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On the Performance of Geometric Charts with Estimated Control Limits

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Abstract

The control chart based on geometric distribution (geometric chart) has been shown to be competitive to *p*- or *np*- charts for monitoring proportion nonconforming, especially for applications in high quality manufacturing environment. However, implementing a geometric chart often assumes the process parameter to be known or accurately estimated. For a high quality process, an accurate parameter estimate may require a very large sample size that is seldom available. In this paper we investigate the sample size effect when the process parameter needs to be estimated. It is shown that the estimated control limits create dependence among the monitoring events. Analytical approximation is derived to compute shift detection probabilities and run length distributions. It is found that, when there is no shift in proportion nonconforming, the false alarm probability increases as the sample size decreases and the effect can be significant even with sample size as large as 10,000. However, the in-control average run length is only affected mildly. On the other hand, when there is a process shift, the out-of-control average run length can be significantly affected by the estimated control limits, even with very large sample sizes. In practice, the quantitative results of the paper can be used to determine the minimum number of items required for estimating the control limits of a geometric chart so that certain average run length requirements are met.

Keywords: Geometric chart, estimated control limits, false alarm probability, average run length, sample size determination, high-quality processes, cumulative count of conforming items

1. Introduction

Statistical process control has been shown to be very effective in monitoring and improving manufacturing and service processes. In practice, implementing a control chart often assumes the process parameters to be known or accurately estimated so that the control limits can be computed. For high quality processes, an accurate parameter estimate may require a very large sample size that is seldom available. It is important to understand the impact of the estimated control limits when the process parameters have to be estimated so that the size of the inspected sample can be determined.

For monitoring normal data, Hillier (1969) pioneered the research on the estimated control limits for the $\overline{X}$ and R charts. Quesenberry (1993) further studied the effects of the sample size on estimated limits for $\overline{X}$ and X charts. Other related studies can be found in Proschan and Savage (1960), Nelson (1984), Quesenberry (1991), Montgomery (1996), Chen (1997, 1998) and Braun (1999).

This paper investigates the effect of estimating control limits on the geometric chart, a statistical control chart that is shown to be particularly useful for high quality processes (Kaminsky *et al*., 1992, Glushkovsky, 1994, Nelson, 1994, Woodall, 1997, Xie and Goh, 1992, 1997). By monitoring cumulative count of conforming items between two nonconforming ones, we could detect further process improvement (Goh and Xie, 1994), avoid the problem of fixing the rational sample size, and reduce the amount of plotting. However, it is especially important to study the effect of estimated control limit in this case as for high quality processes, nonconforming items are rare and a very large sample size is needed to obtain a reasonably accurate estimate of the parameters. The error in the estimated fraction nonconforming p could possibly lead the estimated control limits to be far from the true limits. Under such circumstances, the misleading results as indicated by the geometric chart may result in wrong management decisions. In fact, the study of estimated control limits is a general research issue of importance (Woodall and Montgomery, 1999).

The paper is organized as follows. First the geometric chart with a known or estimated process parameter is described and the problem of estimation error is discussed. The effect of estimating control limits is then investigated and compared with the case of known limits for geometric chart. Explicit equations for false alarm probability and run length distribution are derived. The minimum sample size that provides the geometric chart the protection against giving misleading results is then discussed.

2. The Geometric Charts with Known or Estimated Model Parameter

Let Y_i be the cumulative count of conforming items between the (i-1)th and ith nonconforming items, from a stable process running in an automated manufacturing environment with the probability of having a nonconforming item p_0. As a stable process is just a sequence of independent Bernoulli trials with the same probability of success p_0, $Y_i + 1$ is distributed as geometric with parameter p_0. Hence the probability mass function of Y_i is

$$g(y_i) = (1-p_0)^{y_i} p_0, \; y_i = 0, 1, \ldots,$$

with $P(Y_i \geq y_i) = (1-p_0)^{y_i}$.

Let LCL and UCL be, respectively, the lower and upper probability control limits for Y_i. Then LCL and UCL must satisfy,

$$\sum_{y_i=0}^{LCL-1}(1-p_0)^{y_i} p_0 = \frac{\alpha}{2} \text{ and } \sum_{y_i=UCL+1}^{\infty}(1-p_0)^{y_i} p_0 = \frac{\alpha}{2},$$

which are equivalent to

$$1-(1-p_0)^{LCL} = \frac{\alpha}{2} \text{ and } (1-p_0)^{UCL+1} = \frac{\alpha}{2}.$$

Thus, the control limits for the geometric chart have the form:

$$LCL = \frac{\ln(1-\alpha/2)}{\ln(1-p_0)}, \text{ and} \tag{1}$$

$$UCL = \frac{\ln(\alpha/2)}{\ln(1-p_0)} - 1. \tag{2}$$

It should be noted that these limits produce roughly the desired α for sufficient small p_0, but not exact due to the discrete nature of the geometric distribution. In this sense, α may be understood as the overall desired false alarm rate. The true false alarm rate may deviate a bit from it, but our calculations (Table 1) show that this deviation is negligible.

These formulas assume that the fraction nonconforming p_0 is known. If it is not known, a common practice is to inspect an initial number of nonconforming items and estimate the fraction nonconforming value. An accurate estimate requires a large sample size, and usually because of the estimation error due to small sample size, the estimated control limits could be far from the exact values.

When p_0 is unknown but an initial sample is taken, the traditional estimate is

$$\hat{p}_0 = \frac{N}{m},$$

where N is the number of nonconforming items among a total of m items sampled. Clearly, $N \sim \text{Binomial}(m,\ p_0)$.

Using $\hat{p}_0$ as an estimate of p_0, we obtain estimated control limits for the geometric chart:

$$L\hat{C}L(N) = \frac{\ln(1-\alpha/2)}{\ln(1-N/m)}, \text{ and} \tag{3}$$

$$U\hat{C}L(N) = \frac{\ln(\alpha/2)}{\ln(1-N/m)} - 1. \tag{4}$$

The aim in this paper is to investigate the performance of (3) and (4).

3. The Alarm Rate When the Parameters are Estimated

Let Y^i be a future observation (number of conforming items between two adjacent nonconforming items) from a process (possibly shifted from p_0 to p). Define the event,

$$B_i = \{\, Y^i > U\hat{C}L(N) \text{ or } Y^i < L\hat{C}L(N) \,\}.$$

Then, $P(B_i)$ is the actual alarm rate (AR), which becomes the actual false alarm rate (FAR) when $p = p_0$. Following a conditional argument, we obtain

$$P(B_i) = \sum_{n=0}^{m} P(B_i \mid N = n) P(N = n) = \sum_{n=0}^{m} P(B_i \mid N = n) \binom{m}{n} p_0^n (1 - p_0)^{m-n} \tag{5}$$

where

$$\begin{aligned} P(B_i \mid N = n) &= P\{ Y^i > U\hat{C}L(N) \mid N = n\} + P\{Y^i < L\hat{C}L(N) \mid N = n\} \\ &= (1-p)^{\ln(\alpha/2)/\ln(1-n/m)} - (1-p)^{\ln(1-\alpha/2)/\ln(1-n/m)} + 1. \end{aligned} \tag{6}$$

Notice that the events B_i's are dependent of each other, as they all depend on the same estimated control limits. Lai *et al.* (1998) considered the effects of data correlation on the geometric chart procedure, but did not consider the effect of estimation.

So, the actual AR or FAR can be calculated using (5). To simplify the computation, a truncation procedure is given in the Appendix and it has been shown to be accurate and efficient. Table1 provides FAR values for different combinations of p_0 and *m*, and Table 2 provides AR values when the process parameter is shifted from p_0 = 0.0005.

It is apparent from Table 1 that the actual FAR can deviate a lot from its desired value of 0.0027 when the true process fraction nonconforming p_0 is estimated from *m* sampled items, especially when p_0 is very small. For a fixed p_0 value, increasing *m* will significantly reduce the amount of deviation. This is because the variability in the estimated p_0 gets smaller as *m* increases. When sample size *m* is large enough (a smaller p_0 requires a larger *m*), the FAR can be very close to 0.0027, and hence the estimation effects on the false alarm probability of the geometric chart can be neglected. For example, when p_0 = 0.0001 and *m* = 800,000, we have FAR = 0.00297 and when p_0 = 0.001 and *m* = 100,000, we have FAR = 0.00292. Thus, for a high quality manufacturing process, i.e., when it is known that p_0 is small, Table 1 may be used for deciding an appropriate sample size for implementing a G-chart. Finally, it should be pointed out that even if the true p_0 value is used in the control limits, the FAR is still not exactly 0.0027 as the distribution is discrete.

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Table 1. Values of FAR for G-Chart with Estimated Control Limits, $\alpha = 0.0027$.

$m \setminus p_0$	0.0001	0.0002	0.0003	0.0004	0.0005	0.0006	0.0007	0.0008	0.0009	0.001
10000	0.38651	0.14718	0.05911	0.02623	0.01371	0.00878	0.00671	0.00575	0.00524	0.00492
20000	0.14719	0.02623	0.00878	0.00575	0.00492	0.00452	0.00425	0.00406	0.00391	0.00379
50000	0.01372	0.00492	0.00415	0.00379	0.00357	0.00343	0.00332	0.00325	0.00319	0.00314
100000	0.00492	0.00379	0.00343	0.00325	0.00314	0.00306	0.00301	0.00297	0.00294	0.00292
200000	0.00378	0.00324	0.00306	0.00297	0.00291	0.00288	0.00286	0.00283	0.00282	0.00281
300000	0.00343	0.00306	0.00294	0.00288	0.00284	0.00282	0.00281	0.00279	0.00278	0.00277
400000	0.00324	0.00297	0.00288	0.00283	0.00281	0.00279	0.00278	0.00277	0.00276	0.00276
500000	0.00314	0.00292	0.00285	0.00281	0.00279	0.00277	0.00276	0.00276	0.00275	0.00275
600000	0.00306	0.00288	0.00282	0.00279	0.00279	0.00277	0.00276	0.00275	0.00274	0.00274
700000	0.00301	0.00286	0.0028	0.00278	0.00276	0.00275	0.00275	0.00274	0.00274	0.00273
800000	0.00297	0.00284	0.00279	0.00277	0.00276	0.00275	0.00274	0.00274	0.00273	0.00273
900000	0.00294	0.00282	0.00278	0.00276	0.00274	0.00274	0.00274	0.00273	0.00273	0.00273
1000000	0.00292	0.00281	0.00277	0.00276	0.00274	0.00273	0.00273	0.00273	0.00273	0.00272
2000000	0.00281	0.00276	0.00274	0.00273	0.00272	0.00272	0.00272	0.00272	0.00271	0.00271
∞	0.00270	0.00270	0.00270	0.00270	0.00270	0.00270	0.00270	0.00270	0.00270	0.00270

From Table 2, we see that for a given $p \neq p_0$ (i.e., a given shifted process), the AR behaves similarly to the FAR as m increases, i.e., converging to its known value. For a given m, the AR increases as p decreases, but as p increases it first decreases and then increases very slowly. This means that G-chart is able to detect a process improvement (p decreases), but is unable to detect a process deterioration unless the deterioration amount is large enough. This seems to be a common problem for a probability chart of a quantity having a skewed distribution (See Xie and Goh, 1997), irrespective whether the control limits are known or estimated. The explanation is simple for a G-chart of a high yield process. As p_0 is small, the LCL is usually very small (e.g., 2.7 for α = 0.0027 and p_0 = 0.0005). When p is decreased, the plotted quantity Y tends to be larger, hence more chance to exceed UCL, resulting in an increased AR. When p is increased, the plotted quantity tends to be smaller, hence less chance to exceed UCL. At the same time, the chance for the plotted quantity to go below LCL would not increase much as LCL is already very small, unless p is increased by a big amount. This is why AR decreases first and slowly increases when p moves to right of p_0. Other values of p_0 were also examined and the general conclusions reached are consistent.

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Table 2. Values of AR for G-Chart with Estimated Control Limits: **a.** $\alpha = 0.0027$, p_0=0.0005

m \ p	0.0001	0.0002	0.0003	0.0004	**0.0005**	0.0006	0.0007	0.0008	0.0009	0.001
10000	0.25860	0.09028	0.03854	0.02047	**0.01371**	0.01114	0.01023	0.01000	0.01009	0.01031
20000	0.25751	0.07797	0.02648	0.01027	**0.00492**	0.00317	0.00270	0.00270	0.00288	0.00313
50000	0.26268	0.07421	0.02264	0.00786	**0.00357**	0.00241	0.00222	0.00234	0.00257	0.00283
100000	0.26479	0.07296	0.02126	0.00702	**0.00314**	0.00219	0.00209	0.00226	0.00250	0.00276
200000	0.26592	0.07234	0.02054	0.00658	**0.00292**	0.00208	0.00203	0.00222	0.00247	0.00273
300000	0.26631	0.07213	0.02029	0.00640	**0.00285**	0.00205	0.00202	0.00221	0.00246	0.00272
400000	0.26650	0.07203	0.02017	0.00637	**0.00281**	0.00203	0.00201	0.00220	0.00245	0.00271
500000	0.26662	0.07197	0.02010	0.00632	**0.00278**	0.00202	0.00200	0.00220	0.00245	0.00271
600000	0.26670	0.07190	0.02005	0.00629	**0.00277**	0.00201	0.00200	0.00220	0.00245	0.00271
700000	0.26676	0.07190	0.02001	0.00627	**0.00276**	0.00200	0.00200	0.00219	0.00244	0.00271
800000	0.26680	0.07188	0.01998	0.00626	**0.00276**	0.00200	0.00200	0.00219	0.00244	0.00271
900000	0.26683	0.07186	0.01996	0.00624	**0.00274**	0.00200	0.00200	0.00219	0.00244	0.00271
1000000	0.26686	0.07185	0.01995	0.00623	**0.00274**	0.00200	0.00200	0.00219	0.00244	0.00271
∞	0.26707	0.07171	0.01979	0.00614	**0.00270**	0.00198	0.00199	0.00219	0.00244	0.00270

b. $\alpha = 0.0027$, p_0=0.0003

10000	0.17323	0.07859	**0.05911**	0.05439	0.05336	0.05339	0.05376	0.05424	0.05477	0.05531
20000	0.11968	0.02467	**0.00878**	0.00586	0.00559	0.00592	0.00641	0.00695	0.00750	0.00806
50000	0.11317	0.01713	**0.00415**	0.00244	0.00253	0.00293	0.00340	0.00388	0.00436	0.00484
100000	0.11197	0.01521	**0.00342**	0.00218	0.00238	0.00280	0.00326	0.00372	0.00419	0.00465
200000	0.11145	0.01419	**0.00306**	0.00206	0.00232	0.00275	0.00320	0.00366	0.00412	0.00457
300000	0.11129	0.01384	**0.00294**	0.00202	0.00230	0.00273	0.00318	0.00364	0.00409	0.00454
400000	0.11121	0.01366	**0.00288**	0.00200	0.00229	0.00272	0.00317	0.00363	0.00408	0.00453
500000	0.11117	0.01355	**0.00285**	0.00199	0.00229	0.00272	0.00317	0.00362	0.00407	0.00452
600000	0.11111	0.01347	**0.00282**	0.00198	0.00228	0.00271	0.00317	0.00362	0.00407	0.00452
700000	0.11111	0.01343	**0.00280**	0.00197	0.00228	0.00271	0.00316	0.00361	0.00407	0.00452
800000	0.11110	0.01339	**0.00279**	0.00196	0.00228	0.00271	0.00316	0.00361	0.00406	0.00451
900000	0.11109	0.01336	**0.00278**	0.00197	0.00227	0.00271	0.00316	0.00361	0.00406	0.00451
1000000	0.11108	0.01333	**0.00277**	0.00197	0.00228	0.00271	0.00316	0.00361	0.00406	0.00451
∞	0.11100	0.01312	**0.00270**	0.00195	0.00227	0.00270	0.00315	0.00360	0.00405	0.00449

c. $\alpha = 0.0027$, p_0=0.0007

10000	0.36599	0.15356	0.06989	0.03408	0.01790	0.01033	**0.00671**	0.00499	0.00421	0.00390
20000	0.37621	0.15164	0.06444	0.02881	0.01369	0.00712	**0.00425**	0.00302	0.00255	0.00243
50000	0.38394	0.15156	0.06152	0.02585	0.01149	0.00566	**0.00332**	0.00244	0.00216	0.00216
100000	0.38664	0.15166	0.06051	0.02479	0.01070	0.00515	**0.00301**	0.00225	0.00205	0.00208
200000	0.38801	0.15175	0.06002	0.02425	0.01030	0.00489	**0.00286**	0.00216	0.00199	0.00204
300000	0.38847	0.15178	0.05984	0.02408	0.01016	0.00480	**0.00280**	0.00213	0.00197	0.00203
400000	0.38870	0.15179	0.05977	0.02398	0.01009	0.00476	**0.00278**	0.00211	0.00197	0.00202
500000	0.38884	0.15180	0.05972	0.02392	0.01005	0.00473	**0.00276**	0.00210	0.00196	0.00202
600000	0.38893	0.15181	0.05968	0.02389	0.01002	0.00472	**0.00275**	0.00210	0.00196	0.00202
700000	0.38900	0.15182	0.05966	0.02386	0.01000	0.00470	**0.00274**	0.00209	0.00195	0.00201
800000	0.38905	0.15182	0.05964	0.02384	0.00999	0.00469	**0.00274**	0.00209	0.00195	0.00201
900000	0.38908	0.15182	0.05963	0.02383	0.00998	0.00469	**0.00274**	0.00209	0.00195	0.00201
1000000	0.38912	0.15183	0.05962	0.02381	0.00997	0.00468	**0.00273**	0.00209	0.00195	0.00201
∞	0.38939	0.15185	0.05952	0.02370	0.00989	0.00463	**0.00270**	0.00207	0.00194	0.00201

4. Run Length Distribution with Estimated Limits

Let R be the number of points plotted on the chart until an out-of-control signal is given. The R is called the run length and the distribution of R is called the run length distribution. The study of run length distribution for a given control chart is of a great interest to quality professionals, especially when the events are correlated and/or the control limits are estimated. Denote $P(B_i \mid N)$ defined in (6) by $\alpha(N)$. It is easy to see that the conditional distribution of R given N is geometric with parameter $\alpha(N)$. Hence, the unconditional distribution of R is:

$$f_R(r; p_0, p) = \sum_{n=0}^{m} [1-\alpha(n)]^{r-1} \alpha(n) P(N=n)$$

$$= \sum_{n=0}^{m} [1-\alpha(n)]^{r-1} \alpha(n) \binom{m}{n} p_0^n (1-p_0)^{m-n} \tag{7}$$

The run length distribution given in (7) can be easily evaluated and plotted, for any p_0 and p, using Mathematica or some other statistical software.

Figure 1 shows the run length distribution with $p = p_0 = 0.0005$ and three different sample sizes. It is seen that the three run length distributions differ mainly in the left tail, a smaller m results in a taller curve in the left tail. This means that more short runs are to be expected.

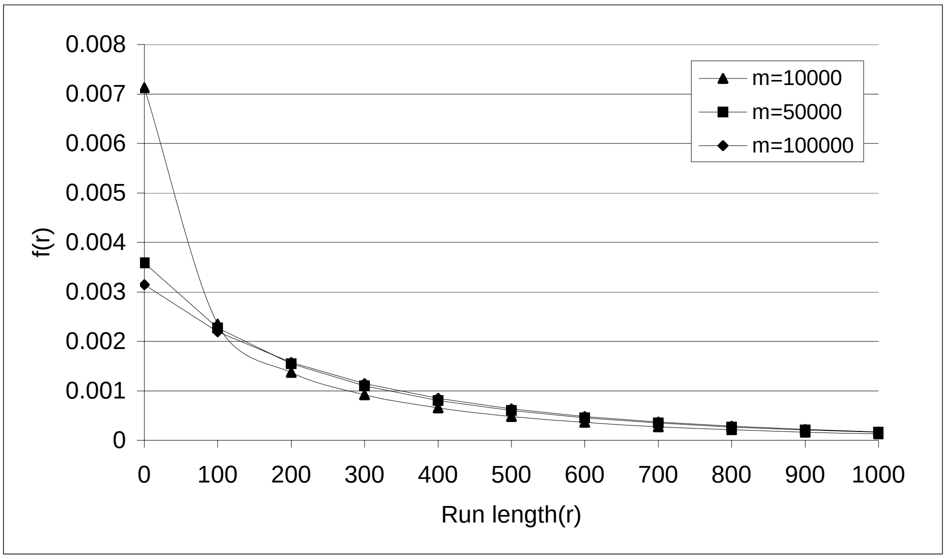

Fig.1: Run length Distribution for three different sample sizes, $p = p_0 = 0.0005$.

Following the similar conditional arguments, the average run length (*ARL*) and the standard deviation of the run length (*SDRL*) can be seen to have the forms:

$$ARL(p_0, p) = E_N[1/\alpha(N)] \text{, and} \tag{8}$$

$$SDRL(p_0, p) = \sqrt{Var_N[1/\alpha(N)] + E_N[(1-\alpha(N))/\alpha^2(N)]}\,. \tag{9}$$

Thus, using the distribution of *N*, the quantities involved in (8) and (9) can be calculated as:

$$E_N[1/\alpha(N)] = \sum_{n=0}^{m} \frac{1}{\alpha(n)} \binom{m}{n} p_0^n (1-p_0)^{m-n} \text{ , and}$$

$$E_N[1/\alpha^2(N)] = \sum_{n=0}^{m} \frac{1}{\alpha^2(n)} \binom{m}{n} p_0^n (1-p_0)^{m-n} \text{ .}$$

The corresponding *ARL* and *SDRL* with known control limits are of the forms

$$ARL_0(p_0, p) = 1/P(A_i) \text{, and} \tag{10}$$

$$SDRL_0(p_0, p) = \sqrt{1-P(A_i)}\,/\,P(A_i) = \sqrt{ARL(ARL-1)}\,, \tag{11}$$

where the events A_i are defined similarly as B_i, but correspond to the known control limits. Thus, a reasonable way to decide when the sample size *m* is large enough for $U\widehat{C}L$ and $L\widehat{C}L$ to be essentially the same as *UCL* and *LCL* is to determine when the *ARL* and *SDRL* with estimated control limits are essentially the same as those given in (10) and (11). For instance, for $\alpha = 0.0027$ and $p = p_0$, one should have $ARL \cong SDRL \cong 370$.

To study the effect of the sample size *m* on the mean and standard deviation (*ARL* and *SDRL*) of the run length distribution, values of *ARL* and *SDRL* are computed from Equations (8) and (9) for a range of values of *m* and *p* and for a fixed $p_0 = 0.0005$. For each value of *p* and *m*, the first value given is the *ARL* and the second is the *SDRL*. For comparison purpose, the exact values for the known *p* case are given in the last row ($m = \infty$).

Comparing the values in Table 3 with their nominal values, certain observations can be noted. First, estimating control limits can cause both *ARL* and *SDRL* larger or smaller than their nominal levels, depending on whether p is smaller or larger than p_0. This is due to the reduced or enlarged probabilities of events $\{B_i\}$ and the dependence among them. Also, for large values of m, *SDRL* can exceed *ARL*, which is in contrast of the case of known control limits where $SDRL = \sqrt{ARL(ARL-1)} < ARL$.

Considering the nature of the run length distribution, it can be seen that when *SDRL* exceeds *ARL*, a large number of short runs that are balanced by a few long runs would be expected, when compared with the standard geometric distribution. This is because when two probability distributions have the same mean, the one with larger standard deviation will have higher probability in its tails. Since the run length takes only the values that are positive integers, which cannot be less than 1, this means that the probability on the lower integers of the distribution with large standard deviation has to be increased and be balanced by an increase in the probability on large integers in the right tail of the distribution. The net effect of the dependence caused by using the estimated control limits $U\hat{C}L$ and $L\hat{C}L$ is that the run length distribution will, for a particular value of *ARL*, have an increased rate of very short runs between alarm signals. There will also be an increased number of long runs between alarm signals. However, the ratio of the number of short runs to the number of long ones also increases and is very large itself. This is an undesirable phenomenon one should be constantly aware of.

Similar to the behavior of AR function with respect to a change in p, the *ARL* and *SDRL* both decrease as p decreases, but when p increases, *ARL* and *SDRL* both increase first and then decrease. The latter phenomenon is undesirable and can be explained in the same way as for *AR* function.

By comparing Tables 2 and 3, it is interesting to note that the impact of estimated control limits are very different on the FAR and the in-control *ARL* (i.e., p_0=0.0005). While the FAR in Table 2 significantly increases as the sample size decreases, the in-control *ARL*'s are only affected mildly by the sample size. On the other hand, the out-of-control *ARL*'s in Table 3 are very significantly affected as the sample size decreases, even with very large sample sizes.

Table 3. Values of *ARL* (upper entry) and *SDRL* (lower entry) for the G-Chart with Estimated Control Limits, $p_0 = 0.0005$

M \ *p*	0.0001	0.0002	0.0003	0.0004	**0.0005**	0.0006	0.0007	0.0008	0.0009	0.001
10000	95.05	173.62	262.73	331.3	374.1	397.5	406.9	406.0	397.1	382.7
	18.55	64.10	138.51	221.4	291.8	339.1	362.3	366.0	356.7	340.1
20000	29.42	108.72	221.54	323.8	391.6	426.2	436.4	429.0	410.1	385.4
	6.56	37.08	114.24	224.1	326.2	391.0	414.4	408.4	386.7	359.5
50000	4.50	34.67	134.84	281.3	398.8	457.4	467.6	447.9	415.1	380.2
	4.18	19.38	78.43	206.8	353.0	442.0	460.8	439.0	403.6	367.9
100000	3.61	19.18	87.26	239.5	394.8	474.9	483.4	454.0	414.0	375.8
	3.93	16.02	63.25	190.4	362.9	467.8	480.9	449.1	407.8	369.4
200000	3.38	15.65	65.55	205.6	387.6	487.1	492.8	456.2	412.4	373.0
	3.83	14.87	56.33	178.1	367.5	484.3	492.0	453.7	409.3	370.0
300000	3.32	14.81	59.63	191.9	383.4	492.2	496.2	456.7	411.7	372.0
	3.80	14.54	54.26	173.3	368.7	490.7	495.8	455.1	409.8	370.1
400000	3.29	14.43	56.96	184.7	380.8	494.9	497.9	456.9	411.3	371.4
	3.79	14.38	53.27	170.8	369.2	494.0	497.8	455.8	410.0	370.1
500000	3.27	14.22	55.45	180.3	379.0	496.7	498.9	457.0	411.1	371.3
	3.78	14.29	52.70	169.2	369.5	496.1	498.9	456.2	410.1	370.2
600000	3.26	14.08	54.47	177.3	377.7	497.9	499.6	457.0	410.9	370.9
	3.77	14.23	52.32	168.2	369.7	497.5	499.7	456.4	410.2	370.2
700000	3.25	13.98	53.79	175.1	376.7	498.8	500.1	457.1	410.8	370.7
	3.77	14.19	52.05	167.4	369.8	498.6	500.3	456.6	410.2	370.2
800000	3.25	13.91	53.29	173.5	376.0	499.5	500.5	457.1	410.7	370.6
	3.77	14.16	51.85	166.8	369.9	499.3	500.7	456.7	410.3	370.2
900000	3.24	13.86	52.91	172.3	375.4	500.0	500.8	457.1	410.6	370.5
	3.76	14.13	51.70	166.4	369.9	500.0	501.00	456.9	410.3	370.2
1000000	3.24	13.81	52.61	171.3	374.9	500.4	501.0	457.1	410.6	370.5
	3.76	14.11	51.58	166.0	370.0	500.4	501.3	456.9	410.3	370.2
2000000	3.75	14.03	51.04	164.4	370.2	502.7	502.4	457.3	410.4	370.3
	3.22	13.62	51.28	166.8	372.5	502.4	502.1	457.1	410.3	370.1
∞	3.74	13.95	50.52	162.8	370.4	505.1	503.1	457.7	410.5	370.3
	3.21	13.44	50.02	162.3	369.9	504.6	503.1	457.2	410.0	369.8

5. Alternative Measures of Run Length

Tables 2 and 3 exhibit some undesirable behavior: the G-chart is unable to detect the process deterioration. It should be pointed out that this is not induced by estimating the control limits, but rather an artifact of the geometric chart. This point can clearly be seen by comparing the results in the last rows of Tables 2 and 3 with results in the other rows.

The run length is defined as the number of plotted points until an out-of-control signal. In other words, it is the number of consecutive nonconforming items until an out-of-control signal. However, what really matters is the total number of checked items R^t until an out-of-control signal. Using the earlier notation, the *i*th plotted point corresponds to $W^i = Y^i+1$ checked items and W^i tends to be larger when p is smaller and smaller when p is larger. Clearly, $R^t = \sum_{i=1}^{R} W^i$ and

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$$E(R^t) = E[E(R^t|N)] = E[E(\textstyle\sum_{i=1}^{R} W^i|N)] = E[E(R|N)E(W^i|N)]\,.$$

The last equation follows Wald's Identity. As W^i is independent of N, we have

$$ARL^t = E(R^t) = E[E(R|N)/p] = ARL/p\,.$$

As R^t is defined in terms of number of inspected items, the ARL^t is referred to as the average run length per item. Using above relation, one can easily convert the results in Table 3 into the values of ARL^t. The converted results are given in Table 4.

From the results in Table 4, we see that for smaller values of m (i.e., less than 100000) average run lengths per item ARL^t decrease after the .0005 entry. For larger values of m, the ARL^t begin to decrease after the .0006 entry. This means that using ARL^t one is able to detect the process deterioration. On the other hand, the ARL^t to the left of the .0005 entry increases when m =10000. This is clearly a consequence of estimation. For larger values of m, the ARL^t to the left of the .0005 entry usually decrease, which is what one would expect.

Similar results can be obtained for alarm rates per item. (Multiply the original alarm rates by p.) The SDRL can also be dealt with but the transformation is a bit more complicated. In summary, G-chart is able to detect both decrease and increase in fraction nonconforming if one uses the run length measures introduced above. Recently, Wu and Spedding (2001) introduced a synthetic control chart for detecting fraction nonconforming increase.

Table 4. The Average Run Length Per Item

$M \setminus p$	0.0001	0.0002	0.0003	0.0004	0.0005	0.0006	0.0007	0.0008	0.0009	0.001
10000	950520	868100	875780	828120	748170	662470	581346	507470	441244	382655
20000	294150	543580	738470	809412	783106	710407	623360	536193	455714	385411
50000	44960	173355	449473	703335	797504	762323	668023	559826	461208	380216
100000	36050	95880	290877	598635	789590	791467	690497	567451	460019	375816
200000	33820	78255	218497	513890	775134	811902	703981	570228	458262	372987
300000	33190	74055	198777	479703	766788	820272	708804	570849	457470	371956
400000	32890	72165	189867	461670	761558	824870	711263	571085	457033	371426
500000	32710	71090	184820	450622	757998	827787	712749	571201	456759	371103
600000	32600	70395	181573	443182	755422	829802	713741	571266	456571	370886
700000	32520	69905	179311	437840	753476	831280	714453	571308	456433	370730
800000	32460	69550	177645	433820	751954	832410	714986	571335	456330	370612
900000	32410	69275	176367	430690	750732	833302	715400	571354	456248	370521
1000000	32380	69055	175356	428180	749728	834025	715733	571369	456181	370447
2000000	37520	70130	170122	410958	740318	837847	717760	571623	456013	370253
∞	37440	69725	168406	406983	740740	841835	719454	572076	456123	370279

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6. Conclusions

Geometric control chart is useful in different context (Kaminsky *et al.*, 1992, Nelson, 1994, Glushkovsky, 1994, Benneyan and Kaminsky, 1994, and Benneyan, 2001). However, it is particularly useful when the cumulative count of conforming items between two nonconforming ones is monitored for high quality process. This poses a great challenge, as the estimation error could be significant when the sample size is not large enough.

In this paper we show that the estimated control limits create dependence among the monitoring events. When there is no shift in proportion nonconforming, the false alarm probability increases as the sample size decreases and the effect can be significant even with very large sample sizes, but the in-control average run length is only affected mildly. However, when there is a process shift, the out-of-control average run length can be significantly affected by the estimated control limits even with very large sample sizes.

It is important to use a right sample size in order to achieve desired performance on the chart. A too small initial sample size will lead to wrong control limits and hence wrong decision to be made. On the other hand, a large sample size can be costly and delays the implementation. To choose the reasonable sample numbers, the exact false alarm probability equation derived in this paper can be used. In practice, Tables 1 to 3 in the paper can be used to determine the minimum number of items required for estimating the control limits so that certain average run length requirements are met.

Alternatively, for a given amount of data, one may consider to adjust the control limits to yield the desired performance such as the in-control ARL. This can be done for the $\overline{X}$ chart with estimated control limits as in this case a known *F*-distribution is involved in the derivation of AR, ARL, etc. There seems, however, no simple ways to do so for the geometric chart. Nevertheless, some further research along this direction should be interesting and worth of pursuing.

Finally, there are other ways of estimating p_0. For example, instead of using N/m, one may consider to add the continuity correction factor to give a possibly better estimate $(N+0.5)/m$. But, the charting performance under this estimate has to be investigated.

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Appendix - Computing the Alarm Rate and Other Related Measures

All the calculations are performed using Mathematica by taking the advantages of its symbolic manipulation capability and build-in Binomial function. The computation of alarm rate involves not only a summation of m terms, but also a combinatorial term related to m. When m is large, exact calculation becomes very slow. We present here a method based on truncation, which is shown to be accurate and efficient even when m is of order of several millions. We have that,

$$P(B_i) = \sum_{n=0}^{m} P(B_i \mid N = n) P(N = n) = \sum_{n=0}^{m} P(B_i \mid N = n) \binom{m}{n} p_0^n (1 - p_0)^{m-n}$$

Since p_0 is a small number as we are considering a high-quality manufacturing process with a very low fraction nonconforming level, both the conditional probability and the binomial probability for large value of n are negligible. In fact, $P(N = n)$ is only of interest for $n \leq cmp_0$ for some value of c. This idea was motivated by the Markov's Inequality (total probability from cmp_0 to m is bounded by $1/c$). In fact, the true truncated probability in the binominal case is much less than this upper bound as demonstrated in many elementary probability books. In the examples above the values of c were chosen to be 5 to 20 and the truncated probability was always less than 10^{-8}. This gives that

$$P(B_i) \approx \sum_{n=0}^{cmp_0} P(B_i \mid N = n) \binom{m}{n} p_0^n (1 - p_0)^{m-n} .$$

The computation reduction can be quite substantial. For example, for m = 1000000, p_0 = 0.0005 and c = 20, we have cmp_0 = 10000. In other words, we only have to sum up to 10000 instead of 1000000.

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