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More on the mis-specification of the shape parameter with Weibull-to-exponential transformation

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Summary

When lifetimes follow Weibull distribution with known shape parameter, a simple power transformation could be used to transform the data to the case of exponential distribution, which is much easier to analyze. Usually, the shape parameter cannot be known exactly and it is important to investigate the effect of mis-specification of this parameter. In a recent article, it was suggested that the Weibull-to-exponential transformation approach should not be used as the confidence interval for the scale parameter has very poor statistical property. However, it would be of interest to study the use of Weibull-to-exponential transformation when the mean time to failure or reliability is to be estimated which is a more common question. In this paper, the effect of mis-specification of Weibull shape parameters on these quantities is investigated. For reliability related quantities such as mean time to failure, percentile lifetime and mission reliability, the Weibull-to-exponential transformation approach is generally acceptable. For the cases when the data is highly censored or when small tail probability is concerned, further studies are needed, but these are known to be difficult statistical problems for which there are no standard solutions.

Keywords. Weibull to exponential transformation, Weibull distribution, shape parameter, mis-specification, confidence interval, coverage probability, mean time to failure.

1. Introduction

In a recent paper by Keats *et al.* (2000), the effect of mis-spefication of the Weibull shape parameter is studied for the case of Weibull-to-exponential transformation. This is a commonly used method when the shape parameter of certain component or system can be obtained from material property or other sources. By using a power-transformation, Weibull random variables could be transformed to exponential ones for which simple and analytical methods for statistical analysis such as hypothesis test and confidence interval are available. However, it is concluded in Keats *et al.* (2000) that the effect could be very large even for a small mis-specification of the Weibull shape parameter. Simulation studies are carried out to support this.

However, it is noticed that the paper specifically deals with the confidence bounds for the scale parameter. In fact, when we "mis-specify" the shape parameter to be a value that is different from the true one and use Weibull-to-exponential transformation, it is expected that the scale parameter will not be estimated correctly and the confidence interval for this inaccurate scale parameter will be wrong. This is because it is known that the estimators of the Weibull shape and scale parameters are correlated.

On the other hand, we are usually interested in the test of whether the mean time to failure will meet certain target value and what the confidence interval of this is. It is usually difficult to carry out hypothesis test in general Weibull case, but when transformed to exponential, such a test is much more straightforward. In this case, when Weibull-to-exponential transformation is used, it could be expected that the test or confidence bound is not too bad as it is based on the total time on test calculation. Similarly, for reliability computed at fixed time or for percentile lifetime, which depends on both the shape and scale parameter, it would be of interest to investigate the effect of mis-specification.

In this paper, the effect of mis-specification of the Weibull shape parameter on the confidence bound for the MTTF based on Weibull-to-exponential transformation is investigated. In fact, for reliability related quantities such as MTTF, percentile lifetime and mission reliability, the Weibull-to-exponential transformation approach is generally

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acceptable, except for cases when the data is highly censored or when small tail probability is to be studied. In these cases, further research is needed. They are known to be difficult statistical problem and there are no simple solutions. In addition to similar studies in Keats *et al*. (2000) which only include the case of β>1, the case of β<1 is also investigated. Furthermore, results for the case of highly censored data are also presented in this paper.

2. The Weibull-to-Exponential Transformation

A random variable T is said to follow a Weibull distribution, denoted by $T \sim \text{WEIB}(\alpha, \beta)$, if its probability density function has the form

$$f(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right],\ t > 0, \tag{1}$$

where α is the scale parameter and β is the shape parameter. It is well-known that T^{β} follows an exponential distribution with parameter α^{β} and such a transformation is called Weibull-to-exponential transformation. This transformation is attractive mainly because the statistical methods for the exponential distribution are simple and well developed (Balakrishnan and Basu, 1995).

Consider first a Type II censored life testing experiment where the test terminates at the rth failure. Let $t_1, t_2, \ldots, t_r$ be the times at which the first r failures occur and n be the total number of items put on test. Then

$$T(\beta) = \sum_{i=1}^{r} t_i^{\beta} + (n-r)\ t_r^{\beta} \tag{2}$$

is the total test time on the transformed scale. From Theorem 3.1.1 of Lawless (1982), we have that

$$2T(\beta)/\alpha^{\beta} \sim \chi^2_{2r}. \tag{3}$$

Hence, if the Weibull shape parameter is known, many statistical inference procedures can be easily developed by an inverse transformation, such as confidence interval for the scale parameter α, confidence interval for the mean time to failure (MTTF), confidence bounds for mission reliability and percentile limits, etc.

Often in practice, the life test is carried out within a fixed period of time, called Type I censored life test. The total test time now becomes

$$T(\beta) = \sum_{i=1}^{r} t_i^{\beta} + (n-r)t_0^{\beta}\,. \qquad (4)$$

where t_0 is the censoring time (or stopping time of the experiment). In this case, the distributional property (3) no longer holds. However, Nelson (1982, p249) pointed out that the Type I censored data are often analyzed as if they were the Type II censored data, which is often satisfactory in practice. Due to this reason, we will concentrate on the Type II censored case in this article.

3. Confidence interval for the MTTF

Usually the aim of life testing is to determine the MTTF or to demonstrate a certain MTTF is achieved with high confidence. The MTTF for the Weibull distribution with parameter (α,β) is

$$\mu = \alpha\Gamma\left(1 + 1/\beta\right). \qquad (5)$$

When β is known, a 100(1$-\delta$)% confidence interval for α is given by

$$\left\{\left(2T(\beta)\Big/\chi^2_{\delta/2,2r}\right)^{1/\beta},\ \ \left(2T(\beta)\Big/\chi^2_{1-\delta/2,2r}\right)^{1/\beta}\right\} \qquad (6)$$

Based on this, a 100(1$-\delta$)% confidence interval for the MTTF μ can be obtained as

$$\left\{ \Gamma\left(1+\frac{1}{\beta}\right)\left(\frac{2T(\beta)}{\chi^2_{\delta/2,2r}}\right)^{1/\beta}, \ \Gamma\left(1+\frac{1}{\beta}\right)\left(\frac{2T(\beta)}{\chi^2_{1-\delta/2,2r}}\right)^{1/\beta} \right\} \tag{7}$$

In practice, one does not know the exact value for β. However, a value could be obtained from historical data, from engineering theory or from experience (Nelson, 1985). Once the β is specified as $\tilde{\beta}$, say, one may replace β in the lower and upper confidence limits in (7) by $\tilde{\beta}$ and calculate the confidence intervals with (7). It is clear that this could be a 'mis-specification' and there must be some effect of this mis-specification on the confidence intervals.

If the confidence interval (7) is denoted by $\{L_\mu(\beta), U_\mu(\beta)\}$, the confidence interval for μ under a mis-specified shape parameter $\tilde{\beta}$ is then $\{L_\mu(\tilde{\beta}), U_\mu(\tilde{\beta})\}$. Studying the effect of mis-specification of β is to quantify the difference between the intervals $\{L_\mu(\beta), U_\mu(\beta)\}$ and $\{L_\mu(\tilde{\beta}), U_\mu(\tilde{\beta})\}$ when $\tilde{\beta}$ is different from the true value β, in terms of the coverage probability. Under Type II censoring, the interval $\{L_\mu(\beta), U_\mu(\beta)\}$ has an exact coverage of $1-\delta$, hence how close the coverage of $\{L_\mu(\tilde{\beta}), U_\mu(\tilde{\beta})\}$ to $1-\delta$ is the question of major concern.

Keats *et al.* (2000) studied such effect on the confidence interval for α (6) and concluded that the effect is large even with a small amount of misspecification in β, 0.10 say. We now examine this effect on the confidence interval (7) for the MTTF based on Monte Carlo simulation.

In this paper, the Weibull random variates are generated using an IMSL subroutine RNWIB. They are sorted in ascending order. The first *r* smallest random numbers are taken as a sample of the Type II censored Weibull data. Each simulated coverage probability is based on 10,000 runs. All the simulations are performed using Fortran 90 on a mainframe computer.

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Two sets of parameter values are considered for generating the Weibull random samples: (α, β) = (1000, 2) and (1000, 0.5), which gives $\mu = \alpha\,\Gamma(1+1/\beta)$ = 886.23 and 2000, respectively. Note that β=2 corresponds to increasing failure rate while β=0.5 corresponds to decreasing failure rate. For each parameter setting, 11 different values are considered for $\tilde{\beta}$, including the true value β = 2 or 0.5. The sample size used is n = 20 with r = 5, 10 and 20. Simulated probabilities for the interval $\{L_\mu(\tilde{\beta}),\ U_\mu(\tilde{\beta})\}$ to contain the true MTTF are summarized in Table 1 for various combinations of values of r, $\tilde{\beta}$, β and 100(1−δ)%.

Table 1. A summary of simulation results for MTTF= MTTF$_0$.

	90% CI			95% CI			99% CI		
$\tilde{\beta}$	r=5	10	20	5	10	20	5	10	20
	β = 2 (MTTF$_0$=886.23)								
1.5	0.8205	0.9049	0.9645	0.8962	0.9620	0.9876	0.9763	0.9949	0.9989
1.6	0.8597	0.9156	0.9561	0.9301	0.9619	0.9842	0.9876	0.9955	0.9979
1.7	0.8887	0.9172	0.9434	0.9421	0.9672	0.9802	0.9876	0.9943	0.9969
1.8	0.8981	0.9155	0.9285	0.9545	0.9616	0.9717	0.9902	0.9928	0.9951
1.9	0.9102	0.9126	0.9154	0.9524	0.9572	0.9635	0.9907	0.9921	0.9941
2.0	**0.8960**	**0.9028**	**0.9007**	**0.9498**	**0.9506**	**0.9492**	**0.9874**	**0.9894**	**0.9905**
2.1	0.8825	0.8861	0.8810	0.9422	0.9384	0.9344	0.9863	0.9857	0.9848
2.2	0.8669	0.8620	0.8547	0.9336	0.9284	0.9176	0.9811	0.9834	0.9786
2.3	0.8454	0.8496	0.8257	0.9062	0.9126	0.8949	0.9731	0.9732	0.9650
2.4	0.8200	0.8281	0.7953	0.8823	0.8970	0.8699	0.9661	0.9669	0.9538
2.5	0.7729	0.8029	0.7526	0.8608	0.8810	0.8304	0.9541	0.9568	0.9333
	β = 0.5 (MTTF=2000)								
.30	0.1233	0.1363	0.4665	0.2126	0.2409	0.6525	0.4472	0.5289	0.9009
.35	0.4209	0.4508	0.7583	0.5570	0.6027	0.8766	0.7881	0.8412	0.9750
.40	0.6981	0.7354	0.8802	0.8055	0.8385	0.9450	0.9421	0.9556	0.9914
.45	0.8620	0.8808	0.9165	0.9208	0.9332	0.9592	0.9837	0.9854	0.9944
.50	**0.9013**	**0.9006**	**0.9005**	**0.9486**	**0.9516**	**0.9529**	**0.9883**	**0.9904**	**0.9891**
.55	0.8487	0.8555	0.8627	0.9160	0.9115	0.9279	0.9788	0.9787	0.9819
.60	0.7437	0.7460	0.8196	0.8342	0.8395	0.8921	0.9457	0.9416	0.9673
.65	0.6202	0.6412	0.7772	0.7353	0.7411	0.8490	0.8844	0.8851	0.9496
.70	0.4881	0.5283	0.7427	0.6023	0.6250	0.8172	0.8063	0.8079	0.9255
.75	0.3681	0.4180	0.6978	0.4857	0.5200	0.7813	0.7068	0.7192	0.8931
.80	0.2743	0.3356	0.6635	0.3827	0.4304	0.7558	0.6011	0.6259	0.8702

From the simulation results, we see that when $\tilde{\beta} = \beta$, the simulated coverage probabilities are all very close to their nominal levels. When $\tilde{\beta} \neq \beta$, the coverage property depends on the value of *r*, the degree of censorship, and on whether $\tilde{\beta}$ is less or larger than β (under- or over- specification). For $\tilde{\beta} > \beta$, over specification of the shape parameter generally lowers the coverage of the interval. When $\tilde{\beta} < \beta$, the under specification may cause the interval coverage to increase, especially for $\beta > 1$. This is especially so when $\tilde{\beta}$ is near β and when data is not censored. This can be explained by the fact that when β is smaller, the variance is smaller, so with a large confidence interval to start with, the coverage probability could be increased when the mis-specification is not large.

When the MTTF of a mis-specified distribution is the same as the true MTTF, mis-specification is expected not to change the coverage probability very much. On the other hand, we should expect the power to be lower when MTTF becomes far away from the original MTTF. To study this, we adopt a similar parameter setting as in Keats *et al.* (2000). As the MTTF is a function of both α and β, two different sets could give the same MTTF. Since the concentration of this study is on the effect of mis-specification of the shape parameter, we will not consider the issue of mis-specifying α. Instead, this kind of Type II error will be investigated from the point of view that a wrong MTTF is used. We compute the probability that

$$\text{MTTF}_0 \in \{ L_\mu(\tilde{\beta}), U_\mu(\tilde{\beta}) \}$$

when the true MTTF is different from MTTF_0. Note that this is not really a hypothesis test as the test depends on the estimate parameter. This is presented here so that the results are comparable with that in Keats *et al.* (2000). The percentage of times that the interval does not contain MTTF_0 is of interest; it should be small for the transformation approach to be acceptable.

The results are shown Table 2 when MTTF is halved or doubled. It shows a low coverage probability when MTTF is not equal to 886.23 which corresponds to α=1000 for β=2, except when we have heavy censoring with a large under-specification.

In general, the simulation results suggest that the effect of mis-specification of the shape parameter on the confidence interval for the Weibull MTTF is not large, especially when the amount of mis-specification is not large and the data is not heavily censored. On the other hand, the mis-specification seems to have larger effect for β=0.5 compared with β=2. More simulations have been performed for some other parameter settings, and the results are consistent with the observations made above.

Table 2. Simulation of coverage probability when MTTF $\neq$ MTTF$_0$=886.23

	90% CI			95% CI			99% CI		
$\tilde{\beta}$	r=5	10	20	5	10	20	5	10	20
				MTTF=443.12 (α=500 for β=2)					
1.5	0.0321	0.0037	0.0008	0.0552	0.0089	0.0018	0.1273	0.0330	0.0065
1.6	0.0418	0.0052	0.0002	0.0678	0.0100	0.0012	0.1506	0.0315	0.0041
1.7	0.0535	0.0068	0.0001	0.0809	0.0103	0.0004	0.1722	0.0313	0.0014
1.8	0.0598	0.0058	0.0001	0.0907	0.0111	0.0000	0.1762	0.0339	0.0010
1.9	0.0722	0.0070	0.0000	0.1032	0.0116	0.0004	0.1975	0.0357	0.0001
2.0	**0.0838**	**0.0090**	**0.0000**	**0.1128**	**0.0116**	**0.0000**	**0.2124**	**0.0328**	**0.0002**
2.1	0.0918	0.0079	0.0000	0.1301	0.0129	0.0000	0.2224	0.0285	0.0003
2.2	0.1068	0.0079	0.0000	0.1406	0.0115	0.0000	0.2346	0.0286	0.0000
2.3	0.1099	0.0071	0.0000	0.1435	0.0115	0.0000	0.2353	0.0296	0.0000
2.4	0.1209	0.0090	0.0000	0.1611	0.0137	0.0000	0.2490	0.0289	0.0000
2.5	0.1261	0.0091	0.0000	0.1642	0.0113	0.0000	0.2605	0.0280	0.0000
				MTTF = 1772.46 (α=2000 for β=2)					
1.5	0.7165	0.1039	0.0001	0.8544	0.2471	0.0000	0.9758	0.6491	0.0018
1.6	0.5531	0.0526	0.0000	0.7331	0.1346	0.0000	0.9427	0.4696	0.0005
1.7	0.4024	0.0192	0.0000	0.5992	0.0651	0.0000	0.8803	0.3277	0.0002
1.8	0.2682	0.0097	0.0000	0.4612	0.0364	0.0000	0.7904	0.2071	0.0000
1.9	0.1750	0.0035	0.0000	0.3315	0.0161	0.0000	0.6906	0.1333	0.0002
2.0	0.1062	0.0017	0.0000	0.2257	0.0075	0.0000	0.5755	0.0758	0.0000
2.1	0.0618	0.0005	0.0000	0.1553	0.0037	0.0000	0.4508	0.0443	0.0001
2.2	0.0382	0.0004	0.0000	0.0997	0.0030	0.0000	0.3314	0.0239	0.0001
2.3	0.0213	0.0002	0.0000	0.0598	0.0006	0.0000	0.2452	0.0131	0.0001
2.4	0.0111	0.0002	0.0000	0.0330	0.0006	0.0000	0.1804	0.0076	0.0000
2.5	0.0070	0.0000	0.0000	0.0201	0.0000	0.0000	0.1190	0.0042	0.0001

4. Percentile Lifetime

In practice, we are often interested in the time that certain percentage of items will fail or the percentile lifetime, t_p. For the Weibull distribution, this can be found by solving

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$$F(t_p) = 1 - \exp\{-(t_p/\alpha)^{\beta}) = p, \tag{8}$$

which gives

$$t_p = \alpha[-\ln(1-p)]^{1/\beta}. \tag{9}$$

From (6), a 100(1−δ)% confidence interval for the percentile lifetime t_p for the case of β known is:

$$\left\{\left(-\frac{2T(\beta)}{\chi^2_{\delta/2,2r}}\ln(1-p)\right)^{1/\beta},\ \left(-\frac{2T(\beta)}{\chi^2_{1-\delta/2,2r}}\ln(1-p)\right)^{1/\beta}\right\}, \tag{10}$$

Denote this interval by $\{L_P(\beta), U_P(\beta)\}$. When β is unknown and replaced by $\tilde{\beta}$, the resulted confidence interval for t_p becomes $\{L_P(\tilde{\beta}), U_P(\tilde{\beta})\}$. The true probability that the wrong interval $\{L_P(\tilde{\beta}), U_P(\tilde{\beta})\}$ covers t_p is to be investigated and the discrepancy between this and (1−δ)% reflects the effect of mis-specification of β on the percentile lifetime (10).

Similar parameter settings as in the case of MTTF are used. Six t_p values, corresponding to p = .01, .05, .10, .25, .50 and .75, are considered. The first three are of interest as they are related to tail probability and the effect of misspecification could be large. The simulated coverage probabilities for the interval $\{L_P(\tilde{\beta}), U_P(\tilde{\beta})\}$, i.e., the proportion of $\{L_P(\tilde{\beta}), U_P(\tilde{\beta})\}$ containing the value t_p in 10,000 runs, are summarized in Table 3 for β = 2 and Table 4 for β = 0.5.

(Insert Table 3 and Table 4 near here)

Several observations can be made based on the simulation results. The effect of mis-specification of the Weibull shape parameter on the percentile limits depends mainly on the values of r and p. Generally speaking, the smaller the r is, the smaller the effect is or the more robust of the interval $\{L_P(\tilde{\beta}), U_P(\tilde{\beta})\}$ against misspecification. This is understandable as a larger *r* value means more information available and β could be estimated more precisely based on data. This latter argument tells that the same amount of misspecification could mean differently to the case of a small *r* and to the case of a larger *r*. Also, the effect seems larger at low percentile level as compared to the high percentile level. Simulation is also performed for the 99% confidence intervals and the results are consistent with the above observations.

5. Bounds on Reliability

Reliability at a given time point, or mission reliability, is another important quantity in practice. For the Weibull distribution, simple bounds for reliability do not seem to exist, especially for the case of censored data. However, if the shape parameter is known, one can have a very simple set of confidence bounds for the Weibull reliability. The reliability function of the Weibull distribution is

$$R(t) = \exp\left\{-(t/\alpha)^{\beta}\right\}. \qquad (11)$$

A 100(1−δ)% confidence interval, denoted by $\{L_R(\beta), U_R(\beta)\}$, for the case of β known can be obtained from (6):

$$\left\{\exp\left(-\frac{t^{\beta}\chi^2_{\delta/2,2r}}{2T(\beta)}\right),\ \exp\left(-\frac{t^{\beta}\chi^2_{1-\delta/2,2r}}{2T(\beta)}\right)\right\}. \qquad (12)$$

When β is unknown and replaced by its specified value $\tilde{\beta}$, the resulted bounds on reliability becomes $\{L_R(\tilde{\beta}), U_R(\tilde{\beta})\}$. Again, how well this interval can cover the true reliability under different specified values $\tilde{\beta}$ is of interest. The simulations are carried out in a very much similar setting as before and the simulated coverage probabilities are summarized in Table 5 for β = 2 and Table 6 for β = 0.5. Similar observation can be made as in the percentile case.

(Insert Table 5 and Table 6 near here)

6. Conclusions

In general, when Weibull-to-exponential transformation is to be used, it is important to understand the purpose of the analysis. As the estimators for Weibull shape and scale parameters are usually highly correlated, it has to be expected that when one is mis-specified, the other will also be wrongly estimated. Hence it is important not to simply specify one parameter and estimate the other with the transformation approach, as pointed out in Keats *et al.* (2000). On the other hand, for quantities such as the MTTF, percentile life and mission reliability that depend on both the scale and shape parameters, the joint effect could be small.

In fact, when the MTTF is to be estimated or tested, it is acceptable to use the Weibull-to-exponential transformation except for highly censored data. The transformation is also suitable for reliability estimation and percentile estimation except for the case of tail probability. The problem with censored data and tail probability are known to be difficult and hence warrant further study. On the other hand, even for cases when the confidence intervals for MTTF, percentile lifetime, and mission reliability are not so good, the effect seems to be much less than in Keats *et al.* (2000).

It should be pointed out that when actual data is used, estimates of the shape parameter are usually inaccurate. In fact, it could be highly biased and a bias of 10% is not uncommon. For example, when the mean rank is used with Weibull probability plot, Drapella and Kosznik (1999) reported an average biased of 15% for the shape parameter.

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Table 3. Simulation Results for Percentile Limits, n = 20, α = 1000, β = 2

$\tilde{\beta}$	r=5			r=10			r=20		
	p =.01	.05	.10	.01	.05	.10	.01	.05	.10
				90% CI					
1.5	0.5696	0.8921	0.9472	0.0263	0.4844	0.7493	0.0000	0.0031	0.0647
1.6	0.7131	0.9110	0.9512	0.1868	0.6518	0.8268	0.0001	0.0594	0.2477
1.7	0.8155	0.9204	0.9421	0.4664	0.7854	0.8750	0.0272	0.2910	0.5121
1.8	0.8819	0.9261	0.9337	0.7266	0.8608	0.9052	0.3027	0.6243	0.7375
1.9	0.9111	0.9166	0.9190	0.8688	0.9109	0.9148	0.7438	0.8418	0.8707
2.0	**0.9010**	**0.8979**	**0.9000**	**0.8984**	**0.8999**	**0.9025**	**0.8999**	**0.8972**	**0.9025**
2.1	0.8436	0.8706	0.8754	0.7908	0.8511	0.8555	0.6541	0.7674	0.8099
2.2	0.7522	0.8227	0.8479	0.5903	0.7516	0.7908	0.2700	0.5100	0.6194
2.3	0.6295	0.7714	0.8175	0.3717	0.6223	0.7158	0.0707	0.2652	0.4282
2.4	0.5087	0.7243	0.7835	0.2123	0.4972	0.6328	0.0132	0.1190	0.2496
2.5	0.4112	0.6668	0.7526	0.1111	0.3822	0.5529	0.0026	0.0463	0.1332
				95% CI					
1.5	0.7645	0.9535	0.9822	0.0916	0.6810	0.8724	0.0000	0.0173	0.1618
1.6	0.8526	0.9590	0.9777	0.3453	0.8043	0.9102	0.0010	0.1508	0.4057
1.7	0.9045	0.9621	0.9737	0.6392	0.8774	0.9349	0.0690	0.4572	0.6707
1.8	0.9401	0.9675	0.9731	0.8396	0.9323	0.9533	0.4640	0.7614	0.8461
1.9	0.9632	0.9633	0.9615	0.9376	0.9522	0.9588	0.8463	0.9122	0.9333
2.0	**0.9474**	**0.9508**	**0.9506**	**0.9483**	**0.9500**	**0.9489**	**0.9522**	**0.9499**	**0.9485**
2.1	0.9110	0.9288	0.9275	0.8627	0.9074	0.9184	0.7588	0.8457	0.8852
2.2	0.8267	0.8970	0.9115	0.6857	0.8316	0.8776	0.3578	0.6089	0.7217
2.3	0.7137	0.8530	0.8874	0.4643	0.7140	0.7996	0.1029	0.3481	0.5144
2.4	0.5992	0.7948	0.8663	0.2769	0.5982	0.7197	0.0237	0.1619	0.3210
2.5	0.4862	0.7485	0.8260	0.1469	0.4739	0.6370	0.0034	0.0671	0.1829
$\tilde{\beta}$	p =.25	.50	.75	.25	.50	.75	.25	.50	.75
				90% CI					
1.5	0.9706	0.8971	0.7695	0.9411	0.9637	0.8585	0.5213	0.8978	0.9690
1.6	0.9573	0.9118	0.8293	0.9404	0.9554	0.8880	0.6757	0.9092	0.9600
1.7	0.9496	0.9161	0.8618	0.9355	0.9386	0.9048	0.7851	0.9170	0.9491
1.8	0.9309	0.9113	0.8934	0.9332	0.9338	0.9084	0.8627	0.9243	0.9349
1.9	0.9189	0.9100	0.9029	0.9170	0.9144	0.9034	0.9026	0.9134	0.9181
2.0	**0.9023**	**0.8995**	**0.9008**	**0.9009**	**0.9012**	**0.8970**	**0.8961**	**0.9058**	**0.8915**
2.1	0.8860	0.8917	0.8866	0.8749	0.8799	0.8826	0.8505	0.8744	0.8786
2.2	0.8737	0.8693	0.8597	0.8496	0.8611	0.8622	0.7572	0.8313	0.8633
2.3	0.8448	0.8502	0.8315	0.8227	0.8574	0.8436	0.6393	0.7847	0.8469
2.4	0.8359	0.8259	0.7930	0.7874	0.8289	0.8080	0.5053	0.7378	0.8209
2.5	0.8129	0.8057	0.7534	0.7529	0.8107	0.7725	0.3995	0.6688	0.7987
				95% CI					
1.5	0.9892	0.9555	0.8652	0.9776	0.9866	0.9300	0.6929	0.9551	0.9902
1.6	0.9858	0.9595	0.9037	0.9763	0.9833	0.9488	0.8086	0.9604	0.9851
1.7	0.9761	0.9613	0.9276	0.9708	0.9775	0.9510	0.8866	0.9667	0.9826
1.8	0.9649	0.9595	0.9443	0.9699	0.9706	0.9584	0.9256	0.9623	0.9664
1.9	0.9584	0.9551	0.9508	0.9623	0.9578	0.9538	0.9557	0.9619	0.9602
2.0	**0.9489**	**0.9498**	**0.9500**	**0.9500**	**0.9497**	**0.9479**	**0.9560**	**0.9481**	**0.9445**
2.1	0.9443	0.9424	0.9395	0.9347	0.9401	0.9393	0.9160	0.9322	0.9336
2.2	0.9289	0.9278	0.9286	0.9125	0.9256	0.9244	0.8441	0.9021	0.9259
2.3	0.9159	0.9127	0.9031	0.8794	0.9085	0.9000	0.7397	0.8616	0.9109
2.4	0.9005	0.8973	0.8691	0.8572	0.9019	0.8783	0.6092	0.8113	0.8911
2.5	0.8867	0.8827	0.8255	0.8273	0.8852	0.8537	0.4868	0.7514	0.8739

Table 4. Simulation Results for Percentile Limits, n = 20, α = 1000, β = 0.5

$\tilde{\beta}$	r=5			r=10			r=20		
	p =.01	.05	.10	.01	.05	.10	.01	.05	.10
					90% CI				
.30	0.0439	0.7872	0.9504	0.0000	0.0171	0.3517	0.0000	0.0000	0.0000
.35	0.3928	0.8671	0.9519	0.0006	0.2844	0.6512	0.0000	0.0000	0.0061
.40	0.7213	0.9088	0.9512	0.1848	0.6535	0.8196	0.0002	0.0632	0.2503
.45	0.8885	0.9257	0.9363	0.7232	0.8646	0.9040	0.3091	0.6188	0.7444
.50	**0.8983**	**0.9028**	**0.8997**	**0.9029**	**0.8984**	**0.8993**	**0.9000**	**0.9010**	**0.9013**
.55	0.7486	0.8273	0.8541	0.5953	0.7475	0.7967	0.2746	0.5099	0.6241
.60	0.5096	0.7162	0.7875	0.2114	0.4933	0.6354	0.0137	0.1206	0.2373
.65	0.3158	0.6056	0.7125	0.0589	0.2861	0.4779	0.0003	0.0166	0.0637
.70	0.1789	0.5034	0.6490	0.0131	0.1545	0.3386	0.0000	0.0017	0.0177
.75	0.1077	0.4152	0.5955	0.0044	0.0793	0.2374	0.0000	0.0002	0.0033
.80	0.0676	0.3375	0.5457	0.0013	0.0463	0.1596	0.0000	0.0000	0.0010
					95% CI				
.30	0.1974	0.9209	0.9854	0.0000	0.0991	0.6085	0.0000	0.0000	0.0000
.35	0.6193	0.9416	0.9807	0.0059	0.5027	0.8123	0.0000	0.0001	0.0263
.40	0.8460	0.9602	0.9783	0.3564	0.8094	0.9107	0.0009	0.1508	0.4226
.45	0.9401	0.9655	0.9693	0.8442	0.9295	0.9539	0.4619	0.7570	0.8469
.50	**0.9484**	**0.9524**	**0.9536**	**0.9510**	**0.9507**	**0.9532**	**0.9482**	**0.9507**	**0.9480**
.55	0.8257	0.8910	0.9125	0.6850	0.8283	0.8703	0.3569	0.6056	0.7233
.60	0.5939	0.8042	0.8574	0.2749	0.5913	0.7195	0.0223	0.1688	0.3269
.65	0.3815	0.6977	0.8003	0.0824	0.3626	0.5596	0.0004	0.0273	0.0973
.70	0.2243	0.5875	0.7385	0.0210	0.1970	0.4111	0.0000	0.0032	0.0232
.75	0.1362	0.4918	0.6710	0.0035	0.1138	0.2863	0.0000	0.0005	0.0046
.80	0.0826	0.4021	0.6145	0.0015	0.0594	0.1952	0.0000	0.0000	0.0014
$\tilde{\beta}$	p =.25	.50	.75	.25	.50	.75	.25	.50	.75
					90% CI				
.30	0.9933	0.8358	0.4683	0.9388	0.9833	0.7133	0.0585	0.8398	0.9928
.35	0.9800	0.8856	0.6855	0.9427	0.9728	0.8332	0.3361	0.8786	0.9775
.40	0.9574	0.9115	0.8295	0.9432	0.9505	0.8865	0.6700	0.9082	0.9622
.45	0.9296	0.9080	0.8908	0.9352	0.9320	0.9060	0.8655	0.9218	0.9295
.50	**0.8982**	**0.9006**	**0.8933**	**0.8980**	**0.9038**	**0.8991**	**0.9022**	**0.8924**	**0.8976**
.55	0.8667	0.8719	0.8649	0.8556	0.8697	0.8637	0.7574	0.8327	0.8627
.60	0.8297	0.8281	0.7933	0.7784	0.8327	0.8041	0.5096	0.7368	0.8288
.65	0.8061	0.7743	0.7015	0.7222	0.7944	0.7365	0.2914	0.6144	0.7804
.70	0.7643	0.7196	0.5961	0.6408	0.7617	0.6707	0.1505	0.4934	0.7346
.75	0.7329	0.6664	0.5085	0.5776	0.7349	0.5934	0.0699	0.3701	0.6866
.80	0.7079	0.6064	0.4018	0.5222	0.6890	0.5283	0.0368	0.2974	0.6331
					95% CI				
.30	0.9985	0.9291	0.6393	0.9808	0.9970	0.8441	0.1768	0.9431	0.9980
.35	0.9951	0.9512	0.8060	0.9783	0.9929	0.9130	0.5402	0.9529	0.9930
.40	0.9829	0.9615	0.8989	0.9759	0.9816	0.9467	0.8104	0.9573	0.9858
.45	0.9702	0.9600	0.9447	0.9690	0.9675	0.9550	0.9247	0.9607	0.9672
.50	**0.9516**	**0.9499**	**0.9483**	**0.9479**	**0.9487**	**0.9466**	**0.9549**	**0.9506**	**0.9483**
.55	0.9303	0.9304	0.9214	0.9173	0.9267	0.9272	0.8364	0.9009	0.9252
.60	0.8999	0.8934	0.8721	0.8546	0.8987	0.8806	0.6021	0.8119	0.8953
.65	0.8706	0.8594	0.7914	0.7918	0.8723	0.8205	0.3705	0.6953	0.8573
.70	0.8457	0.8043	0.7101	0.7279	0.8360	0.7476	0.2014	0.5819	0.8125
.75	0.8161	0.7615	0.6101	0.6542	0.8167	0.6884	0.0992	0.4591	0.7749
.80	0.7785	0.7052	0.5193	0.5980	0.7876	0.6155	0.0492	0.3624	0.7147

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Table 5. Simulation results for Bounds on reliability, n = 20, α = 1000, β = 2

	r=5			r=10			r=20		
$\tilde{\beta}$	R(t)=.99	.95	.90	.99	.95	.90	.99	.95	.90
					90% CI				
1.5	0.5730	0.8897	0.9507	0.0301	0.4834	0.7468	0.0000	0.0031	0.0654
1.6	0.7217	0.9102	0.9435	0.1858	0.6521	0.8239	0.0000	0.0617	0.2476
1.7	0.8238	0.9178	0.9422	0.4689	0.7894	0.8737	0.0302	0.2901	0.5132
1.8	0.8778	0.9272	0.9338	0.7270	0.8644	0.9049	0.3071	0.6144	0.7390
1.9	0.9152	0.9252	0.9180	0.8695	0.8972	0.9153	0.7413	0.8447	0.8769
2.0	**0.9024**	**0.9041**	**0.9037**	**0.9014**	**0.8994**	**0.8999**	**0.8989**	**0.9010**	**0.8985**
2.1	0.8409	0.8686	0.8755	0.8046	0.8417	0.8636	0.6671	0.7661	0.8051
2.2	0.7437	0.8219	0.8497	0.5858	0.7432	0.7920	0.2749	0.5093	0.6188
2.3	0.6375	0.7786	0.8175	0.3780	0.6246	0.7190	0.0695	0.2767	0.4210
2.4	0.5099	0.7191	0.7861	0.2166	0.5038	0.6375	0.0127	0.1205	0.2527
2.5	0.3964	0.6621	0.7586	0.1124	0.3826	0.5554	0.0021	0.0474	0.1332
					95% CI				
1.5	0.7622	0.9569	0.9809	0.0937	0.6780	0.8730	0.0000	0.0134	0.1653
1.6	0.8540	0.9591	0.9795	0.3463	0.8065	0.9116	0.0007	0.1487	0.4222
1.7	0.9099	0.9662	0.9750	0.6474	0.8796	0.9376	0.0790	0.4588	0.6735
1.8	0.9426	0.9690	0.9682	0.8427	0.9257	0.9536	0.4619	0.7607	0.8499
1.9	0.9557	0.9623	0.9636	0.9318	0.9514	0.9563	0.8481	0.9086	0.9303
2.0	**0.9524**	**0.9458**	**0.9513**	**0.9499**	**0.9506**	**0.9511**	**0.9498**	**0.9527**	**0.9500**
2.1	0.9077	0.9274	0.9291	0.8678	0.9073	0.9254	0.7499	0.8540	0.8820
2.2	0.8277	0.8931	0.9157	0.6846	0.8269	0.8782	0.3629	0.6265	0.7265
2.3	0.7137	0.8522	0.8845	0.4667	0.7174	0.8094	0.1063	0.3597	0.5078
2.4	0.5976	0.8058	0.8570	0.2690	0.5925	0.7274	0.0196	0.1671	0.3246
2.5	0.4864	0.7519	0.8363	0.1516	0.4665	0.6363	0.0026	0.0718	0.1810
$\tilde{\beta}$	R(t)=.75	.50	.25	.75	.50	.25	.75	.50	.25
					90% CI				
1.5	0.9700	0.9025	0.7654	0.9434	0.9677	0.8587	0.5206	0.8991	0.9714
1.6	0.9588	0.9119	0.8205	0.9406	0.9540	0.8831	0.6698	0.9134	0.9574
1.7	0.9454	0.9124	0.8659	0.9388	0.9441	0.8970	0.7877	0.9203	0.9464
1.8	0.9316	0.9133	0.8955	0.9290	0.9309	0.9080	0.8658	0.9205	0.9291
1.9	0.9131	0.9114	0.9029	0.9221	0.9145	0.9087	0.9070	0.9129	0.9188
2.0	**0.9025**	**0.8976**	**0.8963**	**0.8993**	**0.8966**	**0.8974**	**0.9032**	**0.8981**	**0.8997**
2.1	0.8812	0.8875	0.8910	0.8711	0.8838	0.8869	0.8598	0.8769	0.8751
2.2	0.8663	0.8678	0.8599	0.8473	0.8710	0.8594	0.7484	0.8337	0.8676
2.3	0.8559	0.8514	0.8370	0.8213	0.8570	0.8414	0.6365	0.7841	0.8449
2.4	0.8342	0.8241	0.7972	0.7779	0.8320	0.8059	0.5113	0.7302	0.8232
2.5	0.8140	0.8007	0.7402	0.7487	0.8169	0.7742	0.3898	0.6760	0.8015
					95% CI				
1.5	0.9885	0.9588	0.8602	0.9765	0.9883	0.9317	0.6982	0.9558	0.9893
1.6	0.9856	0.9608	0.9007	0.9771	0.9837	0.9474	0.8056	0.9625	0.9849
1.7	0.9797	0.9560	0.9259	0.9778	0.9781	0.9558	0.8725	0.9633	0.9774
1.8	0.9715	0.9574	0.9428	0.9711	0.9682	0.9565	0.9312	0.9600	0.9722
1.9	0.9611	0.9532	0.9509	0.9600	0.9596	0.9542	0.9519	0.9603	0.9622
2.0	**0.9452**	**0.9489**	**0.9495**	**0.9479**	**0.9482**	**0.9510**	**0.9512**	**0.9522**	**0.9491**
2.1	0.9435	0.9419	0.9397	0.9304	0.9381	0.9388	0.9116	0.9287	0.9322
2.2	0.9251	0.9293	0.9238	0.9100	0.9314	0.9308	0.8415	0.8987	0.9242
2.3	0.9139	0.9142	0.9038	0.8851	0.9157	0.9017	0.7345	0.8648	0.9103
2.4	0.8987	0.8947	0.8739	0.8658	0.9000	0.8894	0.6094	0.8172	0.8874
2.5	0.8881	0.8780	0.8379	0.8322	0.8842	0.8550	0.4821	0.7658	0.8719

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Table 6. Simulation results for Bounds on reliability, n = 20, α = 1000, β = 0.5

	r=5			r=10			r=20		
$\tilde{\beta}$	R(t)=.99	.95	.90	.99	.95	.90	.99	.95	.90
				90% CI					
.30	0.0469	0.7996	0.9551	0.0000	0.0177	0.3568	0.0000	0.0000	0.0000
.35	0.3835	0.8666	0.9515	0.0005	0.2829	0.6362	0.0000	0.0000	0.0063
.40	0.7261	0.9106	0.9484	0.1809	0.6611	0.8229	0.0001	0.0643	0.2526
.45	0.8821	0.9263	0.9376	0.7162	0.8677	0.9005	0.3067	0.6157	0.7413
.50	**0.9017**	**0.8972**	**0.8999**	**0.8962**	**0.8993**	**0.9029**	**0.9014**	**0.9017**	**0.8967**
.55	0.7515	0.8269	0.8498	0.5882	0.7571	0.7968	0.2770	0.5207	0.6199
.60	0.5170	0.7185	0.7798	0.2171	0.4938	0.6380	0.0123	0.1154	0.2454
.65	0.3089	0.6140	0.7234	0.0567	0.2877	0.4708	0.0005	0.0159	0.0685
.70	0.1877	0.5069	0.6461	0.0136	0.1544	0.3344	0.0000	0.0015	0.0148
.75	0.1052	0.4186	0.5853	0.0035	0.0833	0.2336	0.0000	0.0001	0.0029
.80	0.0665	0.3399	0.5452	0.0009	0.0416	0.1642	0.0000	0.0000	0.0005
				95% CI					
.30	0.1948	0.9219	0.9844	0.0000	0.1003	0.5999	0.0000	0.0000	0.0000
.35	0.6237	0.9449	0.9838	0.0072	0.4983	0.8101	0.0000	0.0001	0.0293
.40	0.8469	0.9633	0.9789	0.3445	0.7993	0.9087	0.0009	0.1451	0.4164
.45	0.9439	0.9681	0.9684	0.8335	0.9324	0.9504	0.4645	0.7517	0.8435
.50	**0.9515**	**0.9456**	**0.9526**	**0.9491**	**0.9511**	**0.9557**	**0.9494**	**0.9525**	**0.9497**
.55	0.8250	0.8896	0.9089	0.6886	0.8277	0.8768	0.3644	0.6066	0.7160
.60	0.5968	0.7997	0.8630	0.2801	0.5907	0.7276	0.0245	0.1709	0.3200
.65	0.3841	0.6938	0.8008	0.0815	0.3648	0.5591	0.0005	0.0217	0.0918
.70	0.2318	0.5699	0.7273	0.0208	0.2057	0.3962	0.0001	0.0039	0.0223
.75	0.1350	0.4834	0.6769	0.0060	0.1107	0.2898	0.0000	0.0004	0.0061
.80	0.0854	0.4040	0.6084	0.0006	0.0577	0.2022	0.0000	0.0000	0.0011
$\tilde{\beta}$	R(t)=.75	.50	.25	.75	.50	.25	.75	.50	.25
				90% CI					
.30	0.9926	0.8379	0.4593	0.9436	0.9862	0.7085	0.0592	0.8510	0.9924
.35	0.9762	0.8845	0.6911	0.9424	0.9719	0.8306	0.3360	0.8819	0.9798
.40	0.9554	0.9061	0.8259	0.9400	0.9539	0.8857	0.6665	0.9116	0.9580
.45	0.9283	0.9161	0.8957	0.9303	0.9267	0.9037	0.8647	0.9165	0.9324
.50	**0.9017**	**0.8972**	**0.8999**	**0.8962**	**0.8993**	**0.9029**	**0.9014**	**0.9017**	**0.8967**
.55	0.8643	0.8723	0.8693	0.8514	0.8638	0.8663	0.7616	0.8361	0.8602
.60	0.8357	0.8227	0.7955	0.7837	0.8340	0.8043	0.5103	0.7342	0.8163
.65	0.7927	0.7779	0.7047	0.7128	0.7956	0.7371	0.2895	0.6106	0.7820
.70	0.7609	0.7175	0.5950	0.6457	0.7655	0.6620	0.1506	0.4904	0.7294
.75	0.7342	0.6592	0.5122	0.5779	0.7310	0.5938	0.0775	0.3810	0.6847
.80	0.6961	0.6095	0.3987	0.5169	0.7051	0.5165	0.0340	0.2995	0.6483
				95% CI					
.30	0.9978	0.9273	0.6309	0.9819	0.9961	0.8461	0.1803	0.9430	0.9986
.35	0.9938	0.9496	0.8138	0.9785	0.9922	0.9168	0.5407	0.9506	0.9944
.40	0.9862	0.9608	0.9092	0.9769	0.9815	0.9490	0.8119	0.9602	0.9841
.45	0.9727	0.9574	0.9413	0.9691	0.9690	0.9559	0.9288	0.9628	0.9685
.50	**0.9515**	**0.9456**	**0.9526**	**0.9491**	**0.9511**	**0.9557**	**0.9494**	**0.9525**	**0.9497**
.55	0.9233	0.9315	0.9247	0.9153	0.9285	0.9200	0.8389	0.9034	0.9238
.60	0.9043	0.8955	0.8781	0.8643	0.8971	0.8822	0.6121	0.8130	0.8901
.65	0.8754	0.8555	0.7948	0.8015	0.8690	0.8220	0.3625	0.7037	0.8561
.70	0.8463	0.8183	0.7113	0.7195	0.8410	0.7642	0.2017	0.5840	0.8206
.75	0.8197	0.7582	0.6085	0.6569	0.8115	0.6859	0.0963	0.4685	0.7644
.80	0.7845	0.7008	0.5142	0.5978	0.7794	0.6228	0.0495	0.3631	0.7265