algebra leads to

$$T_n(0) = nR_n^2 \stackrel{H_0}{=} \frac{ne_n' M_{1n}(I_n - M_n)M_{1n}e_n}{e_n' M_{1n} e_n},$$

where $M_{1n} = I_n - X_{1n}(X_{1n}'X_{1n})^{-1}X_{1n}'$, I_n is an $n \times n$ identity matrix, and $M_n = I_n - X_n(X_n'X_n)^{-1}X_n'$ with $X_n = \{X_{1n}, X_{2n}\}$. Clearly, (i) nR_n^2 is invariant of $\theta = \{\beta_1', \sigma\}'$, (ii) $\hat{\theta}_n$ is consistent for θ, but $\tilde{\theta}_n$ is consistent only when H_0, and (iii) nR_n^2 is an F exactly if $\mathcal{F}$ is exact normal, otherwise it is a chi-square asymptotically (Cameron and Trivedi, 2005, p. 93; Wooldridge, 2010, p. 62). The bootstrap DGP that mimics the real world null DGP is: $Y_n^* = X_{1n}\ddot{\beta}_{1n} + \ddot{\sigma}_n e_n^*$ where the elements of e_n^* are iid $\ddot{\mathcal{F}}_n$. Estimating the null model using the bootstrap data (Y_n^*, X_{1n}) leads to the bootstrap analogue of the LM statistic

$$T_n^*(0) = nR_n^{*2} = \frac{ne_n^{*\prime} M_{1n}(I_n - M_n)M_{1n}e_n^*}{e_n^{*\prime} M_{1n} e_n^*}.$$

As $T_n^*(0)$ is invariant of $\ddot{\theta}_n \equiv \{\ddot{\beta}_{1n}', \ddot{\sigma}_n\}'$, either $\tilde{\theta}_n$ or $\hat{\theta}_n$ can be used for $\ddot{\theta}_n$. Now, the EDF $\hat{\mathcal{F}}_n$ of the (standardized) unrestricted residuals $\hat{e}_n = \hat{\sigma}_n^{-1}(Y_n - X_n\hat{\beta}_n)$ is consistent for $\mathcal{F}$ because $(\hat{\beta}_n, \hat{\sigma}_n)$ is consistent for (β, σ), thus drawing e_n^* from $\hat{\mathcal{F}}_n$ leads to a valid bootstrap procedure. The EDF $\tilde{\mathcal{F}}_n$ of the (standardized) restricted residuals $\tilde{e}_n = \tilde{\sigma}_n^{-1}(Y_n - X_{1n}\tilde{\beta}_{1n})$ is inconsistent for $\mathcal{F}$ when H_0 is false because $\tilde{\theta}_n$ is inconsistent for θ (Cameron and Trivedi, 2005, p. 93; Godfrey, 2009, p. 108). However, LM is robust against misspecification in $\mathcal{F}$, hence LM* is also robust against the misspecification in $\ddot{\mathcal{F}}_n$. It follows that using $\tilde{\mathcal{F}}_n$ for $\ddot{\mathcal{F}}_n$ also leads to a valid bootstrap procedure in obtaining the finite sample critical values of the LM statistic for testing $H_0 : \beta_2 = 0$. Monte Carlo results (not reported for brevity) confirm these points.[6] Many LM tests exhibit the same phenomenon as the above LM test for regression coefficients, e.g., Moran's I or Burridge's (1980) tests of spatial error dependence (to be discussed in detail in the next section), LM test for heteroscedasticity (Breusch and Pagan, 1979), LM test for random effects (Breusch and Pagan, 1980), etc.

In practice, however, many LM tests do not enjoy the nice properties as the above LM test does, i.e., the statistic itself is invariant of the nuisance parameters or the restricted estimates of the nuisance parameters are consistent even when H_0 is false; and/or the EDF of the restricted residuals is consistent even when H_0 is false or the statistic itself is robust against the misspecification in the error distribution. We now present some general arguments for the validity of the propositions stated above. Based on the general model specified in (1) and (2), the general hypothesis stated therein, and the LM statistic $T_n(\lambda_0)$,

[6]Monte Carlo results also show that the LM test nR_n^2 referring to asymptotic critical values performs quite well in finite samples, it is thus less attractive to consider the bootstrap method in these scenarios. Furthermore, due to its invariance to the nuisance parameters and it robustness to the distributional misspecification, comparing various bootstrap procedures based on this type of LM tests may not be informative.

we have by (2) and under H_0, i.e., under the real world null DGP: $Y_n = h(X_n, W_n, \theta, \lambda_0; e_n)$,

$$\begin{aligned} T_n(\lambda_0)|_{H_0} &\equiv T_n(Y_n, X_n, W_n; \lambda_0) \\ &= T_n[h(X_n, W_n, \theta, \lambda_0; e_n), X_n, W_n; \lambda_0] \\ &\equiv T_n(X_n, W_n, \theta, \lambda_0; e_n). \end{aligned}$$

The bootstrap DGP that mimics the real world null DGP is $Y_n^* = h(X_n, W_n; \ddot{\theta}_n, \lambda_0; e_n^*)$, where $e_n^* \overset{iid}{\sim} \ddot{\mathcal{F}}_n$. Based on the bootstrap data (Y_n^*, X_n, W_n), estimating the null model and computing the bootstrap analogue of $T_n(\lambda_0)$, we have

$$\begin{aligned} T_n^*(\lambda_0) &\equiv T_n(Y_n^*, X_n, W_n; \lambda_0) \\ &= T_n[h(X_n, W_n, \ddot{\theta}_n, \lambda_0; e_n^*), X_n, W_n; \lambda_0] \\ &\equiv T_n(X_n, W_n, \ddot{\theta}_n, \lambda_0; e_n^*). \end{aligned}$$

$T_n(\lambda_0)$ and $T_n^*(\lambda_0)$ are identical in structure. Thus, it is obvious that if $\ddot{\theta}_n$ and $\ddot{\mathcal{F}}_n$ are both consistent, then $T_n^*(\lambda_0)$ is consistent for $T_n(\lambda_0)$ in the sense that the difference between the two is of oder $o_p(1)$. As one does not know in practice whether or not H_0 is true, consistency of $T_n^*(\lambda_0)$ would occur or is guaranteed to occur in general only by using $\hat{\theta}_n$ for $\ddot{\theta}_n$ and $\hat{\mathcal{F}}_n$ for $\ddot{\mathcal{F}}_n$. This justifies the general statements in Proposition 1. Obviously, if $\tilde{\theta}_n$ is consistent in general or $T_n(\lambda_0)$ is invariant of θ, then $\tilde{\theta}_n$ can be used in place of $\hat{\theta}_n$. Further, if $\tilde{\mathcal{F}}_n$ is consistent in general or $T_n(\lambda_0)$ is robust against misspecification in the distribution of e_n, $\tilde{\mathcal{F}}_n$ can be used in place of $\hat{\mathcal{F}}_n$. These justify the statements in Propositions 2-4. While we emphasize that only $\texttt{RS}_{uu}$ and $\texttt{RS}_{uf}$ are valid in general, $\texttt{RS}_{rr}$ should be used whenever it is valid to do so, due to the simplicity of the restricted estimates and EDF (and perhaps stability too).

The above arguments give general principles on the proper ways to set up the bootstrap DGP in bootstrapping the critical values of LM tests, and settle the debate on the choice of residuals (e.g., van Giersbergen and Kiviet (2002), MacKinnon (2002), and Godfrey (2009)) within the LM framework. For the reason on why bootstrap is able to provide more accurate approximation to the finite sample null distribution of $T_n(\lambda_0)$, we direct the readers to Beran (1988), Horowitz (1994) and Hall and Horowitz (1996).[7]

In what follows, the set of notation used above will be followed closely. Specifically, Y_n denotes an $n \times 1$ vector of response values, X_n an $n \times k$ matrix containing the values of regressors with its first column being a column of ones, W_n is an $n \times n$ spatial weight matrix, $\mathcal{F}$ the CDF of the standardized errors $\{e_{n,i}\}$, '˜' means `restricted` and 'ˆ' means `unrestricted`, ' * ' corresponds to the bootstrap DGP, etc.

[7] Godfrey (2009, p. 82) remarked that there are many published results on the asymptotic refinements associated with bootstrap tests. This literature is technical and sometimes involves relatively complex asymptotic analysis. However, it is not always the case that such asymptotic analysis seems to provide a good explanation of what is observed in finite samples. See also Davidson (2007) for some similar remarks.

3 Bootstrap LM Tests for Spatial Dependence

In this section, we consider several popular spatial LM tests to demonstrate the general methodology described in the last section. These include the LM tests for spatial error dependence (SED), the LM tests for spatial lag dependence (SLD), and the LM tests for spatial error components (SEC), presented respectively in Subsections 3.1-3.3. In each subsection, we present the LM tests (existing or new), arguments for the validity of the five bootstrap methods to supplement the general theoretical arguments presented in Section 2, and Monte Carlo results to support these arguments. Some theoretical results are given in Appendix A. The general setting of the Monte Carlo experiment (methods for generating the regressors, error distributions and spatial layouts) is described in Appendix B.

We adopt the usual notation: E^*, Var^*, $\xrightarrow{D^*}$, $\xrightarrow{p^*}$, $o_{p^*}(\cdot)$, etc., to indicate that the expectation, variance, convergence in distribution, convergence in probability, smaller order of magnitude in probability, etc., are with respect to the bootstrap error distribution ($\ddot{\mathcal{F}}_n$), to distinguish them from the usual notation corresponding the error distribution $\mathcal{F}$. Further, let $\mathrm{tr}(A)$ denote the trace and $\mathrm{diagv}(A)$ the column vector formed by the diagonal elements of a square matrix A.

3.1 Linear Regression with Spatial Error Dependence

We consider the LM test of Burridge (1980) (or Moran's I) and the standardized LM test of Baltagi and Yang (2010). As shown in Kelejian and Prucha (2001), and Baltagi and Yang (2010), these tests are robust against normality. Further, they are invariant of the nuisance parameters. Hence, according the general principles laid in Section 2, all bootstrap methods are valid when applied to those tests, and thus the one under $\texttt{RS}_{rr}$ is recommended. The purposes of using this model are two fold: one is to provide theoretical justifications for the validity of various bootstrap procedures when applied to the above-mentioned tests, and the other is to demonstrate that these tests indeed can have much better size and power properties when referring to the bootstrap critical values than when referring to asymptotic ones, in particular when one-sided alternatives are considered. While the literature does contain some works on bootstrap tests for this model (Lin et al., 2010) it seems to be lacking on both theoretical justifications and detailed comparisons on various bootstrap methods.

3.1.1 The model and the LM tests.

The linear regression model with spatial error dependence takes the form:

$$Y_n = X_n\beta + u_n, \quad u_n = \rho W_n u_n + \varepsilon_n, \quad \varepsilon_n = \sigma e_n \tag{4}$$

where the elements of the error vector e_n are iid with zero mean, unit variance, and CDF $\mathcal{F}$, ρ is the spatial parameter, β is $k\times 1$ vector of regression coefficients, and σ is the error standard deviation. It is evident that this model falls into the general framework of Model (1) with $\theta = \{\beta', \sigma\}'$, $\lambda = \rho$, $e_n = q(Y_n, X_n, W_n; \theta, \lambda) = B_n(\rho)(Y_n - X_n\beta)/\sigma$, and its inverse $Y_n = X_n\beta + \sigma B^{-1}(\rho)e_n$, where $B_n(\rho) = I_n - \rho W_n$. Given ρ, the restricted QMLEs of β and σ^2 under the Gaussian likelihood are $\tilde{\beta}_n(\rho) = [X_n' B_n(\rho)' B_n(\rho) X_n]^{-1} X_n' B_n(\rho)' B_n(\rho) Y_n$ and $\tilde{\sigma}_n^2(\rho) = \frac{1}{n} Y_n' B_n(\rho)' M_n(\rho) B_n(\rho) Y_n$, where $M_n(\rho) = I_n - B_n(\rho) X_n [X_n' B_n(\rho)' B_n(\rho) X_n]^{-1} X_n' B_n(\rho)'$. Maximizing the concentrated quasi Gaussian likelihood of ρ numerically leads to the unrestricted QMLE $\hat{\rho}_n$ of ρ, which upon substitutions gives the unrestricted QMLEs $\hat{\beta}_n \equiv \tilde{\beta}_n(\hat{\lambda}_n)$ and $\hat{\sigma}_n^2 \equiv \tilde{\sigma}_n^2(\hat{\lambda}_n)$ of the nuisance parameters β and σ^2.

We are interested in testing the lack of SED in the model, i.e., $H_0 : \rho = 0$ vs $H_a : \rho \neq 0$ $(< 0, > 0)$, based on the LM principle. To allow for a certain degree of generality, we derive an LM statistic under a general value of ρ (details in Appendix A):

$$\mathrm{LM}_{\mathtt{SED}}(\rho) = \frac{n}{\sqrt{K_n(\rho)}} \frac{\tilde{\varepsilon}_n(\rho)' Q_n^{\circ}(\rho) \tilde{\varepsilon}_n(\rho)}{\tilde{\varepsilon}_n(\rho)' \tilde{\varepsilon}_n(\rho)}, \tag{5}$$

where $\tilde{\varepsilon}_n(\rho) = B_n(\rho)(Y_n - X_n \tilde{\beta}_n(\rho))$, $Q_n^{\circ}(\rho) = Q_n(\rho) - \frac{1}{n}\mathrm{tr}[Q_n(\rho)] I_n$, $Q_n(\rho) = W_n B_n^{-1}(\rho)$, and $K_n(\rho) = \mathrm{tr}[Q_n^{\circ}(\rho)^2 + Q_n^{\circ}(\rho)' Q_n^{\circ}(\rho)]$. The LM statistic (5) can be used to perform a general test for a possible value of ρ or can be used to construct confidence intervals for ρ without having to estimate it. When $\rho = 0$, it reduces to the LM test of Burridge (1981):

$$\mathrm{LM}_{\mathtt{SED}} = \frac{n}{\sqrt{K_n}} \frac{\tilde{\varepsilon}_n' W_n \tilde{\varepsilon}_n}{\tilde{\varepsilon}_n' \tilde{\varepsilon}_n}, \tag{6}$$

where $\tilde{\varepsilon}_n \equiv \tilde{\varepsilon}_n(0)$ and $K_n \equiv K_n(0) = \mathrm{tr}(W_n' W_n + W_n W_n)$.

To improve the finite sample performance and to enhance the robustness of the LM statistic (5), we derive an standardized version of it by centering and rescaling its numerator (details in Appendix A). The standardized LM statistic for ρ takes the form:

$$\mathrm{SLM}_{\mathtt{SED}}(\rho) = \frac{\tilde{\varepsilon}_n(\rho)' C_n(\rho) \tilde{\varepsilon}_n(\rho)}{\tilde{\sigma}_n^2(\rho)[K_n^{\dagger}(\rho) + \tilde{\kappa}_n(\rho) a_n'(\rho) a_n(\rho)]^{\frac{1}{2}}}, \tag{7}$$

where $C_n(\rho) = Q_n(\rho) - \frac{1}{n-k}\mathrm{tr}[M_n(\rho) Q_n(\rho)] M_n(\rho)$, $K_n^{\dagger}(\rho) = \mathrm{tr}[M_n(\rho) C_n(\rho) M_n(\rho)(C_n(\rho) + C_n(\rho)')]$, $a_n(\rho) = \mathrm{diagv}[M_n(\rho) Q_n(\rho) M_n(\rho)]$, $\tilde{\kappa}_n(\rho)$ is the excess kurtosis of $\tilde{\varepsilon}_n(\rho)$. Again, the SLM given in (7) can be used to perform a general test or construct a confidence interval for ρ, which is expected to perform better due to standardization. However, the main purpose of introducing the general statistics (5) and (7) here is for the implementation of the bootstrap method with $\mathtt{RS_{uf}}$. When $\rho = 0$, $\mathrm{SLM}_{\mathtt{SED}}(\rho)$ reduces to the standardized LM test of Baltagi and Yang (2010):

$$\mathrm{SLM}_{\mathtt{SED}} = \frac{n}{\sqrt{K_n^{\dagger} + \tilde{\kappa}_n a_n' a_n}} \frac{\tilde{\varepsilon}_n' C_n \tilde{\varepsilon}_n}{\tilde{\varepsilon}_n' \tilde{\varepsilon}_n}, \tag{8}$$

where $C_n \equiv C_n(0) = W_n - \frac{1}{n-k}\mathrm{tr}(W_n M_n) M_n$, $M_n = M_n(0)$, $K_n^{\dagger} = K_n^{\dagger}(0)$, and similarly defined for the other quantities.

3.1.2 Validity of the bootstrap methods

To see the validity of the various bootstrap methods presented in Section 2, we concentrate on $\text{LM}_{\texttt{SED}}$. Under the real world null DGP: $Y_n = X_n\beta + \sigma e_n$, $\tilde{\varepsilon}_n = \sigma M_n e_n$, and

$$\text{LM}_{\texttt{SED}}|_{H_0} = \frac{n}{\sqrt{K_n}} \frac{e_n' M_n W_n M_n e_n}{e_n' M_n e_n}. \tag{9}$$

which shows $\text{LM}_{\text{SED}}|_{H_0}$ is invariant of the nuisance parameters, and thus a pivot if $\mathcal{F}$ is known, e.g., $N(0,1)$. In this situation, one can simply use Monte Carlo method to find the finite sample critical values of $\text{LM}_{\text{SEC}}|_{H_0}$ to any level of accuracy. To be exact, one draws e_n from the known distribution $\mathcal{F}$ repeatedly to give a sequence of values for $\text{LM}_{\text{SED}}|_{H_0}$, and then find the quantiles of this sequence that serve as approximations to the finite sample quantiles of $\text{LM}_{\text{SEC}}|_{H_0}$. When $\mathcal{F}$ is unknown or is misspecified, however, $\text{LM}_{\text{SED}}|_{H_0}$ is not an exact pivot, hence the Monte Carlo method just described does not work and bootstrap needs to be called for. Now, it is easy to show that $\text{LM}_{\text{SED}}|_{H_0}$ is an asymptotic pivot by using the fact that the diagonal elements of W_n are zero.[8] Thus, all the five bootstrap procedures are valid. Some details are as follows.

In the bootstrap world, the bootstrap DGP that mimics the real world null DGP is $Y_n^* = X_n\ddot{\beta}_n + \ddot{\sigma}_n e_n^*$, where the elements of e_n^* are random draws from $\ddot{\mathcal{F}}_n$, the EDF of standardized residuals. Based on the bootstrap data (Y_n^*, X_n), computing the OLS estimates of $(\ddot{\beta}_n, \ddot{\sigma}_n)$, the OLS residuals and the LM test (6), we have the bootstrap analogue of $\text{LM}_{\texttt{SED}}|_{H_0}$:

$$\text{LM}_{\texttt{SED}}^* = \frac{n}{\sqrt{K_n}} \frac{e_n^{*\prime} M_n W_n M_n e_n^*}{e_n^{*\prime} M_n e_n^*}, \tag{10}$$

which shows that $\text{LM}_{\texttt{SED}}^*$ is invariant of $\ddot{\beta}_n$ and $\ddot{\sigma}_n^2$. Thus, whether $\ddot{\beta}_n$ and $\ddot{\sigma}_n^2$ correspond to the restricted or unrestricted estimates of β and σ makes no difference on the performance of the bootstrap procedures (in fact, any values within the parameter space can be used).

Comparing (10) with (9), it is intuitively quite clear that if e_n^* are drawn from an EDF $\ddot{\mathcal{F}}$ that consistently estimates $\mathcal{F}$ whether or not H_0 is true, then the EDF of LM_{SED}^* offers a consistent estimate of the finite sample distribution of $\text{LM}_{\text{SED}}|_{H_0}$. This is just like the Monte Carlo approach under a known $\mathcal{F}$, but with $\ddot{\mathcal{F}}$, the finite sample distribution of $\text{LM}_{\texttt{SED}}|_{H_0}$ is estimated nonparametrically. With this in mind, the attractiveness of the bootstrap approach becomes clearer. Now, for the choice of $\ddot{\mathcal{F}}$, assume in general $\ddot{\mathcal{F}}$ has a zero mean and a unit variance (which is achievable by centering and rescaling the residuals), and let $\ddot{\kappa}_n$ be the excess kurtosis of $\ddot{\mathcal{F}}_n$. We have by Lemma A1(iii) in Appendix A,

$$\begin{aligned} \text{E}^*(e_n^{*\prime} M_n W_n M_n e_n^*) &= \text{tr}(M_n W_n); \text{ and} \\ \text{Var}^*(e_n^{*\prime} M_n W_n M_n e_n^*) &= \text{tr}(B_n' B_n + B_n^2) + \ddot{\kappa}_n b_n' b_n, \end{aligned}$$

[8] See Kelejan and Prucha (2001) and Baltagi and Yang (2010) for indirect or direct proofs of this results. Note that this type of results may not hold in general (see Section 3.3).

where E^* and Var^* are the expectation and variance operators with respect to an arbitrary bootstrap error distribution $\ddot{\mathcal{F}}_n$, $B_n = M_n W_n M_n$ and $b_n = \mathrm{diagv}(B_n)$. It is easy to verify that $\lim_{n\to\infty} K_n^{-1}\mathrm{tr}(M_n W_n) = 0$, and that $\lim_{n\to\infty} K_n^{-1}\mathrm{tr}(B_n' B_n + B_n^2) = 1$. By the central limit theorem for linear-quadratic forms given in Kelejian and Prucha (2001), or its simpler version given in Appendix A, we have,

$$\frac{e_n^{*\prime} M_n W_n M_n e_n^*}{\sqrt{K_n}} \xrightarrow{D^*} N(0, 1+\tau^2),$$

where $\tau^2 = \lim_{n\to\infty} \ddot{\kappa}_n \frac{b_n' b_n}{K_n}$. Now, by law of large numbers, $e_n^{*\prime} M_n e_n^*/n \xrightarrow{p^*} 1$. It follows by Slutsky's theorem,

$$\mathrm{LM}_{\mathtt{SED}}^* \xrightarrow{D^*} N(0, 1+\tau^2).$$

Obviously, for the bootstrap distribution of $\mathrm{LM}_{\mathtt{SED}}^*$ (hence its quantiles) to mimic the finite sample distribution of $\mathrm{LM}_{\mathtt{SED}}|_{H_0}$ (hence its quantiles), it is necessary that their asymptotic distributions agree. This occurs only when $\tau = 0$ which implies either $\ddot{\kappa}_n = o_{p*}(1)$ or $\frac{b_n' b_n}{K_n} = o_{p*}(1)$. The former is true when $\ddot{\mathcal{F}}_n = \hat{\mathcal{F}}_n$ but not when $\ddot{\mathcal{F}}_n = \tilde{\mathcal{F}}_n$, and the latter is always true, implying that inconsistency of $\tilde{\mathcal{F}}_n$ is compensated by the smaller magnitude of $\frac{b_n' b_n}{K_n}$, which makes $\tilde{\mathcal{F}}_n$ a valid candidate for $\ddot{\mathcal{F}}_n$.

3.1.3 Monte Carlo Results

The Monte Carlo experiments are carried out based on the following DGP:

$$Y_n = \beta_0 1_n + X_{n1}\beta_1 + X_{n2}\beta_2 + u_n, \quad u_n = \rho W_n u_n + \sigma e_n.$$

The parameter values are set at $\beta = \{5, 1, 1\}'$ and $\sigma = 1$ or 2. Four different sample sizes are considered, i.e., $n = 50, 100, 200$, and 500. All results are based on M=2,000 Monte Carlo samples, and B=699 bootstrap samples for each Monte Carlo sample. The methods of generating spatial layouts, error distributions, and regressors's values are described in Appendix B. The regressors are treated as fixed in the experiments.

For $\rho = \{-0.5, -0.25, 0, 0.25, 0.5\}$, two types of Monte Carlo results are recorded: (a) the means and standard deviations of the bootstrap critical values, and (b) the rejection frequencies of the LM and SLM tests. As the tests are invariant of the nuisance parameters, the results under $\mathtt{RS}_{ur}$ coincide with those under $\mathtt{RS}_{rr}$, and the results under $\mathtt{RS}_{ru}$ are identical to those under $\mathtt{RS}_{uu}$. Also, the results under $\mathtt{RS}_{uf}$ are very close to those under $\mathtt{RS}_{uu}$, and hence are not reported for brevity. Furthermore, the bootstrap results for $\mathtt{SLM}_{\mathtt{SED}}$ are also not reported as the rejection frequencies are almost identical to those for $\mathtt{LM}_{\mathtt{SED}}$, and the critical values, though different from those for $\mathtt{LM}_{\mathtt{SED}}$, show the same degree of stability and agreement with the 'true' finite sample critical values by Monte Carlo methods. Finally, a small sets of results are reported in Table 3.1a for the (average) bootstrap critical values and 3.1b for rejection frequencies. General observations are summarized as follows:

1. The (average) bootstrap critical values are all very close to the 'true' finite sample critical values (obtained by Monte Carlo simulation), but can all be far from their asymptotic critical values which are ± 1.6449 and ± 1.96. The implication of this is clear: the use of asymptotic critical values may lead to large distortions on size and power of the tests. Working with SLM improves in this regard from a two-sided test point of view, but it is still not satisfactory if one sided tests are desired;
2. The bootstrap critical values for both LM and SLM under all resamling methods are all very stable with respect to the alternative values of ρ, confirming the validity of all the five bootstrap methods.
3. The standard deviations of the bootstrap critical values (not reported for brevity) are all small, in the magnitudes of (0.0425, 0.0376, 0.1042, 0.1363) for the four critical values of the $\texttt{LM}_{\texttt{SED}}$ test under normal errors. They increase a little bit when errors are nonnormal or when $\texttt{SLM}_{\texttt{SED}}$ is used; they don't change much with n but decrease when B increases (both are as expected). As far as the rejection frequency is concerned we found that using $B = 699$ is sufficient;
4. Use of the bootstrap critical values significantly improves the size of the LM tests, and the power of the left-tailed LM tests.
5. When the regressors are generated under the iid setting (`XVAL-A`), the finite sample distribution of $\texttt{LM}_{\texttt{SED}}$ is more skewed to the right, which makes the left-tail rejection frequencies much lower than their nominal values. Use of a denser spatial weight matrix worsens this problem. However, in all these scenarios, the standardization method helps and the bootstrap methods work bell.

`<< Insert Table 3.1a Here >>`

A note in passing to read Table 3.1b is that the values under the column of $|\rho|$ should read as negative if L2.5% and L5%, i.e., the left-tailed 2.5% and 5% tests, are concerned. All results in Table 3.1b correspond to $\texttt{LM}_{\texttt{SED}}$, except the rows labeled with `ACR*` which correspond to $\texttt{SLM}_{\texttt{SED}}$ referring to the asymptotic critical values.

`<< Insert Table 3.1b Here >>`

3.2 Linear Regression with Spatial Lag Dependence

We now present a case where the restricted estimates of nuisance parameters are inconsistent when the null hypothesis is false, but the LM statistic at the null is still robust against nonnormality. According to the general results presented in Section 2, only the bootstrap methods under $\texttt{RS}_{uu}$ and $\texttt{RS}_{uf}$ are valid. As this case is more involved, a more detailed study is given. This study contributes to the spatial econometrics literature by

(i) providing theoretical justifications on the validity of various bootstrap methods with respect to the choice of bootstrap parameters and the choice of bootstrap error distribution, and (ii) providing detailed Monte Carlo results to support these theoretical arguments, in particular the results on the bootstrap critical values. Common Monte Carlo study on the performance of bootstrap tests typically reports the empirical rejection frequencies (size and power). This study reveals that judging a bootstrap test only based on power may be misleading as in reality one does not know whether or not the null hypothesis is true, and hence the seeming higher power of certain tests may not be achievable. Some related works can be found in Lin et al. (2007, 2009).

3.2.1 The Model and the LM Tests.

The linear regression model with spatial lag dependence (SLD), also known as the spatial autoregressive (SAR) model, takes the following form:

$$Y_n = \lambda W_n Y_n + X_n\beta + \varepsilon_n, \quad \varepsilon_n = \sigma e_n \tag{11}$$

where the elements of the error vector e_n are iid with zero mean, unit variance and CDF $\mathcal{F}$, λ is the spatial parameter, and β is a $k \times 1$ vector of regression coefficient. Clearly, Model (11) fits into the general framework of Model (1) with $\theta = \{\beta', \sigma^2\}'$, $e_n = q(Y_n, X_n, W_n; \theta, \lambda) = [A_n(\lambda)Y_n - X_n\beta]/\sigma$, and its inverse $Y_n = h(X_n, W_n; \theta, \lambda; e_n) = A_n^{-1}(\lambda)(X_n\beta + \sigma e_n)$, where $A_n(\lambda) = I_n - \lambda W_n$. Given λ, the restricted QMLEs of β and σ^2 based on the Gaussian likelihood are, respectively, $\tilde\beta_n(\lambda) = (X_n'X_n)^{-1}X_n'A_n(\lambda)Y_n$, and $\tilde\sigma_n^2(\lambda) = \frac{1}{n}Y_n'A_n'(\lambda)M_nA_n(\lambda)Y_n$, where $M_n = I_n - X_n(X_n'X_n)^{-1}X_n'$. Maximizing the concentrated Gaussian likelihood for λ gives the unrestricted QMLE $\hat\lambda_n$, which upon substitutions gives the unrestricted QMLEs of β and σ^2 as $\hat\beta_n \equiv \tilde\beta_n(\hat\lambda_n)$, and $\hat\sigma_n^2 \equiv \tilde\sigma_n^2(\hat\lambda_n)$.

We are interested in making inferences for the spatial parameter λ in the SAR model. In particular, we are interested in the score-based inferences as they do not require the estimation of the spatial parameter, and thus avoid the numerical optimization which can be computationally demanding for large sample sizes and general spatial weight matrices. The classical inferences of this type under normality assumption are readily available based on the results of Anselin (1988a,b) and Lee (2004). In particular, the score or LM test for testing the lack of spatial lag dependence, i.e., $H_0: \lambda = 0$ vs $H_a: \lambda \neq 0$ (< 0, > 0), in the regression model is given in Anselin (1988a):

$$\mathrm{LM}_{\mathtt{SLD}} = \frac{\tilde\varepsilon_{n0}' W_n Y_n}{\tilde\sigma_{n0}\sqrt{\tilde\eta_{n0}' M_n \tilde\eta_{n0} + \tilde\sigma_{n0}^2 K_{n0}}}, \tag{12}$$

where $K_{n0} = \mathrm{tr}(W_n'W_n + W_nW_n)$, $\tilde\eta_{n0} = W_nX_n\tilde\beta_n(0)$, $\tilde\sigma_{n0}^2 = \tilde\sigma_n^2(0)$, and $\tilde\varepsilon_{n0} = Y_n - X_n\tilde\beta_n(0)$. Yang and Shen (2011) generalized (12) to give a general test statistic for λ,

$$\mathrm{LM}_{\mathtt{SLD}}(\lambda) = \frac{\tilde\varepsilon_n(\lambda)' G_n^\circ(\lambda) A_n(\lambda) Y_n}{\tilde\sigma_n\sqrt{\tilde\eta_n' M_n \tilde\eta_n + \tilde\sigma_n^2 K_n(\lambda)}}, \tag{13}$$

where $K_n(\lambda) = \text{tr}[G_n^\circ(\lambda)^2 + G_n^\circ(\lambda)' G_n^\circ(\lambda)]$, $G_n^\circ(\lambda) = G_n(\lambda) - \frac{1}{n}\text{tr}(G_n(\lambda))I_n$, $G_n(\lambda) = W_n A_n^{-1}(\lambda)$, $\tilde{\eta}_n \equiv \tilde{\eta}_n(\lambda) = G_n(\lambda)X_n\tilde{\beta}_n(\lambda)$, $\tilde{\sigma}_n^2 \equiv \tilde{\sigma}_n^2(\lambda)$, and $\tilde{\varepsilon}_n(\lambda) = A_n(\lambda)Y_n - X_n\tilde{\beta}_n(\lambda)$. This statistic can be used to test a general hypothesis $H_0 : \lambda = \lambda_0$ or to construct a test-based confidence interval for λ. The purpose of presenting this general LM statistic here is for the implementation of the bootstrap method with $\texttt{RS}_{\texttt{uf}}$. Yang and Shen (2011) show that $\text{LM}_{\texttt{SLD}}(\lambda)$ is robust against non-normality of the error distribution when $\lambda = 0$, but may not be so when $\lambda \neq 0$, and that it is sensitive to the choice of W_n and the value of σ. For these reasons, they provide a robust version of the LM statistic by centering and rescaling $\tilde{\varepsilon}_n(\lambda)' G_n^\circ(\lambda)A_n(\lambda)Y_n$, the key quantity in the concentrated score function for λ,

$$\text{SLM}_{\texttt{SLD}}(\lambda) = \frac{\tilde{\varepsilon}_n'(\lambda)D_n(\lambda)A_n(\lambda)Y_n}{\tilde{\sigma}_n\sqrt{\tilde{\eta}_n' M_n\tilde{\eta}_n + \tilde{\sigma}_n^2 K_n^\dagger(\lambda) + \tilde{\sigma}_n^2\tilde{\kappa}_n d_n' d_n + 2\tilde{\sigma}_n\tilde{\gamma}_n\tilde{\eta}_n' M_n d_n}}, \tag{14}$$

where $K_n^\dagger(\lambda) = \text{tr}[M_n(D_n(\lambda) + D_n'(\lambda))M_n D_n(\lambda)]$, $D_n(\lambda) = G_n(\lambda) - \frac{1}{n-k}\text{tr}(M_n G_n(\lambda))I_n$, $d_n \equiv d_n(\lambda) = \text{diagv}(M_n D_n(\lambda))$, and $\tilde{\gamma}_n$ and $\tilde{\kappa}_n$ are, respectively, the sample skewness and excess kurtosis of the restricted residuals $\tilde{\varepsilon}_n(\lambda)$. When $\lambda = 0$, $\text{SLM}_{\texttt{SLD}}(\lambda)$ reduces to the standardized LM test for testing $H_0 : \lambda = 0$, denoted as $\text{SLM}_{\texttt{SLD}}$. All tests are referred to standard normal as their null distribution. The Monte Carlo results provided in Yang and Shen (2011) show that $\text{SLM}_{\texttt{SLD}}(\lambda)$ is superior to $\text{LM}_{\texttt{SLD}}(\lambda)$ in its finite sample performance.

As discussed in the previous section, for bootstrap to provide asymptotic refinements to the critical values, one needs to work on an asymptotically pivotal statistic. Thus, this standardized LM statistic $\text{SLM}_{\texttt{SLD}}(\lambda)$ may also be useful in bootstrapping the critical values when tests concern a non-zero value of λ. However, our purpose of introducing (13) and (14) is for the implementation of the bootstrap method with $\texttt{RS}_{\texttt{uf}}$.

3.2.2 Validity of the Bootstrap Methods

To study the validity of various resampling schemes when bootstrapping the critical values of the LM and SLM tests of spatial lag dependence, we concentrate on the test $\text{LM}_{\texttt{SLD}}$. Under the real world null DGP: $Y_n = X_n\beta + \sigma e_n$, we have after some algebra,

$$\text{LM}_{\texttt{SLD}}|_{H_0} = \frac{\sigma e_n' M_n W_n e_n + e_n' M_n \eta_n}{(e_n' M_n e_n/n)^{\frac{1}{2}}\left\{\eta_n' M_n\eta_n + \sigma^2 Q(e_n) + 2\sigma e_n' P_n W_n M_n \eta_n\right\}^{\frac{1}{2}}}, \tag{15}$$

where $Q(e_n) = \frac{1}{n}K_{n0}e_n' M_n e_n + e_n' P_n' W_n' M_n W_n e_n$, $\eta_n = W_n X_n \beta$, and $P_n = I_n - M_n$. This shows that $\text{LM}_{\texttt{SLD}}|_{H_0} = f(e_n, X_n, W_n, \beta, \sigma)$, meaning that $\text{LM}_{\texttt{SLD}}|_{H_0}$ is not an exact pivot whether or not $\mathcal{F}$ is known as its finite sample null distribution is governed by $\mathcal{F}$, the CDF of $\{e_{ni}\}$, and the values of the nuisance parameters β and σ, given X_n and W_n. The dependence of $\text{LM}_{\texttt{SLD}}|_{H_0}$ on (β, σ) is expected to impose constrains on the choices of their estimates to be used as parameters in the bootstrap DGP. On the other hand, the limiting distribution of $\text{LM}_{\texttt{SLD}}|_{H_0}$ does not depend on (β, σ) and $\mathcal{F}$ (Kelejian and Prucha, 2001; Yang and Shen, 2011), suggesting (as in Section 3.1.2) that in bootstrapping the critical values of $\text{LM}_{\texttt{SLD}}|_{H_0}$, either the restricted or unrestricted residuals can be used for constructing the bootstrap error distribution. These two points can be seen as follows.

Under the bootstrap world, the bootstrap DGP that mimics the real world null DGP is $Y_n^* = X_n\ddot{\beta}_n + \ddot{\sigma}_n e_n^*$, where the elements of e_n^* are random draws from $\ddot{\mathcal{F}}$. Based on the bootstrap data (Y_n^*, X_n), estimating the bootstrap model and computing the test statistic lead to the bootstrap analogue of $\text{LM}_{\texttt{SLD}}|_{H_0}$:

$$\text{LM}_{\texttt{SLD}}^* = \frac{\ddot{\sigma}_n e_n^{*\prime} M_n W_n e_n^* + e_n^{*\prime} M_n \ddot{\eta}_n}{(e_n^{*\prime} M_n e_n^*/n)^{\frac{1}{2}} \{\ddot{\eta}_n' M_n \ddot{\eta}_n + \ddot{\sigma}_n^2 Q(e_n^*) + 2\ddot{\sigma}_n e_n^{*\prime} P_n W_n M_n \ddot{\eta}_n\}^{\frac{1}{2}}}, \tag{16}$$

where $\ddot{\eta}_n = W_n X_n \ddot{\beta}_n$. By Lemma A1(ii) in Appendix A, we have

$$\begin{aligned} \text{E}^*(\ddot{\sigma}_n e_n^{*\prime} M_n W_n e_n^* + e_n^{*\prime} M_n \ddot{\eta}_n) &= \ddot{\sigma}_n \text{tr}(M_n W_n); \text{ and} \\ \text{Var}^*(\ddot{\sigma}_n e_n^{*\prime} M_n W_n e_n^* + e_n^{*\prime} M_n \ddot{\eta}_n) &= \ddot{\sigma}_n^2 S_n + \ddot{\sigma}_n^2 \ddot{\kappa}_n a_n' a_n + \ddot{\eta}_n' M_n \ddot{\eta}_n + 2\ddot{\sigma}_n \ddot{\gamma}_n a_n' M_n \ddot{\eta}_n, \end{aligned}$$

where $S_n = \text{tr}(W_n' M_n W_n + M_n W_n M_n W_n)$ and $a_n = \text{diagv}(M_n W_n)$. Under regularity conditions, we can easily show that $e_n^{*\prime} M_n e_n^*/n \xrightarrow{p^*} 1$, and that

$$\frac{\ddot{\eta}_n' M_n \ddot{\eta}_n + \ddot{\sigma}_n^2 Q(e_n^*) + 2\ddot{\sigma}_n e_n^{*\prime} P_n W_n M_n \ddot{\eta}_n}{\ddot{\sigma}_n^2 S_n + \ddot{\sigma}_n^2 \ddot{\kappa}_n a_n' a_n + \ddot{\eta}_n' M_n \ddot{\eta}_n + 2\ddot{\sigma}_n \ddot{\gamma}_n a_n' M_n \ddot{\eta}_n} \xrightarrow{p^*} 1.$$

It follows by Slustky's theorem that $\text{LM}_{\texttt{SLD}}^* \xrightarrow{D^*} N(0,1)$. This shows that in bootstrapping the finite sample distribution of $\text{LM}_{\texttt{SLD}}|_{H_0}$, whether using $\tilde{\mathcal{F}}_n$ or $\hat{\mathcal{F}}_n$ as bootstrap error distribution makes no difference. What is more important is the finite sample difference between $\text{LM}_{\texttt{SLD}}^*$ and $\text{LM}_{\texttt{SLD}}|_{H_0}$with respect to the choice of $(\ddot{\beta}_n, \ddot{\sigma}_n)$ in generating the values of $\text{LM}_{\texttt{SLD}}^*$. Now, writing $\ddot{\beta}_n = \beta + \ddot{b}_n$ and $\ddot{\sigma}_n = \sigma + \ddot{s}_n$, we have

$$\text{LM}_{\texttt{SLD}}^{0*} = \frac{\sigma e_n^{*\prime} M_n W_n e_n^* + e_n^{*\prime} M_n \eta_n + c_1(\ddot{b}_n, \ddot{s}_n)}{(e_n^{*\prime} M_n e_n^*/n)^{\frac{1}{2}} \left\{\eta' M_n \eta + \sigma^2 Q(e_n^*) + 2\sigma e_n^{*\prime} P_n W_n M_n \eta_n + c_2(\ddot{b}_n, \ddot{s}_n)\right\}^{\frac{1}{2}}},$$

where $c_1(\ddot{b}_n, s_n) = \ddot{s}_n e_n^{*\prime} M_n W_n e_n^* + e_n^{*\prime} M_n \ddot{b}_n$ and $c_2(\ddot{b}_n, \ddot{s}_n)$ is a similar function of $\ddot{b}_n$ and $\ddot{s}_n$ but with a long expression. If $\ddot{\beta}_n = \hat{\beta}_n$ and $\ddot{\sigma}_n = \hat{\sigma}_n$, i.e., the unrestricted estimates are used as the parameters in the bootstrap DGP, then $\ddot{b}_n = o_p(1)$ and $\ddot{s}_n = o_p(1)$, whether or not H_0 is true, making the order of $c_1(\ddot{b}_n, \ddot{s}_n)$ and $c_2(\ddot{b}_n, \ddot{s}_n)$ smaller than the corresponding neighboring terms. Hence, $\text{LM}_{\texttt{SLD}}^{0*} \sim \text{LM}_{\texttt{SLD}}^{0}$, and the EDF of $\{\text{LM}_{\texttt{SLD}}^{0*}\}$ and therefore its critical values approximate the finite sample distribution of $\text{LM}_{\texttt{SLD}}^{0}$ and therefore its critical values. However, if $\ddot{\beta}_n = \tilde{\beta}_n$ and $\ddot{\sigma}_n = \tilde{\sigma}_n$, i.e., the restricted estimates are used in the bootstrap DGP, it can be easily show that both b_n and s_n do not converge to zero if H_0 is false, with their values dependent upon the true value of λ. Hence, the EDF/quantile of $\{\text{LM}_{\texttt{SLD}}^{0*}\}$ also depends on the true value of λ. This makes the use of restricted estimates in the bootstrap DGP invalid.

From the above discussions we conclude that the bootstrap procedures based on the unrestricted estimates are valid, i.e., $\texttt{RS}_{uu}$, $\texttt{RS}_{ur}$, and $\texttt{RS}_{uf}$ provide valid resampling schemes.

3.2.3 Monte Carlo Results.

The finite sample performance of $\text{LM}^0_{\texttt{SLD}}$ and $\text{SLM}^0_{\texttt{SLD}}$ for testing $H_0^\lambda : \lambda = 0$ vs $H_a^\lambda : \lambda < 0$ or $H_a^\lambda : \lambda > 0$, when referring to the asymptotic critical values and the bootstrap critical values under various resampling schemes, are investigated in terms of accuracy and stability of the bootstrap critical values with respect to the true value of λ, and the size and power of the tests. The Monte Carlo experiments are carried out based on the following data generating process:

$$Y_n = \lambda W_n Y_n + \beta_0 1_n + X_{n1}\beta_1 + X_{n2}\beta_2 + \epsilon_n$$

where the methods for generating W_n, X_n and ϵ_n are described in Appendix B. The regressors are treated as fixed in the experiments. The parameter values are set at $\beta = \{5, 1, 1\}'$ and $\sigma = 1$ or 2, and sample sizes used are $n = (50, 100, 200, 500)$. All results reported below are based on $M = 2,000$ Monte Carlo samples, and $B = 699$ bootstrap samples for each Monte Carlo sample generated. The bootstrap critical values are bench-marked against the Monte Carlo (MC) critical values obtained based on $M = 30,000$ Monte Carlo samples.

Bootstrap critical values. We first report in Table 3.2a the averages of 2,000 bootstrap critical values of $\texttt{LM}^0_{\texttt{SLD}}$ and $\texttt{SLM}^0_{\texttt{SLD}}$ based on the restricted resampling scheme $\texttt{RS}_{\texttt{rr}}$ and the unrestricted resampling scheme $\texttt{RS}_{\texttt{uu}}$. The results with $\texttt{RS}_{\texttt{ru}}$ are very similar to those with $\texttt{RS}_{\texttt{rr}}$ and the results with $\texttt{RS}_{\texttt{ur}}$ and $\texttt{RS}_{\texttt{uf}}$ are very similar to those with $\texttt{RS}_{\texttt{uu}}$, thus, are not reported for saving space. These unreported results show that whether to use the restricted or unrestricted residuals does not affect much the bootstrap critical values, which is consistent with the fact that both $\texttt{LM}^0_{\texttt{SLD}}|_{\texttt{H}_0}$ and $\texttt{SLM}^0_{\texttt{SLD}}|_{\texttt{H}_0}$ are asymptotically pivotal under a general $\mathcal{F}$. Furthermore, Monte Carlo results clearly reveal the following.

1. The bootstrap critical values can be quite different from the corresponding asymptotic critical values, showing the necessity of using finite sample critical values for testing the existence of spatial lag dependence in a linear regression model;
2. The bootstrap critical values based on $\texttt{RS}_{\texttt{rr}}$ (and $\texttt{RS}_{\texttt{ru}}$) vary significantly with λ. This suggests that, if when H_0 is true the bootstrap critical values and the resulted sizes of the tests are accurate (indeed they are), then when H_0 is false, the bootstrap critical values cannot be as accurate and the resulted powers of the tests cannot be reliable;
3. The bootstrap critical values based on $\texttt{RS}_{\texttt{uu}}$ are very stable with respect to λ, and are very accurate as they agree very well with the corresponding Monte Carlo critical values obtained by imposing H_0 and using $M = 30,000$, and with the bootstrap critical values under $\texttt{RS}_{\texttt{rr}}$ and H_0 (considered as an ideal situation).

The bootstrap critical values do not depend much on the error distributions due to the fact that the LM tests involved are asymptotically pivotal at the null under a general $\mathcal{F}$.

As sample size n increases, the bootstrap critical values move closer to their limiting values, but the instability of those based on restricted estimates still exists. The above observations are consistent with the theoretical results: while the tests are asymptotically pivotal, their finite sample distributions depends on the nuisance parameter and the restricted estimates of the nuisance parameters are not consistent when null is false, which make the bootstrap methods based on the restricted estimates invalid.

`<< Insert Table 3.2a Here >>`

Size and power of the tests. We now report in Tables 3.2b and 3.2c the size and power of the one-sided LM tests based on the asymptotic critical values (ACR) and the bootstrap critical values with $\texttt{RS}_{\texttt{rr}}$ and $\texttt{RS}_{\texttt{uu}}$. Again the results based on other three resampling schemes $\texttt{RS}_{\texttt{ru}}$, $\texttt{RS}_{\texttt{ur}}$ and $\texttt{RS}_{\texttt{uf}}$ (not reported for brevity and clarity of presentation) are very close to those based on either $\texttt{RS}_{\texttt{rr}}$ or $\texttt{RS}_{\texttt{uu}}$, showing again the type of residuals to be used in resampling does not affect much the performance of the bootstrap methods. The results further reveal the following.

1. The tests referring to the asymptotic critical values can have severe size-distortion, and more so with heavier spatial dependence. Referring to bootstrap critical values effective remove the size distortions under any resampling method, but one must bear in mind that this is unachievable with bootstrap under the restricted estimates as in practice whether H_0 is true of false remains unknown.
2. Table 3.2a shows that the bootstrap critical values of the LM statistic based on the restricted estimates tend to be larger than the 'true' ones when $\lambda > 0$. As a result, the power is slightly lower compared with that based on the unrestricted estimates. When $\lambda < 0$ the results show other way around. Table 3.2c shows that the bootstrap critical values for SLM statistic based on the restricted estimates is always smaller in magnitude than the true values whenever $\lambda \neq 0$, thus the power for the bootstrap SLM test based on $\texttt{RS}_{\texttt{uu}}$ is always lower than that based on $\texttt{RS}_{\texttt{rr}}$. However, the latter corresponds to a larger size as the underlining bootstrap critical values are smaller.
3. As the original LM test is already asymptotically pivotal and robust, standardization does not provide further improvements on the bootstrap critical values in that the use of restricted estimates still lead to bootstrap critical values that vary with λ.

To summarize, using the restricted estimates of the nuisance parameters in the bootstrap DGP results in bootstrap critical values that can be either larger or smaller than the 'true' ones, leading to a test with either higher or lower power than it supposes to be. In contrast, using the unrestricted parameter estimates leads to test with 'realizable or achievable' power.

`<< Insert Table 3.2b and Table 3.2c Here >>`

Biasness of Restricted Estimators. The biasness of the restricted estimators of the regression coefficients and the error standard deviation when H_0 is false is investigated as it is the major cause of instability of the bootstrap critical values. Under the same setup for the above results, we report in Table 3.2d the empirical means of the restricted and unrestricted estimators. From the result we see that the unrestricted estimators of the regression coefficients can indeed very much biased, and the bias does not go away with larger sample sizes. In contrast, the unrestricted estimators are nearly unbiased, and more so with larger samples. The restricted estimator of σ is also biased when H_0 is false, but in a lesser magnitude compared with the restricted estimators of β.

`<< Insert Table 3.2d Here >>`

3.3 Linear Regression with Spatial Error Components

In this section, we present a case where the usual LM test is not robust against the misspecification of the error distribution $\mathcal{F}$, but its finite sample distribution is invariant of the nuisance parameters (also the restricted estimators of the nuisance parameters are consistent in general). According to the general theories presented in Section 2, the bootstrap methods under $\texttt{RS}_{ru}$, $\texttt{RS}_{uu}$ and $\texttt{RS}_{uf}$ are valid. The results presented in this section contribute to the spatial econometrics literature by providing theoretical justifications and empirical evidence concerning the validity of various bootstrap methods applied to LM and SLM tests of spatial error components.

3.3.1 Model and LM Tests

Kelejian and Robinson (1995) proposed a spatial error components model which provides a useful alternative to the traditional spatial models with a spatial autoregressive (SAR) or a spatial moving average (SMA) error process, in particular in the situation where the range of spatial autocorrelation is constrained to close neighbors, e.g., spatial spillovers in the productivity of infrastructure investments (Kelejian and Robinson, 1997; Anselin and Moreno, 2003). The model takes the following form:

$$Y_n = X_n\beta + u_n, \quad \text{with } u_n = W_n v_n + \varepsilon_n, \text{and } \varepsilon_n = \sigma e_n \tag{17}$$

where Y_n, X_n, W_n and β are as above, v_n is an $n\times 1$ vector of errors that together with W_n incorporates the spatial dependence, and ε_n is an $n\times 1$ vector of location specific disturbance terms. The error components v_n and ε_n are assumed to be independent, with iid elements of mean zero and variances σ_v^2 and σ^2, respectively.

Let $\lambda = \sigma_v^2/\sigma^2$, and $\Omega_n(\lambda) = I_n + \lambda W_n W_n'$, we have $\text{Var}(u_n) = \sigma^2\Omega_n(\lambda)$, and the Gaussian loglikelihood: $\ell_n(\beta, \sigma^2, \lambda) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2}\log|\Omega_n(\lambda)| - \frac{1}{2\sigma^2}(Y_n - X_n\beta)'\Omega_n^{-1}(\lambda)(Y_n - X_n\beta)$.

Given λ, ℓ_n is maximized at $\tilde{\beta}_n(\lambda) = [X_n'\Omega_n^{-1}(\lambda)X_n]^{-1}X_n'\Omega_n^{-1}(\lambda)Y_n$ and $\tilde{\sigma}_n^2(\lambda) = \frac{1}{n}[Y_n - X_n\tilde{\beta}_n(\lambda)]'\Omega_n^{-1}(\lambda)[Y_n - X_n\tilde{\beta}_n(\lambda)]$, giving the concentrated Gaussian log-likelihood for λ: $\ell_n^c(\lambda) = -\frac{n}{2}(\log(2\pi)+1) - \frac{1}{2}\log|\Omega_n(\lambda)| - \frac{n}{2}\tilde{\sigma}_n^2(\lambda)$. Maximizing $\ell_n^c(\lambda)$ gives the unrestricted QMLE $\hat{\lambda}_n$ for λ, which upon substitutions gives the unrestricted QMLEs for β and σ^2 as $\hat{\beta}_n \equiv \tilde{\beta}_n(\hat{\lambda}_n)$ and $\hat{\sigma}_n^2 \equiv \tilde{\sigma}_n^2(\hat{\lambda}_n)$. Although this model is not in the standard form used in Section 2, it can be 'turned' into that form as indicated in the footnote therein. In this case, simply write $u_n = \sigma\Omega_n^{1/2}(\lambda)e_n$, where $\Omega_n^{1/2}(\lambda)$ is the square-root matrix of $\Omega_n(\lambda)$, and $e_n \sim (0, I_n)$ though it is not exactly the same as the e_n in (17) but equivalent in distribution. As far as bootstrap methods is concerned, all it is important is to be able to get a set of residuals whose EDF consistently estimates the true distribution of $e_{n,i}$.

For this model the null hypothesis of no spatial effect can be either $H_0 : \sigma_\nu^2 = 0$, or $\lambda = \sigma_\nu^2/\sigma^2 = 0$. The alternative hypothesis can only be one-sided as σ_ν^2 cannot be negative, i.e., $H_a : \sigma_\nu^2 > 0$, or $\lambda > 0$. Anselin (2001) derived an LM test based on the assumptions that errors are normally distributed, which can be rewritten in a simpler form

$$\mathrm{LM_{SEC}} = \frac{n}{\sqrt{K_n}} \frac{\tilde{\varepsilon}_n' H_n \tilde{\varepsilon}_n}{\tilde{\varepsilon}_n'\tilde{\varepsilon}_n} \tag{18}$$

where $H_n = W_nW_n' - \frac{1}{n}\mathrm{tr}(W_nW_n')I_n$, $K_n = 2\mathrm{tr}(W_nW_n'W_nW_n') - \frac{2}{n}\mathrm{tr}^2(W_nW_n')$, and $\tilde{\varepsilon}_n$ is the vector of OLS residuals. The limiting null distribution of $\mathrm{LM_{SEC}}$ is $N(0,1)$.

Anselin and Moreno (2003) conducted Monte Carlo experiments to assess the finite sample behavior of Anselin's test, the GMM test of Kelejian and Robinson (1995) and Moran's (1950) I test, and found that none seems to perform satisfactorily in general. They recognized that the LM test for spatial error components of Anselin (2001) is sensitive to distributional misspecifications and the spatial layouts. Yang (2010) found out the exact cause of it and provided a robust/standardized LM test, which can be rewritten as:

$$\mathrm{SLM_{SEC}} = \frac{n}{\sqrt{K_n^\dagger + \tilde{\kappa}_n a_n' a_n}} \frac{\tilde{\varepsilon}_n' H_n^\dagger \tilde{\varepsilon}_n}{\tilde{\varepsilon}_n'\tilde{\varepsilon}_n} \tag{19}$$

where $H_n^\dagger = W_nW_n' - \frac{1}{n-k}\mathrm{tr}(W_nW_n'M_n)I_n$, $K_n^\dagger = 2\mathrm{tr}(A_n^2)$, $a_n = \mathrm{diagv}(A_n)$, and $A_n = M_nW_nW_n'M_n - \frac{1}{n-k}\mathrm{tr}(W_nW_n'M_n)M_n$.

Comparing (18) and (19) with (6) and (8), we see that they possess very similar structure. The major difference is that the diagonal elements of W_n in (6) are zero and as a result the quantity $a_n'a_n$ in (8) is of smaller order than $K_n^\dagger$, but the diagonal elements of W_nW_n' in (18) are not zero and as a result the quantity $a_n'a_n$ in (19) can be of the same order as $K_n^\dagger$ therein. This gives the exact reason on why $\mathrm{SLM_{SEC}}$ is robust against the distributional misspecification (or nonnormality) in errors but $\mathrm{LM_{SEC}}$ is not.

3.3.2 Validity of the bootstrap methods

Note that under H_0, $\tilde{\varepsilon}_n' = M_n\varepsilon_n = \sigma M_n e_n$, where $M_n = I_n - X_n(X_n'X_n)X_n'$ and I_n is an $n \times n$ identity matrix. The statistics can be written as

$$\text{LM}_{\text{SEC}}|_{H_0} = \frac{n}{\sqrt{K_n}} \frac{e_n' M_n H_n M_n e_n}{e_n' M_n e_n} \tag{20}$$

which shows that $\text{LM}_{\text{SEC}}|_{H_0}$ is invariant of the nuisance parameters, and thus a pivot if $\mathcal{F}$ is known. In this case one can again, as for the SED model, simply use Monte Carlo method to find the finite sample critical values of $\text{LM}_{\text{SEC}}|_{H_0}$ to any level of accuracy. However, when $\mathcal{F}$ is unknown or is misspecified, $\text{LM}_{\text{SEC}}|_{H_0}$ is not even an asymptotic pivot. Then, what is the consequence of ignoring this when conducting bootstrap?

The bootstrap DGP that mimics the real world null DGP is again: $Y_n^* = X_n\ddot{\beta}_n + \ddot{\sigma}_n e_n^*$. Based on the bootstrap data (Y_n^*, X_n), compute the OLS of $(\ddot{\beta}_n, \ddot{\sigma}_n)$, the corresponding OLS residuals, and the statistic (18). Some algebra leads to the bootstrap analogue of (20):

$$\text{LM}_{\text{SEC}}^* = \frac{n}{\sqrt{K_n}} \frac{e_n^{*\prime} M_n H_n M_n e_n^*}{e_n^{*\prime} M_n e_n^*}. \tag{21}$$

Repeating the arguments of Section 3.1.2 (see Appendix A for supporting theories), we have

$$\text{LM}_{\text{SEC}}^* \xrightarrow{D^*} N(0, 1 + \ddot{\tau}^2),$$

where $\ddot{\tau}^2 = \lim_{n\to\infty} \ddot{\kappa}_n b_n' b_n / K_n$, and $b_n = \text{diagv}(M_n H_n M_n)$. Yang (2010) showed that when the group sizes are fixed and when there exist group size variations, $\lim_{n\to\infty} b_n' b_n / K_n$ is strictly positive. Thus, in general $\ddot{\tau}^2$ vanishes only when $\ddot{\kappa}_n$ does. The latter can possibly occur only when $\mathcal{F}$ is $N(0, 1)$ or simply its excess kurtosis $\kappa = 0$.

Now, consider the case where $\mathcal{F}$ is $N(0, 1)$ or $\kappa = 0$. It is easy to verity that $\text{LM}_{\text{SEC}}|_{H_0} \xrightarrow{D} N(0, 1)$. Similarly, if $\ddot{\mathcal{F}}_n = \hat{\mathcal{F}}_n$, then $\text{LM}_{\text{SEC}}^* \xrightarrow{D^*} N(0, 1)$ as $\hat{\kappa}_n$, the excess kurtosis of $\hat{\mathcal{F}}_n$ approaches to zero. However, if $\ddot{\mathcal{F}}_n = \tilde{\mathcal{F}}_n$, then $\text{LM}_{\text{SEC}}^* \xrightarrow{D^*} N(0, 1 + \ddot{\tau}^2)$ as $\tilde{\kappa}_n$, the excess kurtosis of $\tilde{\mathcal{F}}_n$ does not approach to zero in general due to the inconsistency of $\tilde{\mathcal{F}}_n$. These arguments show that even if the errors are normal, use of the restricted residuals may not give correct bootstrap critical values; in contrast, use of unrestricted residuals does.

Consider the case where $\mathcal{F}$ is not $N(0, 1)$, in particular $\kappa \neq 0$. Following the same arguments for the limiting distributions, we have $\text{LM}_{\text{SEC}}|_{H_0} \xrightarrow{D} N(0, 1 + \tau^2)$, where $\tau^2 = \kappa \lim_{n\to\infty} b_n' b_n / K_n$. Thus, $\text{LM}_{\text{SEC}}|_{H_0}$ is not asymptotically pivotal as its limiting distribution depends on the parameter κ. Similarly $\text{LM}_{\text{SEC}}^* \xrightarrow{D^*} N(0, 1 + \ddot{\tau}^2)$, where $\ddot{\tau}^2 = \lim_{n\to\infty} \ddot{\kappa}_n b_n' b_n / K_n$. Note that $\ddot{\tau}^2 = \tau^2$ only when $\hat{\mathcal{F}}_n = \hat{\mathcal{F}}_n$. In this case, bootstrapping LM_{SEC} does not lead to asymptotic refinements on the bootstrap critical values according to Beran (1988) and Horowitz (1994), and a standardization is necessary. Similar algebra as for $\text{LM}_{\text{SEC}}|_{H_0}$ and its bootstrap analogue LM_{SEC}^* gives

$$\text{SLM}_{\text{SEC}}|_{H_0} = \frac{n}{\sqrt{K_n^\dagger + \kappa(e_n) a_n' a_n}} \frac{e_n' M_n H_n^\dagger M_n e_n}{e_n' M_n e_n} \xrightarrow{D} N(0, 1), \tag{22}$$

where $\kappa(e_n)$ is the excess kurtosis of $M_n e_n$, and its bootstrap analogue

$$\text{SLM}^*_{\text{SEC}} = \frac{n}{\sqrt{K_n^\dagger + \kappa(e_n^*) a_n' a_n}} \frac{e_n^{*\prime} M_n H_n^\dagger M_n e_n^*}{e_n^{*\prime} M_n e_n^*} \xrightarrow{D^*} N(0,1). \tag{23}$$

The latter is true whether $\hat{\mathcal{F}}_n$ or $\tilde{\mathcal{F}}_n$ is used to generate e_n^*. The implication of these results is that when using the standardized LM test for testing for the existence of spatial error components in a linear regression model, either the unrestricted or restricted residuals leads to correct bootstrap critical values. Combining its invariance property, we conclude that all the bootstrap methods discussed in Section 2 are valid when SLM_{SEC} is used.

3.3.3 Monte Carlo results

The finite sample performance of $\text{LM}_{\texttt{SEC}}$ and $\text{SLM}_{\texttt{SEC}}$ for testing $H_0 : \lambda = 0$ vs $H_a : \lambda > 0$, when referring to the asymptotic critical values and the bootstrap critical values under various resampling schemes, are investigated in terms of the accuracy and stability of the bootstrap critical values with respect to the true value of λ, and the size and power of the tests. The Monte Carlo experiments are carried out based on the following data generating process:

$$Y_n = \beta_0 1_n + X_{n1}\beta_1 + X_{n2}\beta_2 + W_n v_n + \epsilon_n$$

where $\{v_{n,i}\}$ are iid draws from $\sqrt{0.6}t_5$, and the methods for generating W_n, X_n and ϵ_n are described in Appendix B. The regressors are treated as fixed in the experiments. The parameter values are set at $\beta = \{5, 1, 1\}'$ and $\sigma = 1$, and sample sizes used are $n = (54, 108, 216, 513)$. All results reported below are based on $M = 2,000$ Monte Carlo samples, and $B = 699$ bootstrap samples for each Monte Carlo sample generated. The bootstrap critical values are bench-marked against the Monte Carlo (`MC`) critical values obtained based on $M = 30,000$ Monte Carlo samples under H_0.

Similar to the LM tests for SED model considered earlier, the LM tests for SEC model are also invariant of the nuisance parameters, thus the bootstrap methods with $\texttt{RS}_{\texttt{ur}}$ and $\texttt{RS}_{\texttt{ur}}$ are omitted as the former produces identical results as $\texttt{RS}_{\texttt{rr}}$ and the latter produces identical results as $\texttt{RS}_{\texttt{uu}}$. We also omit the $\texttt{RS}_{\texttt{uf}}$ method in this study as it requires the derivation of the test statistics for a general value of λ, and concentrate on $\texttt{RS}_{\texttt{rr}}$ and $\texttt{RS}_{\texttt{uu}}$.

Bootstrap critical values. We first report in Tables 3.3a and 3.3b the bootstrap critical values for $\text{LM}_{\texttt{SEC}}$ and $\text{SLM}_{\texttt{SEC}}$. As discussed above, $\text{LM}_{\texttt{SEC}}$ is sensitive to the distributional misspecification, thus it is expected to produce bootstrap critical values that vary with λ when $\tilde{\mathcal{F}}_n$ is used, even if $\mathcal{F}$ is $N(0,1)$. Indeed this is observed from the results under $\texttt{RS}_{rr}$ and `Normal Error` though the change is not big. In contrast, the bootstrap critical values in all other cases with normal error are very stable.

When error distribution is not normal and is unknown, $\text{LM}_{\texttt{SEC}}|_{H_0}$ is no longer a pivot, and not even an asymptotic pivot as both its finite and limiting distribution depend on $\mathcal{F}$. It is thus expected that bootstrap critical values based on $\text{LM}_{\texttt{SEC}}$ would vary more with λ whether $\texttt{RS}_{\texttt{rr}}$ or $\texttt{RS}_{\texttt{uu}}$ is followed. Again we see from the table that this is very much true and in fact the bootstrap critical values change (drop) much more significantly as λ increases. In contrast, if we bootstrap $\text{SLM}_{\texttt{SEC}}$, the bootstrap critical values become much more stable. In both cases, the method with $\texttt{RS}_{\texttt{uu}}$ performs better.

`<< Insert Table 3.3a and Table 3.3b Here >>`

Rejection Frequencies. We now report rejection frequencies in Tables 3.3c and 3.3d corresponding to $n = 216$ and 513, respectively. From the results including those unreported, we observe following.

1. When errors (ε_n) are normal, all other tests improve upon the $\texttt{LM}_{\texttt{SEC}}$ test referring to the asymptotic critical values, in particular when sample size is small;
2. When errors (ε_n) are nonnormal, $\texttt{LM}_{\texttt{SEC}}$ referring to the asymptotic critical values failed, but very interestingly $\texttt{LM}_{\texttt{SEC}}$ referring to the bootstrap critical values performs reasonably well although a clear sign of deterioration is observed for the cases of nonnormal errors;
3. $\texttt{SLM}_{\texttt{SEC}}$ performs well whether to refer to the asymptotic critical values or the bootstrap critical values, but the former is outperformed by the latter, in particular when the error distribution is skewed.

A final and an important remark is as follows. The bootstrap LM test seems offer higher power than does the bootstrap SLM test. However, as cautioned earlier, such a higher power is built upon the 'hidden' lower critical values which is unachievable as in practice one does not know whether or not the null is true. Once again, we stress on that the performance of a bootstrap test should be judged based on whether it can offer critical values which are stable with respect to the change in the value of the parameters of interest.

`<< Insert Table 3.3c and Table 3.3d Here >>`

4 Conclusions and Discussions

In bootstrapping the critical values of an LM test, one faces two important issues: one is on the choice of the type of estimates of nuisance parameters to be used as parameters in the bootstrap data generating process, and the other is on choice of the type of residuals to be used to construct the bootstrap error distribution. We argue in general and show through three popular spatial regression models that the choice that is correct in general is the one which uses the unrestricted estimates and the unrestricted residuals. However, if the test statistic is invariant of the nuisance parameters or the restricted estimates of the nuisance parameters are consistent in general, the restricted estimates can be used in place of the unrestricted estimates; if the test statistic at the null is robust against the distributional misspecification, the restricted residuals can be used in place of the unrestricted residuals.

It is emphasized that comparison on the performance of various bootstrap methods should not be made based on the size and power of the tests, instead it should be made based on the stability of the bootstrap critical values with respect to the change in the value of the parameters of interest. The main reason is that in reality, one does not know whether or not the null hypothesis is true, thus the size of the bootstrap tests based on restricted estimates and/or residuals (which imposes the null) may not be achievable if (i) the null hypothesis is false and (ii) the bootstrap critical values change with the value of the parameters of interest. The same issue applies to the power of the bootstrap tests based on the restricted estimate and/or residuals: the power in this situation tends to be higher (than that based on unrestricted resampling) if the underlining bootstrap critical values are smaller than the true ones, or lower if the underlining bootstrap critical values are larger.

That the bootstrap is able to improve the finite sample performance of the LM tests (in terms of size and power) is demonstrated in this paper through three popular spatial regression models. This is particularly useful for the LM tests of spatial dependence as often the shape of the finite sample distribution of a spatial LM test is 'twisted' significantly by the existence of a strong spatial dependence. With the general principles laid out in this paper, it would be interesting to proceed to study the properties of the bootstrap LM tests for other spatial models - linear or nonlinear.

Appendix A: Some Fundamental Results

To facilitate the theoretical discussions in Section 3, we present two lemmas adapted from the work of Kelejian and Prucha (2001) and the work of Lee (2004). Also, we sketch the derivations of the tests $\texttt{LM}_{\texttt{SED}}(\rho)$ and $\texttt{SLM}_{\texttt{SED}}(\rho)$ given in (5) and (7).

Lemma A1: *Let A_n and B_n be $n \times n$ matrices and c_n be an $n \times 1$ vector. Let ε_n be an $n \times 1$ vector of iid elements with mean zero, variance σ^2, skewness γ, and excess kurtosis κ. Let $P_n = \varepsilon_n' A_n \varepsilon_n + c_n' \varepsilon_n$ and $Q_n = \varepsilon_n' B_n \varepsilon_n$. Then,*

(*i*) $\mathrm{E}(P_n) = \sigma^2 \mathrm{tr}(A_n)$, *and* $\mathrm{E}(Q_n) = \sigma^2 \mathrm{tr}(B_n)$,
(*ii*) $\mathrm{Var}(P_n) = \sigma^4 \mathrm{tr}(A_n' A_n + A_n^2) + \sigma^4 \kappa a_n' a_n + \sigma^2 c_n' c_n + 2\sigma^3 \gamma a_n' c_n$,
(*iii*) $\mathrm{Var}(Q_n) = \sigma^4 \mathrm{tr}(B_n' B_n + B_n^2) + \sigma^4 \kappa b_n' b_n$,
(*iv*) $\mathrm{Cov}(P_n, Q_n) = \sigma^4 \mathrm{tr}[(A_n' + A_n) B_n] + \sigma^4 \kappa a_n' b_n + \sigma^3 \gamma b_n' c_n$,

where $a_n = \mathrm{diagv}(A_n)$ and $b_n = \mathrm{diagv}(B_n)$.

Lemma A2: (Central Limit Theorem for Linear-Quadratic Forms) *Let A_n be $n \times n$ matrices which is bounded uniformly in row and column sums. Let c_n be an $n \times 1$ vector such that $n^{-1} \sum_{i=1}^n |c_{n,i}^{2+\eta_1}| < \infty$ for some $\eta_1 > 0$. Let ε_n be an $n \times 1$ vector of iid elements with mean zero, variance σ^2, skewness γ, and excess kurtosis κ. Assume $E|\varepsilon_{n,i}^{4+\eta_2}| < \infty$ for some $\eta_2 > 0$. Then,*

$$\frac{\varepsilon_n' A_n \varepsilon_n + c_n' \varepsilon_n - \mathrm{tr}(A_n)}{\{\sigma^4 \mathrm{tr}(A_n' A_n + A_n^2) + \sigma^4 \kappa a_n' a_n + \sigma^2 c_n' c_n + 2\sigma^3 \gamma a_n' c_n\}^{\frac{1}{2}}} \xrightarrow{D} N(0, 1).$$

Derivations of $\texttt{LM}_{\texttt{SED}}(\rho)$ and $\texttt{SLM}_{\texttt{SED}}$ in Section 3.1: The loglikelihood function of the spatial error model: $\ell_n(\beta, \sigma^2, \rho) = -\frac{2}{n}(2\pi\sigma^2) + \log|B_n(\rho)| - \frac{1}{2\sigma^2}(Y_n - X_n\beta)'(Y_n - X_n\beta)$. The score functions are: $\frac{\partial \ell_n}{\partial \beta} = \frac{1}{\sigma^2}(Y_n - X_n\beta)' B_n'(\rho) B_n(\rho) X_n$, $\frac{\partial \ell_n}{\partial \sigma^2} = -\frac{n}{\sigma^2} + \frac{1}{\sigma^4}(Y_n - X_n\beta)' B_n'(\rho) B_n(\rho)(Y_n - X_n\beta)$, and $\frac{\partial \ell_n}{\partial \rho} = -\mathrm{tr}(Q_n(\rho)) + \frac{1}{2\sigma^2}(Y_n - X_n\beta)' B_n'(\rho) B_n(\rho)(Y_n - X_n\beta)$. Plugging $\tilde{\beta}_n(\rho)$ and $\tilde{\sigma}_n^2(\rho)$ into the last expression gives the concentrated score for ρ:

$$S_n^c(\rho) = \tilde{\sigma}_n^{-2}(\rho)[Y_n - X_n\tilde{\beta}_n(\rho)]' B_n' Q_n^\circ(\rho)(\rho) B_n(\rho)[Y_n - X_n\tilde{\beta}_n(\rho)].$$

The expected information matrix is:

$$I_n(\beta, \sigma^2, \rho) = \begin{pmatrix} \frac{1}{\sigma^2} X_n' B_n'(\rho) B_n(\rho) X_n, & 0, & 0 \\ 0, & \frac{n}{2\sigma^4}, & \frac{1}{\sigma^2}\mathrm{tr}[Q_n(\rho)] \\ 0, & \frac{1}{\sigma^2}\mathrm{tr}[Q_n(\rho)], & \mathrm{tr}[Q_n'(\rho) Q_n(\rho) + Q_n^2(\rho)] \end{pmatrix}.$$

Thus, $\mathrm{AVar}[S_n^c(\rho)] = I_{n,22} - I_{n,21} I_{n1}^{-1} I_{n,12} = \mathrm{tr}[Q_n^\circ(\rho)' Q_n^\circ(\rho) + Q_n^\circ(\rho)^2$, where $I_{n,ij}, i, j = 1, 2$ are the submatrices of I_n partitioned according to (β, σ^2) and ρ. Putting the two together gives $\texttt{LM}_{\texttt{SED}}(\rho)$ in (5). For $\texttt{SLM}_{\texttt{SED}}(\rho)$, find the mean and variance (without assuming normality) of $[Y_n - X_n\tilde{\beta}_n(\rho)]' B_n' Q_n^\circ(\rho)(\rho) B_n(\rho)[Y_n - X_n\tilde{\beta}_n(\rho)]$, and standardize.

Appendix B: Settings for Monte Carlo Experiments

We now describe the methods for generating the regressors values, the spatial weight matrices, and the errors, to be used in the Monte Carlo experiments. All the DGPs used in our Monte Carlo experiments contain two regressors.

B.1 Regressors Values

The simplest method for generating the values for the regressors is to make random draws from a certain distribution, i.e., the values $\{x_{1i}\}$ of X_{n1} and the values $\{x_{2i}\}$ of X_{n2} in the Monte Carlo experiments are generated according to:

$$\texttt{XVal-A:}\ \{x_{1i}\} \overset{iid}{\sim} N(0,1), \text{ and } \{x_{2i}\} \overset{iid}{\sim} N(0,1),$$

where X_{n1} and X_{n2} are independent. Alternatively, to allow for the possibility that there might be systematic differences in X values across the different sets of spatial units, e.g., spatial groups, spatial clusters, etc., the ith value in the rth 'group' $\{x_{1,ir}\}$ of X_{n1}, and the ith value in the rth group $\{x_{2,ir}\}$ of X_{n2} are generated as follows:

$$\texttt{XVal-B:}\ \{x_{1,ir}\} = (2z_r + z_{ir})/\sqrt{5}, \text{ and } \{x_{2,ir}\} = (2v_r + v_{ir})/\sqrt{5},$$

where $\{z_r, z_{ir}, v_r, v_{ir}\} \overset{iid}{\sim} N(0,1)$, across all i and r. Apparently, unlike the `XVal-A` scheme that gives iid X values, the `XVal-B` scheme gives non-iid X values, or different group means in terms of group interaction (Lee 2004).

B.2 Spatial Weight Matrix

The spatial weight matrices used in the Monte Carlo experiments are generated according to `Rook Contiguity`, `Queen Contiguity` and `Group Interactions`. In the first two cases, the number of neighbors for each spatial unit stays the same (2-4 for Rook and 3-8 for Queen) and does not change when sample size n increases, whereas in the last case, the number of neighbors for each spatial unit increases with the increase of sample size but at a slower rate, and changes from group to group.

`Rook or Queen Contiguity.` The details for generating the W_n matrix under Rook contiguity are as follows: (i) index the n spatial units by $\{1, 2, \cdots, n\}$, randomly permute these indices and then allocate them into a lattice of $r \times m (\geq n)$ squares, (ii) let $W_{n,ij} = 1$ if the index j is in a square which is on the immediate left, or right, or above, or below the square which contains the index i, otherwise $W_{n,ij} = 0$, and (iii) divide each element of W_n by its row sum. The W matrix under Queen contiguity is generated in a similar way, but with additional neighbors which share a common vertex with the unit of interest.

`Group Interaction.` To generate the W_n matrix according to the group interaction scheme, (i) calculate the number of groups according to $g = \text{Round}(n^\delta)$, and the approximate average group size $m = n/g$, (ii) generate the group sizes $(n_1, n_2, \cdots, n_g)$ according to a discrete uniform distribution from 2 to $m-2$, (iii) adjust the group sizes so that $\sum_{i=1}^{g} n_i = n$, and (iv) define $W_n = \text{diag}\{W_i/(n_i-1), i = 1, \cdots, g\}$, a matrix formed by placing the submatrices W_i along the diagonal direction, where W_i is an $n_i \times n_i$ matrix with ones on the off-diagonal positions and zeros on the diagonal positions. In our Monte Carlo experiments, we choose $\delta = 0.3$, 0.5 and 0.7, representing respectively the situations where (i) there are few groups and many spatial units in a group, (ii) the number of groups and the sizes of the groups are of the same order of magnitude, and (iii) there are many groups with few elements in each.

Clearly, under Rook or Queen contiguity, the number of neighbor each spatial unit has is bounded, whereas under group interaction it is divergent with rate $n^{1-\delta}$.[9]

B.3 Error Distributions

The reported Monte Carlo results correspond to the following three error distributions: (i) standard normal, (ii) mixture normal, standardized to have mean zero and variance 1, and (iii) log-normal, also standardized to have mean zero and variance one. The standardized normal-mixture variates are generated according to

$$\varepsilon_{n,i} = ((1-\xi_i)Z_i + \xi_i \tau Z_i)/(1 - p + p * \tau^2)^{0.5},$$

where ξ is a Bernoulli random variable with probability of success p and Z_i is standard normal independent of ξ. The parameter p in this case also represents the proportion of mixing the two normal populations. In our experiments, we choose $p = 0.05$, meaning that 95% of the random variates are from standard normal and the remaining 5% are from another normal population with standard deviation τ. We choose $\tau = 4$ to simulate the situation where there are gross errors in the data. The standardized lognormal random variates are generated according to

$$\epsilon_{n,i} = [\exp(Z_i) - \exp(0.5)]/[\exp(2) - \exp(1)]^{0.5}.$$

This gives an error distribution that is both skewed and leptokurtic. The normal mixture gives an error distribution that is still symmetric like normal but leptokurtic. Other non-normal distributions, such as normal-gamma mixture and chi-squared, may also be considered.

[9] Clearly, this spatial layout covers the scenario considered in Case (1991). Lee (2007) shows that the group size variation plays an important role in the identification and estimation of econometric models with group interactions, contextual factors and fixed effects. Yang (2010) shows that it also plays an important role in the robustness of the LM test of spatial error components.

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Table 3.1a. Bootstrap and MC Critical Values for Burridge's LM Test of SED
`spatial Layout: Group Interaction with` $g = n^{0.5}$; $H_0: \rho = 0; \sigma = 2$

		$n = 100$				$n = 200$			
Method	ρ	2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
		Normal Error							
$\texttt{RS}_{rr}$	0.0	-1.8626	-1.7168	1.0968	1.5089	-1.8827	-1.7292	1.1658	1.5802
	0.5	-1.8618	-1.7168	1.1031	1.5158	-1.8833	-1.7292	1.1655	1.5820
$\texttt{RS}_{uu}$	0.0	-1.8628	-1.7169	1.0961	1.5095	-1.8827	-1.7297	1.1654	1.5801
	0.5	-1.8623	-1.7161	1.1026	1.5179	-1.8819	-1.7285	1.1659	1.5795
MC		-1.8720	-1.7221	1.1129	1.5050	-1.8510	-1.6921	1.2217	1.6476
		Normal Mixture Error							
$\texttt{RS}_{rr}$	0.0	-1.8776	-1.7008	1.0387	1.4463	-1.8770	-1.7141	1.1295	1.5382
	0.5	-1.8789	-1.7040	1.0471	1.4555	-1.8788	-1.7166	1.1317	1.5391
$\texttt{RS}_{uu}$	0.0	-1.8783	-1.7010	1.0388	1.4476	-1.8769	-1.7146	1.1296	1.5393
	0.5	-1.8799	-1.7014	1.0393	1.4495	-1.8781	-1.7146	1.1271	1.5367
MC		-1.9158	-1.7221	1.0062	1.4028	-1.8553	-1.6909	1.1945	1.5752
		Log-Normal Error							
$\texttt{RS}_{rr}$	0.0	-1.8242	-1.6512	1.0014	1.4404	-1.8157	-1.6589	1.0872	1.5218
	0.5	-1.8343	-1.6628	1.0102	1.4467	-1.8279	-1.6686	1.0959	1.5267
$\texttt{RS}_{uu}$	0.0	-1.8260	-1.6521	1.0031	1.4413	-1.8160	-1.6597	1.0872	1.5212
	0.5	-1.8232	-1.6512	0.9999	1.4422	-1.8177	-1.6596	1.0875	1.5266
MC		-1.8626	-1.6644	0.9836	1.4493	-1.7862	-1.6239	1.1133	1.5424

$\texttt{RS}_{rr}$ and $\texttt{RS}_{uu}$: Average bootstrap critical values based on $M = 2,000$ and $B = 699$;
MC: Monte Carlo critical values based on $M = 30,000$; Regressors generated according to XVal-B

Table 3.1b. Rejection Frequencies for One-Sided LM Test of Spatial Error Dependence
`spatial Layout: Group Interaction with` $g = n^{0.5}$; $H_0 : \rho = 0$

Method	$\|\rho\|$	$n = 100$				$n = 200$			
		L2.5%	L5%	R5%	R2.5%	L2.5%	L5%	R5%	R2.5%
					`Normal Error`				
`ACR`	0.00	0.0155	0.0690	0.0175	0.0110	0.0200	0.0740	0.0200	0.0115
	0.25	0.0700	0.2440	0.2295	0.1760	0.0930	0.2740	0.3245	0.2625
	0.50	0.1925	0.4735	0.7865	0.7400	0.3005	0.6000	0.8860	0.8585
`RS`$_{rr}$	0.00	0.0270	0.0530	0.0485	0.0215	0.0290	0.0565	0.0445	0.0235
	0.25	0.1165	0.2000	0.3550	0.2520	0.1370	0.2270	0.4450	0.3410
	0.50	0.2715	0.4105	0.8570	0.8050	0.3875	0.5245	0.9280	0.8915
`RS`$_{uu}$	0.00	0.0265	0.0530	0.0470	0.0210	0.0300	0.0555	0.0435	0.0235
	0.25	0.1170	0.2020	0.3595	0.2560	0.1375	0.2235	0.4430	0.3390
	0.50	0.2740	0.4060	0.8565	0.8030	0.3845	0.5275	0.9280	0.8915
`ACR*`	0.00	0.0015	0.0170	0.0705	0.0420	0.0300	0.0555	0.0435	0.0235
	0.25	0.0145	0.0825	0.4035	0.3440	0.1375	0.2235	0.4430	0.3390
	0.50	0.0555	0.2180	0.8785	0.8465	0.3845	0.5275	0.9280	0.8915
					`Normal Mixture Error`				
`ACR`	0.00	0.0165	0.0605	0.0150	0.0090	0.0155	0.0540	0.0205	0.0135
	0.25	0.0710	0.2110	0.2305	0.1735	0.0945	0.2635	0.3325	0.2625
	0.50	0.2045	0.4460	0.7815	0.7390	0.2940	0.5850	0.9020	0.8705
`RS`$_{rr}$	0.00	0.0250	0.0530	0.0480	0.0230	0.0215	0.0415	0.0520	0.0245
	0.25	0.0930	0.1730	0.3705	0.2710	0.1355	0.2255	0.4575	0.3600
	0.50	0.2580	0.3915	0.8690	0.8110	0.3595	0.5290	0.9410	0.9120
`RS`$_{uu}$	0.00	0.0245	0.0510	0.0475	0.0235	0.0200	0.0415	0.0515	0.0245
	0.25	0.0925	0.1780	0.3700	0.2735	0.1365	0.2235	0.4570	0.3575
	0.50	0.2550	0.3935	0.8675	0.8105	0.3665	0.5335	0.9410	0.9105
`ACR*`	0.00	0.0045	0.0170	0.0600	0.0370	0.0065	0.0165	0.0690	0.0425
	0.25	0.0325	0.0790	0.4070	0.3410	0.0260	0.0995	0.5050	0.4270
	0.50	0.0960	0.2145	0.8855	0.8535	0.0860	0.3045	0.9480	0.9325
					`Log-Normal Error`				
`ACR`	0.00	0.0125	0.0490	0.0180	0.0090	0.0150	0.0530	0.0210	0.0110
	0.25	0.0735	0.1975	0.2190	0.1630	0.0820	0.2605	0.3115	0.2395
	0.50	0.2120	0.4350	0.7910	0.7340	0.2805	0.5605	0.9180	0.8900
`RS`$_{rr}$	0.00	0.0295	0.0440	0.0485	0.0240	0.0250	0.0495	0.0660	0.0285
	0.25	0.1155	0.1950	0.3600	0.2605	0.1525	0.2485	0.4540	0.3460
	0.50	0.2860	0.4235	0.8870	0.8165	0.4090	0.5460	0.9525	0.9255
`RS`$_{uu}$	0.00	0.0290	0.0445	0.0490	0.0250	0.0255	0.0495	0.0635	0.0290
	0.25	0.1155	0.1965	0.3650	0.2580	0.1560	0.2520	0.4550	0.3470
	0.50	0.2905	0.4255	0.8865	0.8170	0.4110	0.5525	0.9530	0.9230
`ACR*`	0.00	0.0045	0.0140	0.0535	0.0375	0.0015	0.0165	0.0720	0.0470
	0.25	0.0310	0.0760	0.3915	0.3140	0.0255	0.0870	0.4825	0.4025
	0.50	0.1170	0.2185	0.9015	0.8630	0.0985	0.2925	0.9570	0.9430

Note: L = Left tail ($\rho < 0$), R = Right tail ($\rho > 0$); Regressors generated according to `XVal-B`

Table 3.2a. Bootstrap Critical Values for LM and SLM Tests of Spatial Lag Dependence

`Spatial Layout: Group Interaction with` $g = n^{0.5}$; $n = 100$; $\sigma = 1$; `XVal-B`

		LM Test				SLM Test			
Method	λ	L2.5%	L5%	U5%	U2.5%	L2.5%	L5%	U5%	U2.5%
		Normal Error							
$\mathtt{RS_{rr}}$	-0.5	-2.0718	-1.8294	1.2718	1.6270	-1.8282	-1.5691	1.7465	2.1265
	-0.3	-2.0872	-1.8313	1.2960	1.6438	-1.8529	-1.5813	1.7331	2.1033
	0.0	-2.1064	-1.8372	1.3469	1.6844	-1.8904	-1.6090	1.7195	2.0722
	0.3	-2.1144	-1.8318	1.4030	1.7303	-1.9238	-1.6322	1.7031	2.0407
	0.5	-2.1135	-1.8245	1.4375	1.7608	-1.9383	-1.6417	1.6994	2.0307
$\mathtt{RS_{uu}}$	-0.5	-2.1034	-1.8378	1.3510	1.6849	-1.8908	-1.6133	1.7145	2.0635
	-0.3	-2.1030	-1.8312	1.3507	1.6870	-1.8905	-1.6072	1.7121	2.0638
	0.0	-2.1064	-1.8363	1.3559	1.6924	-1.8949	-1.6127	1.7163	2.0682
	0.3	-2.1099	-1.8376	1.3563	1.6908	-1.8982	-1.6139	1.7183	2.0667
	0.5	-2.1049	-1.8366	1.3578	1.6898	-1.8929	-1.6132	1.7184	2.0655
MC	0.0	-2.1190	-1.8415	1.3262	1.6512	-1.9018	-1.6117	1.7002	2.0447
		Normal Mixture Error							
$\mathtt{RS_{rr}}$	-0.5	-2.0640	-1.8098	1.2502	1.6027	-1.8228	-1.5513	1.7074	2.0825
	-0.3	-2.0809	-1.8167	1.2730	1.6198	-1.8494	-1.5695	1.6954	2.0620
	0.0	-2.0941	-1.8170	1.3308	1.6675	-1.8818	-1.5923	1.6900	2.0411
	0.3	-2.1066	-1.8191	1.3962	1.7254	-1.9197	-1.6235	1.6859	2.0250
	0.5	-2.1095	-1.8175	1.4302	1.7542	-1.9361	-1.6367	1.6885	2.0196
$\mathtt{RS_{uu}}$	-0.5	-2.0972	-1.8206	1.3424	1.6743	-1.8888	-1.6003	1.6899	2.0362
	-0.3	-2.1001	-1.8210	1.3401	1.6761	-1.8918	-1.6008	1.6887	2.0385
	0.0	-2.0959	-1.8175	1.3414	1.6763	-1.8872	-1.5971	1.6898	2.0389
	0.3	-2.0978	-1.8204	1.3428	1.6777	-1.8900	-1.6009	1.6899	2.0368
	0.5	-2.0975	-1.8229	1.3425	1.6761	-1.8886	-1.6023	1.6913	2.0389
MC	0.0	-2.1175	-1.8320	1.3125	1.6077	-1.9059	-1.6033	1.6781	1.9927
		Log-Normal Error							
$\mathtt{RS_{rr}}$	-0.5	-2.0232	-1.7734	1.2626	1.6337	-1.7806	-1.5159	1.6860	2.0766
	-0.3	-2.0374	-1.7797	1.2960	1.6574	-1.8064	-1.5353	1.6806	2.0586
	0.0	-2.0556	-1.7869	1.3500	1.6995	-1.8455	-1.5663	1.6759	2.0381
	0.3	-2.0807	-1.7979	1.4160	1.7513	-1.8982	-1.6079	1.6794	2.0233
	0.5	-2.0947	-1.8026	1.4362	1.7671	-1.9235	-1.6251	1.6797	2.0169
$\mathtt{RS_{uu}}$	-0.5	-2.0612	-1.7899	1.3612	1.7118	-1.8549	-1.5735	1.6780	2.0391
	-0.3	-2.0592	-1.7883	1.3631	1.7083	-1.8530	-1.5722	1.6782	2.0348
	0.0	-2.0608	-1.7884	1.3581	1.7057	-1.8545	-1.5721	1.6764	2.0344
	0.3	-2.0667	-1.7921	1.3664	1.7162	-1.8626	-1.5780	1.6790	2.0388
	0.5	-2.0614	-1.7901	1.3601	1.7104	-1.8553	-1.5743	1.6762	2.0373
MC	0.0	-2.0276	-1.7597	1.3454	1.6944	-1.8154	-1.5290	1.6663	2.0354

$\mathtt{RS}_{rr}$ and $\mathtt{RS}_{uu}$: Average bootstrap critical values based on $M = 2,000$ and $B = 699$;

`MC`: Monte Carlo critical values based on $M = 30,000$; Regressors generated according to `XVal-B`

Table 3.2b. Rejection Frequencies for LM Tests of Spatial Lag Dependence

`Spatial Layout: Group Interaction with` $g = n^{0.5}$; $\sigma = 1$; `XVal-B`

		$n = 50$				$n = 100$			
Method	$\vert\lambda\vert$	L2.5%	L5%	U5%	U2.5%	L2.5%	L5%	U5%	U2.5%
					`Normal Error`				
`ACR`	0.0	0.0435	0.0970	0.0190	0.0085	0.0430	0.0875	0.0235	0.0095
	0.1	0.1010	0.1905	0.0905	0.0550	0.1405	0.2300	0.1240	0.0805
	0.2	0.2150	0.3510	0.2885	0.1985	0.2955	0.4400	0.4510	0.3430
	0.3	0.3585	0.5420	0.6110	0.4990	0.4705	0.6410	0.8535	0.7690
RS_{rr}	0.0	0.0285	0.0565	0.0485	0.0260	0.0305	0.0540	0.0445	0.0235
	0.1	0.0655	0.1220	0.1640	0.0975	0.1045	0.1725	0.1960	0.1190
	0.2	0.1555	0.2455	0.3975	0.2890	0.2405	0.3505	0.5495	0.4310
	0.3	0.2870	0.4175	0.7135	0.6055	0.4075	0.5390	0.8920	0.8340
RS_{uu}	0.0	0.0270	0.0575	0.0555	0.0280	0.0290	0.0555	0.0475	0.0245
	0.1	0.0605	0.1195	0.1715	0.1030	0.0995	0.1755	0.2015	0.1255
	0.2	0.1415	0.2440	0.4070	0.3020	0.2325	0.3500	0.5590	0.4420
	0.3	0.2610	0.4025	0.7260	0.6220	0.3955	0.5350	0.8935	0.8410
					`Normal Mixture Error`				
`ACR`	0.0	0.0445	0.0860	0.0160	0.0075	0.0335	0.0765	0.0250	0.0125
	0.1	0.1045	0.1925	0.0975	0.0520	0.1265	0.2285	0.1355	0.0800
	0.2	0.2290	0.3795	0.3070	0.2160	0.2995	0.4355	0.4630	0.3400
	0.3	0.3800	0.5505	0.6380	0.5335	0.5035	0.6625	0.8415	0.7780
RS_{rr}	0.0	0.0295	0.0545	0.0470	0.0215	0.0245	0.0520	0.0485	0.0255
	0.1	0.0745	0.1335	0.1730	0.1015	0.0995	0.1705	0.2070	0.1300
	0.2	0.1860	0.2705	0.4200	0.3105	0.2515	0.3610	0.5520	0.4400
	0.3	0.3140	0.4440	0.7375	0.6350	0.4480	0.5790	0.8850	0.8285
RS_{uu}	0.0	0.0280	0.0525	0.0515	0.0235	0.0240	0.0510	0.0495	0.0275
	0.1	0.0675	0.1325	0.1820	0.1055	0.0985	0.1680	0.2180	0.1325
	0.2	0.1720	0.2685	0.4360	0.3215	0.2425	0.3535	0.5660	0.4500
	0.3	0.2935	0.4390	0.7460	0.6485	0.4260	0.5755	0.8890	0.8345
					`Log-Normal Error`				
`ACR`	0.0	0.0275	0.0715	0.0140	0.0070	0.0310	0.0800	0.0270	0.0130
	0.1	0.1165	0.2020	0.1175	0.0655	0.1355	0.2375	0.1630	0.1030
	0.2	0.2725	0.4065	0.3895	0.2795	0.3275	0.4620	0.5045	0.4020
	0.3	0.4380	0.5925	0.7130	0.6260	0.5360	0.6880	0.8675	0.8060
RS_{rr}	0.0	0.0175	0.0480	0.0375	0.0150	0.0255	0.0550	0.0420	0.0225
	0.1	0.0880	0.1540	0.1860	0.1110	0.1145	0.1800	0.2290	0.1515
	0.2	0.2255	0.3260	0.4865	0.3825	0.3020	0.4035	0.5980	0.4725
	0.3	0.3945	0.5045	0.8000	0.7055	0.4970	0.6185	0.9030	0.8520
RS_{uu}	0.0	0.0165	0.0415	0.0450	0.0185	0.0235	0.0495	0.0460	0.0230
	0.1	0.0745	0.1420	0.1935	0.1155	0.1115	0.1815	0.2390	0.1565
	0.2	0.2050	0.3160	0.5010	0.3935	0.2895	0.3990	0.6075	0.4860
	0.3	0.3750	0.4920	0.8080	0.7160	0.4810	0.6165	0.9080	0.8570

Table 3.2c. Rejection Frequencies for SLM Tests of Spatial Lag Dependence

`Spatial Layout: Group Interaction with` $g = n^{0.5}$; $\sigma = 1$; `XVal-B`

		$n = 50$				$n = 100$			
Method	$\lvert\lambda\rvert$	L2.5%	L5%	U5%	U2.5%	L2.5%	L5%	U5%	U2.5%
					`Normal Error`				
ACR	0.0	0.0230	0.0475	0.0635	0.0355	0.0235	0.0495	0.0510	0.0280
	0.1	0.0520	0.1100	0.1865	0.1160	0.0915	0.1610	0.2065	0.1350
	0.2	0.1265	0.2290	0.4345	0.3325	0.2050	0.3340	0.5740	0.4575
	0.3	0.2275	0.3755	0.7500	0.6505	0.3500	0.5055	0.9050	0.8510
RS_{rr}	0.0	0.0280	0.0565	0.0520	0.0280	0.0300	0.0535	0.0450	0.0240
	0.1	0.0655	0.1215	0.1695	0.0980	0.1050	0.1715	0.1955	0.1190
	0.2	0.1540	0.2440	0.4005	0.2950	0.2380	0.3505	0.5525	0.4355
	0.3	0.2865	0.4145	0.7190	0.6120	0.4075	0.5350	0.8925	0.8340
RS_{rr}	0.0	0.0235	0.0520	0.0515	0.0235	0.0250	0.0525	0.0450	0.0220
	0.1	0.0575	0.1105	0.1650	0.0925	0.0960	0.1690	0.1930	0.1175
	0.2	0.1340	0.2270	0.4005	0.2920	0.2240	0.3395	0.5485	0.4220
	0.3	0.2485	0.3910	0.7165	0.6060	0.3820	0.5230	0.8905	0.8320
					`Normal Mixture Error`				
ACR	0.0	0.0230	0.0475	0.0565	0.0285	0.0185	0.0460	0.0540	0.0290
	0.1	0.0540	0.1145	0.1960	0.1245	0.0825	0.1535	0.2220	0.1470
	0.2	0.1450	0.2395	0.4525	0.3520	0.2080	0.3240	0.5760	0.4705
	0.3	0.2475	0.3945	0.7610	0.6750	0.3925	0.5400	0.8940	0.8430
RS_{rr}	0.0	0.0290	0.0530	0.0505	0.0235	0.0245	0.0515	0.0495	0.0275
	0.1	0.0730	0.1280	0.1780	0.1075	0.0985	0.1680	0.2110	0.1295
	0.2	0.1845	0.2690	0.4305	0.3190	0.2520	0.3580	0.5530	0.4445
	0.3	0.3085	0.4415	0.7445	0.6440	0.4460	0.5775	0.8875	0.8305
RS_{uu}	0.0	0.0265	0.0505	0.0505	0.0220	0.0215	0.0490	0.0475	0.0265
	0.1	0.0640	0.1200	0.1765	0.1050	0.0920	0.1560	0.2115	0.1255
	0.2	0.1595	0.2550	0.4275	0.3135	0.2320	0.3395	0.5580	0.4390
	0.3	0.2775	0.4180	0.7430	0.6395	0.4160	0.5610	0.8855	0.8290
					`Log-Normal Error`				
ACR	0.0	0.0120	0.0350	0.0480	0.0235	0.0165	0.0415	0.0485	0.0285
	0.1	0.0605	0.1295	0.2055	0.1315	0.0910	0.1570	0.2395	0.1670
	0.2	0.1800	0.2890	0.5125	0.4175	0.2455	0.3605	0.6145	0.5025
	0.3	0.3250	0.4535	0.8160	0.7400	0.4180	0.5670	0.9090	0.8655
RS_{rr}	0.0	0.0160	0.0460	0.0440	0.0185	0.0235	0.0525	0.0450	0.0230
	0.1	0.0830	0.1505	0.1935	0.1175	0.1125	0.1770	0.2330	0.1555
	0.2	0.2185	0.3200	0.4990	0.3985	0.2990	0.3995	0.6040	0.4815
	0.3	0.3875	0.4975	0.8110	0.7160	0.4925	0.6135	0.9045	0.8550
RS_{uu}	0.0	0.0135	0.0355	0.0445	0.0185	0.0195	0.0465	0.0430	0.0215
	0.1	0.0685	0.1345	0.1920	0.1135	0.1050	0.1700	0.2340	0.1520
	0.2	0.1960	0.3010	0.5010	0.3935	0.2785	0.3845	0.6045	0.4770
	0.3	0.3565	0.4730	0.8115	0.7170	0.4675	0.5970	0.9065	0.8530

Table 3.2d. Empirical Means of Parameter Estimates for SAR Model
`Group Interaction with` $g = n^{0.5}$; $\sigma = 1$; `XVal-B`; $M = 10,000$

DGP	λ	Restricted Estimates β_0	β_1	β_2	σ	Unrestricted Estimates β_0	β_1	β_2	σ
$n = 100$									
1	-0.5	3.3477	0.7586	0.6996	1.0405	5.1982	1.0269	1.0353	0.9731
	-0.3	3.8572	0.8303	0.7919	1.0090	5.2246	1.0290	1.0400	0.9738
	0.0	4.9996	0.9981	0.9992	0.9836	5.2532	1.0321	1.0432	0.9768
	0.3	7.1240	1.3202	1.3899	1.0243	5.2597	1.0358	1.0445	0.9770
	0.5	9.9595	1.7482	1.9161	1.1931	5.2825	1.0388	1.0509	0.9793
2	-0.5	3.3490	0.7612	0.6970	1.0272	5.1915	1.0287	1.0306	0.9594
	-0.3	3.8569	0.8300	0.7906	0.9983	5.2092	1.0262	1.0353	0.9633
	0.0	5.0005	1.0005	1.0013	0.9705	5.2344	1.0321	1.0418	0.9643
	0.3	7.1248	1.3207	1.3930	1.0136	5.2611	1.0371	1.0465	0.9655
	0.5	9.9586	1.7529	1.9134	1.1816	5.2658	1.0386	1.0461	0.9635
3	-0.5	3.3487	0.7589	0.6983	0.9916	5.1696	1.0246	1.0269	0.9244
	-0.3	3.8569	0.8322	0.7907	0.9614	5.1934	1.0272	1.0317	0.9264
	0.0	5.0013	0.9995	0.9991	0.9341	5.2160	1.0278	1.0356	0.9283
	0.3	7.1250	1.3215	1.3917	0.9791	5.2365	1.0329	1.0413	0.9285
	0.5	9.9568	1.7450	1.9168	1.1535	5.2584	1.0314	1.0469	0.9280
$n = 200$									
1	-0.5	3.3361	0.8380	0.7911	1.0307	5.1601	1.0116	1.0185	0.9873
	-0.3	3.8480	0.8867	0.8534	1.0107	5.1816	1.0150	1.0199	0.9883
	0.0	4.9998	0.9996	1.0000	0.9916	5.1780	1.0139	1.0199	0.9885
	0.3	7.1383	1.2237	1.2786	1.0261	5.1899	1.0177	1.0210	0.9886
	0.5	9.9891	1.5264	1.6568	1.1657	5.1929	1.0180	1.0217	0.9898
2	-0.5	3.3370	0.8389	0.7899	1.0242	5.1603	1.0130	1.0169	0.9806
	-0.3	3.8479	0.8860	0.8526	1.0048	5.1796	1.0138	1.0189	0.9824
	0.0	5.0007	0.9988	0.9999	0.9845	5.1872	1.0140	1.0209	0.9814
	0.3	7.1393	1.2194	1.2821	1.0205	5.1972	1.0152	1.0246	0.9828
	0.5	9.9905	1.5231	1.6588	1.1589	5.2048	1.0177	1.0250	0.9830
3	-0.5	3.3364	0.8394	0.7901	1.0021	5.1544	1.0123	1.0163	0.9578
	-0.3	3.8470	0.8859	0.8524	0.9738	5.1603	1.0118	1.0164	0.9515
	0.0	5.0001	0.9990	1.0004	0.9615	5.1753	1.0129	1.0199	0.9585
	0.3	7.1391	1.2237	1.2788	0.9984	5.1782	1.0162	1.0197	0.9592
	0.5	9.9922	1.5219	1.6566	1.1382	5.1942	1.0159	1.0218	0.9572

Table 3.3a. Bootstrap Critical Values for LM and SLM Tests of Spatial Error Components

Group Sizes $\{2, 3, 4, 5, 6, 7\}$, $m = 8$, $\sigma = 1$, XVAL-B

		Normal Error			Normal Mixture			Lognormal		
Method	λ	10%	5%	1%	10%	5%	1%	10%	5%	1%
					LM Test					
RS_{rr}	0.0	1.0763	1.4706	2.2198	1.6682	2.3600	3.6943	2.1365	3.2755	5.5770
	0.5	1.0766	1.4684	2.2162	1.5400	2.1660	3.3711	1.9092	2.9030	4.8534
	1.0	1.0784	1.4699	2.2376	1.4653	2.0475	3.1833	1.8069	2.7365	4.5543
	1.5	1.0836	1.4811	2.2416	1.4126	1.9668	3.0347	1.6942	2.5609	4.2301
	2.0	1.0935	1.4932	2.2571	1.3744	1.9066	2.9449	1.6207	2.4350	4.0063
RS_{uu}	0.0	1.0754	1.4690	2.2184	1.6453	2.3256	3.6383	2.0866	3.1835	5.3784
	0.5	1.0738	1.4649	2.2097	1.5392	2.1640	3.3659	1.9024	2.8723	4.7672
	1.0	1.0709	1.4609	2.2217	1.4829	2.0749	3.2285	1.8312	2.7751	4.5934
	1.5	1.0710	1.4632	2.2140	1.4439	2.0192	3.1225	1.7438	2.6375	4.3598
	2.0	1.0732	1.4657	2.2190	1.4137	1.9705	3.0440	1.6968	2.5611	4.2373
MC	0.0	1.0772	1.4737	2.2308	1.7310	2.4793	4.0564	2.2162	3.4827	7.4663
					SLM Test					
RS_{rr}	0.0	1.3219	1.7255	2.4923	1.3693	1.8443	2.7315	1.4028	1.9818	2.9503
	0.5	1.3204	1.7213	2.4860	1.3578	1.8204	2.6939	1.3953	1.9451	2.8880
	1.0	1.3181	1.7185	2.4993	1.3520	1.8043	2.6625	1.3877	1.9264	2.8542
	1.5	1.3175	1.7208	2.4910	1.3498	1.7939	2.6297	1.3729	1.8944	2.8019
	2.0	1.3218	1.7272	2.4974	1.3463	1.7834	2.6192	1.3654	1.8749	2.7717
RS_{uu}	0.0	1.3215	1.7251	2.4921	1.3675	1.8399	2.7248	1.3998	1.9700	2.9357
	0.5	1.3202	1.7212	2.4856	1.3581	1.8205	2.6921	1.3954	1.9418	2.8843
	1.0	1.3176	1.7182	2.4977	1.3543	1.8077	2.6717	1.3900	1.9348	2.8701
	1.5	1.3169	1.7186	2.4882	1.3529	1.8049	2.6488	1.3783	1.9076	2.8291
	2.0	1.3197	1.7224	2.4938	1.3505	1.7988	2.6390	1.3748	1.8983	2.8169
MC	0.0	1.3189	1.7238	2.5153	1.3714	1.8843	2.8192	1.3823	2.0921	3.1531

MC: Monte Carlo Critical values based on $M = 50,000$.

Table 3.3b. Bootstrap Critical Values for LM and SLM Tests of Spatial Error Components

Group Sizes $\{2,3,4,5,6,7\}$, $m=19$; $\sigma=1$, XVAL-B

		Normal Error			Normal Mixture			Lognormal		
Method	λ	10%	5%	1%	10%	5%	1%	10%	5%	1%
					LM Test					
RS_{rr}	0.0	1.1502	1.5338	2.2687	1.8199	2.4772	3.7743	2.6510	4.0188	6.6391
	0.5	1.1526	1.5372	2.2731	1.6764	2.2758	3.4514	2.3728	3.6083	5.9152
	1.0	1.1558	1.5418	2.2794	1.5888	2.1559	3.2507	2.1695	3.2701	5.2968
	1.5	1.1612	1.5485	2.2905	1.5333	2.0724	3.1247	2.0451	3.0671	4.9338
	2.0	1.1710	1.5607	2.3023	1.4866	2.0145	3.0414	1.9375	2.8884	4.6049
RS_{uu}	0.0	1.1499	1.5333	2.2678	1.8084	2.4606	3.7472	2.6015	3.9204	6.4371
	0.5	1.1495	1.5341	2.2673	1.6880	2.2929	3.4773	2.3740	3.5833	5.8370
	1.0	1.1489	1.5332	2.2617	1.6219	2.2022	3.3299	2.2332	3.3549	5.4274
	1.5	1.1473	1.5295	2.2640	1.5833	2.1467	3.2393	2.1538	3.2382	5.2166
	2.0	1.1525	1.5362	2.2629	1.5435	2.0858	3.1581	2.0716	3.1130	4.9923
MC	0.0	1.1569	1.5445	2.2472	1.8325	2.5278	3.9093	2.6464	4.1103	8.5357
					SLM Test					
RS_{rr}	0.0	1.3026	1.6901	2.4312	1.3369	1.7769	2.6263	1.3836	1.9416	2.9125
	0.5	1.3029	1.6909	2.4326	1.3286	1.7583	2.5835	1.3722	1.9220	2.8692
	1.0	1.3003	1.6882	2.4274	1.3259	1.7512	2.5635	1.3661	1.8927	2.8153
	1.5	1.3011	1.6880	2.4293	1.3239	1.7435	2.5493	1.3579	1.8737	2.7870
	2.0	1.3048	1.6925	2.4299	1.3194	1.7361	2.5346	1.3549	1.8583	2.7535
RS_{uu}	0.0	1.3024	1.6899	2.4311	1.3360	1.7745	2.6227	1.3820	1.9352	2.9025
	0.5	1.3026	1.6911	2.4319	1.3287	1.7610	2.5867	1.3742	1.9187	2.8649
	1.0	1.3010	1.6895	2.4243	1.3274	1.7571	2.5783	1.3696	1.9025	2.8324
	1.5	1.3000	1.6862	2.4279	1.3280	1.7526	2.5666	1.3657	1.8932	2.8177
	2.0	1.3045	1.6926	2.4266	1.3238	1.7442	2.5579	1.3643	1.8821	2.8027
MC	0.0	1.3033	1.6967	2.4031	1.3209	1.7774	2.6576	1.3432	2.0206	3.0694

MC: Monte Carlo Critical values based on $M=50,000$.

Table 3.3c. Rejection Frequencies for LM and SLM Tests of Spatial Error Components

Group Sizes $\{2,3,4,5,6,7\}$, $m = 8$, $\sigma = 1$, XVAL-B

		Normal Error			Normal Mixture			Lognormal		
Method	λ	10%	5%	1%	10%	5%	1%	10%	5%	1%
					LM Test					
ACR	0.0	0.0690	0.0330	0.0070	0.1480	0.1075	0.0575	0.1790	0.1420	0.0960
	0.5	0.5845	0.4640	0.2490	0.5550	0.4590	0.2980	0.5795	0.5015	0.3665
	1.0	0.9005	0.8460	0.6780	0.8540	0.7870	0.6470	0.8110	0.7635	0.6525
	1.5	0.9815	0.9695	0.9180	0.9650	0.9415	0.8680	0.9140	0.8890	0.8205
	2.0	0.9960	0.9910	0.9665	0.9850	0.9750	0.9340	0.9530	0.9375	0.9010
$\mathtt{RS}_{rr}$	0.0	0.1010	0.0465	0.0120	0.1045	0.0555	0.0135	0.1180	0.0625	0.0180
	0.5	0.6560	0.5215	0.2760	0.4890	0.3505	0.1505	0.4735	0.3275	0.1610
	1.0	0.9330	0.8720	0.7045	0.8140	0.6980	0.4520	0.7190	0.5945	0.3850
	1.5	0.9890	0.9765	0.9310	0.9535	0.9045	0.7405	0.8600	0.7820	0.6115
	2.0	0.9960	0.9935	0.9720	0.9805	0.9545	0.8570	0.9215	0.8560	0.7130
$\mathtt{RS}_{uu}$	0.0	0.1010	0.0480	0.0115	0.1065	0.0605	0.0205	0.1215	0.0685	0.0395
	0.5	0.6570	0.5230	0.2840	0.4835	0.3490	0.1540	0.4690	0.3245	0.1580
	1.0	0.9330	0.8740	0.7055	0.8090	0.6850	0.4320	0.7145	0.5820	0.3670
	1.5	0.9890	0.9775	0.9300	0.9520	0.8980	0.7020	0.8555	0.7640	0.5780
	2.0	0.9960	0.9930	0.9715	0.9795	0.9490	0.8295	0.9160	0.8395	0.6735
					SLM Test					
ACR	0.0	0.1025	0.0525	0.0130	0.1090	0.0660	0.0255	0.1160	0.0795	0.0440
	0.5	0.6640	0.5465	0.3210	0.4985	0.3875	0.2085	0.4685	0.3735	0.2225
	1.0	0.9340	0.8845	0.7485	0.8210	0.7320	0.5420	0.7160	0.6320	0.4745
	1.5	0.9890	0.9780	0.9445	0.9540	0.9185	0.8155	0.8480	0.8005	0.6925
	2.0	0.9965	0.9950	0.9805	0.9810	0.9590	0.9010	0.9045	0.8725	0.7900
$\mathtt{RS}_{rr}$	0.0	0.1015	0.0465	0.0120	0.0970	0.0515	0.0105	0.1040	0.0590	0.0170
	0.5	0.6535	0.5205	0.2755	0.4730	0.3380	0.1435	0.4400	0.3105	0.1585
	1.0	0.9330	0.8715	0.7050	0.8045	0.6825	0.4510	0.6870	0.5705	0.3765
	1.5	0.9885	0.9765	0.9310	0.9490	0.8980	0.7290	0.8335	0.7590	0.6120
	2.0	0.9960	0.9935	0.9715	0.9785	0.9510	0.8530	0.8955	0.8335	0.7080
$\mathtt{RS}_{uu}$	0.0	0.1000	0.0485	0.0110	0.0975	0.0525	0.0110	0.1035	0.0595	0.0210
	0.5	0.6550	0.5220	0.2780	0.4750	0.3405	0.1460	0.4420	0.3105	0.1560
	1.0	0.9320	0.8730	0.7020	0.8050	0.6795	0.4470	0.6875	0.5670	0.3715
	1.5	0.9890	0.9770	0.9295	0.9480	0.8970	0.7240	0.8325	0.7560	0.5995
	2.0	0.9960	0.9935	0.9715	0.9780	0.9485	0.8475	0.8940	0.8330	0.7010

Table 3.3d. Rejection Frequencies for LM and SLM Tests of Spatial Error Components

`Group Sizes` $\{2,3,4,5,6,7\}$, $m=19$, $\sigma=1$, `XVAL-B`

		Normal Error			Normal Mixture			Lognormal		
Method	λ	10%	5%	1%	10%	5%	1%	10%	5%	1%
					LM Test					
ACR	0.0	0.0850	0.0425	0.0080	0.1680	0.1205	0.0630	0.2025	0.1705	0.1200
	0.5	0.8835	0.8175	0.6270	0.8205	0.7510	0.5775	0.7595	0.6995	0.5670
	1.0	0.9980	0.9945	0.9795	0.9900	0.9805	0.9340	0.9520	0.9330	0.8785
$\mathtt{RS}_{rr}$	0.0	0.1070	0.0560	0.0085	0.0955	0.0525	0.0100	0.1105	0.0615	0.0140
	0.5	0.9125	0.8420	0.6400	0.7415	0.5895	0.3340	0.5735	0.4145	0.2165
	1.0	0.9985	0.9960	0.9810	0.9790	0.9405	0.8095	0.8590	0.7565	0.5485
$\mathtt{RS}_{uu}$	0.0	0.1085	0.0550	0.0085	0.0975	0.0535	0.0145	0.1130	0.0695	0.0295
	0.5	0.9115	0.8450	0.6400	0.7365	0.5825	0.3255	0.5685	0.4105	0.2110
	1.0	0.9985	0.9960	0.9820	0.9765	0.9375	0.7895	0.8530	0.7440	0.5185
					SLM Test					
ACR	0.0	0.1105	0.0585	0.0125	0.1010	0.0610	0.0205	0.1055	0.0765	0.0360
	0.5	0.9135	0.8510	0.6760	0.7520	0.6220	0.3980	0.5650	0.4600	0.2965
	1.0	0.9985	0.9970	0.9845	0.9805	0.9510	0.8555	0.8465	0.7870	0.6485
$\mathtt{RS}_{rr}$	0.0	0.1075	0.0565	0.0080	0.0935	0.0490	0.0105	0.0975	0.0560	0.0120
	0.5	0.9120	0.8425	0.6385	0.7310	0.5845	0.3295	0.5360	0.3985	0.2105
	1.0	0.9985	0.9960	0.9820	0.9765	0.9375	0.8120	0.8265	0.7330	0.5480
$\mathtt{RS}_{uu}$	0.0	0.1065	0.0555	0.0080	0.0920	0.0515	0.0125	0.0980	0.0570	0.0180
	0.5	0.9115	0.8450	0.6385	0.7300	0.5820	0.3240	0.5345	0.3965	0.2095
	1.0	0.9985	0.9960	0.9815	0.9760	0.9385	0.8065	0.8275	0.7305	0.5385