Bootstrap LM Tests for Higher Order Spatial Effects in Spatial Linear Regression Models

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Abstract

This paper first extends the methodology of Yang (2015, J. of Econometrics 185, 33-59) to allow for non-normality and/or unknown heteroskedasticity in obtaining asymptotically refined critical values for the LM-type tests through bootstrap. Bootstrap refinements in critical values require the LM test statistics to be asymptotically pivotal under the null hypothesis, and for this we provide a set of general methods for constructing LM and robust LM tests. We then give detailed treatments for two general higher order spatial linear regression models: namely the SARAR(p, q) model and the MESS(p, q) model, by providing a complete set of non-normality robust LM and bootstrap LM tests for higher order spatial effects, and a complete set of LM and bootstrap LM tests robust against both unknown heteroskedasticity and non-normality. Monte Carlo experiments are run, and results show an excellent performance of the bootstrap LM-type tests.

Key Words: Asymptotic pivot; Bootstrap; Heteroskedasticity; LM test; Spatial lag; Spatial error; Matrix exponential; Wild bootstrap; Bootstrap critical values.

JEL classifications: C12, C15, C18, C21.

1. Introduction

Since the birth of *spatial econometrics* in the early 1970s, various forms of spatial interactions have been incorporated into a linear regression model to give what is called in the paper the *spatial linear regression* (SLR) model with one or more of the following: spatial lag dependence (SLD), spatial error dependence (SED), and spatial Durbin effect (SDE). The SLD effect can be in either the spatial autoregressive (SAR) form or the matrix exponential spatial specification (MESS), and the SED effect can be in the same form as the SLD effect and, in addition, it can be in the form of the spatial moving average (SMA) or the spatial error components (SEC). These give a rich class of SLR model of 'order' one.[1] Recently, theory and

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[1]See Anselin (1988b) and Anselin and Bera (1998) for various specifications of the SLR models with SLD, SED, SEC or SDE; and LeSage and Pace (2007) and Debarsy et al. (2015) for the SLR model with MESS.

applications have advanced the SLR model to contain various spatial effects of order higher than one, e.g., the SLR model with a `SAR` response of order p and a `SAR` error of order q, referred to in the literature as `SARAR`(p, q); the SLR model with a pth order `MESS` in response and a qth order `MESS` in error, referred to in the literature as `MESS`(p, q).[2]

Evidently, it is desirable to have simple tools that help practitioners choose an appropriate SLR model. The LM test has been an important tool for identifying the existence of various types of spatial effects in a linear regression model as it requires only the estimation of the null model, often an ordinary linear regression model. However, the usual LM tests may not have a satisfactory finite sample performance, and various refined LM tests for SLR models involving `SED`, `SLD`, or `SEC` have been introduced, such as the LM tests based on standardization (Yang, 2010; Baltagi and Yang, 2013a), the LM tests based on Edgeworth correction (Robinson and Rossi, 2014, 2015a,b), and the LM tests based on bootstrap critical values (Yang, 2015; Jin and Lee, 2015). Standardization improves the finite sample performance of a two-sided test, but it may not be able to do so for a one-sided test as the spatial dependence drives the finite sample null distribution of the test statistic skewed, and more so with a denser spatial weights matrix (Yang, 2015). Edgeworth correction method is rather limited (Horowitz, 1994; Hall and Horowitz, 1996), and it may be feasible only when the null model is an ordinary least square (OLS) regression, due to the complications in deriving the Edgeworth expansion. In contrast, bootstrap critical values are very easy to obtain, and more importantly they give a second-order approximation to the finite sample critical values of the LM test statistic if it is asymptotically pivotal when the null hypothesis is true. The method is rather general as well, as it works in the same way when the null model contains *nonlinear parameters* (parameters need to be estimated through numerical optimization) as when the null model is an OLS regression, except that it incurs some additional computation. See Section 4 for details.

Furthermore, the usual LM tests are derived under Gaussian likelihoods and the assumption that the errors are independent and identically distributed (iid). Neither assumption can be realistic, in particular the assumption of homoskedasticity (see Anselin, 1988b). Regular LM tests for `SED` (including Moran's I test given in Moran (1950)) and/or `SLD` are shown to be robust against distributional misspecification (Baltagi and Yang, 2013a), but the regular LM test for `SEC` is not (Yang, 2010). In general, the regular LM test are not robust against unknown heteroskedasticity. Born and Breitung (2011) gave a set of heteroskedasticity and non-normality robust LM tests for the SLR models with `SED` and/or `SLD`. Baltagi and Yang (2013b) followed up with a set of 'standardized' heteroskedasticity and non-normality robust LM tests which are shown to have much improved finite sample property.[3] However, these tests are again very likely to suffer from the problem of finite sample size-distortion for one-sided tests. The LM tests referring to the 'one-sided' bootstrap critical values may offer finite

[2]The basic motivation for a higher order SLR model is that spatial units may subject to different types of interactions (e.g., geographical distance, social relationship, peer effects). See Badinger and Egger (2011) and Elhorst et al. (2012) for `SARAR`(p, q) and the supplement file to Debarsy et al. (2015) for `MESS`(p, q).

[3]See also the generalized Moran I tests given by Kelejian and Prucha (2001) and Liu and Prucha (2016).

sample refinements, but this issue is not formally examined though it was raised in Yang (2015). Also, for certain SLR models such as the SLR model with `SEC` and the SLR model with `MESS`, neither LM tests nor bootstrap LM (BLM) tests that are robust to unknown heteroskedasticity and non-normality are available. An SLR model specification with both `SLD` and `SEC` does not seem to have appeared in the literature, and so are the corresponding LM and BLM tests. Finally, most of the LM tests available in the literature test the spatial effects of 'order one', and the LM tests for higher order spatial effects, in particular the BLM tests, are not available. This paper will fill in some of these gaps. We will focus on the LM-type tests that are either robust against nonnormality or robust against both non-normality and unknown heteroskedasticity. It is seen that once the general principles are clear and strictly followed, the actual implementations of the BLM tests are quite straightforward.

The rest of the paper is organized as follows. Section 2 presents the bootstrap method, discusses its validity, and presents some simple examples. Section 3 outlines the general principles for constructing LM and robust LM tests. Section 4 presents LM and BLM tests and their robust versions for the `SARAR`(p, q) model, and Section 5 presents the same set of tests for the `MESS`(p, q) model. Section 6 presents some Monte Carlo results, showing an excellent performance of the BLM tests. Section 7 concludes the paper with discussions.

2. Bootstrap LM Tests: General Methods and Validity

2.1. The models

All the SLR models discussed in the introduction, except the models with `SEC`, fall into the following general model specification:

$$A_n(\lambda)Y_n = \mathbf{X}_n\beta + u_n, \quad B_n(\rho)u_n = \varepsilon_n, \tag{2.1}$$

where Y_n is an $n \times 1$ vector of response values, $\mathbf{X}_n$ is an $n \times k$ matrix that contain the values of exogenous regressors and may contain some spatial Durbin terms, $A_n(\lambda) \equiv A_n(\mathbb{W}_\ell, \lambda)$ is an $n \times n$ matrix inducing `SLD` of order p in `SAR` or `MESS` form and $B_n(\rho) \equiv B_n(\mathbb{W}_e, \rho)$ is an $n \times n$ matrix inducing `SED` of order q in `SAR` or `SMA` or `MESS` form, $\mathbb{W}_\ell = \{W_{\ell 1} \cdots, W_{\ell p}\}$ and $\mathbb{W}_e = \{W_{e1}, \cdots, W_{eq}\}$ with $W_{\ell j}$ and W_{ej} being the given $n \times n$ spatial weights matrices, β is a $k \times 1$ vector of regression coefficients, λ is a $p \times 1$ vector of spatial lag parameters, ρ is a $q \times 1$ vector of spatial error parameters, and ε_n is an $n \times 1$ vector of idiosyncratic errors, iid with mean 0 and variance σ_ε^2, or independent but not identically distributed (inid) with mean zero and variances $h_i\sigma_\varepsilon^2, i = 1, \cdots, n$ where $\{h_i\}$ represent the unknown heteroskedasticity. Most of the tests in the literature correspond to the models with $p = 1$ and $q = 1$. In this paper, they are extended for testing the higher-order spatial effects.

For example, for an SLR model with pth order `SLD` and qth order `SED` both in the *autoregressive* form, we have $A_n(\lambda) = I_n - \sum_{j=1}^{p} \lambda_j W_{\ell j}$ and $B_n(\rho) = I_n - \sum_{j=1}^{q} \rho_j W_{ej}$ where I_n is an

$n \times n$ identity matrix, leading to the `SARAR`(p,q) model. For an SLR model with pth order `MESS` in response and qth order `MESS` in error, or the `MESS`(p,q) model, $A_n(\lambda) = \exp(\sum_{j=1}^{p} \lambda_j W_{\ell j})$ and $B_n(\rho) = \exp(\sum_{j=1}^{q} \rho_j W_{ej})$ as in the literature, or the proposed forms in Section 5 to overcome the difficulty in finding the partial derivatives. For an SLR model with `SDE`, it is typical that $\mathbf{X}_n = (X_n, W_{1n}X_{1n})$ where X_{1n} is a submatrix of X_n, excluding at least the columns corresponding to the intercept and dummy variables to avoid multicollinearity problem.

An alternative model specification is to replace $B_n(\rho)\varepsilon_n$ on the right-hand side of (2.1) by $W_e v_n + \varepsilon_n$, to give an SLR model with `SLD` of order p (in `SAR` or `MESS` form) and `SEC`:

$$A_n(\lambda)Y_n = \mathbf{X}_n\beta + W_e v_n + \varepsilon_n, \tag{2.2}$$

where the spatial error component $W_e v_n$ is independent of ε_n with the elements of v_n being iid of mean zero and variance σ_v^2 (see Kelejian and Robinson, 1995). These types of model specifications have not been considered in the literature and a full study of them is interesting but beyond the scope of this paper. In this paper, we will concentrate on Model (2.1) with either `SARAR`(p,q) or `MESS`(p,q), and offer discussions relating to Model (2.2).

2.2. Bootstrap methods and their validity

To give a general procedure for bootstrapping the critical values of an LM test, let θ be the parameter vector that corresponds to the null model, and φ be the parameter vector of which the value is specified by the null hypothesis as zero. As we are mainly interested in tests for spatial effects, the φ vector would include all or some of the spatial parameters, which are λ and ρ under Model (2.1), and λ and σ_v^2 under Model (2.2); and the θ vector would always include β and σ_ε^2, and may contain some spatial parameters. The most interesting tests would be ones of which the null hypotheses specify that all the spatial parameters in the model are zero, as in these cases, the null models are simply the ordinary least squares (OLS) regression models. Some tests of model reduction are also interesting, e.g., from `SARAR`(p,q) to `SARAR`$(1,1)$. The Durbin terms act as some additional regressors and hence are treated together with the regular regressors although they may create an additional problem of parameter identification (see Elhorst, 2014, and Lee and Yu, 2016).

Consider the following general hypothesis:

$$H_0 : \varphi = 0,$$

and let the corresponding LM test be

$$\mathrm{LM}_n \equiv \mathrm{LM}_n(Y_n, \mathbf{X}_n, \mathbf{W}_n),$$

where $\mathbf{W}_n = (\mathbb{W}_\ell, \mathbb{W}_e)$. Note that under Model (2.2), we consider the cases where φ contains σ_v^2 so that under H_0 the spatial error component v_n vanishes. Thus, the above set-up covers both models. Clearly, the statistic LM_n is an explicit function of $(Y_n, \mathbf{X}_n, \mathbf{W}_n)$ when the null model is an OLS regression, or an implicit function when the null model contains some spatial parameters whose estimates at the null are implicit functions of $(Y_n, \mathbf{X}_n, \mathbf{W}_n)$.

Case I: iid errors. Consider first the case of iid errors and let $\mathcal{F}$ be the cumulative distribution function (CDF) of ε_{ni}, the ith element of ε_n. Write the null models as

$$Y_n = h(\mathbf{X}_n, \mathbf{W}_n;\ \theta;\ \varepsilon_n). \tag{2.3}$$

Let $\hat{\theta}_n$ be an estimate of θ, consistent whether or not the null hypothesis is true. Let $\hat{\varepsilon}_n$ be an estimate of ε_n **centered** to have mean zero, and $\hat{\mathcal{F}}_n$ be the corresponding empirical distribution function (EDF), also 'consistent' whether or not the null hypothesis is true.[4]

For the bootstrap critical values to achieve second-order approximation to the finite sample critical values of an LM test, Yang (2015) laid out the following fundamental principles:

(i) the bootstrap data generating process (DGP) must be set up so that it is able to mimic the null model, (ii) the LM statistic must be asymptotically pivotal under the null, (iii) the estimates of the nuisance parameters, to be used as parameters in the bootstrap DGP, must be consistent whether or not the null hypothesis is true, (iv) the EDF of the residuals consistently estimates the error distribution whether or not the null hypothesis is true, and (v) calculation of the bootstrapped values of the LM statistic is done under the null hypothesis.

Note that the null model is determined by the pair $(\theta, \mathcal{F})$, so is the finite sample null distribution of LM_n. We are interested in the finite sample CDF $\mathcal{G}_n(\theta, \mathcal{F})$ of LM_n under H_0 (or $\mathrm{LM}_n|_{H_0}$), in particular the finite sample critical values $c_n(\alpha; \theta, \mathcal{F})$ of $\mathrm{LM}_n|_{H_0}$, $0 < \alpha < 1$. With the fundamental principles given above, the bootstrap DGP must be set up as follows:

$$Y_n^* = h(\mathbf{X}_n, \mathbf{W}_n;\ \hat{\theta}_n;\ \varepsilon_n^*),\ \ \{\varepsilon_{ni}^*\} \overset{iid}{\sim} \hat{\mathcal{F}}_n, \tag{2.4}$$

where $\hat{\theta}_n$ acts as parameters and $\hat{\mathcal{F}}_n$ acts as error distribution, called, respectively, the *bootstrap parameters* and the *bootstrap error distribution*, so that it is able to mimic the real world null DGP given in (2.3). With the bootstrap data $(Y_n^*, \mathbf{X}_n, \mathbf{W}_n)$ so generated through (2.4), the bootstrap analogue of LM_n is given as $\mathrm{LM}_n^* \equiv \mathrm{LM}_n(Y_n^*, \mathbf{X}_n, \mathbf{W}_n)$. It follows that the bootstrap CDF of LM_n^* must have the form $\mathcal{G}_n(\hat{\theta}_n, \hat{\mathcal{F}}_n)$, identical in structure to $\mathcal{G}_n(\theta, \mathcal{F})$, and that the bootstrap critical value must be $c_n(\alpha; \hat{\theta}_n, \hat{\mathcal{F}}_n)$, identical in form to $c_n(\alpha; \theta, \mathcal{F})$. These can be seen more clearly from the following identical structures:

$$\mathrm{LM}_n|_{H_0} \equiv \mathrm{LM}_n(Y_n, \mathbf{X}_n, \mathbf{W}_n) = \mathrm{LM}_n(h(\mathbf{X}_n, \mathbf{W}_n; \theta; \varepsilon_n), \mathbf{X}_n, \mathbf{W}_n) \equiv \mathrm{LM}_n(\mathbf{X}_n, \mathbf{W}_n;\ \theta;\ \varepsilon_n),$$

$$\mathrm{LM}_n^* \equiv \mathrm{LM}_n(Y_n^*, \mathbf{X}_n, \mathbf{W}_n) = \mathrm{LM}_n(h(\mathbf{X}_n, \mathbf{W}_n; \hat{\theta}_n; \varepsilon_n^*), \mathbf{X}_n, \mathbf{W}_n) \equiv \mathrm{LM}_n(\mathbf{X}_n, \mathbf{W}_n; \hat{\theta}_n; \varepsilon_n^*).$$

Under certain conditions $c_n(\alpha; \hat{\theta}_n, \hat{\mathcal{F}}_n)$ gives a second-order approximation to $c_n(\alpha; \theta, \mathcal{F})$. However, in real applications, the true bootstrap critical value is infeasible as it is numerically too demanding to exhaust all the possible bootstrap samples. The following algorithm summarizes the steps leading to approximate bootstrap critical values.

[4]A natural choice for the pair $(\hat{\theta}_n, \hat{\mathcal{F}}_n)$ in connection with the LM tests would be the quasi maximum likelihood estimates (QMLEs) of the full model containing θ and φ, but it is not restricted to the full QMLEs. In fact, any pair of $\sqrt{n}$-consistent estimates, such as GMM estimates, of the full model can be used.

Algorithm 2.1. (iid bootstrap)

(a) *Draw a random sample* ε_n^* *from* $\hat{\mathcal{F}}_n$*;*

(b) *Compute* $Y_n^* = h(\mathbf{X}_n, \mathbf{W}_n;\ \hat{\theta}_n;\ \varepsilon_n^*)$*, to obtain the bootstrap data* $\{Y_n^*, \mathbf{X}_n, \mathbf{W}_n\}$*;*

(c) *Estimate the null model based on* $\{Y_n^*, \mathbf{X}_n, \mathbf{W}_n\}$*, and then compute a bootstrapped value* LM_n^* *of* LM_n*;*

(d) *Repeat (a)-(c) B times to obtain bootstrap values* $\{\mathrm{LM}_n^b\}_{b=1}^B$ *of* LM_n*, and the* α*-quantile* $c_n^B(\alpha; \hat{\theta}_n, \hat{\mathcal{F}}_n)$ *of* $\{\mathrm{LM}_n^b\}_{b=1}^B$ *gives a bootstrap approximation to* $c_n(\alpha; \theta, \mathcal{F})$*.*

The validity of the above bootstrap procedure needs to be addressed. First, as B can be made arbitrarily large, the approximation from $c_n^B(\alpha; \hat{\theta}_n, \hat{\mathcal{F}}_n)$ to $c_n(\alpha; \hat{\theta}_n, \hat{\mathcal{F}}_n)$ can be made arbitrarily accurate, and hence such an approximation will be ignored in the following discussions. What is left is to argue that $c_n(\alpha; \hat{\theta}_n, \hat{\mathcal{F}}_n) - c_n(\alpha; \theta, \mathcal{F}) = O_p(n^{-1})$. The following assumptions are adapted from Yang (2015).

Assumption A1. *The errors* $\{\varepsilon_{ni}\}$ *are iid* $(0, \sigma_\varepsilon^2)$ *with CDF* $\mathcal{F}$*, known or unknown.*

Assumption A2. *The LM-type statistic* LM_n *is asymptotically pivotal under* H_0*, whether or not* $\mathcal{F}$ *is correctly specified.*

Assumption A3. $(\hat{\theta}_n, \hat{\mathcal{F}}_n)$ *is* $\sqrt{n}$*-consistent for* $(\theta, \mathcal{F})$ *whether or not* H_0 *is true, and whether or not* $\mathcal{F}$ *is correctly specified.*

Assumption A4. *For* $(\vartheta, \mathbb{F}) \in \mathcal{N}_{\theta,\mathcal{F}}$*, a neighborhood of* $(\theta, \mathcal{F})$*, the null CDF* $\mathcal{G}_n(\cdot, \vartheta, \mathbb{F})$ *converges weakly to a limit null CDF* $\mathcal{G}(\cdot, \vartheta, \mathbb{F})$ *as* n *increases, and admits the following asymptotic expansion uniformly in* t *and locally uniformly for* $(\vartheta, F) \in \mathcal{N}_{\theta,\mathcal{F}}$*:*

$$\mathcal{G}_n(t, \vartheta, \mathbb{F}) = \mathcal{G}(t, \vartheta, \mathbb{F}) + n^{-\frac{1}{2}} g(t, \vartheta, \mathbb{F}) + O(n^{-1}), \tag{2.5}$$

where $\mathcal{G}(\cdot, \vartheta, \mathbb{F})$ *is differentiable and strictly monotone over its support, and* $g(t, \vartheta, \mathbb{F})$ *is a functional of* $(t, \vartheta, \mathbb{F})$ *differentiable in* $(\vartheta, \mathbb{F})$*.*

Assumption A1 is standard and the existence of higher-order moments is implied by the assumptions that follow. Assumption A2 suggests that the limiting CDF of $\mathrm{LM}_n|_{H_0}$ is $\mathcal{G}(\cdot)$, free from $(\theta, \mathcal{F})$. This is generally true if $\mathcal{F}$ is known or correctly specified, but may not be true if $\mathcal{F}$ is unknown or misspecified. In the latter case, some modification on the usual LM statistic is necessary to make it robust against distributional misspecification. Assumption A3 unifies the different cases considered in Yang (2015). It was stressed in Yang (2015) that the bootstrap parameters $\hat{\theta}_n$ must be a consistent estimator of the population parameters θ in the null model **whether or not the null hypothesis is true**, as in real applications one does not know whether or not H_0 is true. This is an important point, but did not seem to have been emphasized in the literature until Yang (2015), and instead, the literature seemed pointed to a contrary (see, e.g., van Giersbergen and Kiviet, 2002; Godfrey, 2009, Ch. 3; MacKinnon, 2002). Clearly, from (2.3) and (2.4) one sees that Y_n^* would not be able to mimic Y_n when H_0 is false but the restricted (inconsistent) estimate $\tilde{\theta}_n$ is used in (2.4),

unless the null distribution of the test statistic does not depend on θ as in certain special cases such as (2.14) and (2.18). See Yang (2015) for detailed discussions on this. Assumption A4 is a general technical assumption given in Yang (2015) by adapting a similar condition given Beran (1988). In an important special case where the test statistic is asymptotically $N(0,1)$, the asymptotic expansion (2.5) reduces to the following Edgeworth expansion:

$$\mathcal{G}_n(t,\theta,\mathcal{F}) = \Phi(t) + n^{-\frac{1}{2}}\phi(t)p(t,\theta,\mathcal{F}) + O(n^{-1}), \quad (2.6)$$

where Φ and ϕ are, respectively, the CDF and pdf of $N(0,1)$, $p(t,\theta,\mathcal{F}) = -k_{1,2} + \frac{1}{6}k_{3,1}(1-t^2)$, and $k_{1,2}$ and $k_{3,1}$ are the n-free polynomials defined in the following expansion for the jth cumulant $\boldsymbol{\kappa}_{j,n} \equiv \boldsymbol{\kappa}_{j,n}(\theta,\mathcal{F})$ of $\text{LM}_n|_{H_0}$ (Hall, 1992, Sec. 2.3):

$$\boldsymbol{\kappa}_{j,n} = n^{-\frac{j-2}{2}}(k_{j,1} + n^{-1}k_{j,2} + n^{-2}k_{j,3} + \cdots). \quad (2.7)$$

Obviously, $k_{1,1} = 0$ and $k_{2,1} = 1$ in connection with the facts that $\boldsymbol{\kappa}_{1,n} \to 0$ and $\boldsymbol{\kappa}_{2,n} \to 1$, as $n \to \infty$. Developing an asymptotic or Edgeworth expansion for a general LM statistic is by no means an easy task. Fortunately, the bootstrap method (discussed in this paper) itself does not require the derivation of asymptotic or Edgeworth expansions. It is just that (quoting Hall, 1992, p. v): "*Methods based on Edgeworth expansion can help explain the performance of bootstrap methods, and on the other hand, the bootstrap provides strong motivation for reexamining the theory of Edgeworth expansion.*"

Yang (2015) developed Edgeworth expansions specific to the SLR models with `SED`, or `SLD` or `SEC`, for the purpose of formal justifications on the validity of the `iid bootstrap` method given above. Jin and Lee (2015) developed Edgeworth expansion under normality and Edgeworth expansions for martingales under nonnormal errors for the CDF of a test statistic that can be approximated by linear quadratic forms. Robinson and Rossi (2014, 2015a,b) derived Edgeworth expansions for the pure SAR model and the SLR model with `SED` under normality, for analytically correcting the distributions of the LM statistics.

Proposition 2.1. *Under Assumptions A1-A4, the bootstrap critical value given in Algorithm 2.1 is such that* $c_n(\alpha;\hat{\theta}_n,\hat{\mathcal{F}}_n) - c_n(\alpha;\theta,\mathcal{F}) = O(n^{-1})$*; in contrast* $c(\alpha) - c_n(\alpha;\theta,\mathcal{F}) = O(n^{-\frac{1}{2}})$ *where* $c(\alpha)$ *is the corresponding critical value of the limiting distribution* $\mathcal{G}(\cdot)$.

Proof. Only a sketch of the proof is given here in the general form. Details corresponding to some particular models can be found in Yang (2015). Under Assumption A2, the limiting distribution of $\text{LM}_n|_{H_0}$ must be such that $\mathcal{G}(t,\theta,\mathcal{F}) = \mathcal{G}(t)$, and under Assumptions A3 and A4, that of LM_n^* must be $\mathcal{G}(t,\hat{\theta}_n,\hat{\mathcal{F}}_n) = \mathcal{G}(t)$ as well based on the triangular-array convergence (see Beran, 1988). Thus, under Assumptions A1-A4, the CDF of $\text{LM}_n|_{H_0}$ and the bootstrap CDF of LM_n^* possess the following stochastic expansions:

$$\mathcal{G}_n(t,\theta,\mathcal{F}) = \mathcal{G}(t) + n^{-\frac{1}{2}}g(t,\theta,\mathcal{F}) + O(n^{-1}), \quad (2.8)$$

$$\mathcal{G}_n(t,\hat{\theta}_n,\hat{\mathcal{F}}_n) = \mathcal{G}(t) + n^{-\frac{1}{2}}g(t,\hat{\theta}_n,\hat{\mathcal{F}}_n) + O_p(n^{-1}). \quad (2.9)$$

Taking the difference, we have

$$\mathcal{G}_n(t,\hat{\theta}_n,\hat{\mathcal{F}}_n)-\mathcal{G}_n(t,\theta,\mathcal{F})=n^{-\frac{1}{2}}[g(t,\hat{\theta}_n,\hat{\mathcal{F}}_n)-g(t,\theta,\mathcal{F})]+O_p(n^{-1})=O_p(n^{-1}),$$

where the last equality follows from the $\sqrt{n}$ consistency of $(\hat{\theta}_n,\hat{\mathcal{F}}_n)$ and the differentiability of $g(t,\vartheta,\mathbb{F})$ in $(\vartheta,\mathbb{F})\in\mathcal{N}_{\theta,\mathcal{F}}$. ■

Remark 2.1. From the proof of Proposition 2.1, it is clear that without $\sqrt{n}$-consistency of $(\hat{\theta}_n,\hat{\mathcal{F}}_n)$, the term $g(t,\hat{\theta}_n,\hat{\mathcal{F}}_n)-\mathcal{G}_n(t,\theta,\mathcal{F})$ would not be $O_p(n^{-\frac{1}{2}})$, and the second-order approximation of $\mathcal{G}_n(t,\hat{\theta}_n,\hat{\mathcal{F}}_n)$ to $\mathcal{G}_n(t,\theta,\mathcal{F})$ would not be achieved.

Case II: inid errors. Consider again the real world null DGP: $Y_n=h(\mathbf{X}_n,\mathbf{W}_n;\ \theta;\ \varepsilon_n)$ defined in (2.3). When $\{\varepsilon_{ni}\}$ are inid with mean zero and variances $h_i\sigma_\varepsilon^2$, Algorithm 2.1 based on the bootstrap DGP (2.4) is no longer valid, as the iid draws lost the heteroskedasticity structure. Based on the fundamental principles laid out below (2.3), a valid bootstrap DGP able to capture the unknown heteroskedasticity must be that (*i*) $\hat{\theta}_n$ is a consistent estimator of θ whether or not H_0 is true and is robust against non-normality (NN) and unknown heteroskedasticity (UH), (*ii*) the estimated residuals $\{\hat{\varepsilon}_{ni}\}$ are consistent with $\{\varepsilon_{ni}\}$, i.e. $\hat{\varepsilon}_{ni}=\varepsilon_{ni}+o_p(1)$, whether or not H_0 is true, and (*iii*) bootstrap samples based on the estimated residuals are able to mimic (consistently estimate) ε_n. Clearly, a valid choice of $\hat{\theta}_n$ and $\hat{\varepsilon}_{ni}$ is from the NN-UH robust estimation of the full model (see, e.g., Kelejian and Prucha (2010), Lin and Lee (2010), and Liu and Yang (2015).

The modified bootstrap DGP, the *wild bootstrap*, takes the form:

$$Y_n^*=h(\mathbf{X}_n,\mathbf{W}_n;\ \hat{\theta}_n;\ \varepsilon_n^*),\ \ \hat{\varepsilon}_{ni}^*=\hat{\varepsilon}_{ni}v_i, \tag{2.10}$$

where $\{v_i\}_{i=1}^n$ are n iid draws from a distribution $\mathcal{H}(\cdot)$ with mean 0 and higher-order moments all 1, independent of ε_{ni}.[5] See, e.g., Wu (1986), Liu (1988), Mammen (1993), Godfrey (2007), and Davidson and Flachaire (2008), for an account on wild bootstrap.

To facilitate the discussion on the validity of the wild bootstrap procedure, denote the NN-UH robust LM-type statistic by LMR_n, and the bootstrap analogue of $\text{LMR}_n|_{H_0}$ by LMR_n^*. Clearly, $\text{LMR}_n|_{H_0}\equiv\text{LMR}_n(\mathbf{X}_n,\mathbf{W}_n;\ \theta;\ \varepsilon_n)$, and $\text{LMR}_n^*\equiv\text{LMR}_n^*(\mathbf{X}_n,\mathbf{W}_n;\ \hat{\theta}_n;\ \varepsilon_n^*)$. Assume the CDF of $h_i^{-\frac{1}{2}}\varepsilon_{ni}$ is $\mathcal{F}(\cdot)$ with higher-order moments $\boldsymbol{\mu}=\{\mu_3,\mu_4,\ldots\}$. Thus, ε_{ni} has CDF $\mathcal{F}(\cdot/\sqrt{h_i})$, and the moments $\mu_{ir}=h_i^{r/2}\mu_r, r=1,2,\ldots$. The bootstrap distribution of $\hat{\varepsilon}_{ni}^*$ is $\overline{\mathcal{H}}(\cdot/\hat{\varepsilon}_{ni})$, which is $\mathcal{H}(\cdot/\hat{\varepsilon}_{ni})$ if $\hat{\varepsilon}_{ni}>0$; $1-\mathcal{H}(\cdot/\hat{\varepsilon}_{ni})$ o.w. Its moments are $\hat{\mu}_{ir}^*=\mathrm{E}^*(\hat{\varepsilon}_{ni}^r v_i^r)=\hat{\varepsilon}_{ni}^r\mathrm{E}(v_i^r)$. Define the product measures $\mathbf{F}_n=\prod_{i=1}^n\mathcal{F}(\cdot/\sqrt{h_i})$ and $\hat{\mathbf{H}}_n=\prod_{i=1}^n\overline{\mathcal{H}}(\cdot/\hat{\varepsilon}_{ni})$. Let $\mathcal{G}_n(\cdot,\theta,\mathbf{F}_n)$ be the finite sample CDF of $\text{LMR}_n|_{H_0}$ and $c_n(\alpha;\theta,\mathbf{F}_n)$ be its αth quantile; let $\mathcal{G}_n(\cdot,\hat{\theta}_n,\hat{\mathbf{H}}_n)$ be the bootstrap CDF of LMR_n^* and $c_n(\alpha;\hat{\theta}_n,\hat{\mathbf{H}}_n)$ be its αth quantile.

[5] In fact, such a distribution does not exist but, often, being able to match up to 3rd or 4th moments suffices (see Mammen, 1993, and Remark 2.2 below). The well-known **Rademacher** distribution ($v_i=\pm1$ with equal probabilities) has all the odd moments being 0, and all the even moments being 1. This is an ideal distribution when the original errors are symmetrically distributed. Another popular choice is Mammen's (1993) two-point distribution: $P\{v_i=-(\sqrt{5}-1)/2\}=(\sqrt{5}+1)/(2\sqrt{5})$, and $P\{v_i=(\sqrt{5}+1)/2\}=(\sqrt{5}-1)/(2\sqrt{5})$, which gives $\mathrm{E}(v_i)=0$, $\mathrm{E}(v_i^2)=1$, and $\mathrm{E}(v_i^3)=1$, but $\mathrm{E}(v_i^4)=2$.

A feasible bootstrap procedure for obtaining $c_n(\alpha; \hat{\theta}_n, \mathbf{H}_n)$ is summarized below.

Algorithm 2.2. (wild bootstrap)

(a) Draw a random sample $\{v_i\}_{i=1}^n$ *from the chosen* $\mathcal{H}$*, to give* $\varepsilon_n^* = \{\hat{\varepsilon}_{ni} v_i\}_{i=1}^n$*;*

(b) Compute $Y_n^* = h(\mathbf{X}_n, \mathbf{W}_n;\ \hat{\theta}_n;\ \varepsilon_n^*)$*, to obtain the bootstrap data* $\{Y_n^*, \mathbf{X}_n, \mathbf{W}_n\}$*;*

(c) Perform an NN-UH robust estimation of the null model based on $\{Y_n^*, \mathbf{X}_n, \mathbf{W}_n\}$*, and then compute a bootstrapped value* LMR_n^* *of* LMR_n*;*

(d) Repeat (a)-(c) B times to obtain bootstrap values $\{\text{LMR}_n^b\}_{b=1}^B$ *of* LMR_n*, and the* α*-quantile* $c_n^B(\alpha; \hat{\theta}_n, \hat{\mathbf{H}}_n)$ *of* $\{\text{LMR}_n^b\}_{b=1}^B$ *gives a bootstrap approximation to* $c_n(\alpha; \theta, \mathbf{F}_n)$*.*

Assumption B1. *The errors* $\{\varepsilon_{ni}\}$ *are inid* $(0, \sigma_\varepsilon^2 h_i)$ *where* $h_i > 0$ *and* $\frac{1}{n}\sum_{i=1}^n h_i = 1$*. The CDF of* $h_i^{-\frac{1}{2}}\varepsilon_{ni}$ *is* $\mathcal{F}$*, with necessary higher-order moments* $\boldsymbol{\mu} = \{\mu_3, \mu_4, \cdots\}$ *being finite.*

Assumption B2. *The LM-type statistic,* LMR_n*, is asymptotically pivotal under* H_0*, and is robust against misspecification in* $\mathcal{F}$ *and unknown heteroskedasticity.*

Assumption B3. *Whether or not* H_0 *is true, (i)* $\hat{\theta}_n$ *is* $\sqrt{n}$*-consistent for* θ *and is robust against misspecification in* $\mathcal{F}$ *and unknown heteroskedasticity, and (ii)* $\hat{\varepsilon}_{ni} = \varepsilon_{ni} + O_p(n^{-1/2})$*.*

Assumptions B1-B3 extends Assumptions A1-A3 to cater the unknown heteroskedasticity. These extensions seem straightforward. Assumption B1 essentially assumes that $\varepsilon_{ni} = h_i^{\frac{1}{2}} e_{ni}$ and $\{e_{ni}\}$ are iid as in Liu and Prucha (2016). With Assumptions B2 and B3, it is reasonable to assume that $\mathcal{G}_n(\cdot, \hat{\theta}_n, \hat{\mathbf{H}}_n)$ converges to the same limit as does $\mathcal{G}_n(\cdot, \theta, \mathbf{F}_n)$. No doubt, proving the existence of Edgeworth/asymptotic expansions for the case of inid errors is even more challenging than the already challenging case of iid errors in spatial models. To simplify the discussions, we put up the following higher-level assumptions compared with Assumption A4 for the case of iid errors, and details may be learnt based on, e.g., (2.20) and (2.21).

Assumption B4. *The null CDF* $\mathcal{G}_n(\cdot, \theta, \mathbf{F}_n)$ *and the bootstrap CDF* $\mathcal{G}_n(\cdot, \hat{\theta}_n, \hat{\mathbf{H}}_n)$ *admit the following asymptotic expansions:*

$$\mathcal{G}_n(t, \theta, \mathbf{F}_n) = \mathcal{G}(t) + n^{-\frac{1}{2}} g(t, \theta, \mathbf{F}_n) + O(n^{-1}), \tag{2.11}$$

$$\mathcal{G}_n(t, \hat{\theta}_n, \hat{\mathbf{H}}_n) = \mathcal{G}(t) + n^{-\frac{1}{2}} g(t, \hat{\theta}_n, \hat{\mathbf{H}}_n) + O(n^{-1}), \tag{2.12}$$

where $g(t, \theta, \mathbf{F}_n)$ *is a functional of* $(t, \theta, \mathbf{F}_n)$ *differentiable in* $(\theta, \mathbf{F}_n)$*.*

Proposition 2.2. *Under Assumptions B1-B3, the bootstrap critical value given in Algorithm 2.2 is such that* $c_n(\alpha; \hat{\theta}_n, \hat{\mathbf{H}}_n) - c_n(\alpha; \theta, \mathbf{F}_n) = O(n^{-1})$*; in contrast* $c(\alpha) - c_n(\alpha; \theta, \mathbf{F}_n) = O(n^{-\frac{1}{2}})$ *where* $c(\alpha)$ *is the corresponding critical value of the limiting distribution* $\mathcal{G}(\cdot)$*.*

Proof. Parallel to the proof of Proposition 2.1. ■

Remark 2.2. As in the case of iid errors, an Edgeworth expansion similar to (2.6) can be obtained for a univariate test, from which one sees that $g(t, \theta, \mathbf{F}_n)$ depends only on the leading terms in the expansions for the first three cumulants of $\text{LMR}_n|_{H_0}$. Furthermore, it can be shown that in a number of special tests, $g(t, \theta, \mathbf{F}_n)$ depends on $\mathbf{F}_n$ only through the first three moments of ε_{ni} as in Yang (2015) for the case of iid errors. This is important as it says that as long as the wild bootstrap is able to capture the first three moments of ε_{ni}, a full second-order refinement on the critical values can be achieved.

2.3. Examples

To help appreciating the general methods given above, we present several simple tests concerning the `SARAR(1,1)` effects, and concerning `SEC` effect.

Example 2.1. *Joint and conditional tests for SARAR(1,1).* To test for the existence of 1st order `SLD` or 1st order `SED` or both in a linear regression model, the three LM tests due to Burridge (1980) for the first one and Anselin (1988a,b) are given as follows:

$$\text{LM}_{\texttt{SLD}} = \frac{n}{\sqrt{\tilde{J}_n + K_n^{\ell\ell}}} \frac{\tilde{\varepsilon}_n' W_\ell Y_n}{\tilde{\varepsilon}_n' \tilde{\varepsilon}_n}, \tag{2.13}$$

$$\text{LM}_{\texttt{SED}} = \frac{n}{\sqrt{K_n^{ee}}} \frac{\tilde{\varepsilon}_n' W_e \tilde{\varepsilon}_n}{\tilde{\varepsilon}_n' \tilde{\varepsilon}_n}, \tag{2.14}$$

$$\text{LM}_{\texttt{SARAR}} = \frac{(\tilde{\varepsilon}_n' W_\ell Y_n)^2 K_n^{ee} - 2(\tilde{\varepsilon}_n' W_\ell Y_n)(\tilde{\varepsilon}_n' W_e \tilde{\varepsilon}_n) K_n^{\ell e} + (\tilde{\varepsilon}_n' W_e \tilde{\varepsilon}_n)^2 (J_n + K_n^{\ell\ell})}{\tilde{\sigma}_n^4 [J_n K_n^{ee} + K_n^{\ell\ell} K_n^{ee} - (K_n^{\ell e})^2]}, \tag{2.15}$$

where $\tilde{\varepsilon}_n$ is the vector of OLS residuals from regressing Y_n on $\mathbf{X}_n$, $\tilde{J}_n = \tilde{\sigma}_n^{-2} \tilde{\eta}_n' \mathbf{M}_n \tilde{\eta}_n$, $\tilde{\eta}_n = W_\ell \mathbf{X}_n \tilde{\beta}_n$, $\tilde{\beta}_n$ and $\tilde{\sigma}_n^2$ are the null estimates of β and σ^2, $\mathbf{M}_n = I_n - \mathbf{X}_n(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{X}_n'$, $K_n^{\ell\ell} = \text{tr}[(W_\ell + W_\ell')W_\ell]$, $K_n^{ee} = \text{tr}[(W_e + W_e')W_e]$, and $K_n^{\ell e} = \text{tr}[(W_\ell + W_\ell')W_e]$. When $W_\ell = W_e \equiv W_n$ is assumed in (2.15), $K_n^{\ell\ell} = K_n^{ee} = K_n^{\ell e} \equiv K_n$ and the test simplifies to

$$\text{LM}^0_{\texttt{SARAR}} = \frac{(\tilde{\varepsilon}_n' W_n Y_n - \tilde{\varepsilon}_n' W_n \tilde{\varepsilon}_n)^2}{\tilde{\sigma}_n^2 \tilde{\eta}_n' \mathbf{M}_n \tilde{\eta}_n} + \frac{(\tilde{\varepsilon}_n' W_n \tilde{\varepsilon}_n)^2}{\tilde{\sigma}_n^4 K_n}. \tag{2.16}$$

These three tests can easily be shown to be robust against nonnormality by verifying the (asymptotic) equivalence in variance-covariance matrices; see Section 4 for details under the general $\texttt{SARAR}(p,q)$ model. However, their finite sample performance when referred to the asymptotic critical values (ACVs) can be poor. Use of bootstrap critical values (BCVs) can greatly reduce the size distortion. As the three tests correspond to the same null model, the bootstrap DGP has the same form: $Y_n^* = \mathbf{X}_n \hat{\beta}_n + \varepsilon_n^*$, where $\{\varepsilon_{ni}^*\}$ are n random draws from centered *unrestricted residuals* $\{\hat{\varepsilon}_{ni}\}$, i.e, the residuals calculated from the estimation of the 'full' model, being respectively `SARAR(0,1)`, `SARAR(1,0)`, and `SARAR(1,1)`, and $\hat{\beta}_n$ is the unrestricted estimate of β. Clearly, $\hat{\sigma}_n^2$ does not play a role in generating the bootstrap data.

To obtain the BCVs of the LM tests given above, take for example the test $\text{LM}^0_{\texttt{SARAR}}$ given in (2.16). Based on the bootstrap data $(Y_n^*, \mathbf{X}_n, W_n)$, one estimates the null model and then calculates the test statistic to obtain its bootstrap value or its bootstrap analogue:

$$\text{LM}^{0*}_{\texttt{SARAR}} = \frac{(\tilde{\varepsilon}_n^{*\prime} W_n Y_n^* - \tilde{\varepsilon}_n^{*\prime} W_n \tilde{\varepsilon}_n^*)^2}{\tilde{\sigma}_n^{*2} \tilde{\eta}_n^{*\prime} \mathbf{M}_n \tilde{\eta}_n^*} + \frac{(\tilde{\varepsilon}_n^{*\prime} W_n \tilde{\varepsilon}_n^*)^2}{\tilde{\sigma}_n^{*4} K_n}, \tag{2.17}$$

where $\tilde{\eta}_n^* = \tilde{\sigma}_n^{*-1} W_n \mathbf{X}_n \tilde{\beta}_n^*$, $\tilde{\beta}_n^*$ and $\tilde{\sigma}_n^{*2}$ are the bootstrap estimates of β and σ^2, i.e., from regressing Y_n^* on X_n, and $\tilde{\varepsilon}_n^* = Y_n^* - \mathbf{X}_n \tilde{\beta}_n^* = \mathbf{M}_n \varepsilon_n^*$. Repeated draws from $\{\hat{\varepsilon}_{ni}\}$ give sequences of bootstrap values of $\text{LM}_{\texttt{SARAR}}^{0*}$, and their quantiles give the BCVs for $\text{LM}_{\texttt{SARAR}}^0$, as in Algorithm (2.1). Similarly, one obtains the BCVs for $\text{LM}_{\texttt{SLD}}$, $\text{LM}_{\texttt{SED}}$, and $\text{LM}_{\texttt{SARAR}}$. The results of Yang (2015) imply that the BCVs for the above tests give a second-order approximation to the finite sample critical values (FCVs) of these tests. In contrast the asymptotic critical values (ACVs) only approximate the FCVs to the first order.[6]

Example 2.2. *Testing for the existence of SEC.* Consider the SLR-SEC model: $Y_n = \mathbf{X}_n \beta + W_n v_n + \varepsilon_n$. The test for SEC amounts to test $H_0 : \sigma_\nu^2 = 0$, or $\lambda = \sigma_\nu^2/\sigma^2 = 0$. An LM test is given in Anselin (2001), and a robust LM test is given in Yang (2010):

$$\text{LMR}_{\text{SEC}} = \frac{n}{\sqrt{K_n^\dagger + \tilde{\kappa}_{4n} a_n' a_n}} \frac{\tilde{\varepsilon}_n' H_n^\dagger \tilde{\varepsilon}_n}{\tilde{\varepsilon}_n' \tilde{\varepsilon}_n}, \tag{2.18}$$

where $H_n^\dagger = W_n W_n' - \frac{1}{n-k}\text{tr}(W_n W_n' M_n) I_n$, $K_n^\dagger = 2\text{tr}(\mathcal{A}_n^2)$, $a_n = \text{diagv}(\mathcal{A}_n)$, $\mathcal{A}_n = M_n H_n^\dagger M_n$, $\tilde{\kappa}_{4n}$ is the 4th cumulant of $\tilde{\sigma}_n^{-1}\tilde{\varepsilon}_n$, and $\tilde{\sigma}_n^2$ is the estimate of σ^2 under H_0. The null model and the bootstrap DGP are again OLS regressions. The bootstrap analogue of (2.18) is thus:

$$\text{LMR}_{\text{SEC}}^* = \frac{n}{\sqrt{K_n^\dagger + \tilde{\kappa}_{4n}^* a_n' a_n}} \frac{\tilde{\varepsilon}_n^{*\prime} H_n^\dagger \tilde{\varepsilon}_n^*}{\tilde{\varepsilon}_n^{*\prime} \tilde{\varepsilon}_n^*}, \tag{2.19}$$

where $\tilde{\varepsilon}_n^*$ is the vector of residuals from regressing Y_n^* on $\mathbf{X}_n$, $\tilde{\kappa}_{4n}^*$ is the 4th cumulant of $\tilde{\sigma}_n^{*-1}\tilde{\varepsilon}_n^*$, and $\tilde{\sigma}_n^{*2}$ is the bootstrap estimate of σ^2 from the same regression. Yang (2015) show that the BCVs give second-order approximations to the FCVs.

Example 2.3. *Heteroskedasticity and non-normality robust LM tests for SARAR(1,1).* While the LM tests given in (2.13)-(2.15) are robust against non-normality of the error distribution, they are not robust against unknown heteroskedasticity. Their robust versions (against both non-normality and heteroskedasticity) are given in Born and Breitung (2011):

$$\text{LMR}_{\texttt{SLD}} = \frac{\tilde{\varepsilon}_n' W_\ell Y_n}{(\tilde{\varepsilon}_n^{2\prime}\, \tilde{\xi}_{1n}^2)^{\frac{1}{2}}}, \tag{2.20}$$

$$\text{LMR}_{\texttt{SED}} = \frac{\tilde{\varepsilon}_n' W_e \tilde{\varepsilon}_n}{(\tilde{\varepsilon}_n^{2\prime}\, \tilde{\xi}_{2n}^2)^{\frac{1}{2}}}, \tag{2.21}$$

$$\text{LMR}_{\texttt{SARAR}} = \begin{pmatrix} \tilde{\varepsilon}_n' W_\ell Y_n \\ \tilde{\varepsilon}_n' W_e \tilde{\varepsilon}_n \end{pmatrix}' \begin{pmatrix} \tilde{\varepsilon}_n^{2\prime} \tilde{\xi}_{1n}^2, & \tilde{\varepsilon}_n^{2\prime}(\tilde{\xi}_{1n} \odot \tilde{\xi}_{2n}) \\ \sim, & \tilde{\varepsilon}_n^{2\prime} \tilde{\xi}_{2n}^2 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\varepsilon}_n' W_\ell Y_n \\ \tilde{\varepsilon}_n' W_e \tilde{\varepsilon}_n \end{pmatrix}, \tag{2.22}$$

[6] As $\frac{1}{n}\sum_{i=1}^n \hat{\varepsilon}_{ni} = 0$, $\{\hat{\varepsilon}_{ni}\}$ are automatically centered. Apparently, $\text{LM}_{\texttt{SED}}^*$ is invariant of $\hat{\beta}_n$ and hence any estimate of β can be used in the bootstrap DGP. The estimate of the scale parameter σ also does not play a role and is absorbed in $\hat{\varepsilon}_n$ and ε_n^*. However, the use of the unrestricted residuals is necessary as the 'finite sample' distribution of $\text{LM}_{\texttt{SED}}|_{H_0^e}$ depends on the third moment of ε_{ni} and only with the unrestricted residuals the third moment of $\hat{\mathcal{F}}_n$ is $\sqrt{n}$-consistent for the third moment of $\mathcal{F}$. See Yang (2015) for details.

where $\tilde{\xi}_{1n} = (W_\ell^{u\prime} + W_\ell^l)\tilde{\varepsilon}_n + \mathbf{M}_n\tilde{\eta}_n$, $\tilde{\xi}_{2n} = (W_e^{u\prime} + W_e^l)\tilde{\varepsilon}_n$, A_n^u and A_n^l denote the upper and lower triangular matrices of a matrix A_n, '$\odot$' denotes the Hadamard product, $a^2 = a \odot a$ for a vector a, and $\tilde{\varepsilon}_n$ and $\tilde{\eta}_n$ are as in (2.13)-(2.15). Baltagi and Yang (2013b) show that these tests referring ACVs can perform very poorly and present standardized versions of them.

The three tests have the same null DGP: $Y_n = \mathbf{X}_n\beta + \varepsilon_n$. The wild bootstrap DGP is $Y_n^* = \mathbf{X}_n\hat{\beta}_n + \varepsilon_n^*$, $\varepsilon_n^* = \hat{\varepsilon}_n \odot v_n$, where v_n is an n-vector of iid draws from an auxiliary distribution with mean zero and higher moments 1, completely independent of original data. With the bootstrap data $(Y_n^*, \mathbf{X}_n, W_n)$, one computes $\tilde{\varepsilon}_n^*$, $\tilde{\eta}_n^*$, and hence $\tilde{\xi}_{1n}^*$ and $\tilde{\xi}_{2n}^*$, giving bootstrap values of the test statistics and hence the BCVs.

3. Construction of LM and Robust LM Tests

In this section, we outline the general procedures for constructing LM tests, robust LM tests against nonnormality, and robust LM tests against both nonnormality and unknown heteroskedasticity. Then, in the subsequent sections, we apply these general procedures to introduce LM/BLM and robust LM/BLM tests, respectively, for the $\texttt{SARAR}(p,q)$ model and the $\texttt{MESS}(p,q)$ model. We endeavor to present the results in a practical manner so the applied researchers can easily apply these BLM tests.

Recall: θ represents parameter vector in the null model, φ the additional vector of parameters appeared in the full model, and the null hypothesis specifies $\varphi = 0$. Let $\psi = (\theta', \varphi')'$ and ψ_0 (θ_0 and φ_0) be the true value of ψ (θ and φ), $S_n(\psi)$ the score vector based on the normality assumption on ε_n, and $\Sigma_n(\psi_0) = -\mathrm{E}[\frac{\partial}{\partial\psi'}S_n(\psi_0)]$. Corresponding to θ and φ, denote the subvectors of $S_n(\psi)$ by $S_{n,\theta}(\theta,\varphi)$ and $S_{n,\varphi}(\theta,\varphi)$, and the submatrices of $\Sigma_n(\psi)$ by $\Sigma_{n,\theta\theta}(\theta,\varphi)$, $\Sigma_{n,\varphi\varphi}(\theta,\varphi)$ and $\Sigma_{n,\theta\varphi}(\theta,\varphi)$. **The LM test** of $H_0 : \varphi = 0$ takes the form:

$$\mathrm{LM}_n = \tilde{S}_{n,\varphi}'\big(\tilde{\Sigma}_n^{-1}\big)_{\varphi\varphi}\tilde{S}_{n,\varphi}, \tag{3.1}$$

with its limiting null distribution being $\chi^2_{\dim(\varphi)}$, where $\tilde{S}_{n,\varphi} = S_{n,\varphi}(\tilde{\theta}_n, 0)$, $\tilde{\Sigma}_n = \Sigma_n(\tilde{\theta}_n, 0)$, $\tilde{\theta}_n$ is the MLE of the null model, and $(\,\cdot\,)_{\varphi\varphi}$ denotes the φ-φ block of the corresponding matrix. By the definition of $\Sigma_n(\psi_0)$, it is clear that $\tilde{\Sigma}_n$ can simply be $-\frac{\partial}{\partial\psi'}S_n(\tilde{\theta}_n, 0)$.

LM test robust against non-normality (NN). Let $\Gamma_n(\psi_0) = \mathrm{Var}[S_n(\psi_0)]$, with submatrices $\Gamma_{n,\theta\theta}(\theta,\varphi)$, $\Gamma_{n,\varphi\varphi}(\theta,\varphi)$ and $\Gamma_{n,\theta\varphi}(\theta,\varphi)$. By Taylor expansion,

$$\tfrac{1}{\sqrt{n}}S_{n,\varphi}(\tilde{\theta}_n, 0) = \tfrac{1}{\sqrt{n}}S_{n,\varphi}(\theta_0, 0) - \tfrac{1}{\sqrt{n}}\Pi_n(\theta_0)S_{n,\theta}(\theta_0, 0) + o_p(1), \tag{3.2}$$

where $\Pi_n(\theta_0) = \Sigma_{n,\varphi\theta}(\theta_0, 0)\Sigma_{n,\theta\theta}^{-1}(\theta_0, 0)$. It follows that $\mathrm{Var}[\frac{1}{\sqrt{n}}S_{n,\varphi}(\tilde{\theta}_n, 0)] = \frac{1}{n}\big[\Gamma_{n,\varphi\varphi}(\theta_0, 0) - \Gamma_{n,\varphi\theta}(\theta_0, 0)\Pi_n'(\theta_0) - \Pi_n'(\theta_0)\Gamma_{n,\theta\varphi}(\theta_0, 0) + \Pi_n(\theta_0)\Gamma_{n,\theta\theta}(\theta_0, 0)\Pi_n'(\theta_0)\big] + o(1)$. An NN-robust LM test, with its limiting null distribution being $\chi^2_{\dim(\varphi)}$, takes the form:

$$\mathrm{LMN}_n^0 = \tilde{S}_{n,\varphi}'\big(\tilde{\Gamma}_{n,\varphi\varphi} - 2\tilde{\Gamma}_{n,\varphi\theta}\tilde{\Pi}_n' + \tilde{\Pi}_n\tilde{\Gamma}_{n,\theta\theta}\tilde{\Pi}_n'\big)^{-1}\tilde{S}_{n,\varphi}, \tag{3.3}$$

where $\tilde{S}_{n,\varphi} = S_{n,\varphi}(\tilde{\theta}_n, 0)$, $\tilde{\Gamma}_{n,\varphi\varphi}$, $\tilde{\Gamma}_{n,\varphi\theta}$ and $\tilde{\Gamma}_{n,\theta\theta}$ are the submatrices of $\Gamma_n(\tilde{\theta}_n, 0)$, and $\tilde{\Pi}_n = \Pi_n(\tilde{\theta}_n)$. Clearly, when $\{\varepsilon_{ni}\}$ are iid normal, $\Gamma_n(\psi_0) = \Sigma_n(\psi_0)$ (information matrix equality, or IME) and LMN_n reduces to LM_n. When $\{\varepsilon_{ni}\}$ are iid but non-normal, the IME does not hold and the explicit expression of $\Gamma_n(\theta_0, 0)$ is required in order to implement LMN_n^0. It is typical that $\Gamma_n(\theta_0, 0)$ involves higher-order moments of the model errors which are estimated based on the null residuals defined by $\tilde{\theta}_n$.

An **alternative way** to estimate $\Gamma(\theta_0, 0)$ is via the *outer-product-of-martingale-difference* (OPMD) (Baltagi and Yang, 2013b; Yang, 2017). **If** $S_n(\theta_0, 0)$ has a martingale difference (MD) representation: $S_n(\theta_0, 0) = \sum_{i=1}^n g_{ni}(\theta_0)$, where $\{g_{ni}(\theta_0)\}$ form an MD sequence, then $\text{Var}[S_n(\theta_0, 0)] = \sum_{i=1}^n \text{Var}[g_{ni}(\theta_0)] = \sum_{i=1}^n \text{E}[g_{ni}(\theta_0)g'_{ni}(\theta_0)]$. Hence, $\sum_{i=1}^n \tilde{g}_{ni}\tilde{g}'_{ni}$, the sum of the estimated OPMDs, thus gives a consistent estimate of $\text{Var}[S_n(\theta_0, 0)]$ in the sense that $\frac{1}{n}[\sum_{i=1}^n \tilde{g}_{ni}\tilde{g}'_{ni} - \text{Var}[S_n(\theta_0, 0)] \xrightarrow{p} 0$, where $\tilde{g}_{ni} = g_{ni}(\tilde{\theta}_n)$. Replacing $\tilde{\Gamma}_{n,\varphi\varphi}$, $\tilde{\Gamma}_{n,\varphi\theta}$ and $\tilde{\Gamma}_{n,\theta\theta}$ in (3.3) by the submatrices of $\sum_{i=1}^n \tilde{g}_{ni}\tilde{g}'_{ni}$, we obtain an OPMD form of NN-robust LM test:

$$\text{LMN}_n = \tilde{S}'_{n,\varphi}\big[\sum_{i=1}^n(\tilde{g}_{ni,\varphi} - \tilde{\Pi}_n\tilde{g}_{ni,\theta})(\tilde{g}_{ni,\varphi} - \tilde{\Pi}_n\tilde{g}_{ni,\theta})'\big]^{-1}\tilde{S}_{n,\varphi}. \tag{3.4}$$

Equivalently, (3.4) can be obtained as follows. By (3.2) and the MD representation $S_n(\theta_0, 0) = \sum_{i=1}^n g_{ni}(\theta_0)$, $S_{n,\theta}(\tilde{\theta}_n, 0)$ has the following asymptotic MD representation:

$$\tfrac{1}{\sqrt{n}}S_{n,\varphi}(\tilde{\theta}_n, 0) = \tfrac{1}{\sqrt{n}}\sum_{i=1}^n[g_{ni,\varphi} - \Pi_n(\theta_0)g_{ni,\theta}] + o_p(1), \tag{3.5}$$

where $\{g_{ni,\varphi} - \Pi_n(\theta_0)g_{ni,\theta}\}$ form an MD sequence with $(g'_{ni,\theta},\ g'_{ni,\varphi})' = g_{ni} \equiv g_{ni}(\theta_0)$. Thus, $\text{Var}[\frac{1}{\sqrt{n}}S_{n,\varphi}(\tilde{\theta}_n, 0)] = \frac{1}{n}\sum_{i=1}^n \text{E}[(g_{ni,\varphi} - \Pi_n(\theta_0)g_{ni,\theta})(g_{ni,\varphi} - \Pi_n(\theta_0)g_{ni,\theta})'] + o(1)$, leading to (3.4).

The advantages of using the OPMD estimate of $\Gamma_n(\theta_0, 0)$ are: (*i*) it avoids the analytical expression of $\Gamma_n(\psi_0)$ containing higher order moments, and (*ii*) it is also robust against unknown heteroskedasticity (UH) besides being robust against NN. These are crucial in developing LM tests that are both NN and UH robust, as seen below. With these, an OPMD alternative to the LM test (3.1) can be easily developed.

LM test robust against NN and UH. Neither LM_n nor LMN_n (or LMN_n^0) is robust against UH in model errors. To derive an LM-type test that is UH-robust, adjust $S_n(\psi)$ so that the adjusted score vector $S_n^\circ(\psi)$ is such that $\text{E}[S_n^\circ(\psi_0)|_{H_0}] = 0$ or $\frac{1}{n}S_n^\circ(\psi_0)|_{H_0} \xrightarrow{p} 0$ as $n \to \infty$ under UH. Let $\tilde{\theta}_n^\circ = \arg\{S_{n,\theta}^\circ(\theta, 0) = 0\}$, the UH-robust estimator of the null model. Let $\Sigma_n^\circ(\psi_0) = -\text{E}[\frac{\partial}{\partial\psi'}S_n^\circ(\psi_0)]$ and $\Gamma_n^\circ(\psi_0) = \text{Var}[S_n^\circ(\psi_0)]$, partitioned similarly as $\Sigma_n(\psi_0)$ and $\Gamma_n(\psi_0)$. **If** $S_n^\circ(\theta_0, 0)$ has a martingale difference (MD) representation: $S_n^\circ(\theta_0, 0) = \sum_{i=1}^n g_{ni}^\circ(\theta_0)$ where $\{g_{ni}^\circ(\theta_0)\}$ form an MD sequence, then, similar to $S_{n,\theta}(\tilde{\theta}_n, 0)$ in (3.5), $S_{n,\theta}^\circ(\tilde{\theta}_n^\circ, 0)$ has the following asymptotic MD representation:

$$\tfrac{1}{\sqrt{n}}S_{n,\varphi}(\tilde{\theta}_n^\circ, 0) = \tfrac{1}{\sqrt{n}}\sum_{i=1}^n[g_{ni,\varphi}(\theta_0) - \Pi_n^\circ(\theta_0)g_{ni,\theta}(\theta_0)] + o_p(1),$$

where $\{g_{ni,\varphi}^\circ - \Pi_n^\circ(\theta_0)g_{ni,\theta}^\circ\}$ form an MD sequence, $(g_{ni,\theta}^{\circ\prime}, g_{ni,\varphi}^{\circ\prime})' = g_{ni}^\circ \equiv g_{ni}^\circ(\theta_0)$ and $\Pi_n^\circ(\theta_0) =$

$\Sigma^{\circ}_{n,\varphi\theta}(\theta_0,0)\Sigma^{\circ-1}_{n,\theta\theta}(\theta_0,0)$. An OPMD-based and NNUH-robust LM test takes a similar form:

$$\text{LMNH}_n = \tilde{S}^{\circ\prime}_{n,\varphi}\big[\textstyle\sum_{i=1}^n(\tilde{g}^{\circ}_{ni,\varphi}-\tilde{\Pi}^{\circ}_n\tilde{g}^{\circ}_{ni,\theta})(\tilde{g}^{\circ}_{ni,\varphi}-\tilde{\Pi}^{\circ}_n\tilde{g}^{\circ}_{ni,\theta})'\big]^{-1}\tilde{S}^{\circ}_{n,\varphi}, \tag{3.6}$$

where the *tilde* quantities are the estimates at H_0 of the corresponding quantities. Clearly, the key in developing the UH-robust LM tests is to find the adjusted quasi score function $S^{\circ}_n(\theta_0,0)$ that is UH-robust, and hence the UH-robust estimate $\tilde{\theta}^{\circ}_n = \arg\{S^{\circ}_{n,\theta}(\theta_0,0)=0\}$.

Notation: To proceed with details for each model, some general notation would be helpful: (i) for a matrix C_n, $C^s_n = C_n + C'_n$; (ii) for a square matrix C_n, its upper triangular, lower triangular and diagonal matrices are denoted, respectively, by C^u_n, C^l_n and C^d_n such that $C_n = C^u_n + C^l_n + C^d_n$, and its diagonal elements are denoted by $C_{n,ii}$; (iii) $\{a_j\}$ forms a row vector if a'_js are scalars, or a matrix if a'_js are column vectors, and $\{b_{ij}\}$ denotes a matrix formed by the elements b_{ij}; (iv) 0_m is an $m\times 1$ vector of zeros; (v) a quantity with a *tilde* denotes the restricted (under the null) QMLE of that quantity; and (vi) a parametric quantity, e.g., $\Sigma_n(\psi)$, will be denoted as Σ_n when $\psi=\psi_0$, shall no confusion arises.

4. BLM and Robust BLM Tests for $\texttt{SARAR}(p,q)$ Model

Consider the $\texttt{SARAR}(p,q)$ model given in Section 2.1: $A_n(\lambda)Y_n = \mathbf{X}_n\beta + u_n$, $B_n(\rho)u_n = \varepsilon_n$, where $A_n(\lambda) = I_n - \sum_{j=1}^p \lambda_j W_{\ell j}$ and $B_n(\rho) = I_n - \sum_{j=1}^q \rho_j W_{ej}$, $\mathbb{W}_\ell = \{W_{\ell 1}\cdots, W_{\ell p}\}$ and $\mathbb{W}_e = \{W_{e1},\cdots,W_{eq}\}$. The following hypotheses are of primary interest, for $p,q\geq 2$:

- (a) H^a_0: $\lambda=0$ and $\rho=0$, in $\texttt{SARAR}(p,q)$;
- (b) H^b_0: $\lambda_2=\cdots=\lambda_p=0$ and $\rho_2=\cdots=\rho_q=0$, in $\texttt{SARAR}(p,q)$;
- (c) H^c_0: $\lambda=0$, in $\texttt{SARAR}(p,q)$;
- (d) H^d_0: $\rho=0$, in $\texttt{SARAR}(p,q)$;
- (e) H^e_0: $\lambda=0$, in $\texttt{SARAR}(p,0)$;
- (f) H^f_0: $\rho=0$, in $\texttt{SARAR}(0,q)$.

The generic set-up given in Section 3, where the null hypothesis is denoted by $H_0:\varphi=0$ and the parameter vector in the reduced model by θ, facilitates the general discussion. These tests are all tests of model reduction (from a larger SLR model down to a smaller SLR model). Other tests of model reduction may also be of interest, e.g., tests of $\texttt{SARAR}(1,0)$ vs $\texttt{SARAR}(p,0)$, $\texttt{SARAR}(1,0)$ vs $\texttt{SARAR}(p,q)$, $\texttt{SARAR}(0,1)$ vs $\texttt{SARAR}(0,q)$, $\texttt{SARAR}(0,1)$ vs $\texttt{SARAR}(p,q)$, etc., and can all be handled by the general method introduced below.

Likelihood, score and information matrix. The Gaussian loglikelihood for the $\texttt{SARAR}(p,q)$ model is $\ell_n(\psi) = -\frac{n}{2}\ln(2\pi\sigma^2) + \ln|A_n(\lambda)| + \ln|B_n(\rho)| - \frac{1}{2\sigma^2}\varepsilon'_n(\beta,\delta)\varepsilon_n(\beta,\delta)$, where $\varepsilon_n(\beta,\delta) = B_n(\rho)[A_n(\lambda)Y_n - \mathbf{X}_n\beta]$ and $\delta = (\lambda',\rho')'$. Maximizing $\ell_n(\psi)$ gives the maximum likelihood estimator (MLE) $\hat{\psi}_n$ of ψ when the errors are normally distributed, otherwise quasi MLE (QMLE) if the errors are non-normal. The Gaussian loglikelihood of the reduced model can be easily obtained to give the restricted (Q)MLE $\tilde{\theta}_n$ of the parameter vector θ. Both $\hat{\psi}_n$ and $\tilde{\theta}_n$ are robust against non-normality. The $\hat{\theta}_n$ component of $\hat{\psi}_n$ will be used in the bootstrap procedure as it is $\sqrt{n}$-consistent whether or not the null is true.[7]

The score function, $S_n(\psi) = \frac{\partial}{\partial\psi}\ell_n(\psi)$, has the form:

$$S_n(\psi) = \begin{cases} \frac{1}{\sigma^2}\mathbf{X}_n' B_n'(\rho)\varepsilon_n(\beta,\delta), \\ \frac{1}{2\sigma^4}\varepsilon_n'(\beta,\delta)\varepsilon_n(\beta,\delta) - \frac{n}{2\sigma^2}, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)B_n(\rho)W_{\ell j}Y_n - \operatorname{tr}(C_{jn}(\lambda)),\ j = 1,\cdots,p, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)D_{jn}(\rho)\varepsilon_n(\beta,\delta) - \operatorname{tr}(D_{jn}(\rho)),\ j = 1,\cdots,q, \end{cases} \tag{4.1}$$

where $C_{jn}(\lambda) = W_{\ell j}A_n^{-1}(\lambda)$ and $D_{jn}(\lambda) = W_{ej}B_n^{-1}(\rho)$. The information matrix has the form:

$$\Sigma_n(\psi_0) = \begin{pmatrix} \frac{1}{\sigma_0^2}\mathbf{X}_n' B_n' B_n \mathbf{X}_n, & 0, & \left\{\frac{1}{\sigma_0^2}\mathbf{X}_n' B_n' \eta_{jn}\right\}, & 0 \\ \sim, & \frac{n}{2\sigma_0^2}, & \left\{\frac{1}{\sigma_0^2}\operatorname{tr}(C_{jn})\right\}, & \left\{\frac{1}{\sigma_0^2}\operatorname{tr}(D_{jn})\right\} \\ \sim, & \sim, & \left\{\frac{1}{\sigma_0^2}\eta_{jn}'\eta_{j'n} + \operatorname{tr}(\bar{C}_{jn}\bar{C}_{j'n}^s)\right\}, & \left\{\operatorname{tr}(\bar{C}_{jn}D_{j'n}^s)\right\} \\ \sim, & \sim, & \sim, & \left\{\operatorname{tr}(D_{jn}D_{j'n}^s)\right\} \end{pmatrix}, \tag{4.2}$$

where $\eta_{jn} \equiv \eta_{jn}(\beta_0, \delta_0) = B_n(\rho_0)C_{jn}(\lambda_0)\mathbf{X}_n\beta_0$ and $\bar{C}_{jn} \equiv \bar{C}_{jn}(\delta_0) = B_n C_{jn} B_n^{-1}, j = 1\cdots,p$. Recall the $\{\cdot\}$ notation is defined before the start of this section. Finally, the concentrated score of δ after β and σ^2 being concentrated out has the form:

$$\tilde{S}_{n,\delta}(\delta) = \begin{cases} \frac{1}{\tilde{\sigma}_n^2(\delta)}\tilde{\varepsilon}_n'(\delta)B_n(\rho)W_{\ell j}Y_n - \operatorname{tr}(C_{jn}(\lambda)),\ j = 1,\cdots,p, \\ \frac{1}{\tilde{\sigma}_n^2(\delta)}\tilde{\varepsilon}_n'(\delta)D_{jn}(\rho)\tilde{\varepsilon}_n(\delta) - \operatorname{tr}(D_{jn}(\rho)),\ j = 1,\cdots,q, \end{cases} \tag{4.3}$$

where $\tilde{\varepsilon}_n'(\delta) = \varepsilon_n'(\tilde{\beta}_n(\delta),\delta)$, and $\tilde{\beta}_n(\delta)$ and $\tilde{\sigma}_n^2(\delta)$ are the QMLEs of β and σ^2 at a given δ.

LM and BLM tests. The tests in (a)-(d) are of the same nature as each corresponds to a test of model reduction from the full $\texttt{SARAR}(p,q)$, $p,q \geq 2$ model down to a model with fewer spatial terms. Thus, they take the general form given in (3.1), or the following reduced form given by Liu and Yang (2017):

$$\mathrm{LM}_{\texttt{SARAR}}(\delta) = \tilde{S}_{n,\delta}'(\delta)\begin{pmatrix} \tilde{J}_n(\delta) + K_n^{\ell\ell}(\delta), & K_n^{\ell e}(\delta) \\ K_n^{\ell e\prime}(\delta), & K_n^{ee}(\delta) \end{pmatrix}^{-1}\tilde{S}_{n,\delta}(\delta), \tag{4.4}$$

where $\tilde{S}_{n,\delta}(\delta)$ is in (4.3); $\tilde{J}_n(\delta) = \frac{1}{\tilde{\sigma}_n^2(\delta)}\left\{\tilde{\eta}_{jn}'(\delta)\mathbf{M}_n(\rho)\tilde{\eta}_{j'n}(\delta)\right\}_{p\times p}$, $\tilde{\eta}_{jn}(\delta) = \eta_{jn}(\tilde{\beta}_n(\delta),\delta)$,

[7] For the asymptotic properties of the QMLEs under homoskedasticity, see Lee (2004) for $\texttt{SARAR}$(1,0) model, Jin and Lee (2013) for $\texttt{SARAR}$(1,1) model, and Liu and Yang (2017) for $\texttt{SARAR}(p,q)$ model. For the asymptotic properties of the QMLEs under unknown heteroskedasticity, see Liu and Yang (2015) for $\texttt{SARAR}$(1,0) model, and Liu and Yang (2017) for $\texttt{SARAR}(p,q)$ model. For GMM estimation of the $\texttt{SARAR}(p,q)$ model under homoskedasticity, see Lee and Liu (2010). For GMM estimation of the $\texttt{SARAR}(p,q)$ model under heteroskedasticity, see Lin and Lee (2010), Kelejian and Prucha (2010), and Badinger and Egger (2011).

$j=1,\cdots,p$, $\mathbf{M}_n(\rho)=I_n-\mathbf{X}_n(\rho)[\mathbf{X}_n'(\rho)\mathbf{X}_n(\rho)]^{-1}\mathbf{X}_n'(\rho)$, $\mathbf{X}_n(\rho)=B_n(\rho)\mathbf{X}_n$; and

$$\begin{aligned}K_n^{\ell\ell}(\delta) &= \big\{\mathrm{tr}(\bar{C}_{jn}\bar{C}_{j'n}^s)-2\mathrm{tr}(\bar{C}_{jn})\mathrm{tr}(\bar{C}_{j'n})\big\}_{p\times p},\\ K_n^{\ell e}(\delta) &= \big\{\mathrm{tr}(\bar{C}_{jn}D_{j'n}^s)-2\mathrm{tr}(\bar{C}_{jn})\mathrm{tr}(D_{j'n})\big\}_{p\times q},\\ K_n^{ee}(\delta) &= \big\{\mathrm{tr}(D_{jn}D_{j'n}^s)-2\mathrm{tr}(D_{jn})\mathrm{tr}(D_{j'n})\big\}_{q\times q}.\end{aligned}$$

With the general expression (4.4), the LM test statistics for (a)-(d) are obtained by setting $\delta=0$ for (a), $(\tilde{\lambda}_{1n},0_{p-1}',\tilde{\rho}_{1n},0_{q-1}')'$ for (b), $(0_p',\tilde{\rho}_n')$ for (c), and $(\tilde{\lambda}_n',0_q')$ for (d), where the *tilded* parameters are the constrained QMLEs of the corresponding parameters under the respective null hypothesis. For easy reference, the resulted statistics are denoted, respectively, by $\mathrm{LM}_{\texttt{SARAR}}^{(a)}$, $\mathrm{LM}_{\texttt{SARAR}}^{(b)}$, $\mathrm{LM}_{\texttt{SARAR}}^{(c)}$, and $\mathrm{LM}_{\texttt{SARAR}}^{(d)}$.

Of particular interest is $\mathrm{LM}_{\texttt{SARAR}}^{(a)}$ for testing H_0^a: $\lambda=0$ and $\rho=0$, which takes the form:

$$\mathrm{LM}_{\texttt{SARAR}}^{(a)}=\frac{1}{\tilde{\sigma}_n^4}\begin{pmatrix}\tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n\\ \tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n\end{pmatrix}'\begin{pmatrix}\tilde{J}_n+K_n^{\ell\ell}, & K_n^{\ell e}\\ K_n^{\ell e\prime}, & K_n^{ee}\end{pmatrix}^{-1}\begin{pmatrix}\tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n\\ \tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n\end{pmatrix}, \tag{4.5}$$

where $\tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n$ denotes $(\tilde{\varepsilon}_n'W_{\ell 1}Y_n,\cdots,\tilde{\varepsilon}_n'W_{\ell p}Y_n)'$, $\tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n$ denotes $(\tilde{\varepsilon}_n'W_{e1}\tilde{\varepsilon}_n,\cdots,\tilde{\varepsilon}_n'W_{eq}\tilde{\varepsilon}_n)'$, $\tilde{\varepsilon}_n$, $\tilde{\beta}_n$ and $\tilde{\sigma}_n^2$ are from OLS regression of Y_n on $\mathbf{X}_n$, $\tilde{J}_n=\big\{\frac{1}{\tilde{\sigma}_n^2}\tilde{\eta}_{jn}'\mathbf{M}_n\tilde{\eta}_{j'n}\big\}$, $\tilde{\eta}_{jn}=W_{\ell j}\mathbf{X}_n\tilde{\beta}_n$, $K_n^{\ell\ell}=\big\{\mathrm{tr}(W_{\ell j}W_{\ell j'}^s)\big\}$, $K_n^{\ell e}=\big\{\mathrm{tr}(W_{\ell j}W_{ej'}^s)\big\}$, $K_n^{ee}=\big\{\mathrm{tr}(W_{ej}W_{ej'}^s)\big\}$, and $\mathbf{M}_n=\mathbf{M}_n(0)$. The LM test $\mathrm{LM}_{\texttt{SARAR}}^{(a)}$ can easily be simplified to give LM tests for $H_0^{(e)}$ and H_0^f:

$$\begin{aligned}\mathrm{LM}_{\texttt{SLD}}^{(e)} &= \tilde{\sigma}_n^{-4}(\tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n)'\big(\tilde{J}_n+K_n^{\ell\ell}\big)^{-1}(\tilde{\varepsilon}_n'\mathbb{W}_\ell Y_n), & (4.6)\\ \mathrm{LM}_{\texttt{SED}}^{(f)} &= \tilde{\sigma}_n^{-4}(\tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n)'\big(K_n^{ee}\big)^{-1}(\tilde{\varepsilon}_n'\mathbb{W}_e\tilde{\varepsilon}_n). & (4.7)\end{aligned}$$

These tests generalize the tests given in Example 2.1, and can be shown to be NN-robust by verifying that the 'variance' in (3.3) is (asymptotically) equivalent to $\tilde{\Sigma}_{n,\varphi\varphi}-\tilde{\Sigma}_{n,\varphi\theta}\tilde{\Sigma}_{n,\theta\theta}^{-1}\tilde{\Sigma}_{n,\theta\varphi}$.

Liu and Yang (2017) show that the asymptotic null distributions of the tests for the hypotheses in (a)-(f) are chi-square with degrees of freedom being, respectively, $p+q$, $p+q-2$, p, q, p and q. However, the finite sample performance of these tests when referring to the chi-square critical values can be poor, similar to the tests for SARAR(1,1) model given in (2.13)-(2.15). They went on to derive the finite sample improved versions of these tests by **re-standardizing** the concentrated scores. However, as seen from their work, the method of re-standardization can be complicated when the concentrated scores involve estimates of nonlinear spatial parameters such as the cases (b)-(d), besides the issues related to one-sided tests. In this paper, we demonstrate how the bootstrap provide refined approximation to the finite sample critical values, leading to tests with a second-order accuracy in size. As discussed in Section 2, for bootstrap to achieve second-order accuracy, the test statistic has to be an asymptotic pivotal under the null. In this sense, one can use the simplest form of test statistic without going through the complicated process of restandardization. Furthermore, in cases that tests are univariate and one-sided tests can be carried out, bootstrap method offers an additional advantage of being able to approximate the actual one-side critical values.

The bootstrap versions of the above tests can be obtained in a similar way as in Example 2.1. Let $\hat{\beta}_n$ be the unrestricted QML estimate of β and $\hat{\varepsilon}_n$ the unrestricted QMLE residuals. The bootstrap DGP is $Y_n^* = \mathbf{X}_n\hat{\beta}_n + \varepsilon_n^*$, where ε_n^* is an $n \times 1$ vector of iid draws from the EDF of $\hat{\varepsilon}_{ni}$, taking the same form for all three tests but with $\hat{\beta}_n$ and $\hat{\varepsilon}_n$ corresponding to $\texttt{SARAR}(p,0)$, $\texttt{SARAR}(0,q)$, and $\texttt{SARAR}(p,q)$, respectively. Taking the test $\text{LM}_{\texttt{SLD}}^{(e)}$ given in (4.6) for example, based on the bootstrap data $(Y_n^*, \mathbf{X}_n, \mathbb{W}_\ell)$, the bootstrap analogue of $\text{LM}_{\texttt{SLD}}^{e}$ is $\text{LM}_{\texttt{SLD}}^{(e)*} = \tilde{\sigma}_n^{*-4}(\tilde{\varepsilon}_n^{*\prime}\mathbb{W}_\ell Y_n^*)'\big(\tilde{J}_n^* + K_n^{\ell\ell}\big)^{-1}(\tilde{\varepsilon}_n^{*\prime}\mathbb{W}_\ell Y_n^*)$. Repeated samples from the EDF of $\hat{\varepsilon}_n$ give a sequence of bootstrap values for $\text{LM}_{\texttt{SLD}}$, and hence the BCVs.

To demonstrate further how flexible the bootstrap method is, we use the test statistic $\text{LM}_{\texttt{SARAR}}^{(b)}$, obtained from (4.4) by replacing δ by $\tilde{\delta}_n = (\tilde{\lambda}_{1n}, 0_{p-1}', \tilde{\rho}_{1n}, 0_{q-1}')'$, for testing $\texttt{SARAR}(1,1)$ vs $\texttt{SARAR}(p,q)$. In this case, $\theta = (\beta, \sigma^2, \lambda_1, \rho_1)'$, and $\varphi = (\lambda_2, \ldots, \lambda_p, \rho_2, \ldots, \rho_q)$ which is 0_{p+q-2} under the null. Let $\hat{\theta}_n$ be the MLE of θ and $\hat{\varepsilon}_n$ be the ML residuals from the estimation of the full $\texttt{SARAR}(p,q)$ model, based on the original data. The bootstrap DGP is

$$Y_n^* = (I_n - \hat{\lambda}_{1n}W_{\ell 1})^{-1}[\mathbf{X}_n\hat{\beta}_n + (I_n - \hat{\rho}_{1n}W_{e1})^{-1}\varepsilon_n^*], \tag{4.8}$$

where ε_n^* is a vector of n iid draws from the EDF of $\hat{\varepsilon}_n$. Based on the bootstrap data from the above DGP: $(Y_n^*, \mathbf{X}_n, W_{\ell 1}, W_{e1})$, estimate the null model $\texttt{SARAR}(1,1)$ to give the bootstrap estimates $\tilde{\beta}_n^*$,$\tilde{\sigma}_n^{*2}$, and $\tilde{\delta}_n^* = (\tilde{\lambda}_{1n}^*, 0_{p-1}', \tilde{\rho}_{1n}^*, 0_{q-1}')'$, and then compute the bootstrapped value:

$$\text{LM}_{\texttt{SARAR}}^{(b)*} = \tilde{S}_{n,\delta}'(\tilde{\delta}_n^*)\begin{pmatrix} \tilde{J}_n(\tilde{\delta}_n^*) + K_n^{\ell\ell}(\tilde{\delta}_n^*), & K_n^{\ell e}(\tilde{\delta}_n^*) \\ K_n^{\ell e\prime}(\tilde{\delta}_n^*), & K_n^{ee}(\tilde{\delta}_n^*) \end{pmatrix}^{-1}\tilde{S}_{n,\delta}(\tilde{\delta}_n^*), \tag{4.9}$$

where $\tilde{S}_{n,\delta}(\delta)$ is given in (4.3) and other quantities are defined below (4.4). Repeat this process B times to give a sequence of bootstrapped values of $\text{LM}_{\texttt{SARAR}}^{(b)}$ under the null, and their sample quantiles give the bootstrap critical values.

NN-robust LM and BLM tests. To give LM and BLM tests that are generally robust against non-normality, we follow the OPMD method introduced in Section 3 as this method gives an NN-UH robust estimate of the variance of the score without the need of an analytical expression of it. Also this method is simple and the resulted test statistics are asymptotically pivotal at the null, which is all it is needed for BLM to achieve second-order accuracy.

Writing the element $B_n(\rho)W_{\ell j}Y_n$ in the quasi score function $S_n(\psi)$ given in (4.1) as $\bar{C}_{jn}(\delta)Y_n(\delta)$ where $Y_n(\delta) = B_n(\rho)A_n(\lambda)Y_n$ and noticing that $Y_n(\delta_0) = \mathbf{X}_n(\rho_0)\beta_0 + \varepsilon_n$. Then, at the true parameter value ψ_0, we have, for the score vector given in (4.1),

$$S_n(\psi_0) = \begin{cases} \frac{1}{\sigma_0^2}\mathbf{X}_n'B_n'\varepsilon_n, \\ \frac{1}{2\sigma_0^4}\varepsilon_n'\varepsilon_n - \frac{n}{2\sigma_0^2}, \\ \frac{1}{\sigma_0^2}\varepsilon_n'\bar{C}_{jn}\varepsilon_n + \frac{1}{\sigma_0^2}\varepsilon_n'\eta_{jn} - \text{tr}(C_{jn}),\ j = 1, \cdots, p, \\ \frac{1}{\sigma_0^2}\varepsilon_n'D_{jn}\varepsilon_n - \text{tr}(D_{jn}),\ j = 1, \cdots, q. \end{cases} \tag{4.10}$$

This leads immediately to an MD representation: $S_n(\psi_0) = \sum_{i=1}^n g_{ni}(\psi_0)$, where

$$g_{ni}(\psi_0) = \begin{cases} \frac{1}{\sigma_0^2} x_{bi}\varepsilon_{ni}, \\ \frac{1}{2\sigma_0^4}(\varepsilon_{ni}^2 - \sigma_0^2), \\ \frac{1}{\sigma_0^2}[\varepsilon_{ni}\xi_{jn,i} + \bar{C}_{jn,ii}(\varepsilon_{ni}^2 - \sigma_0^2) + \eta_{jn,i}\varepsilon_{ni}],\ j = 1, \cdots, p, \\ \frac{1}{\sigma_0^2}[\varepsilon_{ni}\zeta_{jn,i} + D_{jn,ii}(\varepsilon_{ni}^2 - \sigma_0^2)],\ j = 1, \cdots, q, \end{cases} \tag{4.11}$$

where x_{bi} is the ith column of $\mathbf{X}_n' B_n'$, $\xi_{jn} = (\bar{C}_{jn}^{u\prime} + \bar{C}_{jn}^l)\varepsilon_n$, and $\zeta_{jn} = (D_{jn}^{u\prime} + D_{jn}^l)\varepsilon_n$.

Equipped with (4.2) and (4.11), and following general principles laid out by (3.4) and the discussions around it, we have the general form of NN-robust LM test:

$$\text{LMN}_{\texttt{SARAR}}^{(\texttt{m})} = \tilde{S}_{n,\varphi}' \big[\textstyle\sum_{i=1}^n (\tilde{g}_{ni,\varphi} - \tilde{\Pi}_n \tilde{g}_{ni,\theta})(\tilde{g}_{ni,\varphi} - \tilde{\Pi}_n \tilde{g}_{ni,\theta})' \big]^{-1} \tilde{S}_{n,\varphi}, \tag{4.12}$$

where $\texttt{m} = a, b, c, d, e, f$, giving the NN-robust tests for the six hypotheses listed above. The $\tilde{\Pi}_n$ can be either the plug-in estimate of $\Pi_n = \Sigma_{n,\varphi\theta}\Sigma_{n,\theta\theta}^{-1}$ based on $\Sigma_n(\psi_0)$ given in (4.2), or the estimate based on the Hessian matrix, with θ and φ defined accordingly. For example, for testing $H_0^a : \delta = 0$, we have $\theta = (\beta', \sigma^2)'$ and $\varphi = \delta$, $\Pi_n = \Sigma_{n,\varphi\theta}\Sigma_{n,\theta\theta}^{-1}$ has only non-zero element at the upper-left corner block: $\{\eta_{jn}' \mathbf{X}_n(\rho)[\mathbf{X}_n'(\rho)\mathbf{X}_n(\rho)]^{-1}\}$, and for tests in (e) and (f), we have $\theta = (\beta', \sigma^2)'$, and $\delta = \lambda$ for (e) and ρ for (f). Liu and Yang (2017) show that the null asymptotic distribution of $\text{LMN}_{\texttt{SARAR}}^{(\texttt{m})}$ is chi-square with degrees of freedom being $\dim(\varphi)$.

Bootstrap critical values for the NN-robust LM tests are obtained in a similar manner. As discussed above, the parameter estimates maximizing the Gaussian likelihood of the SARAR models are robust against non-normality, the bootstrap DGPs take the same form as those for the case of the regular LM tests, e.g., (4.8) for testing SARAR$(1,1)$ vs SARAR(p,q).

NNUH-robust LM and BLM tests. Under UH, i.e., $\varepsilon_{ni} \sim (0, \sigma_0^2 h_i)$. From (4.10), it is easy to see that the δ-component of $\mathrm{E}[S_n(\psi_0)]$, involving $\{h_i\}$, is not zero in general, and that the probability limit of $\frac{1}{n}S_n(\psi_0)$ is not zero in general. Thus, the statistics developed earlier would not converge to central chi-squares limiting distributions under the null, and inferences based on them would be misleading. Define

$$S_n^\circ(\psi) = \begin{cases} \frac{1}{\sigma^2}\mathbf{X}_n' B_n'(\rho)\varepsilon_n(\beta, \delta), \\ \frac{1}{2\sigma^4}\varepsilon_n'(\beta, \delta)\varepsilon_n(\beta, \delta) - \frac{n}{2\sigma^2}, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta, \delta)\bar{C}_{jn}^\circ(\delta)Y_n(\delta),\ j = 1, \cdots, p, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta, \delta)D_{jn}^\circ(\rho)\varepsilon_n(\beta, \delta),\ j = 1, \cdots, q, \end{cases} \tag{4.13}$$

where $\bar{C}_{jn}^\circ(\delta) = \bar{C}_{jn}(\delta) - \mathrm{diag}(\bar{C}_{jn}(\delta))$ and $D_{jn}^\circ(\delta) = D_{jn}(\delta) - \mathrm{diag}(D_{jn}(\delta))$. It is easy to verify that, under UH, $\mathrm{E}[S_n^\circ(\psi_0)] = 0$ and $\frac{1}{n}S_n^\circ(\psi_0) \xrightarrow{p} 0$. Solving $S_n^\circ(\psi) = 0$ leads to NNUH-robust estimator $\hat{\psi}_n^\circ$ for ψ of the full model, and the $\hat{\theta}_n^\circ$ component of $\hat{\psi}_n^\circ$ will be used in the bootstrap procedure discussed below. Similarly, solving $S^\circ_{n,\theta}(\psi)=0$ under $H_0:\varphi=0$ gives the restricted NNUH-robust estimator $\tilde\theta^\circ_n$ of θ. See Liu and Yang (2017) for details.

Similarly, $S^\circ_n(\psi_0)$ has an MD representation: $S^\circ_n(\psi_0)=\sum_{i=1}^n g^\circ_{ni}(\psi_0)$, where

$$g^\circ_{ni}(\psi_0)=\begin{cases}\frac{1}{\sigma_0^2}x_{bi}\varepsilon_{ni},\\ \frac{1}{2\sigma_0^4}(\varepsilon^2_{ni}-\sigma^2_0),\\ \frac{1}{\sigma_0^2}(\varepsilon_{ni}\xi^\circ_{jn,i}+\eta^\circ_{jn,i}\varepsilon_{ni}),\ j=1,\cdots,p,\\ \frac{1}{\sigma_0^2}\varepsilon_{ni}\zeta^\circ_{jn,i},\ j=1,\cdots,q,\end{cases} \tag{4.14}$$

where x_{bi} is as in (4.11), $\xi^\circ_{jn}=(\bar C^{\circ u\prime}_{jn}+\bar C^{\circ l}_{jn})\varepsilon_n$, $\zeta^\circ_{jn}=(D^{\circ u\prime}_{jn}+D^{\circ l}_{jn})\varepsilon_n$, and $\eta^\circ_{jn}=\bar C^\circ_{jn}B_n\mathbf{X}_n\beta_0$.

With $S^\circ_n(\psi_0)$ and its MD representation, letting $\tilde\Pi^\circ_n=[\frac{\partial}{\partial\theta'}S^\circ_{n,\varphi}(\tilde\theta^\circ_n,0)][\frac{\partial}{\partial\theta'}S^\circ_{n,\theta}(\tilde\theta^\circ_n,0)]^{-1}$ be a feasible estimate of $\Pi^\circ_n=\Sigma^\circ_{n,\varphi\theta}\Sigma^{\circ-1}_{n,\theta\theta}$, the LM tests fully robust against NN and UH take the general form as that given in (3.6):

$$\mathrm{LMNH}^{(\mathrm{m})}_{\mathtt{SARAR}}=\tilde S^{\circ\prime}_{n,\varphi}\big[\textstyle\sum_{i=1}^n(\tilde g^\circ_{ni,\varphi}-\tilde\Pi^\circ_n\tilde g^\circ_{ni,\theta})(\tilde g^\circ_{ni,\varphi}-\tilde\Pi^\circ_n\tilde g^\circ_{ni,\theta})'\big]^{-1}\tilde S^\circ_{n,\varphi}, \tag{4.15}$$

where $\tilde S^\circ_{n,\varphi}=S^\circ_{n,\varphi}(\tilde\theta^\circ_n,0)$, $\tilde g^\circ_{ni,\theta}$ and $\tilde g^\circ_{ni,\varphi}$ are the subvectors of $g^\circ_{ni}(\tilde\theta^\circ_n,0)$, and $\mathtt{m}=a,b,c,d,e,f$ corresponding to the six tests defined at the beginning of this section with relevant choice of θ and φ and the related quantities. Liu and Yang (2017) show that the null asymptotic distribution of $\mathrm{LMNH}^{(\mathtt{m})}_{\mathtt{SARAR}}$ is chi-square with degrees of freedom being $\dim(\varphi)$.

We again use the case (b) with the test statistic $\mathrm{LMNH}^{(b)}_{\mathtt{SARAR}}$ to provide details on the bootstrap procedures for obtaining refined approximations to the finite sample critical values of the test statistics. First, the test statistic $\mathrm{LMHN}^{(b)}_{\mathtt{SARAR}}$ is obtained from (4.15) using $\tilde\theta^\circ_n=(\tilde\beta^{\circ\prime}_n,\tilde\sigma^{\circ 2}_n,\tilde\lambda^\circ_{1n},\tilde\rho^\circ_{1n})'$. Based on the unrestricted estimates $\hat\theta^\circ_n=(\hat\beta^{\circ\prime}_n,\hat\sigma^{\circ 2}_n,\hat\lambda^\circ_{1n},\hat\rho^\circ_{1n})'$ and the unrestricted residuals $\hat\varepsilon^\circ_n$ obtained from the UH-robust estimation of the full model, the *wild* bootstrap DGP is set up as follows:

$$Y^*_n=(I_n-\hat\lambda^\circ_{1n}W_{\ell 1})^{-1}[\mathbf{X}_n\hat\beta^\circ_n+(I_n-\hat\rho^\circ_{1n}W_{e1})^{-1}(\hat\varepsilon^\circ_n\odot v^*_n)], \tag{4.16}$$

where v^*_n is a vector of n iid draws from a distribution as discussed in Section 2. Based on the bootstrap data from the above DGP: $(Y^*_n,\mathbf{X}_n,W_{\ell 1},W_{e1})$, estimate the null model $\mathtt{SARAR}(1,1)$ to give the bootstrap estimate $\tilde\theta^{\circ *}_n$, and then compute the bootstrapped value:

$$\mathrm{LMNH}^{(\mathrm{b})*}_{\mathtt{SARAR}}=\tilde S^{\circ *\prime}_{n,\varphi}\big[\textstyle\sum_{i=1}^n(\tilde g^{\circ *}_{ni,\varphi}-\tilde\Pi^{\circ *}_n\tilde g^{\circ *}_{ni,\theta})(\tilde g^{\circ *}_{ni,\varphi}-\tilde\Pi^{\circ *}_n\tilde g^{\circ *}_{ni,\theta})'\big]^{-1}\tilde S^{\circ *}_{n,\varphi}. \tag{4.17}$$

Repeat this process B times to give a sequence of bootstrapped values of $\mathrm{LMNH}^{(b)}_{\mathtt{SARAR}}$ under the null, and their sample quantiles give the bootstrap critical values.[8]

[8]The usual plug-in estimate may not be feasible as the explicit expression of Σ°_n contains the unknown heteroskedasticity $\{h_i\}$ besides the regular parameters. It can easily be verified that the tests $\mathrm{LMN}^{(\mathtt{a})}_{\mathtt{SARAR}}$, $\mathrm{LMN}^{(\mathtt{e})}_{\mathtt{SARAR}}$, and $\mathrm{LMN}^{(\mathtt{f})}_{\mathtt{SARAR}}$ defined in (4.12) are also UH-robust. Assumption B4 may be verified based on

5. BLM and Robust BLM Tests for MESS(p, q) Model

Consider the SLR model with MESS(p, q) effect: $A_n(\lambda)Y_n = \mathbf{X}_n\beta + u_n$, $B_n(\rho)u_n = \varepsilon_n$, where $A_n(\lambda) = \exp(\sum_{j=1}^{p}\lambda_j W_{\ell j})$ and $B_n(\rho) = \exp(\sum_{j=1}^{q}\rho_j W_{ej})$. Similar to the SARAR$(p, q)$ model, the following hypotheses are of primary interest:

- (a) H_0^a: $\delta = 0$, in MESS(p, q);
- (b) H_0^b: $\lambda_2 = \cdots = \lambda_p = 0$ and $\rho_2 = \cdots = \rho_q = 0$, in MESS$(p, q)$;
- (c) H_0^c: $\lambda = 0$, in MESS(p, q);
- (d) H_0^d: $\rho = 0$, in MESS(p, q);
- (e) H_0^e: $\lambda = 0$, in MESS$(p, 0)$;
- (f) H_0^f: $\rho = 0$, in MESS$(0, q)$.

Again, we use the notation δ to denote $(\lambda', \rho')'$, θ to denote the parameters in the null model, and φ to denote the additional parameters in the full model which are specified by the null hypothesis to be zero. Other tests of model reduction concerning δ only can be treated in the same way, using the general methods introduced below.

We will proceed with the details on the construction of the LM and robust LM tests without providing detailed proofs of the results as they are largely implied by the asymptotic results in the supplement file to Debarsy et al. (2015), except the case of NNUH-robust LM tests, of which proofs require the central limit theorems for linear-quadratic form of Kelijian and Prucha (2001) and the weak law of large numbers of, e.g., Davidson (1994).

Likelihood, score and information matrix. The loglikelihood function of the MESS(p, q) model is $\ell_n(\psi) = -\frac{2}{n}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\varepsilon_n'(\beta, \delta)\varepsilon_n(\beta, \delta)$, where $\varepsilon_n(\beta, \delta) = B_n(\rho)[A_n(\lambda)Y_n - X_n\beta]$. Maximizing $\ell_n(\psi)$ gives the unrestricted MLE or QMLE $\hat{\psi}_n$ of ψ in the full model, and maximizing the loglikelihood of the reduced model under the null gives the restricted MLE or QMLE $\tilde{\theta}_n$ of θ. Note that, with MESS specifications, $\ln|A_n(\lambda)| = 0$ and $\ln|B_n(\rho)| = 0$. Thus, the QML estimation of the MESS(p, q) model has a computational advantage over that of a SARAR(p, q) model as it avoids the repeated calculations of the determinants of the two matrices $A_n(\lambda)$ and $B_n(\rho)$ in the optimization process. Another advantage is that the QMLEs of the MESS$(p, 0)$ and MESS$(0, q)$ models are robust against unknown heteroskedasticity, and the QMLEs of the MESS(p, q) model can be robust against unknown heteroskedasticity if $W_{\ell j}W_{ej'} = W_{ej'}W_{\ell j}$, i.e., the two types of spatial weights matrices are commutative. These can easily be seen by showing that the expectation of the score (given below) at the true parameters values under UH are zero.[9] To allow for more generality, we do not assume $W_{\ell j}$ and $W_{ej'}$ to be commutative, and propose a UH-robust QML-type estimators for the MESS(p, q) model and use them in constructing the UH-robust bootstrap LM tests.

univariate tests given in (2.20) and (2.21) alone the lines of Yang (2015) for iid errors.

[9] A further advantage of the QML estimation of the MESS(p, q) model is that its parameter space is unrestricted, whereas the values of the parameters in the SARAR(p, q) must be restricted in a compact space which can be hard to find (Lee and Liu, 2010; Elhorst et al., 2012). Consistency and asymptotic normality of the QMLEs of the general MESS(p, q) model are proved in the supplement file to Debarsy et al. (2015).

The score function of $\psi = (\beta', \sigma^2, \delta')'$ has the forms:

$$S_n(\psi) = \begin{cases} \frac{1}{\sigma^2} X_n' B_n'(\rho)\varepsilon_n(\beta,\delta), \\ \frac{1}{2\sigma^4}\varepsilon_n'(\beta,\delta)\varepsilon_n(\beta,\delta) - \frac{n}{2\sigma^2} \\ -\frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)B_n(\rho)\dot{A}_{nj}(\lambda)Y_n,\ j=1,\cdots,p, \\ -\frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)\dot{B}_{nj}(\rho)[A_n(\lambda)Y_n - \mathbf{X}_n\beta],\ j=1,\cdots,q, \end{cases} \tag{5.1}$$

where $\dot{A}_{nj}(\lambda) = \frac{\partial}{\partial\lambda_j}A_n(\lambda),\ j=1,\dots,p$, and $\dot{B}_{nj}(\rho) = \frac{\partial}{\partial\rho_j}B_n(\rho),\ j=1,\dots,q$. The information matrix has a similar form to that for $\texttt{SARAR}(p,q)$ model given in (4.2).

$$\Sigma_n(\psi_0) = \begin{pmatrix} \frac{1}{\sigma_0^2}\mathbf{X}_n' B_n' B_n \mathbf{X}_n, & 0, & \{\frac{1}{\sigma_0^2}\mathbf{X}_n' B_n' \eta_{jn}\}, & 0 \\ \sim, & \frac{n}{2\sigma_0^2}, & \{\frac{1}{\sigma_0^2}\text{tr}(C_{jn})\}, & \{\frac{1}{\sigma_0^2}\text{tr}(D_{jn})\} \\ \sim, & \sim, & \{\frac{1}{\sigma_0^2}\eta_{jn}'\eta_{j'n} + T_{n,jj'}^{\ell\ell}\}, & \{\text{tr}(\bar{C}_{jn}D_{j'n}^s)\} \\ \sim, & \sim, & \sim, & \{T_{n,jj'}^{ee}\} \end{pmatrix}, \tag{5.2}$$

where $\eta_{jn} = B_n C_{jn}\mathbf{X}_n\beta_0$, $C_{jn} = \dot{A}_{nj}A_n^{-1}$, $D_{jn} = \dot{B}_{jn}B_n^{-1}$, and $\bar{C}_{jn} = B_n C_{jn} B_n^{-1}$; $T_{n,jj'}^{\ell\ell} = \text{tr}(C_{jn}C_{j'n} + \ddot{A}_{n,jj'}A_n^{-1})$, and $T_{n,jj'}^{ee} = \text{tr}(D_{jn}D_{j'n} + \ddot{B}_{n,jj'}B_n^{-1})$; and $\ddot{A}_{n,jj'} = \frac{\partial^2}{\partial\lambda_j\partial\lambda_{j'}}A_n(\lambda_0)$, and $\ddot{B}_{n,jj'} = \frac{\partial^2}{\partial\rho_j\partial\rho_{j'}}A_n(\rho_0)$. The partial derivatives of $A_n(\lambda)$ and $B_n(\rho)$ do not possess closed form expressions, unless $W_{\ell j}$ and $W_{\ell j'}$ are commutative, and W_{ej} and $W_{ej'}$ are commutative. However, the LM type-tests considered in this paper require their expressions only at the null. When the null model is of order $\texttt{MESS}$(1,1) or lower, we have $\dot{A}_{n,j}(\lambda_1, 0_{p-1}) = W_{\ell j}A_n(\lambda_1, 0_{p-1})$ and $\dot{A}_{n,j}(0_p) = W_{\ell j}$, and $\dot{B}_{n,j}(\rho_1, 0_{q-1}) = W_{ej}B_n(\rho_1, 0_{q-1})$ and $\dot{B}_{n,j}(0_q) = W_{ej}$. Hence, LM tests can be constructed using the OPMD estimate of Γ_n (as in this case, $\Gamma_n = \Sigma_n$) so that the second-order partial derivatives are avoided. For more general LM tests, robust LM tests, BLM and robust BLM tests, one may consider to use the following alternative specifications:

$$A_n(\lambda) = \textstyle\prod_{j=1}^p \exp(\lambda_j W_{\ell j}) \text{ and } B_n(\rho) = \prod_{j=1}^q \exp(\rho_j W_{ej}), \tag{5.3}$$

to overcome the difficulties in finding the partial derivatives. It would be interesting to study in detail this alternative $\texttt{MESS}(p,q)$ model, but it is beyond the scope of this paper.

LM and BLM tests. For the LM tests of the first four hypotheses that correspond to the tests of model reduction from $\texttt{MESS}(p,q)$, we adopt the general form given in (3.1):

$$\text{LM}_{\texttt{MESS}}^{(\texttt{m})} = \tilde{S}_{n,\varphi}'(\tilde{\Sigma}_n^{-1})_{\varphi\varphi}\tilde{S}_{n,\varphi}, \tag{5.4}$$

where $\texttt{m} = a, b, c, d$, and correspondingly $\varphi = \delta$, $(\lambda_2,\dots,\lambda_p,\rho_2,\dots,\rho_q)'$, λ, and ρ. The tests (e) and (f) can be obtained from the test (a) by dropping the ρ-components or λ-components.

Of particular interest is $\text{LM}_{\texttt{MESS}}^{(a)}$ for testing H_0^a: $\delta = 0$ in $\texttt{MESS}(p,q)$, and very interestingly it can easily be seen that under (5.3) it takes the identical form as $\text{LM}_{\texttt{SARAR}}^{(a)}$ given in (4.5).

Similarly, the LM test $\text{LM}_{\texttt{MESS}}^{(e)}$ for testing H_0^e: $\lambda = 0$ in $\texttt{MESS}(p, 0_q)$ has the identical form as $\text{LM}_{\texttt{SLD}}^{(e)}$ given in (4.6), and the LM test $\text{LM}_{\texttt{MESS}}^{(f)}$ for testing H_0^f: $\rho = 0$ in $\texttt{MESS}(0_p, q)$ has the identical form as $\text{LM}_{\texttt{SED}}^{(f)}$ given in (4.7). This means that the tests given in (4.5)-(4.7) derived under $\texttt{SARAR}(p, q)$ specification not only have power against the departure from the linear regression in the form of SARAR but also have power against the MESS. It can also be easily seen that these tests have power against the SAR in response and *spatial moving average* (SMA) in the error. Similar properties may hold for the robust versions of these tests.

The tests $\text{LM}_{\texttt{MESS}}^{(b)}$, $\text{LM}_{\texttt{MESS}}^{(c)}$, and $\text{LM}_{\texttt{MESS}}^{(d)}$ are similar to $\text{LM}_{\texttt{SARAR}}^{(b)}$, $\text{LM}_{\texttt{SARAR}}^{(c)}$, and $\text{LM}_{\texttt{SARAR}}^{(d)}$ for the $\texttt{SARAR}(p, q)$ model but not identical. This is because the elements related to the spatial effects remained in the null model are different for different specifications on spatial effects. It is easy to see that when λ is a scalar, or $\{W_{\ell j}\}$ are commutative, or the alternative MESS form given in (5.3) is used, $\text{tr}(C_{jn}) = 0$; similarly for $\text{tr}(C_{jn})$. Hence, the derivation of the LM tests can be done without the σ^2-components of the score and the information matrix. In general, this property may not hold, and thus the σ^2-components are kept.

Bootstrap proceeds in a similar manner. Let $\hat{\theta}_n$ be the MLE of θ and $\hat{\varepsilon}_n$ be the ML residuals from the estimation of the full $\texttt{MESS}(p, q)$ model, based on the original data. Taking for example the test $\text{LM}_{\texttt{MESS}}^{(b)}$ for testing $H_0^b : \varphi = 0$, where $\varphi = (\lambda_2, \ldots, \lambda_p,\ \rho_2, \ldots, \rho_q)' = 0$, the bootstrap DGP is

$$Y_n^* = \exp(-\hat{\lambda}_{1n} W_{\ell 1})[\mathbf{X}_n \hat{\beta}_n + \exp(-\hat{\rho}_{1n} W_{e1}) \varepsilon_n^*], \tag{5.5}$$

where ε_n^* is a vector of n iid draws from the EDF of $\hat{\varepsilon}_n$. Based on the bootstrap data from the above DGP: $(Y_n^*, \mathbf{X}_n, W_{\ell 1}, W_{e1})$, estimate the null model $\texttt{MESS}(1, 1)$ to give the bootstrap estimates $\tilde{\theta}_n^* = (\tilde{\beta}_n^{*\prime}, \tilde{\sigma}_n^{*2}, \tilde{\lambda}_{1n}^*, \tilde{\rho}_{1n}^*)'$, and then compute the bootstrapped value:

$$\text{LM}_{\texttt{MESS}}^{(b)*} = \tilde{S}_{n,\varphi}^{*\prime} (\tilde{\Sigma}_n^{*-1})_{\varphi\varphi} \tilde{S}_{n,\varphi}^*, \tag{5.6}$$

at $\tilde{\theta}_n^*$, ε_n^*, and $\varphi = 0$. Repeat this process B times to give a sequence of bootstrapped values of $\text{LM}_{\texttt{MESS}}^{(b)}$ under the null, and their sample quantiles give the bootstrap critical values.

NN-robust LM and BLM tests. We again use the OPMD form of the LM test given in Section 3. The score at ψ_0 has an identical form as that in (4.10) for the SARAR model:

$$S_n(\psi_0) = \begin{cases} \frac{1}{\sigma_0^2} \mathbf{X}_n' B_n' \varepsilon_n, \\ \frac{1}{2\sigma_0^4} \varepsilon_n' \varepsilon_n - \frac{1}{2\sigma_0^2}, \\ \frac{1}{\sigma_0^2} \varepsilon_n' \bar{C}_{jn} \varepsilon_n + \frac{1}{\sigma_0^2} \varepsilon_n' \eta_{jn} - \text{tr}(C_{jn}),\ j = 1, \cdots, p, \\ \frac{1}{\sigma_0^2} \varepsilon_n' D_{jn} \varepsilon_n - \text{tr}(D_{jn}),\ j = 1, \cdots, q, \end{cases} \tag{5.7}$$

which leads to an identical MD representation for $S_n(\psi_0)$ as that given in (4.11) for the SARAR model, i.e., $S_n(\psi_0) = \sum_{i=1}^n g_{ni}(\psi_0)$, where $g_{ni}(\psi_0)$ is given in (4.11), but replacing the $\bar{C}_{jn}$

and D_{jn} by those defined below (5.2). Thus, the NN-robust test for testing $H_0^{\mathtt{m}}$ is

$$\mathrm{LMN}_{\mathtt{MESS}}^{(\mathtt{m})} = \tilde{S}'_{n,\varphi}\big[\sum_{i=1}^{n}(\tilde{g}_{ni,\varphi} - \tilde{\Pi}_n\tilde{g}_{ni,\theta})(\tilde{g}_{ni,\varphi} - \tilde{\Pi}_n\tilde{g}_{ni,\theta})'\big]^{-1}\tilde{S}_{n,\varphi}, \tag{5.8}$$

where $\mathtt{m} = a, b, c, d, e, f$, which is identical in form to the general test $\mathrm{LMN}_{\mathtt{SARAR}}^{(\mathtt{m})}$ given in (4.12). The bootstrap critical values for the NN-robust LM tests are obtained in a similar manner as those for the NN-robust LM tests for the $\mathtt{SARAR}(p, q)$ model.

NNUH-robust LM and BLM tests. Modify the score function so that it is robust against UH, besides being robust against NN:

$$S_n^{\circ}(\psi) = \begin{cases} \frac{1}{\sigma^2}X_n'B_n'\varepsilon_n(\beta,\delta), \\ \frac{1}{2\sigma^4}\varepsilon_n'(\beta,\delta)\varepsilon_n(\beta,\delta) - \frac{1}{2\sigma_0^2}, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)\bar{C}_{jn}^{\circ}(\delta)B_n(\rho)A_n(\lambda)Y_n,\ j = 1,\cdots,p, \\ \frac{1}{\sigma^2}\varepsilon_n'(\beta,\delta)D_{jn}^{\circ}(\rho)\varepsilon_n(\beta,\delta),\ j = 1,\cdots,q, \end{cases} \tag{5.9}$$

where $\bar{C}_{jn}^{\circ}(\delta) = \bar{C}_{jn}(\delta) - \mathrm{diag}(\bar{C}_{jn}(\delta))$ and $D_{jn}^{\circ}(\rho) = D_{jn}(\rho) - \mathrm{diag}(D_{jn}(\rho))$. The unrestricted NNUH-robust QMLE of ψ is thus

$$\hat{\psi}_n^{\circ} = \arg\{S_n^{\circ}(\psi) = 0\},$$

and its component $\hat{\theta}_n^{\circ}$ is used as the 'parameters' in the bootstrap DGP. The restricted NNUH-robust QMLE for θ under the null hypothesis $H_0 : \varphi = 0$, is thus

$$\tilde{\theta}_n^{\circ} = \arg\{S_{n,\theta}^{\circ}(\theta,\varphi)|_{\varphi=0} = 0\}.$$

Based on $\tilde{\theta}_n^{\circ}$ and the MD representation for $S_n^{\circ}(\psi_0)$, one easily obtains the NNUH-robust LM test $\mathrm{LMNH}_{\mathtt{MESS}}^{(\mathtt{m})}$, which has an identical form as $\mathrm{LMNH}_{\mathtt{SARAR}}^{(\mathtt{m})}$ given in (4.15), for $m = a, b, c, d, e, f$. The bootstrap critical values for the NNUH-robust LM tests are obtained in a similar manner as those for the NNUH-robust LM tests for the $\mathtt{SARAR}(p, q)$ model.

Take for example the test $\mathrm{LMNH}_{\mathtt{MESS}}^{(\mathtt{b})}$. We have $\theta = (\beta', \sigma^2, \lambda_1, \rho_1)'$. Using the unrestricted estimate $\hat{\theta}_n^{\circ}$ and the unrestricted residuals $\hat{\varepsilon}_n^{\circ}$ from the UH-robust estimation of the full model discussed above, the *wild* bootstrap DGP is

$$Y_n^* = \exp(-\hat{\lambda}_{1n}^{\circ}W_{\ell 1})[\mathbf{X}_n\hat{\beta}_n^{\circ} + \exp(-\hat{\rho}_{1n}^{\circ}W_{e1})(\hat{\varepsilon}_n^{\circ} \odot v_n^*)], \tag{5.10}$$

where v_n^* is a vector of n iid draws from a distribution as discussed in Section 2. Based on the bootstrap data from the above DGP: $(Y_n^*, \mathbf{X}_n, W_{\ell 1}, W_{e1})$, estimate the null model $\mathtt{MESS}(1, 1)$ to give the bootstrap estimate $\tilde{\theta}_n^{\circ *}$, and then compute the bootstrapped value:

$$\mathrm{LMNH}_{\mathtt{MESS}}^{(\mathrm{b})*} = \tilde{S}_{n,\varphi}^{\circ *\prime}\big[\sum_{i=1}^{n}(\tilde{g}_{ni,\varphi}^{\circ *} - \tilde{\Pi}_n^{\circ *}\tilde{g}_{ni,\theta}^{\circ *})(\tilde{g}_{ni,\varphi}^{\circ *} - \tilde{\Pi}_n^{\circ *}\tilde{g}_{ni,\theta}^{\circ *})'\big]^{-1}\tilde{S}_{n,\varphi}^{\circ *}. \tag{5.11}$$

Repeat this process B times to give a sequence of bootstrapped values of $\mathrm{LMNH}_{\mathtt{MESS}}^{(b)}$ under the null, and their sample quantiles give the bootstrap critical values.

Some final remarks for the UH-robust QML estimation of the $\texttt{MESS}(p, q)$ model are as follows. It can be proved along the lines of Liu and Yang (2017) that the unrestricted estimator $\hat{\psi}_n^\circ$ is $\sqrt{n}$-consistent under unknown heteroskedasticity, so is the restricted estimator $\tilde{\theta}_n^\circ$ when the restrictions are true. As in Liu and Yang (2017) for the $\texttt{SARAR}(p, q)$ model, one could also pursue the finite sample improved estimators and LM tests robust against UH through re-standardization. However, as discussed in the introduction, in order to obtain the asymptotically refined critical values through bootstrapping, it is sufficient that the test statistic is asymptotic pivotal under the null.

6. Monte Carlo Results

Monte Carlo experiments are run to assess the finite sample performance of the proposed tests, based on a $\texttt{SARAR}(2, 2)$ model and a $\texttt{MESS}(2, 2)$ model, both having two regressors and a constant term with coefficients $\beta = (5, 1, 1)'$. $W_{\ell 1} = W_{e1}$, which are generated from the `Queen contiguity` scheme, and $W_{\ell 2} = W_{e2}$, which are generated from a `group interaction` scheme with an average group size of 10, and the sizes of the groups are generated uniformly from 2 to 18. The errors $\{\varepsilon_{ni}\}$ are iid copies of $Z \sim N(0, 1)$, or `lognormal` (standardized $\exp(Z)$), or `normal mixture` (mixing Z and $4Z$ with probabilities 0.9 and 0.1, and then standardizing). The regressors' values are generated in a non-iid manner as in Yang (2015). To conserve space, only a portion of the Monte Carlo results are presented here in Tables 1 and 2 for tests of H_0^a corresponding to $\texttt{SARAR}(2,\ 2)$ and $\texttt{MESS}(2,\ 2)$, respectively.[10] Each set of results (under a given combination of λ, ρ and n values, and error distribution) are based on 2000 Monte Carlo samples and 699 bootstrap samples within each Monte Carlo sample.

The results (reported and unreported) show that (i) the LM tests are generally undersized, and their bootstrap versions (under iid bootstrap) have a significantly better size property under homoskedastic errors (normal or nonnormal) but not under heteroskedasticity; (ii) the NNUH-robust LM tests can be severely oversized in general, and their bootstrap versions (under *wild* bootstrap with first distribution as in Footnote 5) have empirical sizes very close to their nominal levels, whether the errors are normal or nonnormal, and homoskedastic or heteroskedastic; and (iii) the NN-robust LM tests (the OPMD versions) can also be severely oversized, and by referring to the bootstrap critical values with *wild* bootstrap the size distortions can effectively be removed under both homoskedasticity and heteroskedasticity.

7. Conclusions and Discussions

Methods for bootstrapping the critical values of LM-type tests under non-normality and/or unknown heteroskedasticity are introduced and their validity are justified. Meth-

[10] A more comprehensive set of Monte Carlo results is available from the author upon request.

ods for constructing LM and robust LM tests, in particular the latter, are also introduced, to facilitate the bootstrap methods. The outer-product-of-martingale-difference (OPMD) for estimating the variance of the score function is seen to be a crucial step in achieving robustness against unknown heteroskedasticity. Three versions of the LM-type tests and their bootstrap analogues are given for each of the two general higher order models, namely the SARAR(p, q) and the MESS(p, q). These tests are easy to implement. Monte Carlo results show that the LM-type tests referring to the bootstrap critical values effectively remove the size distortions resulted from referring to the asymptotic critical values.

The same methodology can be applied to give LM and robust LM tests and their bootstrap analogues for the other types of higher order SLR models such as the SLR model with a pth order SAR response and SEC, and the SLR model with a pth order MESS in response and SEC. However, before formal studies on these tests, a detailed and formal study on the SEC model (2.2) with added attributes such as SAR or MESS in response and unknown heteroskedasticity in errors, would be more interesting. Similarly, it would also be more interesting to conduct a detailed study on the higher-order SLR model with the alternative MESS(p, q) form (5.3). Monte Carlo evidence provided by Yang (2015) and in this paper has shown a much improved performance of the LM-type tests referring to the bootstrap critical values over referring to the asymptotic critical values. Thus, it would be interesting to provide some empirical illustrations on these tests to guide the practitioners in their applications, and to provide computer software for the full implementation of these tests.

In studying the asymptotic properties of Moran I test, Kelejian and Prucha (2001) introduced a central limit theorem for linear-quadratic forms, which has over the years become a standard tool in studying the asymptotic properties of various spatial estimators and tests, including the LM-type tests introduced in this paper. Liu and Prucha (2016) generalized Moran I test, allowing for higher-order spatial dependence in the disturbance as well as in the response, heteroskedastic errors and endogenous regressors. One nice feature of the Moran I-type tests is that the exact form of the alternative model is not required. Our tests share many of these features except that endogenous regressors are not allowed. In cases that the null model is an OLS regression or spatial model with spherical errors (as are the cases for the Moran I-type tests), it can be shown that our LM-type tests have the same form under the SAR, MESS and SME (in error) alternatives. Or null model can be more general, e.g., SARAR$(1, 1)$ and MESS$(1, 1)$, corresponding to tests of spatial model reduction from a high-order one. The main focuses of our paper are to introduce the OPMD forms of the LM-type statistics that are NN- or NNUH-robust, and to introduce their bootstrap versions that achieve second-order size approximations. It would be highly interesting, as future research works, to extend our tests to allow for endogenous regressors, or to develop bootstrap versions of the Moran I type of tests given in Liu and Prucha (2016) that may have better size property when the sample size n is not so large, e.g., 50 and 100 instead of 400 as in the paper.

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Table 1 Empirical rejection rates of LM and BLM tests of $H_0^{\mathtt{a}}$: $\lambda = 0$ and $\rho = 0$ in $\mathtt{SARAR}(2, 2)$

		Homoscedastic Errors						Heteroskedastic Errors					
		Normal Errors			Lognormal			Normal Errors			Lognormal		
	λ_1	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
$\mathrm{LM}_{\mathtt{SARAR}}^{(a)}$	0.2	.487	.392	.248	.476	.390	.220	.432	.342	.189	.455	.356	.211
	0.1	.197	.131	.058	.169	.108	.042	.155	.093	.036	.159	.101	.036
	0.0	.082	.042	.005	.059	.024	.005	.059	.029	.007	.058	.026	.007
	-0.1	.083	.035	.007	.070	.022	.003	.046	.019	.001	.051	.014	.002
	-0.2	.151	.055	.008	.162	.060	.005	.102	.041	.002	.128	.056	.005
$\mathrm{LM}_{\mathtt{SARAR}}^{(a)*}$	0.2	.523	.430	.269	.539	.438	.264	.459	.360	.198	.496	.392	.216
	0.1	.229	.153	.064	.219	.141	.050	.172	.109	.041	.183	.124	.041
	0.0	.103	.051	.009	.089	.041	.008	.075	.037	.009	.079	.033	.008
	-0.1	.109	.054	.010	.104	.045	.006	.059	.024	.001	.072	.025	.002
	-0.2	.197	.085	.014	.247	.108	.013	.132	.059	.003	.179	.081	.006
$\mathrm{LMNH}_{\mathtt{SARAR}}^{(a)}$	0.2	.371	.207	.038	.438	.258	.054	.364	.215	.038	.437	.272	.059
	0.1	.222	.114	.020	.200	.102	.018	.210	.106	.015	.203	.102	.018
	0.0	.210	.099	.017	.155	.075	.014	.204	.101	.015	.147	.069	.007
	-0.1	.320	.175	.031	.224	.098	.011	.290	.146	.028	.216	.115	.018
	-0.2	.452	.264	.046	.359	.193	.030	.426	.240	.040	.369	.199	.035
$\mathrm{LMNH}_{\mathtt{SARAR}}^{(a)*}$	0.2	.198	.110	.028	.361	.253	.102	.216	.123	.037	.368	.260	.112
	0.1	.112	.057	.013	.146	.086	.028	.103	.058	.011	.150	.086	.028
	0.0	.096	.052	.010	.097	.056	.015	.102	.055	.016	.101	.053	.011
	-0.1	.169	.090	.022	.149	.073	.017	.145	.081	.024	.158	.084	.028
	-0.2	.260	.136	.035	.264	.163	.046	.237	.139	.033	.283	.167	.056
Upper Panel: $n = 50$; Lower Panel: $n = 200$													
$\mathrm{LM}_{\mathtt{SARAR}}^{(a)}$	0.2	.984	.964	.925	.988	.979	.930	.972	.946	.872	.980	.963	.903
	0.1	.512	.405	.219	.471	.367	.221	.463	.351	.171	.449	.331	.181
	0.0	.098	.043	.013	.077	.038	.009	.081	.038	.008	.065	.035	.007
	-0.1	.421	.262	.076	.430	.253	.080	.333	.184	.047	.330	.182	.050
	-0.2	.935	.860	.586	.934	.861	.605	.870	.760	.461	.881	.781	.480
$\mathrm{LM}_{\mathtt{SARAR}}^{(a)*}$	0.2	.983	.968	.927	.988	.982	.921	.975	.948	.881	.983	.970	.898
	0.1	.518	.417	.231	.504	.383	.215	.473	.368	.179	.481	.355	.176
	0.0	.103	.053	.012	.090	.044	.010	.086	.044	.009	.078	.038	.007
	-0.1	.430	.275	.084	.466	.275	.073	.337	.200	.051	.380	.213	.048
	-0.2	.937	.870	.602	.943	.869	.585	.877	.772	.482	.903	.804	.460
$\mathrm{LMNH}_{\mathtt{SARAR}}^{(a)}$	0.2	.970	.940	.821	.984	.961	.879	.967	.932	.790	.981	.961	.855
	0.1	.466	.317	.111	.517	.358	.135	.430	.292	.110	.492	.337	.122
	0.0	.160	.085	.021	.133	.062	.009	.152	.083	.020	.141	.070	.011
	-0.1	.608	.462	.187	.595	.427	.163	.586	.421	.176	.582	.432	.157
	-0.2	.977	.943	.775	.925	.843	.596	.963	.919	.723	.916	.838	.585
$\mathrm{LMNH}_{\mathtt{SARAR}}^{(a)*}$	0.2	.947	.909	.754	.981	.958	.893	.951	.900	.731	.979	.956	.873
	0.1	.363	.234	.077	.482	.359	.160	.340	.222	.074	.468	.334	.153
	0.0	.105	.050	.012	.105	.054	.009	.104	.055	.011	.119	.063	.015
	-0.1	.515	.355	.125	.557	.415	.173	.476	.331	.128	.563	.422	.189
	-0.2	.954	.897	.665	.918	.842	.632	.935	.870	.630	.915	.842	.628

Note: $\lambda_1 = \lambda_2 = \rho_1 = \rho_2$; Heteroskedasticity $\propto$ group size.

Table 2 Empirical rejection rates of LM and BLM tests of H_0^{a}: $\lambda = 0$ and $\rho = 0$ in MESS(2, 2)

		Homoscedasticity						Heteroskedasticity					
		Normal Errors			Normal Mixture			Normal Errors			Normal Mixture		
	λ_1	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
$\mathrm{LM}^{(a)}_{\mathtt{MESS}}$	0.2	.199	.095	.011	.185	.081	.013	.187	.090	.010	.184	.080	.013
	0.1	.097	.039	.004	.091	.037	.004	.090	.039	.003	.086	.033	.005
	0.0	.077	.033	.004	.075	.038	.009	.070	.029	.003	.073	.036	.009
	-0.1	.190	.108	.040	.173	.106	.028	.167	.098	.035	.167	.097	.034
	-0.2	.419	.321	.176	.427	.319	.161	.410	.312	.168	.421	.317	.156
$\mathrm{LM}^{(a)*}_{\mathtt{MESS}}$	0.2	.237	.132	.020	.231	.116	.021	.221	.122	.019	.226	.115	.019
	0.1	.126	.056	.005	.118	.060	.008	.117	.053	.005	.112	.049	.009
	0.0	.096	.045	.007	.096	.050	.013	.091	.039	.004	.095	.048	.012
	-0.1	.210	.137	.049	.206	.133	.042	.192	.118	.043	.196	.115	.043
	-0.2	.446	.349	.198	.479	.360	.192	.440	.350	.193	.460	.365	.190
$\mathrm{LMNH}^{(a)}_{\mathtt{MESS}}$	0.2	.479	.280	.056	.396	.217	.034	.482	.284	.050	.375	.210	.024
	0.1	.327	.180	.025	.223	.109	.014	.327	.171	.027	.258	.120	.015
	0.0	.212	.095	.014	.158	.072	.007	.206	.100	.012	.163	.078	.010
	-0.1	.226	.118	.019	.195	.089	.013	.217	.119	.021	.186	.087	.009
	-0.2	.339	.190	.033	.365	.214	.029	.340	.197	.034	.374	.210	.038
$\mathrm{LMNH}^{(a)*}_{\mathtt{MESS}}$	0.2	.260	.148	.038	.270	.160	.046	.261	.150	.039	.244	.139	.038
	0.1	.169	.085	.024	.133	.072	.018	.161	.085	.022	.150	.078	.018
	0.0	.090	.045	.008	.088	.045	.010	.093	.045	.009	.101	.046	.012
	-0.1	.106	.056	.015	.117	.064	.020	.107	.060	.015	.112	.061	.012
	-0.2	.178	.098	.029	.262	.165	.054	.187	.106	.025	.262	.166	.056
Upper Panel: $n = 50$; Lower Panel: $n = 200$													
$\mathrm{LM}^{(a)}_{\mathtt{MESS}}$	0.2	.955	.877	.634	.952	.888	.654	.928	.840	.538	.939	.849	.572
	0.1	.448	.299	.091	.421	.276	.091	.371	.224	.052	.342	.197	.047
	0.0	.106	.050	.009	.085	.043	.011	.064	.028	.004	.057	.024	.008
	-0.1	.499	.389	.213	.507	.397	.234	.451	.344	.187	.444	.334	.158
	-0.2	.978	.958	.892	.968	.949	.880	.967	.940	.857	.962	.936	.853
$\mathrm{LM}^{(a)*}_{\mathtt{MESS}}$	0.2	.955	.883	.640	.957	.890	.581	.927	.848	.557	.948	.849	.481
	0.1	.456	.315	.095	.447	.275	.071	.384	.234	.061	.360	.201	.029
	0.0	.110	.055	.013	.095	.045	.008	.065	.031	.003	.061	.026	.006
	-0.1	.505	.400	.217	.522	.400	.207	.459	.352	.197	.457	.341	.143
	-0.2	.979	.958	.891	.968	.947	.847	.968	.939	.862	.965	.938	.819
$\mathrm{LMNH}^{(a)}_{\mathtt{MESS}}$	0.2	.979	.952	.768	.977	.928	.729	.981	.949	.803	.977	.930	.737
	0.1	.611	.467	.189	.610	.436	.178	.635	.489	.225	.615	.447	.177
	0.0	.164	.087	.021	.132	.064	.009	.163	.086	.019	.135	.069	.012
	-0.1	.425	.281	.096	.475	.332	.117	.438	.314	.120	.479	.333	.129
	-0.2	.950	.898	.742	.957	.918	.757	.953	.906	.733	.953	.915	.751
$\mathrm{LMNH}^{(a)*}_{\mathtt{MESS}}$	0.2	.964	.909	.671	.960	.911	.718	.963	.906	.692	.961	.914	.725
	0.1	.511	.354	.130	.539	.388	.167	.523	.379	.146	.544	.393	.159
	0.0	.106	.049	.010	.096	.046	.007	.104	.051	.010	.105	.055	.012
	-0.1	.327	.198	.061	.413	.287	.118	.342	.223	.071	.411	.292	.115
	-0.2	.920	.860	.646	.944	.901	.756	.919	.845	.632	.939	.898	.739

Note: $\lambda_1 = \lambda_2 = \rho_1 = \rho_2$; Heteroskedasticity $\propto$ group size.