# A Supplement to

# "QML Estimation of Dynamic Panel Data Models with Spatial Errors"

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**This supplemental material provides proofs for the technical lemmas in the above paper.**

**Proof of Lemma B.1.** For the proof of the first part, see footnote 20 in Kelejian and Prucha (1999). The second and third parts follow from Lemmas A.7 and A.8 in Lee (2002), respectively. ■

**Proof of Lemma B.2.** (1) Recall that $B = B_n = I_n - \lambda W_n$, $A = (B'B)^{-1}(W'B + B'W)(B'B)^{-1}$, and $\dot{A} = 2(B'B)^{-1}[(W'B + B'W)A - W'W]$. Noting that both $W$ and $B^{-1}$ are all uniformly bounded in both row and column sums under our assumptions, by Lemma B.1(1) $B'B$, $(B'B)^{-1}$, $A$, and $\dot{A}$ are uniformly bounded in both row and column sums. Observing that $J_T \otimes I_n$ has $T$ ones and $(n-1)T$ zeros on each row and each column and thus uniformly bounded in both row and column sums when $T$ is fixed and that $I_T \otimes (B'B)^{-1}$ is a block diagonal matrix uniformly bounded in both row and column sums, by (3.2) $\Omega = \phi_\mu(J_T \otimes I_n) + I_T \otimes (B'B)^{-1}$ is uniformly bounded in both row and column sums. Analogously, we can show that $\Omega^*$ and $\Omega^\dagger$ are uniformly bounded in both row and column sums.

(2) As we argue in the proof of Theorem 4.1, $\Omega = \Omega(\delta)$ has minimum eigenvalue bounded away from zero and maximum eigenvalue bounded away from infinity uniformly in $\delta \in \mathbf{\Delta}$. With this, we can readily show that the eigenvalues of $\Omega^{-1}, \Omega^{-1}(I_T \otimes A)\Omega^{-1}$,and $\Omega^{-1}(J_T \otimes I_n)\Omega^{-1}$ are also bounded away from zero and infinity and all these matrices have trace of order $O(n)$. By the Cauchy-Schwarz inequality, $\frac{1}{n}\text{tr}(D_1\Omega D_2) \leq \{\frac{1}{n}\text{tr}(D_1\Omega D_1')\}^{1/2}\{\frac{1}{n}\text{tr}(D_2'\Omega D_2)\}^{1/2}$. For each $D = \Omega^{-1}, \Omega^{-1}(I_T \otimes A)\Omega^{-1}$, or $\Omega^{-1}(J_T \otimes I_n)\Omega^{-1}$, we have $\frac{1}{n}\text{tr}(D\Omega D') \leq \lambda_{\max}(\Omega)\frac{1}{n}\text{tr}(DD) \leq \lambda_{\max}(\Omega)\lambda_{\max}(D)\frac{1}{n}\text{tr}(D) = O(1)$ as $D$ is symmetric here. In case $D = \Omega^{-1}(I_T \otimes \dot{A})$, we have

$$
\begin{aligned}
\frac{1}{n}\text{tr}(D\Omega D') &= \frac{1}{n}\text{tr}\left(\Omega^{-1}(I_T \otimes \dot{A})\Omega(I_T \otimes \dot{A}')\Omega^{-1}\right) \leq \lambda_{\max}(\Omega)\frac{1}{n}\text{tr}\left(\Omega^{-1}(I_T \otimes \dot{A}\dot{A}')\Omega^{-1}\right) \\
&\leq \lambda_{\max}(\Omega)\left[\lambda_{\min}(\Omega)\right]^{-2}\frac{T}{n}\text{tr}(\dot{A}\dot{A}') = O(1),
\end{aligned}
$$

and similarly $\frac{1}{n}\text{tr}(D'\Omega D) = O(1)$. Consequently, $\frac{1}{n}\text{tr}(D_1\Omega D_2) = O(1)$ for $D_1, D_2 = \Omega^{-1}, \Omega^{-1}(I_T \otimes A)\Omega^{-1}, \Omega^{-1}(J_T \otimes I_n)\Omega^{-1}$, and $\Omega^{-1}(I_T \otimes \dot{A})$. Noting that $\Omega^*$ and $\Omega^\dagger$ also have eigenvalues that are bounded away from zero and infinity, the same conclusion holds when $\Omega$ is replaced by $\Omega^*$ or $\Omega^\dagger$, and $D_1$ and $D_2$ are replaced by their analogs corresponding to the case of $\Omega^*$ or $\Omega^\dagger$.

(3) By the Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{1}{n}\mathrm{tr}(B'^{-1}RB^{-1}) &\leq \left\{\frac{1}{n}\mathrm{tr}(B'^{-1}RR'B^{-1})\right\}^{1/2}\left\{\frac{1}{n}\mathrm{tr}((B'B)^{-1})\right\}^{1/2} \\ &= \left\{\frac{1}{n}\mathrm{tr}(RR'(B'B)^{-1})\right\}^{1/2}\left\{\frac{1}{n}\mathrm{tr}((B'B)^{-1})\right\}^{1/2} \\ &\leq [\lambda_{\min}(B'B)]^{-1}\left\{\frac{1}{n}\mathrm{tr}(RR')\right\}^{1/2} = O(1),\end{aligned}$$

where we use the fact that $\|R\|^2 \leq \|R\|_1 \|R\|_\infty$, where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are maximum column and row sum matrix norms, respectively. ■

**Proof of Lemma B.3.** To show (1), write $E[(a'q_n a)(a'p_n a)] = E(\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n a_i a_j q_{n,ij} a_k a_l p_{n,kl})$. Noting that $E(a_i a_j a_k a_l)$ will not vanish only when $i = j = k = l$, $(i = j) \neq (k = l)$, $(i = k) \neq (j = l)$, and $(i = l) \neq (j = k)$, we have

$$\begin{aligned}E[(a'q_n a)(a'p_n a)] &= E(a_1^4)\sum_{i=1}^n q_{n,ii}p_{n,ii} + \sigma_a^4 \sum_{i=1}^n \sum_{j\neq i}^n (q_{n,ii}p_{n,jj} + q_{n,ij}p_{n,ij} + q_{n,ij}p_{n,ji}) \\ &= \kappa_a \sum_{i=1}^n q_{n,ii}p_{n,ii} + \sigma_a^4 \sum_{i=1}^n \sum_{j=1}^n (q_{n,ii}p_{n,jj} + q_{n,ij}p_{n,ij} + q_{n,ij}p_{n,ji}) \\ &= \kappa_a \sum_{i=1}^n q_{n,ii}p_{n,ii} + \sigma_a^4 [tr(q_n)tr(p_n) + tr(q_n(p_n + p_n'))].\end{aligned}$$

The result (2) follows from the independence between $a'q_n a$ and $b'p_n b$. For (3), $E[(a'q_n b)(a'p_n b)] = E(\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n a_i b_j q_{n,ij} a_k b_l p_{n,kl}) = E(\sum_{i=1}^n \sum_{j=1}^n a_i^2 b_j^2 q_{n,ij} p_{n,ij}) = \sigma_a^2 \sigma_b^2 tr(q_n p_n')$. ■

**Proof of Lemma B.4.** We only sketch the proof of (1) and (2) since it mainly follows from Lemma B.3 and the proof of other claims is similar. First, let $G_{q_n,3} \equiv (\iota_T' \otimes I_n) q_n (I_T \otimes B_0^{-1})$. Then by the independence of $\mu$ and $v$ and Lemma B.3, we have

$$\begin{aligned}E[(u'q_n u)(u'p_n u)] &= E(\mu' G_{q_n,1}\mu\mu' G_{p_n,1}\mu + v' G_{q_n,2} v v' G_{p_n,2} v + \mu' G_{q_n,1}\mu v' G_{p_n,2} v \\ &\quad + v' G_{q_n,2} v \mu' G_{p_n,1}\mu + 2\mu' G_{q_n,3} v \mu' G_{p_n,3} v + 2v' G_{q_n,3}' \mu v' G_{p_n,3}\mu) \\ &= \kappa_\mu \sum_{i=1}^n G_{q_n,1ii} G_{p_n,1ii} + \kappa_v \sum_{i=1}^{nT} G_{q_n,2ii} G_{p_n,2ii} \\ &\quad + \sigma_v^4 [\mathrm{tr}(q_n \Omega_0)\mathrm{tr}(p_n \Omega_0) + 2\mathrm{tr}(q_n \Omega_0 p_n \Omega_0)].\end{aligned}$$

Next, write $a = b + B_0^{-1} c$, where $b = \zeta + \mu(1-\rho_0^m)/(1-\rho_0)$ and $c = \sum_{j=0}^{m-1} \rho_0^j v_{-j}$. Then $b$ and $c$ are iid and mutually independent. It follows that

$$\begin{aligned}E[(u'q_n u)(a'r_n a)] &= E(\mu' G_{q_n,1}\mu b' r_n b + v' G_{q_n,2} v c' B_0'^{-1} r_n B_0^{-1} c + \mu' G_{q_n,1}\mu c' B_0'^{-1} r_n B_0^{-1} c \\ &\quad + v' G_{q_n,2} v b' r_n b) \\ &= \tfrac{\kappa_\mu (1-\rho_0^m)^2}{(1-\rho_0)^2} \sum_{i=1}^n G_{q_n,1ii} r_{n,ii} + \sigma_v^4 [\mathrm{tr}(r_n \omega_{11})\mathrm{tr}(q_n \Omega_0) + 2\mathrm{tr}(\omega_{12} q_n \omega_{21} p_n)].\end{aligned}$$

Similarly, we can prove the other claims. ■

**Proof of Lemma B.5.** Note that $Q_n$ is a linear-quadratic form of $b$ as in Theorem 1 of Kelejian and Prucha (2001). The difference is that the coefficient $a'P_{1n}$ of the linear term is random. The proof proceeds by modifying that of Theorem 1 in Kelejian and Prucha (2001) or Lemma A.13 of Lee (2002). ■

**Proof of Lemma B.6.** Note that $\tilde{X} = (X, Z, Y_{-1})$. By the strict exogeneity of $X$ and $Z$, we can readily show that both $X'\Omega_0^{-1}u$ and $Z'\Omega_0^{-1}u$ have expectations zero. We are left to show $E(Y_{-1}'\Omega_0^{-1}u) = 0$. By (B.3), $E(Y_{-1}'\Omega_0^{-1}u) = E\left[\mu'(l_{\rho_0}' \otimes I_n)\Omega_0^{-1}u\right] + E\left[v'A_v'\Omega_0^{-1}u\right]$. Using $u = (\iota_T \otimes I_n)\mu + (I_T \otimes B_0^{-1})$ and (3.29), we have

$$\begin{aligned} E\left[\mu'(l_{\rho_0}' \otimes I_n)\Omega_0^{-1}u\right] &= E[\mu'(l_{\rho_0}' \otimes I_n)\Omega_0^{-1}(\iota_T \otimes I_n)\mu] = \phi_{\mu 0}\sigma_{v0}^2 \text{tr}[\Omega_0^{-1}((\iota_T l_{\rho_0}') \otimes I_n)] \\ &= \phi_{\mu 0}\sigma_{v0}^2 \text{tr}\left\{(J_T \mathcal{J}_{\rho_0}) \otimes [(B_0'B_0)^{-1} + \phi_{\mu 0} T I_n]^{-1}\right\}, \end{aligned}$$

and

$$\begin{aligned} E\left[v'A_v'\Omega_0^{-1}u\right] &= E[v'A_v'\Omega_0^{-1}(I_T \otimes B_0^{-1})v] \\ &= \sigma_{v0}^2 \text{tr}[\Omega_0^{-1}(I_T \otimes B_0^{-1})(\mathcal{J}_{\rho_0} \otimes B_0'^{-1})] = \sigma_{v0}^2 \text{tr}[\Omega_0^{-1}(\mathcal{J}_{\rho_0} \otimes (B_0'B_0)^{-1})] \\ &= \sigma_{v0}^2 \text{tr}\left\{(T^{-1}J_T \mathcal{J}_{\rho_0}) \otimes [(B_0'B_0)^{-1} + \phi_{\mu 0} T I_n]^{-1}(B_0'B_0)^{-1}\right\} + \sigma_{v0}^2 \text{tr}\left[(\mathcal{J}_{\rho_0} - T^{-1}J_T \mathcal{J}_{\rho_0}) \otimes I_n\right], \end{aligned}$$

where we have used the fact that $E(vv'A_v') = \mathcal{J}_{\rho_0} \otimes B_0'^{-1}$. It follows that $E(Y_{-1}'\Omega_0^{-1}u) = \sigma_{v0}^2 \text{tr}(\mathcal{J}_{\rho_0} \otimes I_n) = \sigma_{v0}^2 \text{tr}(\mathcal{J}_{\rho_0})\text{tr}(I_n) = 0$. ■

**Proof of Lemma B.7.** By the expressions of the Hessian matrix $\frac{\partial \mathcal{L}^r(\psi_0)}{\partial \psi \partial \psi'}$ in Section 4.2, it suffices to prove (*i*) $n^{-1}[\tilde{X}'\Omega_0^{-1}\tilde{X} - E(\tilde{X}'\Omega_0^{-1}\tilde{X})] = o_p(1)$; (*ii*) $n^{-1}[\tilde{X}'Ru - E(\tilde{X}'Ru)] = o_p(1)$ for $R = \Omega_0^{-1}$ and $P_{\omega 0}$ with $\omega = \lambda$ and $\phi_\mu$; (*iii*) $n^{-1}[u'Ru - \sigma_{v0}^2 \text{tr}(R\Omega_0)] = o_p(1)$ for $R = \Omega_0^{-1}$ and $P_{\omega 0}$ with $\omega = \lambda$ and $\phi_\mu$; and (*iv*) $n^{-1}[q_{\omega\bar{\omega}}(u) - E(q_{\omega\bar{\omega}}(u))] = o_p(1)$ for $\omega, \bar{\omega} = \lambda$ and $\phi_\mu$.

Let $\Omega_{\omega\bar{\omega}0} = \frac{\partial^2}{\partial\omega\partial\bar{\omega}}\Omega(\delta_0)$ for $\omega, \bar{\omega} = \lambda$ and $\phi_\mu$. Noting that $\Omega_0$, $\Omega_{\omega 0}$, $P_{\omega 0}$, and $\Omega_{\omega\bar{\omega}0}$ with $\omega, \bar{\omega} = \lambda$ and $\phi_\mu$ are well behaved by Lemmas B.1-B.2 and Assumption G2 and $q_{\omega\bar{\omega}}(u)$ is quadratic in $u$, we can readily show that (*iii*)-(*iv*) hold by straightforward moment calculations, Chebyshev inequality, and Lemma B.4. For example, to show (*iii*), first note that $E(u'Ru) = \sigma_{v0}^2 \text{tr}(R\Omega_0)$. By Lemma B.4,

$$\begin{aligned} \text{Var}(n^{-1}u'Ru) &= n^{-2}\{E(u'Ruu'Ru) - [E(u'Ru)]^2\} \\ &= n^{-2}\kappa_\mu \sum_{i=1}^n G_{R,1ii}^2 + n^{-2}\kappa_v \sum_{i=1}^n G_{R,2ii}^2 + 2n^{-2}\sigma_{v0}^4 \text{tr}(R\Omega_0 R\Omega_0), \end{aligned}$$

where $G_{R,1}$ and $G_{R,2}$ are as defined in Lemma B.4. Using the fact that $R = \Omega_0^{-1}$ or $P_{\omega 0}$ is symmetric and positive definite, we can readily show that $\sum_{i=1}^n G_{R,jii}^2 = O(n)$ for $j = 1, 2$. In addition, $\text{tr}(R\Omega_0 R\Omega_0) \leq \lambda_{\max}(\Omega_0)\text{tr}(R\Omega_0 R) \leq [\lambda_{\max}(\Omega_0)]^2 \lambda_{\max}(R)\text{tr}(R) = O(n)$. It follows that $\text{Var}(n^{-1}u'Ru) = O(n^{-1})$. Then (*iii*) follows by Chebyshev inequality.

To prove (*i*), let $R = \Omega_0^{-1}$. Noticing that $\tilde{X} = (X, Z, Y_{-1})$, it is easy to show that the terms not involving $Y_{-1}$, such as $n^{-1}X'RX$, $n^{-1}X'RZ$, and $n^{-1}Z'RZ$ converge in probability to their expectations. For the terms involving $Y_{-1}$, we first have by (B.3),

$$\begin{aligned} n^{-1}Y_{-1}'RY_{-1} &= n^{-1}[A_x X'\beta_0 + (l_{\rho_0} \otimes I_n)z\gamma_0]'R[A_x X'\beta_0 + (l_{\rho_0} \otimes I_n)z\gamma_0] \\ &\quad + n^{-1}[(l_{\rho_0} \otimes I_n)\mu + A_v v]'R[(l_{\rho_0} \otimes I_n)\mu + A_v v] \\ &\quad + n^{-1}\mathbb{Y}_0'R\mathbb{Y}_0 + 2n^{-1}[A_x X'\beta_0 + (l_{\rho_0} \otimes I_n)z\gamma_0]'R[(l_{\rho_0} \otimes I_n)\mu + A_v v] \\ &\quad + 2n^{-1}[A_x X'\beta_0 + (l_{\rho_0} \otimes I_n)z\gamma_0]'R\mathbb{Y}_0 + 2n^{-1}[(l_{\rho_0} \otimes I_n)\mu + A_v v]'R\mathbb{Y}_0 \\ &\equiv \textstyle\sum_{i=1}^6 A_{ni}, \text{ say.} \end{aligned}$$

It suffices to show that each $A_{ni}$ $(i = 1, \cdots, 6)$ converges in probability to its expectations. Take $A_{n6}$ as an example. $E(A_{n6}) = 0$ because $\mathbb{Y}_0$ is kept fixed here. For the second moment,

$$\begin{aligned}\mathrm{Var}(A_{n6}) &= 4n^{-2}\{E[\mu'(l'_{\rho_0} \otimes I_n)R\mathbb{Y}_0\mathbb{Y}_0'R'(l_{\rho_0} \otimes I_n)\mu] + E(v'A_vR\mathbb{Y}_0\mathbb{Y}_0'R'A_vv)\} \\ &= 4n^{-2}\{\sigma^2_{\mu0}\mathrm{tr}[R\mathbb{Y}_0\mathbb{Y}_0'R'(l_{\rho_0}l'_{\rho_0} \otimes I_n)] + \sigma^2_{v0}\mathrm{tr}(A_vR\mathbb{Y}_0\mathbb{Y}_0'R'A_v)\} = O(n^{-1}),\end{aligned}$$

where the last equality follows from the fact that both matrices in the two trace operators are uniformly bounded in both row and column sums. Similarly, we can show that $n^{-1}X'RY_{-1}$ and $n^{-1}Z'RY_{-1}$ converge to their expectations in probability, and thus $(i)$ follows. Analogously, we can show $(ii)$. ■

**Proof of Lemma B.8.** The key step of the proof is to show that $\frac{1}{\sqrt{nT}}\tilde{X}'\Omega_0^{-1}u \xrightarrow{d} N(0, \Gamma_{r,11})$ where $\Gamma_{r,11} = \mathrm{plim}_{n\to\infty}(nT)^{-1}\tilde{X}'\ \Omega_0^{-1}\tilde{X}$. By Cramér-Wold device, it suffices to show that for any $c = (c_1', c_2', c_3)' \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}$ with $\|c\| = 1$, $(nT)^{-1/2}c'\tilde{X}'\Omega_0^{-1}u \xrightarrow{d} N(0, c'\Gamma_{r,11}c)$. Using (B.3) and $u = (\iota_T \otimes I_n)\mu + (I_T \otimes B_0^{-1})v$, we have $c'\tilde{X}'\Omega_0^{-1}u = c_1'X\Omega_0^{-1}u + c_2'Z\Omega_0^{-1}u + c_3Y_{-1}\Omega_0^{-1}u = \sum_{i=1}^3 T_{ni}$, where

$$\begin{aligned}T_{n1} &= [c_1'X + c_2'Z + c_3\beta_0'XA_x' + c_3\gamma_0'z(l'_{\rho_0} \otimes I_n) + c_3\mathbb{Y}_0']\Omega_0^{-1}(\iota_T \otimes I_n)\mu + c_3\mu'(l'_{\rho_0} \otimes I_n)\Omega_0^{-1}(\iota_T \otimes I_n)\mu, \\ T_{n2} &= [c_1'X + c_2'Z + c_3\beta_0'XA_x' + c_3\gamma_0'z(l'_{\rho_0} \otimes I_n) + c_3\mathbb{Y}_0']\Omega^{-1}(I_T \otimes B_0^{-1})v + c_3v'A_v'\Omega_0^{-1}(I_T \otimes B_0^{-1})v, \\ T_{n3} &= c_3\mu'[(l'_{\rho_0} \otimes I_n)\Omega_0^{-1}(I_T \otimes B_0^{-1}) + (\iota_T' \otimes I_n)\Omega_0^{-1}A_v]v.\end{aligned}$$

It is easy to verify that $E(T_{n3}) = 0$, $E(T_{n1}) = c_3\phi_{\mu0}\sigma^2_{v0}\mathrm{tr}[\Omega_0^{-1}(\iota_Tl'_{\rho_0} \otimes I_n)]$, and thus $E(T_{n2}) = -E(T_{n1})$ by Lemma B.6. Also, we can verify that $\mathrm{Cov}(T_{ni}, T_{nj}) = 0$ for $i \neq j$. It suffices to show that each $T_{ni}$ (after appropriately centered for $T_{n1}$ and $T_{n2}$) is asymptotically normal with mean zero.

Note that $T_{n1}$ and $T_{n2}$ are linear and quadratic functions of $\mu$ and $v$, respectively. For $T_{n3}$, it is a special case of Lemma B.5 since it can be regarded as a linear function of either $\mu$ or $v$, with $\mu$ and $v$ independent of each other. So we can apply Lemma B.5 to $T_{ni}$ to obtain

$$\{T_{ni} - E(T_{ni})\}/\sqrt{\mathrm{Var}(T_{ni})} \xrightarrow{d} N(0, 1) \text{ for } i = 1, 2, 3.$$

Now by the independence of $T_{n1}$ and $T_{n2}$, and the asymptotic independence of $T_{n3}$ with $T_{n1}$ and $T_{n2}$, we have

$$\frac{1}{\sqrt{nT}}c'\tilde{X}'\Omega_0^{-1}u = \frac{1}{\sqrt{nT}}\sum_{i=1}^{3}T_{ni} \xrightarrow{d} N(0, \lim_{n\to\infty}(nT)^{-1}\sum_{i=1}^{3}\mathrm{Var}(T_{ni})),$$

implying that $(nT)^{-1/2}\tilde{X}'\Omega_0^{-1}u \xrightarrow{d} N(0, \Gamma_{r,11})$ because we can readily show that $(nT)^{-1}[\tilde{X}'\Omega_0^{-1}\tilde{X} - \mathrm{Var}(\tilde{X}'\Omega_0^{-1}u)] = o_p(1)$.

Noticing that each component of $\partial\mathcal{L}^r(\psi_0)/\partial\psi$ can be written as linear and quadratic functions of $\mu$ or $v$, the rest of the proof proceeds by following the above steps closely. ■

**Proof of Lemma B.9.** Let $P_j \equiv \rho_0^jB_0^{-1}$. Then $\mathbb{V}_t = \sum_{j=0}^{t+m-1}P_jv_{t-j}$. Noting that $E(v_t'Dv_s) = \sigma^2_{v0}\mathrm{tr}(D)$ for any nonstochastic conformable matrix $D$ if $t = s$ and 0 otherwise, we have

$$\begin{aligned}E(\mathbb{V}_t'R_{ts}\mathbb{V}_s) &= \textstyle\sum_{i=0}^{t+m-1}\sum_{j=0}^{s+m-1}E(v'_{t-i}P_i'R_{ts}P_jv_{s-j}) = \sum_{i=\max(0,t-s)}^{t+m-1}E(v'_{t-i}P_i'R_{ts}P_{s-t+i}v_{t-i}) \\ &= \sigma^2_{v0}\mathrm{tr}(\textstyle\sum_{i=\max(0,t-s)}^{t+m-1}P_i'R_{ts}P_{s-t+i}) = \sigma^2_{v0}\mathrm{tr}(B_0'^{-1}R_{ts}B_0^{-1})\sum_{i=\max(0,t-s)}^{t+m-1}\rho_0^{s-t+2i}.\end{aligned}$$

Next, noting that $\mathbb{X}_t = \sum_{j=0}^{t+m-1} \rho_0^j x_{t-j}$, we have

$$E(\mathbb{X}_t' R_{ts} \mathbb{X}_s) = \sum_{j=0}^{s+m-1} \sum_{k=0}^{t+m-1} \rho_0^{j+k} E(x_{t-k}' R_{ts} x_{s-j}) = \mathrm{tr}(\sum_{j=0}^{s+m-1} \sum_{k=0}^{t+m-1} \rho_0^{j+k} R_{ts} E(x_{s-j} x_{t-k}')).$$

Lastly, $E(\mathbb{X}_t' R_{ts} \mathbb{V}_s) = \sum_{j=0}^{s+m-1} \sum_{k=0}^{t+m-1} \rho_0^{j+k} E(x_{t-k}' R_{ts} B_0^{-1} v_{s-j}) = 0$. ■

**Proof of Lemma B.10.** Let $R_1$ and $R_2$ be arbitrary $n \times n$ nonstochastic matrices. We can show that

$$E[(v_t' R_1 v_s)(v_g' R_2 v_h)] = \begin{cases} \kappa_v \sum_{i=1}^n R_{1,ii} R_{2,ii} + \sigma_{v0}^4 \{\mathrm{tr}(R_1)\mathrm{tr}(R_2) + \mathrm{tr}[R_1(R_2 + R_2')]\} & \text{if } t = s = g = h \\ \sigma_{v0}^4 \mathrm{tr}(R_1)\mathrm{tr}(R_2) & \text{if } t = s \neq g = h \\ \sigma_{v0}^4 \mathrm{tr}(R_1 R_2) & \text{if } t = g \neq s = h \\ \sigma_{v0}^4 \mathrm{tr}(R_1 R_2') & \text{if } t = h \neq s = g \\ 0 & \text{otherwise} \end{cases}.$$

Consequently,

$$\begin{aligned} & E(\mathbb{V}_t' R_{ts} \mathbb{V}_s \mathbb{V}_g' R_{gh} \mathbb{V}_h) \\ = \; & E(\sum_{i=0}^{t+m-1} \sum_{j=0}^{s+m-1} \sum_{k=0}^{g+m-1} \sum_{l=0}^{h+m-1} \rho_0^{i+j+k+l} v_{t-i}' B_0'^{-1} R_{ts} B_0^{-1} v_{s-j} v_{g-k}' B_0'^{-1} R_{gh} B_0^{-1} v_{h-l}) \\ = \; & \sum_{j=\max(0,t-s,t-g,t-h)}^{t+m-1} \rho_0^{(s+g+h-3t+4j)} \{\kappa_v \sum_{i=1}^n (B_0'^{-1} R_{ts} B_0^{-1})_{ii} (B_0'^{-1} R_{gh} B_0^{-1})_{ii} \\ & + \sigma_{v0}^4 [\mathrm{tr}(B_0'^{-1} R_{ts} B_0^{-1}) \mathrm{tr}(B_0'^{-1} R_{gh} B_0^{-1}) + 2\mathrm{tr}(B_0'^{-1} R_{ts} B_0^{-1} (B_0'^{-1} R_{gh} B_0^{-1} + B_0'^{-1} R_{gh}' B_0^{-1}))]\} \\ & + \sigma_{v0}^4 \sum_{i=\max(0,t-s)}^{t+m-1} \rho_0^{s-t+2i} \mathrm{tr}(B_0'^{-1} R_{ts} B_0^{-1}) \sum_{j=\max(0,g-h)}^{g+m-1} \rho_0^{h-g+2j} \mathrm{tr}(B_0'^{-1} R_{gh} B_0^{-1}) 1(j \neq i+g-t) \\ & + \sum_{i=\max(0,t-g)}^{t+m-1} \rho_0^{g-t+2i} \sum_{j=\max(0,s-h)}^{s+m-1} \rho_0^{h-s+2j} \sigma_{v0}^4 \mathrm{tr}(B_0'^{-1} R_{ts} (B_0' B_0)^{-1} R_{gh} B_0^{-1}) 1(j \neq i+s-t) \\ & + \sum_{i=\max(0,t-h)}^{t+m-1} \rho_0^{h-t+2i} \sum_{j=\max(0,s-g)}^{s+m-1} \rho_0^{g-s+2j} \sigma_{v0}^4 \mathrm{tr}(B_0'^{-1} R_{ts} (B_0' B_0)^{-1} R_{gh}' B_0^{-1}) 1(j \neq i+s-t). \end{aligned}$$

Then (1) follows by Lemma B.9. For (2), we have

$$\begin{aligned} \mathrm{Cov}(\mathbb{X}_t' R_{ts} \mathbb{V}_s, \mathbb{X}_g' R_{gh} \mathbb{V}_h) &= E(\mathbb{X}_t' R_{ts} \mathbb{V}_s (\mathbb{X}_t' R_{gh} \mathbb{V}_s)') \\ &= \sum_{i=0}^{t+m-1} \sum_{j=0}^{s+m-1} \sum_{k=0}^{g+m-1} \sum_{l=0}^{h+m-1} \rho_0^{i+j+k+l} E[x_{t-i}' R_{ts} B_0^{-1} v_{s-j} (x_{g-k}' R_{gh} B_0^{-1} v_{h-l})'] \\ &= \sigma_{v0}^2 \mathrm{tr} \left[ \sum_{i=0}^{t+m-1} \sum_{k=0}^{g+m-1} \sum_{j=\max(0,s-h)}^{s+m-1} \rho_0^{i+k+h-s+2j} R_{ts} (B_0' B_0)^{-1} R_{gh}' E(x_{g-k}' x_{t-i}) \right]. \end{aligned}$$

The expression for $\mathrm{Cov}(\mathbb{X}_t' R_{ts} \mathbb{X}_t, \mathbb{X}_g' R_{gh} \mathbb{X}_h)$ is quite complicated, but we can use Lemmas B.1-B.2 to show it is of order $O(n)$, which suffices for our purpose. ■

**Proof of Lemma B.11.** By Lemmas B.1, B.2, B.9, and B.10, we can show that $(nT)^{-1} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} E(\mathbb{V}_t' R_{ts} \mathbb{V}_s) = O(1)$, and $\mathrm{Var}(n^{-1} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} \mathbb{V}_t' R_{ts} \mathbb{V}_s) = n^{-2} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} \sum_{g=0}^{T-1} \sum_{h=0}^{T-1} \mathrm{Cov}(\mathbb{V}_t' R_{ts} \mathbb{V}_s, \mathbb{V}_g' R_{gh} \mathbb{V}_h) = O(n^{-1})$. Then (1) follows from Chebyshev inequality. For (2), we have $E[\frac{1}{nT} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} \mathbb{X}_t' R_{ts} \mathbb{V}_s] = 0$, and

$$\begin{aligned} \mathrm{Var}\left( n^{-1} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} \mathbb{X}_t' R_{ts} \mathbb{V}_s \right) &= n^{-2} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} \sum_{g=0}^{T-1} \sum_{h=0}^{T-1} \mathrm{Cov}(\mathbb{X}_t' R_{ts} \mathbb{V}_s, \mathbb{X}_g' R_{ts} \mathbb{V}_h) \\ &= n^{-2} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} \sum_{g=0}^{T-1} \sum_{h=0}^{T-1} \sigma_{v0}^2 \sum_{i=0}^{t+m-1} \sum_{k=0}^{g+m-1} \sum_{j=\max(0,s-h)}^{s+m-1} \mathrm{tr}[\rho_0^{i+k+h-s+2j} R_{ts} \\ & \quad \times (B_0' B_0)^{-1} R_{gh}' E(x_{g-k} x_{t-i}')] \\ &= O(n^{-1}), \end{aligned}$$

where the last equality follows because $(i)$ $x_{it}$ are independent across $i$ with second moments uniformly bounded in $i$, $(ii)$ $R_{ts}(B_0'B_0)^{-1}R_{gh}'$ are uniformly bounded in both row and column sums by Lemma B.1(1) and Lemma B.2(1), and $(iii)$ elements of $R_{ts}(B_0'B_0)^{-1}R_{gh}'E(x_{g-k}x_{t-i}')$ are uniformly bounded by the same lemmas. Hence the conclusion follows from Chebyshev inequality. (3) follows from Lemma B.10 and Chebyshev inequality. ■

**Proof of Lemma B.12.** Let $u^* = u^*(\theta_0, \rho_0)$ and $u_\rho^* = u_\rho^*(\theta_0, \rho_0) = \frac{\partial}{\partial \rho} u^*(\theta_0, \rho_0)$. Noting that $E\left(X^{*\prime} R u^*\right) = 0$ for any $n(T+1) \times n(T+1)$ nonstochastic matrix $R$ and $X_\rho^*$ is free of $\rho$, by the expressions of the Hessian matrix $\frac{\partial \mathcal{L}^{rr}(\psi_0)}{\partial \psi \partial \psi'}$ in Section 4.2, it suffices to prove

$(i)$ $n^{-1}\left[X^{*\prime}\Omega_0^{*-1}X^* - E\left(X^{*\prime}\Omega_0^{*-1}X^*\right)\right] = o_p(1)$;

$(ii)$ $n^{-1}X^{*\prime}Ru^* = o_p(1)$ for $R = \Omega_0^{*-1}$ and $P_{\omega 0}^*$ with $\omega = \rho$, $\lambda$, $\phi_\mu$, and $\phi_\zeta$;

$(iii)$ $n^{-1}\left[u^{*\prime}Ru^* - E\left(u^{*\prime}Ru^*\right)\right] = o_p(1)$ for $R = \Omega_0^{*-1}$ and $P_{\omega 0}^*$ with $\omega = \rho$, $\lambda$, $\phi_\mu$, and $\phi_\zeta$;

$(iv)$ $n^{-1}\left[X_\rho^{*\prime}\Omega_0^{*-1}u^* - E\left(X_\rho^{*\prime}\Omega_0^{*-1}u^*\right)\right] = o_p(1)$;

$(v)$ $n^{-1}\left[X^{*\prime}\Omega_0^{*-1}u_\rho^* - E\left(X^{*\prime}\Omega_0^{*-1}u_\rho^*\right)\right] = o_p(1)$;

$(vi)$ $n^{-1}\left[u_\rho^{*\prime}Ru^* - E\left(u_\rho^{*\prime}Ru^*\right)\right] = o_p(1)$ for $R = \Omega_0^{*-1}$ and $P_{\omega 0}^*$ with $\omega = \rho$, $\lambda$, $\phi_\mu$, and $\phi_\zeta$;

$(vii)$ $n^{-1}\left[u_{\rho\rho}^{*\prime}\Omega_0^{*-1}u^* - E\left(u_{\rho\rho}^{*\prime}\Omega_0^{*-1}u^*\right)\right] = o_p(1)$;

$(viii)$ $n^{-1}\left[u_\rho^{*\prime}\Omega_0^{*-1}u_\rho^* - E\left(u_\rho^{*\prime}\Omega_0^{*-1}u_\rho^*\right)\right] = o_p(1)$;

$(ix)$ $n^{-1}\left[q_{\omega\bar{\omega}}^*(u^*) - E\left(q_{\omega\bar{\omega}}^*(u^*)\right)\right] = o_p(1)$ for $\omega$, $\bar{\omega} = \rho$, $\lambda$, $\phi_\mu$, and $\phi_\zeta$.

Let $\Omega_{\omega\bar{\omega}0}^* = \frac{\partial^2}{\partial\omega\partial\bar{\omega}}\Omega^*(\delta_0)$ for $\rho$, $\lambda$, $\phi_\mu$, and $\phi_\zeta$. Noting that $\Omega_0^{*-1}$, $\Omega_{\omega 0}^*$, $P_{\omega 0}^*$ and $\Omega_{\omega\bar{\omega}0}^*$ with $\omega, \bar{\omega} = \rho$, $\lambda$, $\phi_\mu$, and $\phi_\zeta$ are uniformly bounded in both row and column sums and $q_{\omega\bar{\omega}}^*(u^*)$ is quadratic in $u^*$, we can readily show that $(i)$-$(iv)$ and $(ix)$ hold by straightforward moment calculations and Chebyshev inequality. Noting that $u_\rho^* = -\begin{pmatrix} \dot{a}_{m0} z\gamma_0 \\ Y_{-1} \end{pmatrix}$ and $u_{\rho\rho}^* = -\begin{pmatrix} \ddot{a}_{m0} z\gamma_0 \\ 0_{nT\times 1} \end{pmatrix}$ with $\dot{a}_{m0} = \frac{d}{d\rho}a_m(\rho_0)$ and $\ddot{a}_{m0} = \frac{d^2}{d\rho^2}a_m(\rho_0)$, we can readily prove $(v)$-$(vii)$ by Chebyshev inequality. In fact, $E\left(u_{\rho\rho}^{*\prime}\Omega_0^{*-1}u^*\right) = 0$ in $(vii)$.

We are left to prove $(viii)$. Write $\Omega_0^{*-1} = \begin{pmatrix} \omega_*^{11} & \omega_*^{12} \\ \omega_*^{12\prime} & \omega_*^{22} \end{pmatrix}$ where $\omega_*^{11}, \omega_*^{12}$, and $\omega_*^{22}$ are $n \times n$, $n \times nT$, and $nT \times nT$ matrices, respectively.

$$
\begin{aligned}
n^{-1}u_\rho^{*\prime}\Omega_0^{*-1}u_\rho^* &= n^{-1}\begin{pmatrix} \dot{a}_{m0} z\gamma_0 \\ Y_{-1} \end{pmatrix}'\begin{pmatrix} \omega_*^{11} & \omega_*^{12} \\ \omega_*^{12\prime} & \omega_*^{22} \end{pmatrix}\begin{pmatrix} \dot{a}_{m0} z\gamma_0 \\ Y_{-1} \end{pmatrix} \\
&= n^{-1}\left(\left(\dot{a}_{m0}\right)^2\gamma_0' z'\omega_*^{11}z\gamma_0 + 2\dot{a}_{m0}\gamma_0' z'\omega_*^{12}Y_{-1} + Y_{-1}'\omega_*^{22}Y_{-1}\right).
\end{aligned}
$$

To show the convergence of $n^{-1}u_\rho^{*\prime}\Omega_0^{*-1}u_\rho^*$ to its expectation, it suffices show each term in the last expression converges to its expectation. We only show $n^{-1}[Y_{-1}'\omega_*^{22}Y_{-1} - E\left(Y_{-1}'\omega_*^{22}Y_{-1}\right)] = o_p(1)$ since the proof that $n^{-1}[(\dot{a}_{m0})^2\gamma_0' z'\omega_*^{11}z\gamma_0 \; - E\left((\dot{a}_{m0})^2\gamma_0' z'\omega_*^{11}z\gamma_0\right)] = o_p(1)$ and that $n^{-1}[\dot{a}_{m0}\gamma_0' z'\omega_*^{12}Y_{-1} - \left(\dot{a}_{m0}\gamma_0' z'\omega_*^{12}Y_{-1}\right)] = o_p(1)$ is similar and simpler. By (B.2)

$$
\begin{aligned}
n^{-1}Y_{-1}'\omega_*^{22}Y_{-1} &= n^{-1}\left(\mathbb{X}_{(-1)}\beta_0 + \left(l_{\rho_0}\otimes I_n\right)z\gamma_0 + \left(l_{\rho_0}\otimes I_n\right)\mu + \mathbb{V}_{(-1)} + \mathbb{Y}_0\right)'\omega_*^{22} \\
&\quad \times \left(\mathbb{X}_{(-1)}\beta_0 + \left(l_{\rho_0}\otimes I_n\right)z\gamma_0 + \left(l_{\rho_0}\otimes I_n\right)\mu + \mathbb{V}_{(-1)} + \mathbb{Y}_0\right).
\end{aligned}
$$

After expressing out the right hand side of the last expression, it has 25 terms, most of which can easily be shown to converge to their respective expectations. The exceptions are terms involving $\mathbb{X}_{(-1)}$

and $\mathbb{V}_{(-1)}$, namely: $n^{-1}\beta_0'\mathbb{X}_{(-1)}'\omega_*^{22}\mathbb{X}_{(-1)}\beta_0$, $n^{-1}\beta_0'\mathbb{V}_{(-1)}'\omega_*^{22}\mathbb{V}_{(-1)}$, $n^{-1}\beta_0'\mathbb{X}_{(-1)}'\omega_*^{22}\mathbb{V}_{(-1)}$, $n^{-1}\beta_0'\mathbb{X}_{(-1)}'\omega_*^{22}(l_{\rho_0}\otimes I_n)\,z\gamma_0$, $n^{-1}\beta_0'\mathbb{X}_{(-1)}'\omega_*^{22}(l_{\rho_0}\otimes I_n)\,\mu$, $n^{-1}\mathbb{V}_{(-1)}'\omega_*^{22}(l_{\rho_0}\otimes I_n)\,z\gamma_0$, $n^{-1}\mathbb{V}_{(-1)}'\,\omega_*^{22}(l_{\rho_0}\otimes I_n)\,\mu$, $n^{-1}\beta_0'\mathbb{X}_{(-1)}'\omega_*^{22}\mathbb{Y}_0$, and $n^{-1}\mathbb{V}_{(-1)}'\omega_*^{22}\mathbb{Y}_0$. The first three terms converge in probability to their expectations by Lemma B.11. We can show the other terms converge in probability to their expectations by similar arguments to those used in proving Lemmas B.9-B.11. ■

**Proof of Lemma B.13.** By Cramér-Wold device, it suffices to show that for any $c=(c_1',c_2,c_3,c_4,c_5,c_6)'\in\mathbb{R}^{p+q+k}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ with $\|c\|=1$, $S_n^*\equiv\frac{1}{\sqrt{nT}}c'\frac{\partial\mathcal{L}^{rr}(\psi_0)}{\partial\psi}\xrightarrow{d}N(0,c'\Gamma_{rr}c)$. Using the expression for elements of $\frac{\partial\mathcal{L}^{rr}(\psi)}{\partial\psi}$ defined in Section 4.2, we can readily obtain

$$\begin{aligned}
S_n^* &= \frac{1}{\sqrt{nT}}\left[c_1'\frac{\partial\mathcal{L}^{rr}(\psi_0)}{\partial\theta'}+c_2\frac{\partial\mathcal{L}^{rr}(\psi_0)}{\partial\sigma_v^2}+c_3\frac{\partial\mathcal{L}^{rr}(\psi_0)}{\partial\rho}+c_4\frac{\partial\mathcal{L}^{rr}(\psi_0)}{\partial\lambda}+c_5\frac{\partial\mathcal{L}^{rr}(\psi_0)}{\partial\phi_\mu}+c_6\frac{\partial\mathcal{L}^{rr}(\psi_0)}{\partial\phi_{\phi_\zeta}}\right]\\
&= \frac{1}{\sqrt{nT}}\left\{\frac{1}{\sigma_{v0}^2}c_1'X^{*\prime}\Omega_0^{*-1}u^*-\frac{c_3}{\sigma_{v0}^2}u_\rho^{*\prime}\Omega_0^{*-1}u^*+\frac{c_2}{2\sigma_{v0}^2}\left[\sigma_{v0}^{-2}u^{*\prime}\Omega_0^{*-1}u^*-n(T+1)\right]\right.\\
&\quad+\frac{c_3}{2\sigma_{v0}^2}\left[u^{*\prime}P_{\rho0}^*u^*-\sigma_{v0}^2\mathrm{tr}(P_{\rho0}^*\Omega_0^*)\right]+\frac{c_4}{2\sigma_{v0}^2}\left[u^{*\prime}P_{\lambda0}^*u^*-\sigma_{v0}^2\mathrm{tr}(P_{\lambda0}^*\Omega_0^*)\right]\\
&\quad\left.+\frac{c_5}{2\sigma_{v0}^2}\left[u^{*\prime}P_{\phi_\mu0}^*u^*-\sigma_{v0}^2\mathrm{tr}(P_{\phi_\mu0}^*\Omega_0^*)\right]+\frac{c_6}{2\sigma_{v0}^2}\left[u^{*\prime}P_{\phi_\zeta0}^*u^*-\sigma_{v0}^2\mathrm{tr}(\Omega_{\phi_\zeta0}^*\Omega_0^*)\right]\right\}\\
&= S_{n1}^*+S_{n2}^*+\left[S_{n3}^*-\mathrm{E}\left(S_{n3}^*\right)\right]
\end{aligned}$$

where $S_{n1}^*=\frac{1}{\sqrt{nT}}\frac{1}{\sigma_{v0}^2}c_1'X^{*\prime}\Omega_0^{*-1}u^*$, $S_{n2}^*=\frac{-1}{\sqrt{nT}}\frac{c_3}{\sigma_{v0}^2}u_\rho^{*\prime}\Omega_0^{*-1}u^*$, $S_{n3}^*=\frac{1}{\sqrt{nT}}\frac{1}{2\sigma_{v0}^2}u^{*\prime}\bar{\Omega}_0^*u^*$ and $\bar{\Omega}_0^*=\frac{c_2}{\sigma_{v0}^2}\Omega_0^{*-1}+c_3P_{\rho0}^*+c_4P_{\lambda0}^*+c_5P_{\phi_\mu0}^*+c_6P_{\phi_\zeta0}^*$. Note that

$$\begin{aligned}
S_{n1}^* &= \frac{1}{\sqrt{nT}}\frac{1}{\sigma_{v0}^2}c_1'X^{*\prime}\begin{pmatrix}\omega_*^{11} & \omega_*^{12}\\ \omega_*^{21} & \omega_*^{22}\end{pmatrix}\begin{pmatrix}\zeta+a_{m0}\mu+\sum_{j=0}^{m-1}\rho_0^jB_0^{-1}v_{-j}\\ (\iota_T\otimes I_n)\mu+(I_T\otimes B_0^{-1})v\end{pmatrix}\\
&= \frac{1}{\sqrt{nT}}\frac{1}{\sigma_{v0}^2}c_1'X^{*\prime}\begin{pmatrix}\omega_*^{11}\\ \omega_*^{21}\end{pmatrix}\zeta+\frac{1}{\sqrt{nT}}\frac{1}{\sigma_{v0}^2}c_1'X^{*\prime}\begin{pmatrix}\omega_*^{11}a_{m0}+\omega_*^{12}(\iota_T\otimes I_n)\\ \omega_*^{21}a_{m0}+\omega_*^{22}(\iota_T\otimes I_n)\end{pmatrix}\mu\\
&\quad+\frac{1}{\sqrt{nT}}\frac{1}{\sigma_{v0}^2}c_1'X^{*\prime}\begin{pmatrix}\omega_*^{12}\\ \omega_*^{22}\end{pmatrix}(I_T\otimes B_0^{-1})v+\frac{1}{\sqrt{nT}}\frac{1}{\sigma_{v0}^2}c_1'X^{*\prime}\begin{pmatrix}\omega_*^{11}\\ \omega_*^{21}\end{pmatrix}\sum_{j=0}^{m-1}\rho_0^jB_0^{-1}v_{-j}\\
&\equiv S_{n1,1}^*+S_{n1,2}^*+S_{n1,3}^*+S_{n1,4}^*,\ \text{say},
\end{aligned}$$

where $S_{n1,1}^*, S_{n1,2}^*$, $S_{n1,3}^*$, and $S_{n1,4}^*$ are linear in $\zeta$, $\mu$, $v$ and $v_{-j}$'s, respectively. Similarly

$$\begin{aligned}
S_{n3}^* &= \frac{1}{\sqrt{nT}}\frac{1}{2\sigma_{v0}^2}\left\{\zeta'\bar{\omega}_*^{11}\zeta+\mu'\left[a_{m0}\bar{\omega}_*^{11}+(\iota_T'\otimes I_n)\bar{\omega}_*^{22}(\iota_T\otimes I_n)+2a_{m0}(\iota_T'\otimes I_n)\bar{\omega}_*^{21}\right]\mu\right.\\
&\quad+v'(I_T\otimes B'^{-1})\bar{\omega}_*^{22}(I_T\otimes B_0^{-1})v+\left(\sum\nolimits_{j=0}^{m-1}\rho_0^jB_0^{-1}v_{-j}\right)'\bar{\omega}_*^{11}\left(\sum\nolimits_{j=0}^{m-1}\rho_0^jB_0^{-1}v_{-j}\right)\\
&\quad+2\left[\left(a_{m0}^2\mu+\sum\nolimits_{j=0}^{m-1}\rho_0^jB_0^{-1}v_{-j}\right)+(\iota_T\otimes I_n)\mu+(I_T\otimes B_0^{-1})v\right]'\bar{\omega}_*^{21}\zeta\\
&\quad+2a_{m0}\mu'\bar{\omega}_*^{11}\sum\nolimits_{j=0}^{m-1}\rho_0^jB_0^{-1}v_{-j}+2\mu'(\iota_T'\otimes I_n)\bar{\omega}_*^{22}(I_T\otimes B_0^{-1})v\\
&\quad\left.+2v'(I_T\otimes B_0'^{-1})\bar{\omega}_*^{21}\left(a_{m0}\mu+\sum\nolimits_{j=0}^{m-1}\rho_0^jB_0^{-1}v_{-j}\right)+2\mu'(\iota_T'\otimes I_n)\bar{\omega}_*^{21}\sum\nolimits_{j=0}^{m-1}\rho_0^jB_0^{-1}v_{-j}\right\}.
\end{aligned}$$

where $\bar{\Omega}_0^{*-1}=\begin{pmatrix}\bar{\omega}_*^{11} & \bar{\omega}_*^{21\prime}\\ \bar{\omega}_*^{21} & \bar{\omega}_*^{22}\end{pmatrix}$ with $\bar{\omega}_*^{11}$, $\bar{\omega}_*^{12}$, and $\bar{\omega}_*^{22}$ being $n\times n$, $nT\times n$, and $nT\times nT$ matrices. Apparently, $S_{n3}^*$ can be written as the summation of five asymptotically independent terms, i.e., $S_{n3}^*=$

$\sum_{j=1}^{5} S_{n3,j}^{*}$, where $S_{n3,1}^{*}$, $S_{n3,2}^{*}$, $S_{n3,3}^{*}$, and $S_{n3,4}^{*}$ are quadratic functions of $\zeta$, $\mu$, $v$, and $v_{-j}$'s, respectively, and $S_{n3,5}^{*}$ is the summation of terms that are bilinear in any two of $\zeta$, $\mu$, $v$, and $v_{-j}$'s. Analogous to the proof of Lemma B.8, we can use $u_{\rho}^{*} = -\left(\dot{a}_{m0}\left(z\gamma_{0}\right)', Y_{-1}'\right)$ and the expression of $Y_{-1}$ in (B.2) to write $S_{n2}^{*} = \sum_{j=1}^{5} S_{n2,j}^{*}$, where $S_{n2,1}^{*}$, $S_{n2,2}^{*}$, and $S_{n2,3}^{*}$ are quadratic functions of $\mu$, $v$, and $v_{-j}$'s, respectively, $S_{n2,4}^{*}$ is a bilinear function that contains summation of terms which are linear in any two of $\zeta$, $\mu$, $v$, and $v_{-j}$'s, and $S_{n2,5}^{*}$ is the summation of terms that are linear in one of $\varsigma$, $\mu$, $v$, and $v_{-j}$'s. Consequently, we can write $S_{n}^{*} = \sum_{j=1}^{6} s_{nj}^{*}$, where $s_{n1}^{*}, ..., s_{n4}^{*}$ are quadratic functions of $\zeta$, $\mu$, $v$, and $v_{-j}$'s, respectively, $s_{n5}^{*}$ is a summation of terms tat are bilinear in any two of $\zeta$, $\mu$, $v$, and $v_{-j}$'s, and $s_{n6}^{*}$ is summation of terms that are linear in $\zeta$, $\mu$, $v$, and $v_{-j}$'s. By the mutual independence of $\zeta$, $\mu$, $v$, and $v_{-j}$'s and their zero mean property, these six terms are either independent or asymptotically independent. By Lemma B.5,

$$\{s_{nj}^{*} - \mathrm{E}(s_{nj}^{*})\}/\sqrt{\mathrm{Var}(s_{nj}^{*})} \xrightarrow{d} N(0,1).$$

It follows that $S_{n}^{*} \xrightarrow{d} N(0, \lim_{n\to\infty} \sum_{j=1}^{6} \mathrm{Var}(s_{nj}^{*}))$, implying that $S_{n}^{*} \xrightarrow{d} N(0, c'\Gamma_{rr}c)$. ■

**Proof of Lemma B.14.** Noting that $E\left(\Delta X^{\dagger\prime} R \Delta u\right) = 0$ for any $nT \times nT$ nonstochastic matrix $R$, by the expressions of the Hessian matrix $\frac{\partial \mathcal{L}^{f}(\psi_{0})}{\partial\psi\partial\psi'}$ in Section 4.3, it suffices to prove

(*i*) $n^{-1}[\Delta X^{\dagger\prime}\Omega_{0}^{\dagger-1}\Delta X^{\dagger} - E(\Delta X^{\dagger\prime}\Omega_{0}^{\dagger-1}\Delta X^{\dagger})] = o_{p}(1)$;
(*ii*) $n^{-1}\Delta X^{\dagger\prime} R \Delta u = o_{p}(1)$ for $R = \Omega_{0}^{\dagger-1}$ and $P_{\omega 0}^{\dagger}$ with $\omega = \rho$, $\lambda$,and $\phi_{\zeta}$;
(*iii*) $n^{-1}[\Delta u' R \Delta u - \sigma_{v0}^{2}\mathrm{tr}(R\Omega_{0}^{\dagger})] = o_{p}(1)$ for $R = \Omega_{0}^{\dagger-1}$ and $P_{\omega 0}^{\dagger}$ with $\omega = \rho$, $\lambda$, and $\phi_{\zeta}$;
(*iv*) $n^{-1}[\Delta X^{\dagger\prime}\Omega_{0}^{\dagger-1}\Delta u_{\rho} - E(\Delta X^{\dagger\prime}\Omega_{0}^{\dagger-1}\Delta u_{\rho})] = o_{p}(1)$;
(*v*) $n^{-1}\left[\Delta u_{\rho}' R \Delta u - E\left(\Delta u_{\rho}' R \Delta u\right)\right] = o_{p}(1)$ for $R = \Omega_{0}^{\dagger-1}$ and $P_{\omega 0}^{\dagger}$ with $\omega = \rho$, $\lambda$, and $\phi_{\zeta}$;
(*vi*) $n^{-1}[\Delta u_{\rho}'\Omega_{0}^{\dagger-1}\Delta u_{\rho} - E\left(\Delta u_{\rho}'\Omega_{0}^{\dagger-1}\Delta u_{\rho}\right)] = o_{p}(1)$;
(*vii*) $n^{-1}[q_{\omega\bar{\omega}}^{\dagger}(\Delta u) - E(q_{\omega\bar{\omega}}^{\dagger}(\Delta u))] = o_{p}(1)$ for $\omega, \bar{\omega} = \rho$, $\lambda$, and $\phi_{\zeta}$.

Let $\Omega_{\omega\bar{\omega}0}^{\dagger} = \frac{\partial^{2}}{\partial\omega\partial\bar{\omega}}\Omega^{\dagger}(\delta_{0})$ for $\rho$, $\lambda$, and $\phi_{\zeta}$. Noting that $\Omega_{0}^{\dagger-1}$, $\Omega_{\omega 0}^{\dagger}$, $P_{\omega 0}^{\dagger}$ and $\Omega_{\omega\bar{\omega}0}^{\dagger}$ with $\omega, \bar{\omega} = \rho$, $\lambda$, and $\phi_{\zeta}$ are uniformly bounded in both row and column sums and $q_{\omega\bar{\omega}}^{\dagger}(\Delta u)$ is quadratic in $\Delta u$, we can show that (*i*)-(*vii*) hold by straightforward moment calculations and Chebyshev inequality. Below we only demonstrate the proof of (*iii*) and (*vi*) since the proof of the other claims is similar or simpler.

Since $E(\Delta u' R \Delta u) = \sigma_{v0}^{2}\mathrm{tr}(R\Omega_{0}^{\dagger})$, by Chebyshev inequality (*iii*) follows provided $\mathrm{Var}(n^{-1}\Delta u' R \Delta u) = o(1)$. Let $\Delta v_{(0)} = B_{0}\zeta + \rho_{0}^{m} v_{-m+1} + \sum_{j=0}^{m-1}\rho_{0}^{j}\Delta v_{1-j}$, $\Delta v_{(1)} = (\Delta v_{2}', ....\Delta v_{T}')'$, and $\Delta v = (\Delta v_{(0)}', \Delta v_{(1)}')'$. Then $\Delta u = (I_{n} \otimes B_{0}^{-1})\Delta v$ and $\Delta u' R \Delta u = \Delta v'(I_{n} \otimes B_{0}^{'-1}) R (I_{n} \otimes B_{0}^{-1})\Delta v = \Delta v' \widetilde{R} \Delta v$, where $\widetilde{R} \equiv (I_{n} \otimes B_{0}^{'-1}) R (I_{n} \otimes B_{0}^{-1})$. Now, write

$$R = \begin{pmatrix} \underset{n\times n}{R_{00}} & \underset{n\times n(T-1)}{R_{01}} \\ \underset{n(T-1)\times n}{R_{10}} & \underset{n(T-1)\times n(T-1)}{R_{11}} \end{pmatrix}$$

and partition $\widetilde{R}$ similarly. Let $C$ be a $(T-1) \times T$ matrix with $C_{ij} = -1$ if $i = j$, $C_{ij} = 1$ if $j = i+1$, and $C_{ij} = 0$ otherwise. Then $\Delta v_{(1)} = (C \otimes I_{n})v$, where $v = (v_{1}', \cdots, v_{T}')'$. So

$$\begin{aligned} \Delta v' \widetilde{R} \Delta v &= \Delta v_{(0)}' \widetilde{R}_{00} \Delta v_{(0)} + \Delta v_{(1)}' \widetilde{R}_{11} \Delta v_{(1)} + \Delta v_{(0)}'(R_{01} + R_{10}')\Delta v_{(1)} \\ &= \Delta v_{(0)}' \widetilde{R}_{00} \Delta v_{(0)} + v'(C' \otimes I_{n}) \widetilde{R}_{11} (C \otimes I_{n}) v + \Delta v_{(0)}'(R_{01} + R_{10}')(C \otimes I_{n}) v \end{aligned}$$

Then by Cauchy-Schwarz inequality

$\mathrm{Var}(\Delta u' R \Delta u) \leq 3\mathrm{Var}(\Delta v_{(0)}' \widetilde{R}_{00} \Delta v_{(0)}) + 3\mathrm{Var}(v'(C' \otimes I_n)\widetilde{R}_{11}(C \otimes I_n)v) + 3\mathrm{Var}(\Delta v_{(0)}'(R_{01} + R_{10}')(C \otimes I_n)v)$.

Write $\Delta v_{(0)} = B_0 \zeta + v_1 + \rho_0^{m-1}(\rho_0 - 1) v_{-m+1} + \sum_{j=0}^{m-2} \rho_0^j (\rho_0 - 1) v_{-j}$. Since $B_0' \widetilde{R}_{00} B_0$ is uniformly bounded in both row and column sums, by Lemma B.3(1)

$$\mathrm{Var}(\zeta' B_0' \widetilde{R}_{00} B_0 \zeta) = \kappa_\zeta \sum_{i=1}^{n} [(B_0' \widetilde{R}_{00} B_0)_{ii}]^2 + \sigma_{\zeta 0}^4 \mathrm{tr}(B_0' \widetilde{R}_{00} B_0 B_0' (\widetilde{R}_{00} + \widetilde{R}_{00}') B_0) = O(n).$$

Similarly, we can show that $\mathrm{Var}(v_1' \widetilde{R}_{00} v_1) = O(n)$, $\mathrm{Var}(v_{-m+1}' \widetilde{R}_{00} v_{-m+1}) = O(n)$, and $\mathrm{Var}(\sum_{j=0}^{m-2} \rho_0^j v_{-j}' \widetilde{R}_{00} \times \sum_{j=0}^{m-2} \rho_0^j v_{-j}) = O(n)$. It follows from Cauchy-Schwarz inequality that $\mathrm{Var}(\Delta v_{(0)}' \widetilde{R}_{00} \Delta v_{(0)}) = O(n)$. By the same token, we can show that $\mathrm{Var}(v'(C' \otimes I_n) \widetilde{R}_{11} (C \otimes I_n) v) = O(n)$, and $\mathrm{Var}(\Delta v_{(0)}'(R_{01} + R_{10}')(C \otimes I_n)v) = O(n)$. This completes the proof of $(iii)$.

Now, we show $(vi)$. Let $\Delta Y^* = (0_{1\times n}, \Delta y_1', \cdots, \Delta y_{T-1}')'$. Then $\Delta u_\rho = -\Delta Y^*$. Let $k_\rho = (0, 1, \rho, \cdots, \rho^{T-2})'$, $\mathcal{X} = (0_{1\times n}, 0_{1\times n}, (\Delta x_2 \beta_0)', \cdots, \sum_{j=0}^{T-3} \rho_0^j (\Delta x_{T-1-j} \beta_0)')$, and $\mathcal{V} = (0_{1\times n}, 0_{1\times n}, (\Delta v_2)', \cdots, \sum_{j=0}^{T-3} \rho_0^j (\Delta v_{T-1-j})')$. Since $\Delta y_1 = \tilde{\Delta} x \pi_0 + \Delta x_1 \beta_0 + \tilde{\Delta} u_1$ and

$$\Delta y_t = \rho_0^{t-1} \Delta y_1 + \sum_{j=0}^{t-2} \rho_0^j \Delta x_{t-j} \beta_0 + \sum_{j=0}^{t-2} \rho_0^j B_0^{-1} \Delta v_{t-j} \text{ for } t = 2, 3, \cdots,$$

we have $\Delta Y^* = k_{\rho_0} \otimes \Delta y_1 + \mathcal{X} + (I_T \otimes B_0^{-1})\mathcal{V}$. It follows that

$$\begin{aligned} \mathrm{Var}\left(\Delta u_\rho' \Omega_0^{\dagger -1} \Delta u_\rho\right) &\leq 3\mathrm{Var}\left(\left(k_{\rho_0}' \otimes \Delta y_1\right) \Omega_0^{\dagger -1} \left(k_{\rho_0} \otimes \Delta y_1\right)\right) + 3\mathrm{Var}\left(\mathcal{X}' \Omega_0^{\dagger -1} \mathcal{X}\right) \\ &\quad + 3\mathrm{Var}\left(\mathcal{V}'(I_T \otimes B_0'^{-1}) \Omega_0^{\dagger -1} (I_T \otimes B_0^{-1}) \mathcal{V}\right) \end{aligned}$$

We can show that each term on the right hand side of the last expression is $O(n)$. Then $(vi)$ follows by Chebyshev inequality. ■

**Proof of Lemma B.15.** By Cramér-Wold device, it suffices to show that for any $c = (c_1', c_2, c_3, c_4, c_5)' \in \mathbb{R}^{p+k} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with $\|c\| = 1$, $S_n^\dagger \equiv \frac{1}{\sqrt{nT}} c' \frac{\partial \mathcal{L}^f(\psi_0)}{\partial \psi} \xrightarrow{d} N(0, c' \Gamma_f c)$. Recall $\Delta u = \Delta u(\theta_0, \rho_0)$. Let $\Delta u_\rho = -(0_{n\times 1}', \Delta y_1', \cdots, \Delta y_{T-1}')'$, and $P_{\omega 0}^\dagger = P_\omega^\dagger(\delta_0)$ for $\omega = \rho, \lambda$, and $\phi_\zeta$. Using the expression for elements of $\frac{\partial \mathcal{L}^f(\psi)}{\partial \psi}$ defined in Section 4.3, we can readily obtain

$$\begin{aligned} S_n^\dagger &= \frac{1}{\sqrt{nT}} \left[ c_1' \frac{\partial \mathcal{L}^f(\psi_0)}{\partial \theta'} + c_2 \frac{\partial \mathcal{L}^f(\psi_0)}{\partial \sigma_v^2} + c_3 \frac{\partial \mathcal{L}^f(\psi_0)}{\partial \rho} + c_4 \frac{\partial \mathcal{L}^f(\psi_0)}{\partial \lambda} + c_5 \frac{\partial \mathcal{L}^f(\psi_0)}{\partial \phi_{\phi_\zeta}} \right] \\ &= \frac{1}{\sqrt{nT}} \left\{ \frac{1}{\sigma_{v0}^2} c_1' \Delta X^{\dagger\prime} \Omega_0^{\dagger -1} \Delta u - \frac{c_3}{\sigma_{v0}^2} \Delta u_\rho' \Omega_0^{\dagger -1} \Delta u \right. \\ &\quad + \frac{c_2}{2\sigma_{v0}^2} \left[ \frac{1}{2\sigma_{v0}^2} \Delta u' \Omega_0^{\dagger -1} \Delta u - nT \right] + \frac{c_3}{2\sigma_{v0}^2} \left[ \Delta u' P_{\rho 0}^\dagger \Delta u - \sigma_{v0}^2 \mathrm{tr}(P_{\rho 0}^\dagger \Omega_0^\dagger) \right] \\ &\quad \left. + \frac{c_4}{2\sigma_{v0}^2} \left[ \Delta u' P_{\lambda 0}^\dagger \Delta u - \sigma_{v0}^2 \mathrm{tr}(P_{\lambda 0}^\dagger \Omega_0^\dagger) \right] + \frac{c_5}{2\sigma_{v0}^2} \left[ \Delta u' P_{\phi_\zeta 0}^\dagger \Delta u - \sigma_{v0}^2 \mathrm{tr}(P_{\phi_\zeta 0}^\dagger \Omega_0^\dagger) \right] \right\} \\ &= S_{n1}^\dagger + S_{n2}^\dagger + \left[ S_{n3}^\dagger - \mathrm{E}\left(S_{n3}^\dagger\right) \right] \end{aligned}$$

where $S_{n1}^\dagger = \frac{1}{\sqrt{nT}} \frac{1}{\sigma_{v0}^2} c_1' \Delta X^{\dagger\prime} \Omega_0^{\dagger -1} \Delta u$, $S_{n2}^\dagger = \frac{-1}{\sqrt{nT}} \frac{c_3}{\sigma_{v0}^2} \Delta u_\rho' \Omega_0^{\dagger -1} \Delta u$, $S_{n3}^\dagger = \frac{1}{\sqrt{nT}} \frac{1}{2\sigma_{v0}^2} \Delta u' \bar{\Omega}_0^\dagger \Delta u$ and $\bar{\Omega}_0^\dagger = \frac{c_2}{\sigma_{v0}^2} \Omega_0^{\dagger -1} + c_3 P_{\rho 0}^\dagger + c_4 P_{\lambda 0}^\dagger + c_5 P_{\phi_\zeta 0}^\dagger$. Analogous to the proof of Lemma B.13, one can write $S_n^\dagger = \sum_{j=1}^{5} s_{nj}^\dagger$, where $s_{n1}^{\dagger}, ..., s_{n3}^{\dagger}$ are quadratic functions of $\zeta$, $v$, and $v_{-j}$'s, respectively, $s_{n4}^{\dagger}$ is a summation of terms tat are bilinear in any two of $\zeta$, $v$, and $v_{-j}$'s, and $s_{n5}^{\dagger}$ is summation of terms that are linear in $\varsigma$, $v$, and $v_{-j}$'s. By the mutual independence of $\zeta$, $v$, and $v_{-j}$'s and their zero mean property, these five terms are either independent or asymptotically independent. By Lemma B.5,

$$\{s_{nj}^{\dagger} - \mathrm{E}(s_{nj}^{\dagger})\}/\sqrt{\mathrm{Var}(s_{nj}^{\dagger})} \xrightarrow{d} N(0,1).$$

It follows that $S_n^{\dagger} \xrightarrow{d} N(0, \lim_{n\to\infty} \sum_{j=1}^{5} \mathrm{Var}(s_{nj}^{\dagger}))$, implying that $S_n^{\dagger} \xrightarrow{d} N(0, c'\Gamma_{rr}c)$. ■

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