Supplementary Appendix

To "Spatial Dynamic Panel Data Models with Correlated Random Effects"

by Liyao Li and Zhenlin Yang

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Summary

In this Supplementary Appendix, we present (*i*) detailed discussion on connection between fixed-effects estimator and Mundlak and Chamberlain approach, (*ii*) a detailed discussion on possible extensions of our work, (*iii*) a more comprehensive set of Monte Carlo results, (*iv*) detailed results related to the empirical application, and (*v*) more details on the proofs of the main results in the paper.

I. FE estimator and Mundlak/Chamberlain Approach

As noted in the paper, in a very simple static panel data model, i.e., setting $\rho, \lambda_1, \lambda_2$, and λ_3 in Model (1.1) to zero, one can show that the CRE-estimator of β (the coefficients of time-varying regressors) under Mundlak's specification reduces to the usual FE-estimator (Cameron and Trivedi, 2005, Sec. 21.4.4; Krishnakumar, 2006; Hsiao, 2014, Sec. 3.4.2.1). This raises an interesting question on how 'far' this equivalence can go. Unfortunately, we show that such an equivalence fails to hold once we move away from these formulations (e.g., only a subset of X_t is correlated with μ), add dynamic terms, or add spatial terms. The following details provide general support to these conclusions.

A.1. Partitioned Regression. To help understanding the issue, we begin by a review of an interesting result from the **partitioned regression**. Write the multiple linear regression model as: $Y = X_1\beta_1 + X_2\beta_2 + \mathbf{v}$, where $\mathbf{v} \sim (0, \sigma^2 I_n)$. We have,

$$\hat{\beta}_{1,\mathtt{OLS}} = (X_1' M_{X_2} X_1)^{-1} X_1' M_{X_2} Y, \tag{I.1}$$

$$\hat{\beta}_{2,\mathtt{OLS}} = (X_2' M_{X_1} X_2)^{-1} X_2' M_{X_1} Y, \tag{I.2}$$

where $M_{X_1} = I - P_{X_1}$, and $P_{X_1} = X_1(X_1'X_1)^{-1}X_1'$; $M_{X_2} = I - P_{X_2}$, and $P_{X_2} = X_2(X_2'X_2)^{-1}X_2'$.

A.2. FE estimator of static panel data model. Consider the following static panel data model with fixed effects (FE):

$$Y = X\beta + 1_T \otimes \mu + \mathbf{v}, \tag{I.3}$$

where $Y = (y_1', y_2', \ldots, y_T')'$ is an $nT \times 1$ vector of dependent variables, $X = (X_1', X_2', \ldots, X_T')$ is an $nT \times p$ matrix containing the values of time varying exogenous regressors, $\mathbf{v} = (v_1', v_2', \ldots, v_T')'$ is an $nT \times 1$ vector of idiosyncratic errors iid with mean 0 and variance σ_v^2, μ is $n \times 1$ vector of individual effects, 1_T is a $T \times 1$ vector of ones, and $\otimes$ denotes the Kronecker product.

When μ is treated as fixed effects, the FE-estimator of β is:

$$\hat{\beta}_{\texttt{FE}} = (X'QX)^{-1}X'QY,$$

where $Q = I_{nT} - P$, $P = \frac{1}{T}J_T \otimes I_n$, and J_T is a $T \times T$ matrix of ones. **Note** that $\hat{\beta}_{\texttt{FE}}$ can be obtained by: (i) OLS or GLS on the Q-transformed model: $QY = QX\beta + Q\mathbf{v}$, or (ii) least squares dummy variables (LSVD) or maximum likelihood estimation on Model (I.3).

A.3.1. Mundlak's approach without time-invariant regressors. Mundlak (1978) suggests to approximate $\mathrm{E}(\mu_i|X_i)$ by a function linear in $\{X_{it}\}$, so that

$$\mu_i = (\tfrac{1}{T}\textstyle\sum_{t=1}^{T} X_{it})\pi + \varepsilon_i, \tag{I.4}$$

where ε_i is independent of X_{it} and v_{it}, and refers to $\{\mu_i\}$ as the *correlated random effects* (CRE). Using this approximation, model (I.3) can be written as

$$Y = X\beta + PX\pi + \mathbf{e} = QX\beta + PX\theta + \mathbf{e}, \tag{I.5}$$

where $\mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{v}$, $\boldsymbol{\varepsilon} = 1_T \otimes \varepsilon$, and $\theta = (\beta', \pi')'$. Model (I.5) can be seen as a random effect model and can be estimated using GLS. We have, $\Sigma = \mathrm{Var}(\mathbf{e}) = \sigma_\varepsilon^2 J_T \otimes I_n + \sigma_v^2 I_n = (T\sigma_\varepsilon^2 + \sigma_v^2)P + \sigma_v^2 Q$, which is a spectral decomposition of Σ. Thus,

$$\Sigma^{-1} = \sigma_c^{-2}P + \sigma_v^{-2}Q, \quad \text{where } \sigma_c^2 = T\sigma_\varepsilon^2 + \sigma_v^2. \tag{I.6}$$

Notice that $P\Sigma^{-1} = \Sigma^{-1}P = \sigma_c^{-2}P$, and $Q\Sigma^{-1} = \Sigma^{-1}Q = \sigma_v^{-2}Q$.

Now, using the results from the partitioned regression given in (I.1), the GLS or Mundlak (MUND) estimator of β based on (I.5) is

$$\begin{aligned}\hat{\beta}_{\texttt{MUND}} &= (X'Q\Sigma^{-1}QX)^{-1}X'Q\Sigma^{-1}(Y - PX\theta)\\ &= (X'QX)^{-1}X'QY = \hat{\beta}_{\texttt{FE}}.\end{aligned}$$

A.3.2. Mundlak's approach with time-invariant regressors. When Model (I.3) contains time-invariant regressors Z, Mundlak's specification of Model (I.3) becomes

$$Y = X\beta + (1_T \otimes Z)\gamma + PX\pi + \mathbf{e} = QX\beta + PX^*\theta + \mathbf{e}, \tag{I.7}$$

where $X^* = [X, 1_T \otimes Z]$. Using (I.1), it is easy to see that $\hat{\beta}_{\texttt{MUND}} = (X'QX)^{-1}X'QY = \hat{\beta}_{\texttt{FE}}$.

A.4.1. Chamberlain's approach without time-invariant regressors. Chamberlain (1982, 1984) suggested the following approximation of the individual effects

$$\mu_i = \textstyle\sum_{t=1}^{T} X_{it}\pi_t + \varepsilon_i. \tag{I.8}$$

Using this approximation, model (I.3) can be written as

$$Y = X\beta + 1_T \otimes (\textstyle\sum_{t=1}^{T} X_t\pi_t) + \mathbf{e} = X\beta + \tilde{X}\Pi + \mathbf{e}, \tag{I.9}$$

where $\Pi = (\pi_1', \pi_2', \ldots, \pi_T')'$ is a $Tk \times 1$ vector, $\tilde{X} = 1_T \otimes X^\circ$ is an $nT \times Tk$ matrix, and

$X^\circ = [X_1, X_2, \ldots, X_T]$ is an $n \times T$ matrix. Note that $P\tilde{X} = \tilde{X}$, and $Q\tilde{X} = 0$.

Let $M^* = I_{nT} - P^*$, and $P^* = \Sigma^{-\frac{1}{2}}\tilde{X}(\tilde{X}'\Sigma^{-1}\tilde{X})^{-1}\tilde{X}'\Sigma^{-\frac{1}{2}}$. We have $\Sigma^{-\frac{1}{2}}P^*\Sigma^{-\frac{1}{2}} = \Sigma^{-1}\tilde{X}(\tilde{X}'\Sigma^{-1}\tilde{X})^{-1}\tilde{X}'\Sigma^{-1} = \sigma_c^{-2}\tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}' = \sigma_c^{-2}P_{\tilde{X}}$. Using the results from partitioned regression given in (I.1), the GLS or Chamberlain (CHAM) estimator of β based on (I.9) is

$$\begin{aligned}
\hat{\beta}_{\texttt{CHAM}} &= (X'\Sigma^{-\frac{1}{2}}M^*\Sigma^{-\frac{1}{2}}X)^{-1}X'\Sigma^{-\frac{1}{2}}M^*\Sigma^{-\frac{1}{2}}Y \\
&= [X'\Sigma^{-1}X - X'\Sigma^{-\frac{1}{2}}P^*\Sigma^{-\frac{1}{2}}X]^{-1}[X'\Sigma^{-1}Y - X'\Sigma^{-\frac{1}{2}}P^*\Sigma^{-\frac{1}{2}}Y] \\
&= [X'\Sigma^{-1}X - \sigma_c^{-2}X'P_{\tilde{X}}X]^{-1}[X'\Sigma^{-1}Y - \sigma_c^{-2}X'P_{\tilde{X}}Y] \\
&= [\sigma_v^{-2}X'QX + \sigma_c^{-2}X'(P - P_{\tilde{X}})X]^{-1}[\sigma_v^{-2}X'QY + \sigma_c^{-2}X'(P - P_{\tilde{X}})Y].
\end{aligned}$$

As $\tilde{X} = 1_T \otimes X^\circ$, we have $P_{\tilde{X}} = \frac{1}{T}J_T \otimes P_{X^\circ}$, where $P_{X^\circ} = X^\circ(X^{\circ\prime}X^\circ)^{-1}X^{\circ\prime}$. So, $P - P_{\tilde{X}} = \frac{1}{T}J_T \otimes (I_n - P_{X^\circ}) = \frac{1}{T}J_T \otimes M_{X^\circ}$. Then, we can write $X'(P - P_{\tilde{X}})X = \frac{1}{T}X'(J_T \otimes M_{X^\circ})X = \mathrm{tr}(X^\circ X^{\circ\prime}M_{X^\circ}) = 0$, and $X'(P - P_{\tilde{X}})Y = 0$. Therefore, we have

$$\hat{\beta}_{\texttt{CHAM}} = (X'QX)^{-1}X'QY = \hat{\beta}_{\texttt{FE}}$$

A.4.2. Chamberlain's approach with time-invariant regressors. When the individual effects μ contains time-invariant regressors Z, Chamberlain's specification of model (I.3) can be written as

$$Y = X\beta + (1_T \otimes Z)\gamma + 1_T \otimes (\textstyle\sum_{t=1}^T X_t\pi_t) + \mathbf{e} = X\beta + X^*\theta + \mathbf{e}, \tag{I.10}$$

where $X^* = 1_T \otimes X^\dagger$, $X^\dagger = [X^\circ, Z]$, and $\theta = (\Pi', \gamma')'$. Note that $PX^* = X^*$, and $QX^* = 0$. Let $P_{X^*} = \frac{1}{T}J_T \otimes P_{X^\dagger}$, and $P_{X^\dagger} = X^\dagger(X^{\dagger\prime}X^\dagger)^{-1}X^{\dagger\prime}$. It is easy to verify that $P_{X^\dagger} = P_{X^\circ} - P^\dagger$, where $P^\dagger = M_{X^\circ}Z(Z'M_{X^\circ}Z)^{-1}Z'M_{X^\circ}$. So, $P - P_{X^*} = \frac{1}{T}J_T \otimes M_{X^\circ} + \frac{1}{T}J_T \otimes P^\dagger$

Using the results from partitioned regression given in (I.1), the GLS or Chamberlain (CHAM) estimator of β based on (I.10) is, similarly,

$$\hat{\beta}_{\texttt{CHAM}} = [\sigma_v^{-2}X'QX + \sigma_c^{-2}X'(P - P_{X^*})X]^{-1}[\sigma_v^{-2}X'QY + \sigma_c^{-2}X'(P - P_{X^*})Y].$$

As $X'J_T \otimes P^\dagger X = \mathrm{tr}(X^\circ X^{\circ\prime}P^\dagger) = \mathrm{tr}(X^\circ X^{\circ\prime}M_{X^\circ}Z(Z'M_{X^\circ}Z)^{-1}Z'M_{X^\circ}) = 0$, and similarly $X'J_T \otimes P^\dagger Y = 0$, we have

$$\hat{\beta}_{\texttt{CHAM}} = (X'QX)^{-1}X'QY = \hat{\beta}_{\texttt{FE}}$$

From the results above, we see that Mundlak's and Chamberlain's formulations specified in (I.4), and (I.8) lead to estimators of the regression coefficients β identical to the various FE-estimators. The equivalence remains with inclusion of time invariant variables in (I.3). Next, we will show that this equivalence fails if we move away from specifications in (I.4), and (I.8), add spatial terms, add dynamic terms, etc.

A.5. Mundlak's approach using a subset of time-varying regressors. Suppose μ only correlates with X_1, which is a subset of X such that $X = (X_1, X_2)$. In this case, Mundlak's specification gives

$$\mu = (\tfrac{1}{T}\textstyle\sum_{t=1}^{T} X_{1t})\pi + \varepsilon, \tag{I.11}$$

Using this approximation, model (I.3) can be written as

$$Y = X\beta + PX_1\pi + \mathbf{e}. \tag{I.12}$$

Let $P_{\bar{X}_1} = PX_1(X_1'PX_1)^{-1}X_1'P$, and $M_{\bar{X}_1} = I_{nT}P_{\bar{X}_1}$. Using the results from partitioned regression given in (I.1), the GLS or Mundlak (MUND) estimator of β based on (I.5) is

$$\begin{aligned}\hat{\beta}_{\texttt{MUND}} &= [\sigma_v^{-2}X'QX + \sigma_c^{-2}X'(P - P_{\bar{X}_1})X]^{-1}[\sigma_v^{-2}X'QY + \sigma_c^{-2}X'(P - P_{\bar{X}_1})Y] \\ &= [\sigma_v^{-2}X'QX + \sigma_c^{-2}X_2'(P - P_{\bar{X}_1})X_2]^{-1}[\sigma_v^{-2}X'QY + \sigma_c^{-2}X_2'(P - P_{\bar{X}_1})Y] \neq \hat{\beta}_{\texttt{FE}}\end{aligned}$$

The result above show that when individual effects only correlates with a subset of time varying regressors, the equivalence between FE estimator and GLS estimator under Mundlak's specification fails. It can be show similarly that in this case, the equivalence between FE estimator and GLS estimator under Chamberlain's specification also fails.

A.6. Pure Random Effects. When μ only contains time invariant regressors and does not correlate with time varying regressors, $\mu = Z\gamma + \varepsilon$, model (I.3) becomes a pure random effects model, we have

$$Y = X\beta + (1_T \otimes Z)\gamma + \mathbf{e}.$$

It is easy to see that the GLS or RE estimator is

$$\bar{\beta}_{\texttt{RE}} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}(Y - (1_T \otimes Z)\gamma) \neq \hat{\beta}_{\texttt{FE}}$$

Thus, we have shown that in static panel data model, the equivalence can easily fail when we move away from Mundlak or Chamberlain's specification. Next, we consider more general models with spatial or dynamic terms.

B. Panel data model with spatial error. We begin with a simple SPD model with spatial error (SE):

$$y_t = X_t\beta + \mu + u_t, \quad u_t = \lambda W u_t + v_t, \quad t = 1, 2, \ldots, T, \tag{I.13}$$

where W is a given $n \times n$ spatial weight matrix. Let $B = I_n - \lambda W$ and $\mathbf{B} = I_T \otimes B$. In vector form, (I.13) becomes: $Y = X\beta + 1_T \otimes \mu + \mathbf{u}$, $\mathbf{u} = \mathbf{B}^{-1}\mathbf{v}$. Noting that $\mathrm{Var}(\mathbf{u}) = \sigma_v^2(\mathbf{B}'\mathbf{B})^{-1}$ and that $Q(\mathbf{B}'\mathbf{B})^{-1} = (\mathbf{B}'\mathbf{B})^{-1}Q$, where Q is defined above, the FE-estimator of β, given λ, can be the GLS estimator based on Q-transformation, the QMLE based directly on (I.13) and a QMLE based on an orthonormal transformation as in Lee and Yu (2010):

$$\hat{\beta}_{\texttt{FE}}(\lambda) = (X^*\mathbf{B}'\mathbf{B}X^{*\prime})^{-1}X^{*\prime}(\mathbf{B}'\mathbf{B})Y^*,$$

where $X^* = QX$ and $Y^* = QY$. We can further show that these three approaches lead to the same estimate of λ but not of σ_v^2 (see Lee and Yu, 2010).

Now, following Mundlak's CRE formulation, Model (I.13) can be written as

$$Y = X^*\beta + PX\theta + \mathbf{e}, \quad \mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}^{-1}\mathbf{v} \tag{I.14}$$

where θ, $\boldsymbol{\varepsilon}$ and $\mathbf{v}$ are defined as in (I.5). Letting $\phi = \sigma_\varepsilon^2/\sigma_v^2$, we have

$$\mathrm{Var}(\mathbf{e}) = \sigma_v^2[\phi(J_T \otimes I_n) + (\mathbf{B}'\mathbf{B})^{-1}] \equiv \sigma_v^2\Omega, \tag{I.15}$$

By Magnus (1982, p.242), $\Omega^{-1} = P_T \otimes [\phi T I_n + (B'B)^{-1}]^{-1} + Q_T \otimes (B'B)$. Using the results for partitioned regression again, we show the *conditional equivalence*, given λ:

$$\hat{\beta}_{\texttt{MUND}}(\lambda) = (X^*\mathbf{B}'\mathbf{B}X^{*\prime})^{-1}X^{*\prime}(\mathbf{B}'\mathbf{B})Y^* \;\equiv\; \hat{\beta}_{\texttt{FE}}(\lambda). \tag{I.16}$$

However, it is easy to see that the unconstrained estimators of β: $\hat{\beta}_{\texttt{MUND}} \equiv \hat{\beta}_{\texttt{MUND}}(\hat{\lambda}_{\texttt{MUND}})$ and $\hat{\beta}_{\texttt{FE}} \equiv \hat{\beta}_{\texttt{FE}}(\hat{\lambda}_{\texttt{FE}})$, are not equivalent, as $\hat{\lambda}_{\texttt{FE}}$ is the maximizer of the concentrated loglikelihood:

$$\ell^c_{\texttt{FE}}(\lambda) = -\tfrac{nT}{2}[\log(2\pi) + 1] + \log|\mathbf{B}| - \tfrac{nT}{2}\log\hat{\sigma}^2_{v,\texttt{FE}}(\lambda), \tag{I.17}$$

where $\hat{\sigma}^2_{v,\texttt{FE}}(\lambda) = \frac{1}{nT}[Y^* - X^*\hat{\beta}_{\texttt{FE}}(\lambda)]'(\mathbf{B}'\mathbf{B})[Y^* - X^*\hat{\beta}_{\texttt{FE}}(\lambda)]$; whereas $\hat{\lambda}_{\texttt{MUND}}$ and $\hat{\phi}_{\texttt{MUND}}$ maximize the concentrated loglikelihood:

$$\ell^*_{\texttt{MUND}}(\delta) = -\tfrac{nT}{2}[\log(2\pi) + 1] - \tfrac{1}{2}\log|\Omega| - \tfrac{nT}{2}\log\hat{\sigma}^2_{v,\texttt{MUND}}(\delta), \tag{I.18}$$

where $\delta = (\lambda', \phi)'$, $\hat{\sigma}^2_{v,\texttt{MUND}}(\delta) = \frac{1}{nT}(Y - X^*\hat{\beta}_{\texttt{MUND}}(\delta) - PX\hat{\theta}_{\texttt{MUND}}(\delta))'\Omega^{-1}(Y - X^*\hat{\beta}_{\texttt{MUND}}(\delta) - PX\hat{\theta}_{\texttt{MUND}}(\delta))$.

C. Panel data model with spatial lag. We next consider an SPD model with spatial lag (SL): $y_t = \lambda W y_t + X_t\beta + \mu + v_t, t = 1, \ldots, T$, which in vector form becomes $\mathbf{B}Y = X\beta + 1_T \otimes \mu + \mathbf{v}$ under the FE-formulation, and $\mathbf{B}Y = QX\beta + PX\theta + \mathbf{e}$ under Mundlak's CRE formulation, where $\mathbf{B}$ is defined as in the SE model. Therefore, given λ the two models are identical to the two regular panel data models discussed above. Therefore, all the conclusions reached there apply, in particular, the constrained FE and Mundlak estimators given λ are equivalent, but the corresponding unconditional estimators of β are not equivalent due to the difference in the unconstrained estimators of λ. Moreover, deviations from the Mundlak's (or Chamberlain's) formulation will further widen the difference between FE and CRE estimators.

Obviously, combining the SE and SL model to give a more general spatial panel data model, all the conclusions apply.

D. Dynamic (spatial) panel data model. Finally, we consider the general spatial dynamic panel data (SDPD) model studied in the paper, written in vector form with α_t being dropped for an easier exposition:

$$\mathbf{B}_1Y = \mathbf{B}_2Y_{-1} + X\beta + (1_T \otimes Z)\gamma + 1_T \otimes \mu + \mathbf{u}, \;\; \mathbf{u} = \mathbf{B}_3\mathbf{v}, \tag{I.19}$$

where $\mathbf{B}_r, r = 1, 2, 3$ depends, respectively, on λ_1, ρ and λ_2, and λ_3. The FE-estimation proceeds by applying certain transformation on (I.19) to wipe out the fixed effects μ. Yang (2018) applied the first-difference transformation. Now, if ρ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)'$ were known, then one can apply the Q-transformation on (I.19) to get an FE-estimator of β, which is the GLS estimator of the same form as the FE-estimator in the panel SE model discussed above:

$$\hat{\beta}_{\texttt{FE}}(\rho, \lambda) = (X^* \mathbf{B}_3' \mathbf{B}_3 X^{*\prime})^{-1} X^{*\prime} (\mathbf{B}_3' \mathbf{B}_3) Y^*(\rho, \lambda_2),$$

where $X^* = QX$, and $Y^*(\rho, \lambda_2) = Q(\mathbf{B}_1 Y - \mathbf{B}_2 Y_{-1})$. Note that the Q-transformation also wipes out Z. Under the Mundlak's CRE formulation for μ, the model takes the form:

$$\mathbf{B}_1 Y = \mathbf{B}_2 Y_{-1} + QX\beta + PX\theta + (1_T \otimes Z)\gamma + \mathbf{e}, \quad \mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}_3 \mathbf{v}, \tag{I.20}$$

where θ and $\mathbf{e}$ are identical to those in Model (I.13). Given ρ, λ, and ϕ (defined in (I.13)), **and if** $\gamma = 0$, Model (I.20) reduces to Model (I.13) with Y being replaced by $\mathbf{B}_1 Y - \mathbf{B}_2 Y_{-1}$, and therefore $\hat{\beta}_{\texttt{MUND}}(\rho, \lambda)$, which is the GLS estimator of β of this reduced model, has an identical expression as $\hat{\beta}_{\texttt{FE}}(\rho, \lambda)$. However, this equivalence immediately fails to hold if we move away from Mundlak or Chamberlain's specifications as show in Section A.5 and A.6. Furthermore, the unconstrained estimates of ρ and λ can be quite different as they solve different sets of estimating equations as shown in Yang (2018) for the FE estimation and the current paper for the CRE estimation.

II. Extensions

Various extensions have been discussed in Section 6 of the main paper. Here we provide some details on possible extensions of our work in relation to issues such as *time dependence*, *time heterogeneity*, and *cross-sectional heteroskedasticity*.

Time dependence. In the paper, we have focused on the case where the idiosyncratic errors $\{v_{it}\}$ are iid. While time dependence is already built in the model as a dynamic lag of the response, it may be important to allow time dependence in $\{v_t\}$ as well. We show that our results can be extended by allowing v_t to follow an MA(1) process:

$$v_t = \nu_t + \tau \nu_{t-1}, \tag{II.1}$$

where $\{\nu_{it}\}$ are $i.i.d(0, \sigma_\nu^2)$. It is easy to see that $\mathrm{E}(v_t v_t') = (1+\tau^2)\sigma_\nu^2 I_n$, and that $\mathrm{E}(v_t v_{t-1}') = \mathrm{E}(v_{t-1} v_t') = \tau \sigma_\nu^2 I_n$, $t = 2, \ldots, T$, so that $\mathrm{E}(\mathbf{v}\mathbf{v}') = \sigma_\nu^2 \Sigma \otimes I_n$, where $\Sigma = (1+\tau^2) I_T + \tau A$ and A is $T \times T$ with its (i,j)th element being 1 if $i = j \pm 1$, and 0 otherwise. Then, letting $\phi = \sigma_\varepsilon^2 / \sigma_\nu^2$, the VC matrix of the composite errors, $\mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}_3^{-1}\mathbf{v}$, takes a similar form:

$$\mathrm{Var}(\mathbf{e}) = \sigma_\nu^2 [\phi (J_T \otimes I_n) + \Sigma \otimes (B_3' B_3)^{-1}] \equiv \sigma_\nu^2 \Omega. \tag{II.2}$$

With the new parameter τ, the vector of model parameters is $\psi = (\boldsymbol{\beta}', \sigma_\nu^2, \phi, \tau, \rho, \lambda')'$, and the conditional quasi-score function becomes:

$$S_{\texttt{SDPD}}(\psi) = \begin{cases} \frac{1}{\sigma_\nu^2}\mathbf{X}'\Omega^{-1}e(\theta), \\ \frac{1}{2\sigma_\nu^4}[e'(\theta)\Omega^{-1}e(\theta)] - \frac{nT}{2\sigma_\nu^2}, \\ \frac{1}{2\sigma_\nu^2}e'(\theta)\Omega^{-1}(J_T \otimes I_n)\Omega^{-1}e(\theta) - \frac{1}{2}\text{tr}[\Omega^{-1}(J_T \otimes I_n)], \\ \frac{1}{2\sigma_\nu^2}e'(\theta)\Omega^{-1}\dot{\Omega}_\tau\Omega^{-1}e(\theta) - \frac{1}{2}\text{tr}(\Omega^{-1}\dot{\Omega}_\tau), \\ \frac{1}{\sigma_\nu^2}e'(\theta)\Omega^{-1}Y_{-1}, \\ \frac{1}{\sigma_\nu^2}e'(\theta)\Omega^{-1}\mathbf{W}_1 Y - \text{tr}(\mathbf{B}_1^{-1}\mathbf{W}_1), \\ \frac{1}{\sigma_\nu^2}e'(\theta)\Omega^{-1}\mathbf{W}_2 Y_{-1}, \\ \frac{1}{2\sigma_\nu^2}e'(\theta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}e(\theta) - \frac{1}{2}\text{tr}(\Omega^{-1}\dot{\Omega}_{\lambda_3}), \end{cases} \tag{II.3}$$

where $\dot{\Omega}_\tau = (2\alpha I_t + A) \otimes (B_3'B_3)^{-1}$, and $\dot{\Omega}_{\lambda_3} = \Sigma \otimes [(B_3'B_3)^{-1}(B_3'W_3 + W_3'B_3)(B_3'B_3)^{-1}]$.

It is easy to see that the τ-component of $E[S_{\texttt{SDPD}}(\psi_0)]$ also has expectation zero, besides the $(\beta, \sigma_\nu^2, \phi, \lambda_3$-components. With the MA(1) structure of v_t, a similar set of results as Lemma 2.1 are obtained,

$$\text{E}(Y_{-1}\mathbf{e}') = \sigma_{\nu 0}^2(\phi_0\mathbf{C}_{-10} + \mathbf{D}_{-10}), \tag{II.4}$$

$$\text{E}(Y\mathbf{e}') = \sigma_{\nu 0}^2(\phi_0\mathbf{C}_0 + \mathbf{D}_0), \tag{II.5}$$

with $\mathbf{C}$ and $\mathbf{C}_{-1}$ being kept the same, but $\mathbf{D}$ and $\mathbf{D}_{-1}$ taking new and slightly more complicated expressions:

$$\mathbf{D} = \begin{pmatrix} D_0 & D_{-1} & 0 & \ldots & 0 & 0 \\ D_1 & D_0 & D_{-1} & \ldots & 0 & 0 \\ D_2 & D_1 & D_0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{T-2} & D_{T-3} & D_{T-4} & \ldots & D_0 & D_{-1} \\ D_{T-1} & D_{T-2} & D_{T-3} & \ldots & D_1 & D_0 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_{-1} = \begin{pmatrix} D_{-1} & 0 & \ldots & 0 & 0 \\ D_0 & D_{-1} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{T-3} & D_{T-4} & \ldots & D_{-1} & 0 \\ D_{T-2} & D_{T-3} & \ldots & D_0 & D_{-1} \end{pmatrix},$$

where $D_{-1} = \tau B_1^{-1}B_3^{-1}\sigma_\nu^2$, $D_0 = \mathcal{B}D_{-1} + (1+\tau^2)B_1^{-1}B_3^{-1}\sigma_\nu^2$, $D_1 = \mathcal{B}D_0 + D_{-1}$, and $D_t = \mathcal{B}^{t-1}D_1, t = 2, 3, \ldots, T-1$. With these results, we have the desired AQS functions:

$$S^*_{\texttt{SDPD}}(\psi) = \begin{cases} \frac{1}{\sigma_\nu^2}\mathbf{X}'\Omega^{-1}e(\theta), \\ \frac{1}{2\sigma_\nu^4}[e'(\theta)\Omega^{-1}e(\theta)] - \frac{nT}{2\sigma_\nu^2}, \\ \frac{1}{2\sigma_\nu^2}e'(\theta)\Omega^{-1}(J_T \otimes I_n)\Omega^{-1}e(\theta) - \frac{1}{2}\text{tr}[\Omega^{-1}(J_T \otimes I_n)], \\ \frac{1}{2\sigma_\nu^2}e'(\theta)\Omega^{-1}\dot{\Omega}_\tau\Omega^{-1}e(\theta) - \frac{1}{2}\text{tr}(\Omega^{-1}\dot{\Omega}_\tau), \\ \frac{1}{\sigma_\nu^2}e'(\theta)\Omega^{-1}Y_{-1} - \text{tr}[(\phi\mathbf{C}_{-1} + \mathbf{D}^\dagger_{-1})\Omega^{-1}], \\ \frac{1}{\sigma_\nu^2}e'(\theta)\Omega^{-1}\mathbf{W}_1 Y - \text{tr}[(\phi\mathbf{C} + \mathbf{D}^\dagger)\Omega^{-1}\mathbf{W}_1], \\ \frac{1}{\sigma_\nu^2}e'(\theta)\Omega^{-1}\mathbf{W}_2 Y_{-1} - \text{tr}[(\phi\mathbf{C}_{-1} + \mathbf{D}^\dagger_{-1})\Omega^{-1}\mathbf{W}_2], \\ \frac{1}{2\sigma_\nu^2}e'(\theta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}e(\theta) - \frac{1}{2}\text{tr}(\Omega^{-1}\dot{\Omega}_{\lambda_3}). \end{cases} \tag{II.6}$$

Solving the AQS equations, $S^*_{\texttt{SDPD}}(\psi) = 0$, gives the M-estimator of ψ. Theorems 2.1 and 2.2 can be extended as the AQS functions can be written as linear combinations of terms linear, quadratic, and bilinear in $\boldsymbol{\nu}$, $\boldsymbol{\nu}_{-1}$, ε, and V_m. Lemma 3.1 and Theorem 3.1 can be extended as well by re-defining the $\mathbf{g}_i$ functions and re-deriving the results in Lemma 3.1. While fundamental ideas are the same, these extensions require additional complicated algebra and deserve a full investigation as a future research.

Time heterogeneity. So far in the paper, the time heterogeneity appears in the model in the form of time-specific effects $\{\alpha_t\}$. It may be of interest to allow more extensive forms of time-heterogeneity such as time-varying regression coefficients, time-varying spatial coefficients, time-varying spatial weight matrices, etc.. From the theoretical developments given in the paper (in particular, Lemma 2.1, and the representations given in (2.14) and (2.17)), we see that our results may be extended to allow for time-varying regression coefficients by breaking the AQS component for β into T terms and reformulating the $\mathbf{X}$ matrix. However, our results may be extended to allow for the other two types of time-heterogeneity simply because β_t's in the past do not enter the 'adjustments' but λ_t's and W_t's in the past do. These points are reflected in the discussions in Sec. 6 of the paper.

Cross-sectional heteroskedasticity. Cross-sectional heteroskedasticity (varying error variances across spatial units) in the CRE-SDPD model is another interesting extension to consider. It requires an entirely different way to adjust the conditional quasi scores so that the AQS functions obtained are not only (asymptotically) unbiased but also robust against unknown cross-sectional heteroskedasticity. These models and methods would be much more challenging than the already quite challenging works presented in this paper, and will be the topics of our future research. Now, we present some details on possible extensions to allow for heteroskedastic in both v_{it} and ε_i. Let $\{v_{it}\}$ be independent and identically distributed (*iid*) across t for each i, and independent but not necessarily identically distribute (*inid*) across i for each t such that $\mathrm{E}(v_{it}) = 0$ and $\mathrm{Var}(v_{it}) = \sigma_v^2 h_{v,ni}$, $i = 1, \ldots, n$, where $h_{v,ni} > 0$ and $\frac{1}{n}\sum_{i=1}^n h_{v,ni} = 1$. Note that σ_v^2 is the average of $\mathrm{Var}(v_{it})$, which can be consistently estimated along with the other model parameters. Let $\mathcal{H}_v = \texttt{diag}(h_{v,n1}, h_{v,n2}, \ldots, h_{v,nn})$, where $\texttt{diag}(\cdot)$ forms a diagonal matrix based on the given the elements or based on the diagonal elements of a given matrix. Let $\{\varepsilon_i\}$ be inid with mean zero and variance $\sigma_\varepsilon^2 h_{\varepsilon,ni}$, where $h_{\varepsilon,ni} > 0$ and $\frac{1}{n}\sum_{i=1}^n h_{\varepsilon,ni} = 1$. Similarly let $\mathcal{H}_\varepsilon = \texttt{diag}(h_{\varepsilon,n1}, h_{\varepsilon,n2}, \ldots, h_{\varepsilon,nn})$. With this specification, the VC matrix of $\mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}_3^{-1}\mathbf{v}$ becomes:

$$\mathrm{Var}(\mathbf{e}) = \sigma_v^2[\phi(J_T \otimes \mathcal{H}_\varepsilon) + I_T \otimes (B_3'\mathcal{H}_v B_3)^{-1}] \equiv \sigma_v^2 \Omega_h. \tag{II.7}$$

The quasi-score functions remain the same but the results in Lemma 2.1 need to be updated:

$$\mathrm{E}(Y_{-1}\mathbf{e}') = \sigma_{v0}^2(\phi_0 \mathbf{C}^*_{-10} + \mathbf{D}^*_{-10}), \tag{II.8}$$

$$\mathrm{E}(Y\mathbf{e}') = \sigma_{v0}^2(\phi_0 \mathbf{C}^*_0 + \mathbf{D}^*_0), \tag{II.9}$$

where $\mathbf{C}^* = [1_T \otimes (C_1^{*\prime}, C_2^{*\prime}, \ldots, C_T^{*\prime})]'$, $\mathbf{C}^*_{-1} = [1_T \otimes (C_0^{*\prime}, C_1^{*\prime}, \ldots, C_{T-1}^{*\prime})]'$, and $C_t^* = (\sum_{i=0}^{t+m-1} \mathcal{B}^i) B_1^{-1} \mathcal{H}_\varepsilon$,

$$\mathbf{D}^* = \begin{pmatrix} D_0^* & 0 & \ldots & 0 & 0 \\ D_1^* & D_0^* & \ldots & 0 & 0 \\ D_2^* & D_1^* & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{T-1}^* & D_{T-2}^* & \ldots & D_1^* & D_0^* \end{pmatrix}, \quad \mathbf{D}_{-1}{}^* = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ D_0^* & 0 & \ldots & 0 & 0 \\ D_1^* & D_0^* & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{T-2}^* & D_{T-3}^* & \ldots & D_0^* & 0 \end{pmatrix},$$

and $D_t^* = \mathcal{B}^t B_1^{-1} B_3^{-1} \mathcal{H}_v (B_3^{-1})'$. With these results we see immediately that the adjustment terms to the quasi score functions would involve the unknown matrices $\mathcal{H}_v$ and $\mathcal{H}_\varepsilon$. Therefore we are not able to adjust the quasi score functions using the method proposed in the paper, and alternative methods are desired. The inference method proposed will not go through either as in this case as we need to estimate the third and fourth moments of v_{it}, which may change with i. However, it is interesting to note that if the unknown heteroskedasticity occurs on ε_i only, i.e., $\mathcal{H}_v = I_n$, out inference method remains valid. In other words, our inference method is robust against unknown heteroskedasticity in ε_i as the decomposed scores are martingale differences w.r.t. the σ-field generated by $\{v_{1t}, \ldots, v_{it}\}$. This raises an interesting question: is there a simple way to adjust the quasi score functions so that they become robust against unknown heteroskedasticity in ε_i as well? Apparently, the **cross-sectional heteroskedasticity** in the CRE-SDPD model is an interesting and important issue, to which no clear answers have been given. We will address this issue in a future research.

III. More Monte Carlo Results

In this section, we report all the Monte Carlo results, including those reported in the paper, and these unreported but mentioned in the paper for general discussions. Among all the Tables 1-9 reported here, Tables 1, 3, 7, 9, 13, and 16 are reported in the main paper.

Table 1. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP1**, $T = 3$, $m = 10$
$W_1 = W_3$: **Queen Contiguity;** W_2: **Group Interaction**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.8634(.417)	1.0514(.428)[.372]	.8819(.398)	1.0673(.411)[.376]	.8615(.394)	1.0472(.412)[.371]
	.26	.3213(.270)	.2599(.263)[.232]	.3083(.272)	.2475(.266)[.232]	.3240(.260)	.2623(.254)[.228]
	.23	.2577(.260)	.2317(.251)[.228]	.2498(.261)	.2242(.253)[.227]	.2604(.260)	.2345(.251)[.227]
	1	.9993(.053)	1.0028(.052)[.050]	.9979(.054)	1.0012(.053)[.050]	.9994(.053)	1.0028(.052)[.050]
	1	.9146(.319)	.9964(.339)[.322]	.9250(.311)	1.0070(.331)[.320]	.9113(.313)	.9922(.332)[.320]
	1	.8564(.163)	.9947(.153)[.136]	.8538(.166)	.9912(.156)[.137]	.8501(.167)	.9866(.157)[.136]
	1	.9948(.163)	.9397(.147)[.135]	1.0081(.261)	.9524(.242)[.205]	1.0038(.235)	.9489(.216)[.188]
	1	.7614(.374)	.9923(.432)[.374]	.7745(.424)	1.0027(.485)[.397]	.7740(.422)	1.0007(.486)[.390]
	.3	.3530(.055)	.2995(.050)[.043]	.3547(.055)	.3016(.050)[.043]	.3543(.055)	.3015(.051)[.043]
	.2	.1887(.052)	.1933(.053)[.049]	.1859(.050)	.1904(.051)[.049]	.1884(.053)	.1931(.054)[.049]
	.2	.1927(.039)	.1970(.040)[.037]	.1911(.039)	.1955(.040)[.037]	.1906(.039)	.1947(.039)[.036]
	.2	.1014(.182)	.0957(.184)[.169]	.0983(.183)	.0928(.188)[.165]	.1059(.174)	.0994(.177)[.165]
100	1	.8798(.269)	1.0172(.279)[.270]	.8851(.274)	1.0227(.284)[.271]	.8902(.274)	1.0275(.285)[.267]
	-.44	-.3924(.192)	-.4432(.186)[.176]	-.3972(.187)	-.4479(.182)[.176]	-.4017(.194)	-.4522(.188)[.173]
	.33	.3991(.197)	.3308(.189)[.179]	.4033(.195)	.3346(.187)[.179]	.3986(.197)	.3303(.190)[.175]
	1	1.0006(.036)	1.0001(.035)[.035]	.9996(.036)	.9991(.036)[.035]	1.0011(.036)	1.0004(.036)[.035]
	1	.9189(.224)	.9966(.237)[.230]	.9160(.217)	.9929(.229)[.231]	.9213(.220)	.9985(.232)[.230]
	1	.8425(.117)	.9940(.106)[.099]	.8489(.119)	1.0000(.109)[.100]	.8465(.118)	.9974(.107)[.100]
	1	1.0350(.117)	.9792(.105)[.100]	1.0389(.188)	.9825(.170)[.155]	1.0336(.171)	.9776(.156)[.143]
	1	.7552(.254)	.9813(.295)[.259]	.7701(.284)	.9958(.322)[.284]	.7660(.278)	.9913(.315)[.280]
	.3	.3547(.037)	.3014(.033)[.030]	.3532(.037)	.3001(.033)[.030]	.3535(.037)	.3004(.033)[.030]
	.2	.1851(.025)	.1979(.026)[.025]	.1849(.026)	.1977(.027)[.025]	.1846(.026)	.1974(.027)[.025]
	.2	.1908(.028)	.1980(.029)[.028]	.1909(.027)	.1980(.028)[.028]	.1892(.028)	.1963(.029)[.028]
	.2	.1648(.118)	.1521(.121)[.116]	.1650(.118)	.1511(.120)[.114]	.1621(.117)	.1498(.120)[.114]
200	1	.9030(.207)	1.0226(.217)[.213]	.9003(.206)	1.0193(.216)[.213]	.9007(.206)	1.0201(.218)[.213]
	-.25	-.2330(.136)	-.2578(.132)[.131]	-.2248(.139)	-.2498(.135)[.130]	-.2240(.138)	-.2493(.135)[.130]
	.30	.3411(.135)	.2996(.130)[.126]	.3400(.133)	.2989(.129)[.125]	.3419(.134)	.3005(.129)[.125]
	1	.9994(.025)	1.0007(.025)[.025]	.9989(.026)	1.0003(.026)[.025]	.9996(.025)	1.0010(.025)[.025]
	1	.9250(.160)	.9969(.171)[.167]	.9290(.156)	1.0005(.167)[.167]	.9265(.160)	.9983(.172)[.167]
	1	.8292(.086)	.9963(.078)[.072]	.8326(.086)	.9979(.077)[.073]	.8328(.087)	.9993(.078)[.072]
	1	1.0502(.085)	.9872(.075)[.072]	1.0458(.129)	.9838(.116)[.113]	1.0523(.122)	.9893(.109)[.105]
	1	.7383(.179)	.9919(.208)[.190]	.7528(.196)	1.0044(.227)[.207]	.7438(.184)	.9961(.210)[.203]
	.3	.3608(.028)	.3010(.024)[.022]	.3595(.028)	.3003(.024)[.023]	.3594(.028)	.2998(.024)[.023]
	.2	.1859(.022)	.1979(.023)[.023]	.1866(.022)	.1984(.023)[.023]	.1862(.023)	.1981(.023)[.023]
	.2	.1846(.023)	.1974(.023)[.022]	.1847(.022)	.1973(.023)[.022]	.1854(.022)	.1982(.023)[.022]
	.2	.1861(.083)	.1761(.085)[.084]	.1854(.084)	.1754(.086)[.083]	.1873(.085)	.1775(.087)[.083]
400	1	.8942(.155)	1.0108(.160)[.158]	.8917(.151)	1.0085(.157)[.158]	.8946(.153)	1.0114(.160)[.157]
	-.21	-.1986(.093)	-.2091(.090)[.088]	-.1938(.092)	-.2045(.089)[.088]	-.1969(.090)	-.2076(.087)[.088]
	.42	.4432(.095)	.4170(.091)[.088]	.4485(.094)	.4222(.090)[.088]	.4428(.091)	.4167(.087)[.088]
	1	.9995(.019)	.9995(.019)[.019]	1.0002(.019)	1.0001(.019)[.019]	1.0007(.019)	1.0006(.018)[.019]
	1	.9231(.110)	1.0016(.118)[.117]	.9166(.112)	.9947(.119)[.117]	.9204(.113)	.9987(.120)[.117]
	1	.8415(.061)	1.0001(.054)[.051]	.8394(.061)	.9982(.055)[.052]	.8388(.061)	.9976(.055)[.052]
	1	1.0532(.060)	.9946(.054)[.051]	1.0538(.094)	.9948(.085)[.082]	1.0538(.088)	.9950(.080)[.075]
	1	.7564(.122)	.9954(.140)[.133]	.7607(.143)	1.0013(.163)[.148]	.7593(.141)	.9989(.160)[.144]
	.3	.3561(.019)	.3000(.017)[.015]	.3565(.019)	.3004(.017)[.016]	.3566(.019)	.3005(.017)[.016]
	.2	.1837(.014)	.1995(.015)[.014]	.1836(.014)	.1995(.014)[.014]	.1831(.014)	.1989(.014)[.014]
	.2	.1889(.021)	.1986(.021)[.021]	.1885(.020)	.1983(.021)[.021]	.1890(.021)	.1988(.021)[.021]
	.2	.2031(.059)	.1870(.060)[.059]	.2050(.059)	.1886(.061)[.059]	.2028(.059)	.1870(.061)[.059]

Note: $\psi = (\breve{\alpha}', \beta', \gamma', \pi', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 2. Empirical sd and average of estimated standard errors of M-Estimator
DGP1, $T = 3, m = 10$, Parameter configurations as in Table 1.

		Normal Error				Normal Mixture				Chi-Square			
n	ψ	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$
50	1	.428	.372	.379	.457	.411	.376	.379	.470	.412	.371	.378	.477
	.26	.263	.232	.234	.270	.266	.232	.234	.287	.254	.228	.234	.292
	.23	.251	.228	.227	.258	.253	.227	.227	.276	.251	.227	.228	.277
	1	.052	.050	.051	.062	.053	.050	.051	.068	.052	.050	.051	.067
	1	.339	.322	.323	.388	.331	.320	.322	.397	.332	.320	.322	.394
	1	.153	.136	.138	.169	.156	.137	.138	.179	.157	.136	.137	.181
	1	.147	.135	.138	.172	.242	.205	.141	.145	.216	.188	.140	.155
	1	.432	.374	.380	.483	.485	.397	.382	.497	.486	.390	.381	.503
	.3	.050	.043	.043	.052	.050	.043	.043	.055	.051	.043	.042	.055
	.2	.053	.049	.050	.062	.051	.049	.050	.064	.054	.049	.050	.063
	.2	.040	.037	.037	.046	.040	.037	.037	.047	.039	.036	.037	.048
	.2	.184	.169	.173	.216	.188	.165	.173	.237	.177	.165	.173	.234
100	1	.279	.270	.271	.293	.284	.271	.271	.298	.285	.267	.271	.304
	-.44	.186	.176	.177	.188	.182	.176	.177	.194	.188	.173	.176	.194
	.33	.189	.179	.179	.188	.187	.179	.178	.194	.190	.175	.178	.195
	1	.035	.035	.035	.039	.036	.035	.035	.041	.036	.035	.035	.041
	1	.237	.230	.230	.250	.229	.231	.230	.254	.232	.230	.230	.252
	1	.106	.099	.100	.110	.109	.100	.100	.114	.107	.100	.100	.114
	1	.105	.100	.101	.112	.170	.155	.102	.085	.156	.143	.101	.093
	1	.295	.259	.264	.297	.322	.284	.266	.301	.315	.280	.265	.299
	.3	.033	.030	.030	.033	.033	.030	.030	.034	.033	.030	.030	.034
	.2	.026	.025	.025	.028	.027	.025	.026	.029	.027	.025	.025	.028
	.2	.029	.028	.028	.031	.028	.028	.028	.031	.029	.028	.028	.031
	.2	.121	.116	.118	.130	.120	.114	.118	.138	.120	.114	.118	.137
200	1	.217	.213	.214	.224	.216	.213	.214	.226	.218	.213	.215	.229
	-.25	.132	.131	.131	.135	.135	.130	.131	.137	.135	.130	.131	.139
	.30	.130	.126	.126	.129	.129	.125	.125	.131	.129	.125	.126	.133
	1	.025	.025	.025	.026	.026	.025	.025	.027	.025	.025	.025	.027
	1	.171	.167	.167	.174	.167	.167	.167	.175	.172	.167	.168	.175
	1	.078	.072	.072	.075	.077	.073	.072	.077	.078	.072	.072	.077
	1	.075	.072	.072	.076	.116	.113	.072	.054	.109	.105	.072	.060
	1	.208	.190	.190	.201	.227	.207	.192	.200	.210	.203	.191	.200
	.3	.024	.022	.022	.023	.024	.023	.022	.024	.024	.023	.022	.024
	.2	.023	.023	.023	.025	.023	.023	.023	.025	.023	.023	.023	.025
	.2	.023	.022	.023	.024	.023	.022	.023	.024	.023	.022	.023	.024
	.2	.085	.084	.084	.088	.086	.083	.084	.091	.087	.083	.084	.091
400	1	.160	.158	.158	.162	.157	.158	.158	.163	.160	.157	.158	.165
	-.21	.090	.088	.088	.089	.089	.088	.088	.090	.087	.088	.088	.090
	.42	.091	.088	.088	.089	.090	.088	.088	.090	.087	.088	.088	.091
	1	.019	.019	.019	.019	.019	.019	.019	.019	.018	.019	.019	.019
	1	.118	.117	.117	.120	.119	.117	.117	.120	.120	.117	.117	.120
	1	.054	.051	.051	.053	.055	.052	.051	.053	.055	.052	.051	.053
	1	.054	.051	.051	.053	.085	.082	.051	.036	.080	.075	.051	.040
	1	.140	.133	.134	.137	.163	.148	.134	.134	.160	.144	.134	.135
	.3	.017	.015	.015	.016	.017	.016	.015	.016	.017	.016	.015	.016
	.2	.015	.014	.014	.015	.014	.014	.014	.015	.014	.014	.014	.015
	.2	.021	.021	.021	.022	.021	.021	.021	.022	.021	.021	.021	.022
	.2	.060	.059	.059	.060	.061	.059	.059	.061	.061	.059	.059	.061

Table 3 Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP1**, $T = 6$, $m = 10$

$W_1 = W_3$: **Queen Contiguity;** W_2: **Group Interaction**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.9942(.335)	1.0273(.338)[.306]	.9962(.331)	1.0287(.333)[.301]	.9882(.323)	1.0211(.327)[.304]
	-.20	-.2026(.253)	-.1951(.252)[.229]	-.1964(.254)	-.1893(.254)[.227]	-.1968(.254)	-.1896(.254)[.229]
	.44	.4376(.258)	.4439(.258)[.232]	.4397(.251)	.4458(.251)[.230]	.4395(.253)	.4457(.252)[.232]
	-.19	-.2114(.256)	-.1953(.255)[.233]	-.1920(.259)	-.1762(.258)[.231]	-.2008(.261)	-.1849(.260)[.234]
	-.31	-.3212(.242)	-.3113(.241)[.230]	-.3200(.252)	-.3102(.250)[.229]	-.3229(.259)	-.3130(.257)[.232]
	.41	.4159(.250)	.4138(.248)[.228]	.4240(.249)	.4220(.247)[.226]	.4086(.255)	.4068(.253)[.228]
	1	.9975(.028)	1.0005(.028)[.028]	.9977(.029)	1.0007(.029)[.027]	.9970(.029)	1.0000(.029)[.027]
	1	.9905(.306)	1.0022(.309)[.295]	.9866(.300)	.9981(.303)[.293]	.9955(.300)	1.0071(.303)[.292]
	1	.9752(.087)	.9965(.086)[.082]	.9786(.087)	.9995(.087)[.082]	.9760(.089)	.9970(.089)[.082]
	1	.9649(.088)	.9606(.088)[.085]	.9711(.160)	.9666(.158)[.144]	.9708(.145)	.9665(.144)[.133]
	1	.9273(.270)	.9603(.277)[.256]	.9385(.441)	.9732(.468)[.364]	.9267(.395)	.9594(.407)[.337]
	.3	.3085(.025)	.2997(.024)[.023]	.3075(.025)	.2988(.025)[.023]	.3085(.025)	.2998(.025)[.023]
	.2	.1948(.040)	.1953(.040)[.036]	.1957(.039)	.1963(.039)[.035]	.1940(.039)	.1946(.039)[.036]
	.2	.1996(.031)	.1977(.031)[.028]	.1987(.030)	.1968(.030)[.028]	.1995(.031)	.1976(.031)[.028]
	.2	.1199(.115)	.1193(.116)[.109]	.1195(.113)	.1187(.113)[.107]	.1239(.113)	.1229(.113)[.107]
100	1	.9828(.245)	1.0173(.246)[.234]	.9758(.239)	1.0100(.240)[.232]	.9883(.243)	1.0232(.246)[.233]
	-.26	-.2657(.176)	-.2655(.176)[.170]	-.2632(.179)	-.2630(.178)[.169]	-.2607(.177)	-.2606(.177)[.169]
	-.30	-.2973(.181)	-.3046(.180)[.172]	-.2919(.185)	-.2992(.185)[.172]	-.2977(.179)	-.3051(.178)[.171]
	-.21	-.2145(.181)	-.2244(.180)[.174]	-.2063(.182)	-.2162(.182)[.174]	-.2050(.184)	-.2150(.184)[.174]
	.17	.1782(.180)	.1662(.179)[.173]	.1801(.187)	.1682(.186)[.173]	.1828(.187)	.1708(.186)[.172]
	.29	.2949(.174)	.2887(.173)[.170]	.3054(.179)	.2991(.178)[.170]	.3028(.179)	.2966(.178)[.169]
	1	.9971(.021)	1.0010(.020)[.020]	.9956(.021)	.9995(.021)[.020]	.9956(.021)	.9995(.021)[.020]
	1	.9849(.222)	.9984(.225)[.213]	.9888(.219)	1.0023(.222)[.212]	.9816(.217)	.9951(.220)[.213]
	1	.9800(.062)	.9997(.062)[.061]	.9814(.062)	1.0011(.062)[.060]	.9817(.064)	1.0014(.064)[.061]
	1	.9856(.064)	.9814(.064)[.062]	.9885(.114)	.9842(.113)[.108]	.9896(.102)	.9853(.102)[.099]
	1	.9451(.187)	.9772(.192)[.186]	.9490(.312)	.9814(.321)[.279]	.9553(.286)	.9879(.295)[.263]
	.3	.3076(.017)	.2990(.017)[.016]	.3083(.017)	.2998(.016)[.016]	.3077(.018)	.2992(.017)[.017]
	.2	.1972(.026)	.1982(.026)[.025]	.1965(.025)	.1974(.025)[.024]	.1970(.026)	.1980(.026)[.024]
	.2	.2002(.026)	.1988(.026)[.026]	.2003(.027)	.1989(.027)[.026]	.1997(.028)	.1983(.028)[.026]
	.2	.1641(.081)	.1630(.081)[.077]	.1644(.080)	.1631(.080)[.076]	.1607(.080)	.1595(.080)[.076]
200	1	.9747(.236)	1.0161(.236)[.230]	.9672(.232)	1.0083(.232)[.228]	.9688(.228)	1.0100(.229)[.228]
	.34	.3413(.127)	.3377(.127)[.123]	.3402(.124)	.3365(.124)[.123]	.3450(.124)	.3412(.124)[.123]
	.15	.1460(.127)	.1474(.127)[.123]	.1383(.125)	.1397(.125)[.122]	.1477(.129)	.1491(.128)[.123]
	.05	.0481(.127)	.0490(.126)[.123]	.0477(.125)	.0485(.124)[.122]	.0495(.124)	.0503(.124)[.122]
	.42	.4185(.126)	.4173(.125)[.124]	.4146(.130)	.4134(.130)[.124]	.4186(.124)	.4174(.123)[.124]
	-.38	-.3868(.128)	-.3826(.128)[.123]	-.3905(.127)	-.3863(.126)[.123]	-.3874(.127)	-.3832(.126)[.123]
	1	.9967(.014)	1.0001(.014)[.014]	.9965(.014)	.9999(.014)[.014]	.9963(.014)	.9997(.014)[.014]
	1	.9904(.154)	1.0022(.155)[.152]	.9897(.153)	1.0014(.155)[.152]	.9893(.152)	1.0010(.154)[.151]
	1	.9818(.045)	1.0012(.045)[.043]	.9799(.045)	.9992(.045)[.043]	.9810(.044)	1.0004(.044)[.043]
	1	.9972(.045)	.9932(.044)[.045]	.9954(.079)	.9914(.078)[.078]	.9990(.073)	.9950(.072)[.072]
	1	.9594(.136)	.9896(.140)[.133]	.9610(.219)	.9911(.225)[.209]	.9521(.193)	.9821(.198)[.189]
	.3	.3074(.012)	.2995(.012)[.011]	.3079(.012)	.2999(.012)[.011]	.3078(.012)	.2999(.012)[.011]
	.2	.1987(.019)	.1992(.019)[.019]	.1992(.019)	.1996(.019)[.019]	.1987(.019)	.1992(.019)[.019]
	.2	.1987(.020)	.1986(.020)[.020]	.1996(.020)	.1995(.020)[.020]	.1990(.020)	.1989(.020)[.020]
	.2	.1805(.055)	.1801(.055)[.055]	.1799(.056)	.1792(.056)[.055]	.1807(.055)	.1799(.055)[.055]
400	1	.9822(.142)	1.0120(.142)[.142]	.9783(.143)	1.0079(.143)[.141]	.9748(.141)	1.0045(.142)[.141]
	-.20	-.2020(.091)	-.2061(.091)[.089]	-.2017(.090)	-.2058(.090)[.089]	-.1994(.090)	-.2036(.090)[.089]
	.16	.1670(.090)	.1598(.090)[.089]	.1642(.089)	.1570(.089)[.089]	.1705(.091)	.1633(.091)[.089]
	-.10	-.0909(.090)	-.0944(.090)[.088]	-.0964(.089)	-.0998(.089)[.088]	-.0929(.091)	-.0963(.091)[.088]
	-.001	-.0007(.089)	-.0015(.088)[.087]	-.0009(.088)	-.0017(.087)[.087]	.0016(.094)	.0007(.094)[.087]
	-.22	-.2254(.090)	-.2249(.089)[.087]	-.2275(.089)	-.2269(.088)[.087]	-.2242(.092)	-.2236(.092)[.087]
	1	.9962(.010)	.9997(.010)[.010]	.9972(.010)	1.0007(.010)[.010]	.9962(.010)	.9997(.010)[.010]
	1	.9842(.106)	.9948(.108)[.108]	.9866(.110)	.9972(.111)[.108]	.9905(.107)	1.0011(.109)[.108]
	1	.9810(.029)	1.0004(.029)[.030]	.9819(.030)	1.0011(.029)[.030]	.9814(.030)	1.0007(.030)[.030]
	1	1.0001(.031)	.9961(.031)[.032]	1.0005(.057)	.9966(.057)[.056]	1.0005(.053)	.9965(.052)[.051]
	1	.9642(.093)	.9943(.095)[.095]	.9657(.153)	.9958(.157)[.152]	.9642(.138)	.9943(.142)[.139]
	.3	.3079(.008)	.3000(.008)[.008]	.3073(.009)	.2994(.008)[.008]	.3077(.008)	.2998(.008)[.008]
	.2	.1985(.012)	.1998(.012)[.012]	.1984(.012)	.1997(.012)[.012]	.1983(.012)	.1996(.012)[.012]
	.2	.1991(.016)	.1989(.016)[.016]	.1999(.016)	.1996(.016)[.016]	.1995(.016)	.1992(.016)[.016]
	.2	.1917(.038)	.1912(.038)[.038]	.1917(.037)	.1912(.038)[.038]	.1919(.039)	.1915(.039)[.038]

Note: $\psi = (\tilde{\alpha}', \beta', \gamma', \pi', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 4. Empirical sd and average of estimated standard errors of M-Estimator
DGP1, $T = 6, m = 10$, Parameter configurations as in Table 3.

		Normal Error				Normal Mixture				Chi-Square			
n	ψ	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$
50	1	.338	.306	.313	.387	.333	.301	.312	.418	.327	.304	.312	.421
	-.20	.252	.229	.229	.264	.254	.227	.229	.277	.254	.229	.230	.275
	.44	.258	.232	.232	.268	.251	.230	.232	.282	.252	.232	.233	.279
	-.19	.255	.233	.234	.271	.258	.231	.234	.285	.260	.234	.235	.280
	-.31	.241	.230	.231	.267	.250	.229	.231	.280	.257	.232	.232	.275
	.41	.248	.228	.228	.260	.247	.226	.227	.274	.253	.228	.229	.272
	1	.028	.028	.028	.034	.029	.027	.028	.036	.029	.027	.028	.036
	1	.309	.295	.296	.361	.303	.293	.293	.392	.303	.292	.293	.379
	1	.086	.082	.084	.106	.087	.082	.084	.114	.089	.082	.084	.112
	1	.088	.085	.087	.109	.158	.144	.088	.077	.144	.133	.087	.084
	1	.277	.256	.264	.344	.468	.364	.267	.301	.407	.337	.264	.316
	.3	.024	.023	.023	.029	.025	.023	.023	.031	.025	.023	.023	.031
	.2	.040	.036	.037	.046	.039	.035	.037	.049	.039	.036	.037	.048
	.2	.031	.028	.030	.039	.030	.028	.029	.042	.031	.028	.029	.040
	.2	.116	.109	.111	.138	.113	.107	.111	.149	.113	.107	.111	.145
100	1	.246	.234	.235	.256	.240	.232	.234	.269	.246	.233	.236	.282
	-.26	.176	.170	.170	.181	.178	.169	.171	.186	.177	.169	.170	.184
	-.30	.180	.172	.173	.184	.185	.172	.173	.188	.178	.171	.173	.188
	-.21	.180	.174	.175	.186	.182	.174	.175	.190	.184	.174	.174	.189
	.17	.179	.173	.173	.183	.186	.173	.173	.189	.186	.172	.173	.187
	.29	.173	.170	.170	.179	.178	.170	.170	.184	.178	.169	.170	.183
	1	.020	.020	.020	.023	.021	.020	.020	.023	.021	.020	.020	.023
	1	.225	.213	.213	.234	.222	.212	.212	.249	.220	.213	.213	.245
	1	.062	.061	.061	.069	.062	.060	.061	.073	.064	.061	.062	.072
	1	.064	.062	.063	.069	.113	.108	.063	.045	.102	.099	.063	.050
	1	.192	.186	.189	.214	.321	.279	.190	.171	.295	.263	.191	.189
	.3	.017	.016	.017	.018	.016	.016	.017	.019	.017	.017	.017	.019
	.2	.026	.025	.025	.028	.025	.024	.025	.029	.026	.024	.025	.029
	.2	.026	.026	.026	.029	.027	.026	.026	.031	.028	.026	.026	.030
	.2	.081	.077	.078	.085	.080	.076	.078	.088	.080	.076	.078	.088
200	1	.236	.230	.230	.241	.232	.228	.229	.248	.229	.228	.230	.254
	.34	.127	.123	.123	.127	.124	.123	.123	.128	.124	.123	.123	.128
	.15	.127	.123	.123	.126	.125	.122	.123	.128	.128	.123	.123	.128
	.05	.126	.123	.123	.126	.124	.122	.122	.127	.124	.122	.123	.127
	.42	.125	.124	.124	.128	.130	.124	.124	.129	.123	.124	.124	.129
	-.38	.128	.123	.123	.126	.126	.123	.123	.128	.126	.123	.123	.128
	1	.014	.014	.014	.015	.014	.014	.014	.015	.014	.014	.014	.015
	1	.155	.152	.152	.159	.155	.152	.152	.165	.154	.151	.151	.163
	1	.045	.043	.043	.046	.045	.043	.043	.047	.044	.043	.043	.047
	1	.044	.045	.045	.047	.078	.078	.045	.029	.072	.072	.045	.033
	1	.140	.133	.134	.143	.225	.209	.134	.105	.198	.189	.133	.120
	.3	.012	.011	.011	.012	.012	.011	.011	.012	.012	.011	.011	.012
	.2	.019	.019	.019	.020	.019	.019	.019	.020	.019	.019	.019	.020
	.2	.020	.020	.020	.021	.020	.020	.020	.021	.020	.020	.020	.021
	.2	.055	.055	.055	.058	.056	.055	.055	.059	.055	.055	.055	.059
400	1	.142	.142	.142	.145	.143	.141	.142	.148	.142	.141	.142	.155
	-.20	.091	.089	.089	.090	.090	.089	.089	.091	.090	.089	.089	.091
	.16	.090	.089	.089	.090	.089	.089	.089	.091	.091	.089	.089	.091
	-.10	.090	.088	.088	.089	.089	.088	.088	.089	.091	.088	.088	.089
	-.001	.088	.087	.087	.089	.087	.087	.087	.089	.094	.087	.087	.089
	-.22	.089	.087	.087	.088	.088	.087	.087	.089	.092	.087	.087	.089
	1	.010	.010	.010	.010	.010	.010	.010	.010	.010	.010	.010	.010
	1	.108	.108	.108	.111	.111	.108	.108	.113	.109	.108	.108	.113
	1	.029	.030	.030	.030	.029	.030	.030	.031	.030	.030	.030	.031
	1	.031	.032	.032	.033	.057	.056	.032	.019	.052	.051	.032	.022
	1	.095	.095	.095	.098	.157	.152	.095	.068	.142	.139	.095	.080
	.3	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008	.008
	.2	.012	.012	.012	.012	.012	.012	.012	.012	.012	.012	.012	.012
	.2	.016	.016	.016	.017	.016	.016	.016	.017	.016	.016	.016	.017
	.2	.038	.038	.039	.039	.038	.038	.039	.040	.039	.038	.039	.040

Table 5. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP1**, $T = 3$, $m = 10$
$W_1 = W_2 = W_3$: **Group Interaction**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.8959(.340)	1.0452(.355)[.310]	.9057(.335)	1.0545(.350)[.315]	.8792(.329)	1.0277(.344)[.309]
	.30	.3776(.294)	.2918(.282)[.248]	.3677(.295)	.2820(.284)[.255]	.3885(.290)	.3022(.277)[.247]
	.24	.2781(.269)	.2393(.255)[.234]	.2648(.272)	.2264(.259)[.237]	.2807(.273)	.2418(.258)[.235]
	1	.9920(.058)	.9982(.057)[.054]	.9943(.057)	1.0003(.057)[.056]	.9937(.059)	.9999(.058)[.054]
	1	.8928(.323)	.9849(.343)[.324]	.9059(.315)	1.0008(.333)[.326]	.8992(.327)	.9931(.349)[.322]
	1	.8579(.161)	1.0003(.155)[.139]	.8465(.162)	.9906(.154)[.142]	.8442(.168)	.9866(.160)[.136]
	1	.9859(.162)	.9295(.150)[.139]	1.0016(.264)	.9422(.240)[.208]	.9974(.241)	.9396(.222)[.186]
	1	.7696(.378)	1.0209(.449)[.388]	.7750(.422)	1.0248(.491)[.410]	.7778(.433)	1.0268(.503)[.407]
	.3	.3580(.056)	.3006(.053)[.045]	.3609(.056)	.3031(.051)[.047]	.3611(.057)	.3037(.053)[.044]
	.2	.1754(.101)	.1957(.096)[.090]	.1721(.104)	.1920(.098)[.099]	.1722(.107)	.1922(.102)[.092]
	.2	.2003(.111)	.1964(.100)[.096]	.2007(.113)	.1974(.102)[.104]	.2020(.116)	.1985(.106)[.097]
	.2	.1124(.206)	.0965(.201)[.180]	.1141(.209)	.0985(.204)[.180]	.1164(.204)	.1009(.199)[.176]
100	1	1.0010(.236)	1.0155(.232)[.218]	.9966(.231)	1.0079(.225)[.230]	.9981(.233)	1.0120(.229)[.232]
	-.47	-.5815(.282)	-.4764(.254)[.243]	-.5891(.285)	-.4809(.257)[.262]	-.5913(.276)	-.4857(.252)[.281]
	-.42	-.4808(.236)	-.4301(.213)[.198]	-.4890(.236)	-.4362(.213)[.211]	-.4823(.236)	-.4318(.214)[.216]
	1	.9933(.040)	.9987(.039)[.038]	.9929(.040)	.9986(.039)[.040]	.9926(.039)	.9981(.038)[.041]
	1	.9001(.227)	.9898(.244)[.236]	.9153(.224)	1.0070(.242)[.238]	.9068(.222)	.9971(.238)[.237]
	1	.8294(.121)	.9959(.111)[.102]	.8335(.120)	.9990(.111)[.103]	.8318(.124)	.9978(.113)[.105]
	1	1.0292(.118)	.9678(.104)[.100]	1.0280(.185)	.9674(.167)[.155]	1.0259(.171)	.9646(.153)[.146]
	1	.7447(.257)	1.0020(.303)[.270]	.7651(.287)	1.0231(.337)[.294]	.7612(.286)	1.0204(.333)[.290]
	.3	.3633(.039)	.3017(.035)[.032]	.3623(.040)	.3009(.036)[.033]	.3631(.041)	.3017(.037)[.035]
	.2	.1433(.121)	.1957(.108)[.109]	.1369(.127)	.1914(.113)[.122]	.1365(.122)	.1893(.111)[.131]
	.2	.2254(.120)	.1996(.105)[.108]	.2322(.125)	.2040(.109)[.119]	.2328(.122)	.2065(.108)[.128]
	.2	.1725(.178)	.1228(.176)[.178]	.1791(.186)	.1274(.184)[.180]	.1868(.171)	.1374(.172)[.183]
200	1	.8902(.157)	1.0022(.161)[.157]	.8967(.157)	1.0087(.162)[.157]	.8924(.152)	1.0048(.159)[.156]
	.14	.1524(.131)	.1426(.127)[.121]	.1493(.132)	.1394(.128)[.120]	.1535(.130)	.1435(.126)[.121]
	.35	.3631(.137)	.3517(.132)[.123]	.3586(.135)	.3472(.131)[.123]	.3576(.135)	.3464(.130)[.123]
	1	.9959(.028)	.9999(.028)[.027]	.9962(.028)	1.0002(.027)[.027]	.9951(.028)	.9990(.028)[.027]
	1	.9104(.156)	.9972(.165)[.165]	.9095(.160)	.9958(.170)[.166]	.9103(.158)	.9977(.167)[.165]
	1	.8464(.081)	.9990(.073)[.070]	.8493(.082)	1.0016(.074)[.070]	.8450(.082)	.9982(.074)[.070]
	1	1.0399(.081)	.9829(.073)[.071]	1.0414(.131)	.9838(.118)[.113]	1.0462(.119)	.9881(.107)[.105]
	1	.7680(.179)	1.0045(.207)[.189]	.7779(.196)	1.0157(.221)[.209]	.7609(.188)	.9969(.214)[.202]
	.3	.3561(.027)	.3004(.024)[.022]	.3549(.027)	.2993(.024)[.022]	.3566(.027)	.3007(.024)[.022]
	.2	.1854(.066)	.1996(.064)[.065]	.1854(.068)	.2000(.066)[.065]	.1841(.069)	.1983(.067)[.065]
	.2	.1847(.069)	.1993(.065)[.066]	.1853(.070)	.1995(.066)[.066]	.1856(.072)	.2003(.068)[.066]
	.2	.1537(.131)	.1464(.129)[.126]	.1516(.137)	.1431(.135)[.127]	.1507(.135)	.1430(.133)[.126]
400	1	.9001(.124)	1.0059(.128)[.123]	.8973(.125)	1.0030(.128)[.122]	.9042(.120)	1.0098(.125)[.121]
	.32	.3528(.101)	.3250(.096)[.096]	.3526(.102)	.3248(.097)[.096]	.3513(.104)	.3237(.099)[.095]
	-.09	-.0897(.094)	-.0849(.089)[.087]	-.0887(.093)	-.0839(.089)[.086]	-.0906(.094)	-.0857(.090)[.086]
	1	.9974(.019)	1.0001(.018)[.018]	.9970(.019)	.9995(.019)[.018]	.9978(.019)	1.0003(.018)[.018]
	1	.9265(.110)	1.0015(.117)[.118]	.9231(.113)	.9980(.120)[.117]	.9215(.112)	.9960(.119)[.117]
	1	.8621(.055)	1.0013(.050)[.049]	.8609(.056)	1.0002(.052)[.049]	.8600(.056)	.9989(.051)[.049]
	1	1.0431(.057)	.9917(.051)[.051]	1.0466(.089)	.9950(.081)[.082]	1.0439(.086)	.9927(.079)[.075]
	1	.7907(.118)	1.0066(.134)[.133]	.7887(.136)	1.0039(.153)[.146]	.7901(.137)	1.0039(.155)[.142]
	.3	.3494(.017)	.2994(.015)[.015]	.3500(.018)	.3000(.016)[.015]	.3500(.017)	.3002(.016)[.015]
	.2	.1744(.061)	.1971(.058)[.059]	.1742(.061)	.1969(.057)[.059]	.1762(.062)	.1985(.059)[.058]
	.2	.2019(.061)	.2020(.057)[.058]	.2019(.061)	.2020(.057)[.059]	.1991(.063)	.1996(.059)[.058]
	.2	.1857(.108)	.1672(.107)[.107]	.1845(.109)	.1660(.108)[.107]	.1787(.108)	.1603(.107)[.107]

Note: $\psi = (\breve{\alpha}', \beta', \gamma', \pi', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 6. Empirical sd and average of estimated standard errors of M-Estimator
DGP1, $T = 3, m = 10$, Parameter configurations as in Table 5.

		Normal Error				Normal Mixture				Chi-Square			
n	ψ	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$
50	1	.355	.310	.314	.380	.350	.315	.314	.386	.344	.309	.314	.394
	.30	.282	.248	.248	.290	.284	.255	.249	.299	.277	.247	.250	.308
	.24	.255	.234	.232	.265	.259	.237	.234	.277	.258	.235	.234	.279
	1	.057	.054	.055	.068	.057	.056	.055	.073	.058	.054	.055	.073
	1	.343	.324	.324	.387	.333	.326	.323	.395	.349	.322	.323	.397
	1	.155	.139	.137	.170	.154	.142	.137	.179	.160	.136	.137	.180
	1	.150	.139	.138	.174	.240	.208	.140	.143	.222	.186	.140	.154
	1	.449	.388	.391	.501	.491	.410	.392	.509	.503	.407	.392	.514
	.3	.053	.045	.044	.055	.051	.047	.044	.058	.053	.044	.044	.058
	.2	.096	.090	.086	.103	.098	.099	.087	.105	.102	.092	.087	.106
	.2	.100	.096	.092	.110	.102	.104	.093	.112	.106	.097	.092	.113
	.2	.201	.180	.175	.210	.204	.180	.175	.216	.199	.176	.174	.220
100	1	.232	.218	.214	.228	.225	.230	.215	.233	.229	.232	.217	.238
	-.47	.254	.243	.231	.240	.257	.262	.234	.251	.252	.281	.237	.247
	-.42	.213	.198	.193	.202	.213	.211	.195	.211	.214	.216	.196	.209
	1	.039	.038	.038	.041	.039	.040	.038	.044	.038	.041	.038	.043
	1	.244	.236	.235	.255	.242	.238	.236	.260	.238	.237	.236	.259
	1	.111	.102	.102	.113	.111	.103	.102	.118	.113	.105	.102	.117
	1	.104	.100	.101	.112	.167	.155	.101	.087	.153	.146	.101	.093
	1	.303	.270	.272	.305	.337	.294	.275	.312	.333	.290	.275	.310
	.3	.035	.032	.032	.035	.036	.033	.032	.037	.037	.035	.032	.036
	.2	.108	.109	.100	.102	.113	.122	.102	.107	.111	.131	.103	.106
	.2	.105	.108	.099	.100	.109	.119	.100	.105	.108	.128	.101	.104
	.2	.176	.178	.167	.174	.184	.180	.167	.184	.172	.183	.166	.178
200	1	.161	.157	.157	.163	.162	.157	.157	.165	.159	.156	.157	.169
	.14	.127	.121	.121	.124	.128	.120	.120	.126	.126	.121	.121	.127
	.35	.132	.123	.123	.125	.131	.123	.122	.128	.130	.123	.123	.128
	1	.028	.027	.027	.028	.027	.027	.027	.029	.028	.027	.027	.029
	1	.165	.165	.165	.172	.170	.166	.166	.174	.167	.165	.165	.173
	1	.073	.070	.070	.073	.074	.070	.070	.075	.074	.070	.070	.075
	1	.073	.071	.072	.075	.118	.113	.072	.053	.107	.105	.072	.059
	1	.207	.189	.190	.201	.221	.209	.192	.199	.214	.202	.189	.197
	.3	.024	.022	.022	.023	.024	.022	.022	.023	.024	.022	.022	.023
	.2	.064	.065	.062	.062	.066	.065	.062	.063	.067	.065	.062	.063
	.2	.065	.066	.063	.064	.066	.066	.063	.065	.068	.066	.064	.065
	.2	.129	.126	.122	.124	.135	.127	.123	.128	.133	.126	.123	.128
400	1	.128	.123	.123	.125	.128	.122	.123	.126	.125	.121	.122	.128
	.32	.096	.096	.095	.096	.097	.096	.095	.097	.099	.095	.094	.096
	-.09	.089	.087	.086	.087	.089	.086	.086	.088	.090	.086	.086	.088
	1	.018	.018	.019	.019	.019	.018	.019	.019	.018	.018	.019	.019
	1	.117	.118	.118	.120	.120	.117	.117	.120	.119	.117	.117	.120
	1	.050	.049	.049	.050	.052	.049	.049	.050	.051	.049	.048	.050
	1	.051	.051	.051	.052	.081	.082	.051	.035	.079	.075	.051	.040
	1	.134	.133	.133	.136	.153	.146	.133	.133	.155	.142	.133	.133
	.3	.015	.015	.015	.015	.016	.015	.015	.015	.016	.015	.015	.015
	.2	.058	.059	.056	.055	.057	.059	.056	.056	.059	.058	.056	.055
	.2	.057	.058	.056	.055	.057	.059	.056	.055	.059	.058	.055	.055
	.2	.107	.107	.103	.101	.108	.107	.103	.102	.107	.107	.103	.102

Table 7. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP1**, $T = 3$, $m = 10$
$W_1 = W_3$**: Group Interaction;** W_2**: Queen Contiguity**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.9775(.299)	1.0164(.309)[.288]	.9880(.294)	1.0257(.303)[.287]	.9727(.304)	1.0120(.314)[.287]
	-.19	-.2419(.272)	-.1893(.266)[.243]	-.2520(.272)	-.2007(.265)[.241]	-.2375(.278)	-.1860(.273)[.245]
	-.10	-.1026(.256)	-.0957(.249)[.231]	-.1100(.257)	-.1037(.250)[.230]	-.0975(.262)	-.0910(.255)[.232]
	1	.9943(.048)	.9997(.048)[.046]	.9980(.049)	1.0027(.048)[.046]	.9953(.051)	1.0010(.050)[.046]
	1	.9061(.330)	.9908(.345)[.327]	.9110(.329)	.9941(.343)[.326]	.8988(.331)	.9830(.346)[.326]
	1	.8827(.141)	.9956(.135)[.123]	.8840(.147)	.9953(.141)[.121]	.8808(.143)	.9930(.136)[.121]
	1	.9896(.158)	.9460(.146)[.136]	.9960(.256)	.9527(.240)[.205]	.9971(.230)	.9530(.214)[.189]
	1	.7841(.351)	.9654(.400)[.357]	.8110(.416)	.9896(.465)[.384]	.7989(.411)	.9800(.462)[.377]
	.3	.3438(.046)	.3017(.043)[.038]	.3430(.048)	.3014(.045)[.038]	.3444(.047)	.3030(.044)[.038]
	.2	.1874(.026)	.1970(.027)[.024]	.1870(.027)	.1966(.027)[.024]	.1866(.027)	.1960(.028)[.024]
	.2	.1862(.037)	.1962(.037)[.035]	.1880(.037)	.1980(.037)[.035]	.1871(.038)	.1970(.038)[.035]
	.2	.1272(.152)	.1226(.152)[.132]	.1220(.157)	.1170(.159)[.129]	.1256(.151)	.1210(.152)[.129]
100	1	.9101(.300)	1.0463(.311)[.289]	.9060(.290)	1.0413(.300)[.288]	.8845(.284)	1.0200(.296)[.288]
	.44	.4569(.188)	.4403(.183)[.169]	.4560(.179)	.4396(.175)[.170]	.4598(.183)	.4430(.178)[.169]
	-.08	-.1073(.187)	-.0813(.181)[.169]	-.1010(.187)	-.0757(.180)[.170]	-.0953(.189)	-.0700(.183)[.171]
	1	.9971(.034)	1.0012(.034)[.033]	.9970(.033)	1.0007(.033)[.033]	.9966(.035)	1.0010(.035)[.033]
	1	.9078(.225)	.9974(.237)[.234]	.9060(.232)	.9950(.243)[.234]	.9082(.228)	.9970(.238)[.235]
	1	.8677(.106)	.9973(.099)[.092]	.8690(.107)	.9981(.098)[.092]	.8688(.108)	.9980(.099)[.092]
	1	1.0233(.114)	.9738(.104)[.099]	1.0300(.184)	.9800(.169)[.154]	1.0236(.169)	.9740(.156)[.143]
	1	.7861(.257)	.9922(.298)[.260]	.7890(.282)	.9916(.318)[.280]	.7982(.277)	1.0050(.313)[.280]
	.3	.3476(.035)	.2999(.032)[.029]	.3480(.034)	.3008(.031)[.029]	.3476(.036)	.3000(.032)[.029]
	.2	.1869(.025)	.1970(.026)[.024]	.1860(.024)	.1957(.025)[.024]	.1874(.024)	.1970(.025)[.024]
	.2	.1911(.030)	.1968(.031)[.029]	.1910(.030)	.1972(.031)[.029]	.1934(.031)	.1990(.031)[.029]
	.2	.1484(.117)	.1454(.118)[.109]	.1540(.117)	.1509(.118)[.108]	.1573(.116)	.1540(.117)[.107]
200	1	.9001(.217)	1.0286(.224)[.213]	.8980(.211)	1.0246(.218)[.213]	.8933(.210)	1.0220(.218)[.212]
	-.40	-.3976(.133)	-.4101(.131)[.125]	-.3980(.136)	-.4100(.134)[.125]	-.3927(.132)	-.4050(.130)[.125]
	.43	.4684(.133)	.4271(.129)[.123]	.4610(.133)	.4204(.129)[.123]	.4672(.128)	.4260(.125)[.122]
	1	.9969(.025)	.9990(.025)[.025]	.9980(.026)	1.0005(.026)[.025]	.9981(.025)	1.0000(.025)[.025]
	1	.9397(.167)	.9963(.177)[.165]	.9490(.157)	1.0061(.165)[.166]	.9411(.165)	.9980(.174)[.165]
	1	.8763(.074)	1.0006(.068)[.066]	.8770(.074)	1.0007(.069)[.067]	.8731(.076)	.9980(.071)[.067]
	1	1.0339(.083)	.9871(.075)[.071]	1.0280(.125)	.9818(.116)[.112]	1.0342(.118)	.9870(.108)[.104]
	1	.7997(.178)	.9956(.201)[.184]	.8150(.197)	1.0109(.220)[.204]	.8011(.185)	.9980(.207)[.198]
	.3	.3447(.023)	.2997(.021)[.020]	.3440(.023)	.2991(.021)[.020]	.3453(.024)	.3000(.021)[.020]
	.2	.1893(.028)	.1950(.029)[.028]	.1910(.029)	.1962(.030)[.028]	.1899(.028)	.1960(.030)[.028]
	.2	.1918(.020)	.1987(.021)[.020]	.1920(.021)	.1986(.021)[.020]	.1919(.021)	.1990(.021)[.020]
	.2	.1649(.095)	.1626(.097)[.094]	.1620(.100)	.1600(.102)[.094]	.1646(.095)	.1620(.097)[.094]
400	1	.9113(.166)	1.0230(.174)[.169]	.9070(.164)	1.0182(.172)[.169]	.9083(.164)	1.0200(.173)[.168]
	.18	.1762(.092)	.1777(.089)[.086]	.1740(.091)	.1758(.088)[.086]	.1725(.092)	.1740(.089)[.086]
	-.23	-.2411(.093)	-.2256(.090)[.086]	-.2420(.091)	-.2261(.087)[.086]	-.2430(.091)	-.2280(.088)[.087]
	1	.9961(.019)	1.0001(.019)[.018]	.9970(.018)	1.0006(.018)[.018]	.9961(.019)	1.0000(.019)[.018]
	1	.9158(.108)	.9956(.116)[.117]	.9240(.110)	1.0036(.116)[.117]	.9172(.109)	.9970(.117)[.117]
	1	.8380(.059)	.9988(.053)[.051]	.8380(.059)	.9987(.053)[.051]	.8396(.062)	1.0000(.056)[.051]
	1	1.0550(.059)	.9945(.053)[.051]	1.0530(.092)	.9922(.083)[.082]	1.0535(.087)	.9930(.078)[.075]
	1	.7505(.121)	.9963(.141)[.134]	.7570(.138)	1.0027(.158)[.148]	.7544(.135)	1.0000(.155)[.145]
	.3	.3582(.019)	.3003(.016)[.016]	.3580(.019)	.3004(.016)[.016]	.3580(.020)	.3000(.017)[.016]
	.2	.1839(.021)	.1970(.022)[.021]	.1840(.021)	.1968(.022)[.021]	.1841(.021)	.1970(.022)[.021]
	.2	.1865(.013)	.1991(.014)[.014]	.1870(.013)	.1991(.014)[.014]	.1870(.013)	.2000(.014)[.014]
	.2	.1816(.079)	.1736(.080)[.077]	.1810(.079)	.1727(.081)[.076]	.1839(.080)	.1760(.082)[.076]

Note: $\psi = (\breve{\alpha}', \beta', \gamma', \pi', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 8. Empirical sd and average of estimated standard errors of M-Estimator
DGP1, $T = 3, m = 10$, Parameter configurations as in Table 7.

		Normal Error				Normal Mixture				Chi-Square			
n	ψ	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$
	-.19	.266	.243	.244	.281	.265	.241	.243	.295	.273	.245	.244	.291
	-.10	.249	.231	.231	.262	.250	.230	.230	.274	.255	.232	.231	.273
	1	.048	.046	.047	.058	.048	.046	.047	.063	.050	.046	.047	.062
	1	.345	.327	.328	.393	.343	.326	.327	.402	.346	.326	.327	.397
	1	.135	.123	.125	.156	.141	.121	.125	.169	.136	.121	.125	.166
	1	.146	.136	.139	.172	.240	.205	.140	.142	.214	.189	.140	.152
	1	.400	.357	.364	.462	.465	.384	.370	.487	.462	.377	.367	.481
	.3	.043	.038	.039	.048	.045	.038	.039	.052	.044	.038	.039	.051
	.2	.027	.024	.025	.032	.027	.024	.025	.033	.028	.024	.025	.033
	.2	.037	.035	.035	.043	.037	.035	.035	.044	.038	.035	.035	.044
	.2	.152	.132	.135	.163	.159	.129	.135	.175	.152	.129	.135	.172
100	1	.311	.289	.290	.314	.300	.288	.290	.320	.296	.288	.291	.324
	.44	.183	.169	.170	.180	.175	.170	.171	.187	.178	.169	.171	.188
	-.08	.181	.169	.168	.177	.180	.170	.170	.185	.183	.171	.170	.182
	1	.034	.033	.034	.037	.033	.033	.034	.039	.035	.033	.034	.039
	1	.237	.234	.235	.256	.243	.234	.234	.258	.238	.235	.235	.258
	1	.099	.092	.092	.102	.098	.092	.092	.106	.099	.092	.092	.105
	1	.104	.099	.100	.111	.169	.154	.101	.084	.156	.143	.100	.091
	1	.298	.260	.263	.295	.318	.280	.263	.297	.313	.280	.266	.298
	.3	.032	.029	.029	.031	.031	.029	.029	.033	.032	.029	.029	.033
	.2	.026	.024	.024	.027	.025	.024	.024	.027	.025	.024	.024	.027
	.2	.031	.029	.030	.033	.031	.029	.030	.033	.031	.029	.030	.033
	.2	.118	.109	.111	.121	.118	.108	.110	.127	.117	.107	.110	.125
200	1	.224	.213	.214	.224	.218	.213	.214	.226	.218	.212	.214	.229
	-.40	.131	.125	.126	.129	.134	.125	.125	.131	.130	.125	.125	.132
	.43	.129	.123	.123	.127	.129	.123	.123	.128	.125	.122	.123	.130
	1	.025	.025	.025	.026	.026	.025	.025	.027	.025	.025	.025	.027
	1	.177	.165	.166	.173	.165	.166	.166	.174	.174	.165	.165	.173
	1	.068	.066	.067	.070	.069	.067	.066	.071	.071	.067	.067	.071
	1	.075	.071	.072	.075	.116	.112	.071	.052	.108	.104	.072	.059
	1	.201	.184	.185	.195	.220	.204	.187	.195	.207	.198	.185	.194
	.3	.021	.020	.020	.021	.021	.020	.020	.021	.021	.020	.020	.021
	.2	.029	.028	.028	.030	.030	.028	.028	.030	.030	.028	.028	.030
	.2	.021	.020	.020	.021	.021	.020	.020	.021	.021	.020	.020	.021
	.2	.097	.094	.095	.099	.102	.094	.095	.103	.097	.094	.095	.101
400	1	.174	.169	.170	.175	.172	.169	.170	.175	.173	.168	.170	.177
	.18	.089	.086	.086	.087	.088	.086	.086	.088	.089	.086	.086	.089
	-.23	.090	.086	.087	.088	.087	.086	.086	.088	.088	.087	.087	.088
	1	.019	.018	.018	.019	.018	.018	.018	.019	.019	.018	.018	.019
	1	.116	.117	.117	.119	.116	.117	.117	.120	.117	.117	.117	.120
	1	.053	.051	.051	.052	.053	.051	.051	.053	.056	.051	.051	.053
	1	.053	.051	.051	.053	.083	.082	.051	.036	.078	.075	.051	.040
	1	.141	.134	.134	.138	.158	.148	.135	.135	.155	.145	.135	.135
	.3	.016	.016	.016	.016	.016	.016	.016	.016	.017	.016	.016	.016
	.2	.022	.021	.021	.021	.022	.021	.021	.021	.022	.021	.021	.021
	.2	.014	.014	.014	.014	.014	.014	.014	.014	.014	.014	.014	.014
	.2	.080	.077	.077	.078	.081	.076	.077	.080	.082	.076	.077	.079

Table 9. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP2**, $T = 3$, $m = 10$
$W_1 = W_3$: **Queen Contiguity;** W_2: **Group Interaction**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.8245(.541)	1.0988(.529)[.481]	.8562(.531)	1.1262(.522)[.486]	.8320(.541)	1.1044(.534)[.483]
	.26	.3172(.272)	.2533(.269)[.238]	.3027(.276)	.2393(.274)[.239]	.3194(.266)	.2550(.262)[.236]
	.23	.2532(.256)	.2302(.252)[.230]	.2453(.259)	.2221(.255)[.229]	.2568(.258)	.2336(.253)[.229]
	1	.9788(.047)	1.0008(.045)[.042]	.9776(.047)	.9995(.045)[.042]	.9770(.047)	.9989(.046)[.042]
	1	.9360(.321)	.9998(.335)[.325]	.9474(.315)	1.0108(.330)[.323]	.9301(.314)	.9931(.329)[.322]
	1	.9706(.153)	.9371(.144)[.134]	.9825(.248)	.9487(.236)[.203]	.9801(.223)	.9470(.213)[.187]
	1	.8608(.373)	1.0237(.424)[.371]	.8748(.419)	1.0366(.474)[.398]	.8732(.416)	1.0323(.469)[.392]
	.3	.3331(.042)	.2970(.040)[.035]	.3342(.041)	.2984(.039)[.036]	.3335(.041)	.2977(.040)[.035]
	.2	.2034(.076)	.1886(.075)[.067]	.1991(.072)	.1844(.072)[.067]	.2028(.076)	.1882(.075)[.068]
	.2	.2043(.061)	.1943(.059)[.055]	.2020(.061)	.1925(.059)[.055]	.2017(.061)	.1916(.058)[.055]
	.2	.0817(.194)	.0949(.192)[.175]	.0820(.191)	.0949(.191)[.171]	.0875(.183)	.0997(.182)[.170]
100	1	.9179(.371)	1.0566(.367)[.362]	.9060(.380)	1.0455(.378)[.362]	.9363(.389)	1.0747(.386)[.361]
	-.44	-.4017(.202)	-.4535(.197)[.187]	-.4042(.202)	-.4563(.197)[.187]	-.4109(.207)	-.4626(.201)[.184]
	.33	.3945(.203)	.3244(.197)[.187]	.4008(.202)	.3299(.197)[.186]	.3983(.199)	.3276(.194)[.184]
	1	.9794(.032)	.9989(.031)[.030]	.9794(.032)	.9990(.031)[.030]	.9794(.032)	.9989(.031)[.029]
	1	.9559(.226)	.9977(.234)[.229]	.9667(.221)	1.0091(.229)[.230]	.9576(.229)	.9994(.237)[.229]
	1	1.0017(.107)	.9739(.103)[.098]	1.0127(.175)	.9841(.168)[.155]	1.0059(.161)	.9777(.154)[.143]
	1	.8693(.244)	.9963(.276)[.252]	.8771(.279)	1.0056(.310)[.278]	.8736(.267)	1.0011(.298)[.272]
	.3	.3289(.026)	.2995(.025)[.023]	.3287(.026)	.2991(.026)[.024]	.3287(.026)	.2992(.025)[.024]
	.2	.1861(.044)	.1951(.044)[.042]	.1864(.044)	.1954(.045)[.042]	.1858(.045)	.1949(.046)[.042]
	.2	.2042(.046)	.1955(.045)[.044]	.2040(.046)	.1951(.045)[.044]	.2000(.047)	.1913(.046)[.044]
	.2	.1630(.126)	.1538(.128)[.120]	.1606(.125)	.1514(.126)[.119]	.1599(.126)	.1509(.128)[.119]
200	1	.9363(.293)	1.0573(.296)[.285]	.9108(.288)	1.0315(.291)[.282]	.9168(.280)	1.0375(.282)[.283]
	-.25	-.2454(.150)	-.2671(.149)[.143]	-.2357(.145)	-.2571(.144)[.141]	-.2357(.145)	-.2571(.144)[.142]
	.30	.3259(.138)	.2972(.136)[.131]	.3237(.139)	.2954(.137)[.129]	.3270(.132)	.2986(.129)[.129]
	1	.9770(.023)	.9989(.022)[.022]	.9781(.024)	.9999(.023)[.022]	.9781(.023)	.9999(.022)[.022]
	1	.9379(.162)	1.0022(.169)[.168]	.9406(.166)	1.0048(.172)[.168]	.9358(.162)	.9996(.169)[.168]
	1	1.0207(.076)	.9905(.073)[.071]	1.0138(.121)	.9837(.116)[.113]	1.0194(.113)	.9893(.108)[.105]
	1	.8613(.169)	.9963(.191)[.180]	.8788(.195)	1.0154(.217)[.202]	.8672(.185)	1.0021(.206)[.195]
	.3	.3314(.019)	.2996(.018)[.017]	.3313(.019)	.2997(.019)[.017]	.3313(.019)	.2996(.019)[.017]
	.2	.1953(.036)	.1940(.037)[.035]	.1990(.036)	.1977(.036)[.035]	.1977(.035)	.1964(.035)[.035]
	.2	.1967(.037)	.1957(.037)[.035]	.1976(.037)	.1966(.036)[.035]	.1979(.036)	.1969(.036)[.035]
	.2	.1783(.089)	.1797(.089)[.088]	.1728(.089)	.1744(.089)[.087]	.1766(.088)	.1780(.088)[.087]
400	1	.9025(.220)	1.0133(.219)[.215]	.9248(.229)	1.0352(.227)[.215]	.9128(.221)	1.0239(.218)[.215]
	-.21	-.1996(.089)	-.2097(.087)[.089]	-.2006(.090)	-.2108(.088)[.088]	-.1978(.092)	-.2082(.090)[.088]
	.42	.4336(.096)	.4152(.093)[.089]	.4352(.092)	.4167(.089)[.089]	.4380(.093)	.4194(.090)[.089]
	1	.9786(.017)	.9998(.016)[.016]	.9794(.017)	1.0006(.016)[.016]	.9786(.017)	.9998(.016)[.016]
	1	.9529(.112)	.9965(.116)[.117]	.9546(.113)	.9980(.117)[.117]	.9568(.114)	1.0005(.118)[.117]
	1	1.0225(.052)	.9921(.050)[.051]	1.0251(.090)	.9947(.086)[.082]	1.0231(.080)	.9928(.076)[.075]
	1	.8665(.116)	1.0025(.132)[.128]	.8683(.136)	1.0035(.152)[.142]	.8703(.131)	1.0056(.146)[.139]
	.3	.3315(.013)	.2997(.013)[.012]	.3309(.013)	.2992(.013)[.012]	.3316(.013)	.2999(.013)[.012]
	.2	.1897(.023)	.1993(.023)[.023]	.1886(.023)	.1981(.024)[.023]	.1887(.023)	.1981(.023)[.023]
	.2	.2027(.033)	.1988(.032)[.031]	.1998(.033)	.1959(.032)[.031]	.2009(.033)	.1970(.032)[.031]
	.2	.1992(.062)	.1882(.063)[.061]	.1957(.062)	.1852(.063)[.061]	.1978(.062)	.1874(.063)[.061]

Note: $\psi = (\breve{\alpha}', \beta', \gamma', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 10. Empirical sd and average of estimated standard errors of M-Estimator
DGP2, $T = 3, m = 10$, Parameter configurations as in Table 9.

		Normal Error				Normal Mixture				Chi-Square			
n	ψ	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$
50	1	.529	.481	.493	.588	.522	.486	.494	.609	.534	.483	.494	.612
	.26	.269	.238	.240	.275	.274	.239	.241	.290	.262	.236	.241	.294
	.23	.252	.230	.229	.258	.255	.229	.229	.273	.253	.229	.230	.274
	1	.045	.042	.043	.051	.045	.042	.043	.054	.046	.042	.043	.054
	1	.335	.325	.325	.380	.330	.323	.324	.389	.329	.322	.324	.387
	1	.144	.134	.137	.167	.236	.203	.139	.138	.213	.187	.139	.148
	1	.424	.371	.379	.468	.474	.398	.382	.486	.469	.392	.380	.482
	.3	.040	.035	.036	.042	.039	.036	.036	.045	.040	.035	.036	.045
	.2	.075	.067	.069	.084	.072	.067	.069	.088	.075	.068	.069	.086
	.2	.059	.055	.056	.068	.059	.055	.056	.071	.058	.055	.057	.071
	.2	.192	.175	.180	.222	.191	.171	.180	.240	.182	.170	.179	.240
100	1	.367	.362	.365	.393	.378	.362	.365	.402	.386	.361	.365	.406
	-.44	.197	.187	.187	.198	.197	.187	.188	.205	.201	.184	.187	.205
	.33	.197	.187	.186	.196	.197	.186	.187	.203	.194	.184	.186	.204
	1	.031	.030	.030	.032	.031	.030	.030	.034	.031	.029	.030	.034
	1	.234	.229	.229	.247	.229	.230	.230	.251	.237	.229	.229	.249
	1	.103	.098	.100	.110	.168	.155	.101	.082	.154	.143	.100	.090
	1	.276	.252	.255	.285	.310	.278	.258	.286	.298	.272	.257	.285
	.3	.025	.023	.024	.026	.026	.024	.024	.027	.025	.024	.024	.026
	.2	.044	.042	.043	.047	.045	.042	.043	.048	.046	.042	.043	.047
	.2	.045	.044	.045	.049	.045	.044	.045	.050	.046	.044	.045	.050
	.2	.128	.120	.123	.135	.126	.119	.123	.143	.128	.119	.123	.142
200	1	.296	.285	.286	.298	.291	.282	.284	.300	.282	.283	.285	.302
	-.25	.149	.143	.143	.147	.144	.141	.141	.148	.144	.142	.142	.149
	.30	.136	.131	.131	.134	.137	.129	.129	.135	.129	.129	.130	.136
	1	.022	.022	.022	.023	.023	.022	.022	.023	.022	.022	.022	.023
	1	.169	.168	.168	.174	.172	.168	.168	.175	.169	.168	.168	.175
	1	.073	.071	.072	.075	.116	.113	.071	.052	.108	.105	.072	.058
	1	.191	.180	.181	.190	.217	.202	.183	.189	.206	.195	.182	.188
	.3	.018	.017	.017	.018	.019	.017	.017	.018	.019	.017	.017	.018
	.2	.037	.035	.035	.037	.036	.035	.035	.038	.035	.035	.035	.037
	.2	.037	.035	.036	.037	.036	.035	.035	.037	.036	.035	.035	.037
	.2	.089	.088	.088	.092	.089	.087	.089	.096	.088	.087	.088	.095
400	1	.219	.215	.216	.221	.227	.215	.216	.223	.218	.215	.216	.224
	-.21	.087	.089	.089	.090	.088	.088	.088	.090	.090	.088	.089	.091
	.42	.093	.089	.089	.090	.089	.089	.089	.091	.090	.089	.089	.091
	1	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016	.016
	1	.116	.117	.117	.119	.117	.117	.117	.119	.118	.117	.117	.119
	1	.050	.051	.051	.052	.086	.082	.051	.035	.076	.075	.051	.039
	1	.132	.128	.128	.131	.152	.142	.128	.127	.146	.139	.128	.129
	.3	.013	.012	.012	.012	.013	.012	.012	.013	.013	.012	.012	.013
	.2	.023	.023	.023	.023	.024	.023	.023	.024	.023	.023	.023	.024
	.2	.032	.031	.031	.032	.032	.031	.031	.032	.032	.031	.031	.032
	.2	.063	.061	.062	.063	.063	.061	.062	.064	.063	.061	.062	.064

Table 11. Empirical Mean(sd)[se] of CQMLE and M-Estimator, **DGP2**, $T = 3$, $m = 10$
$W_1 = W_2 = W_3$**: Group Interaction**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.8801(.415)	1.0891(.424)[.366]	.8940(.414)	1.1002(.423)[.385]	.8672(.407)	1.0750(.416)[.370]
	.30	.3669(.295)	.2779(.285)[.251]	.3571(.299)	.2685(.290)[.260]	.3779(.292)	.2886(.282)[.249]
	.24	.2724(.266)	.2346(.255)[.235]	.2602(.271)	.2227(.261)[.239]	.2767(.269)	.2390(.258)[.234]
	1	.9737(.051)	.9986(.049)[.045]	.9725(.050)	.9973(.049)[.049]	.9721(.052)	.9969(.050)[.045]
	1	.9110(.321)	.9898(.338)[.326]	.9296(.316)	1.0091(.331)[.328]	.9192(.329)	.9984(.347)[.325]
	1	.9698(.151)	.9297(.142)[.135]	.9836(.250)	.9435(.237)[.207]	.9781(.228)	.9378(.214)[.186]
	1	.8418(.371)	1.0420(.440)[.385]	.8510(.421)	1.0471(.492)[.411]	.8580(.427)	1.0573(.498)[.408]
	.3	.3417(.046)	.2987(.045)[.039]	.3435(.045)	.3009(.044)[.041]	.3426(.045)	.2999(.044)[.039]
	.2	.1875(.102)	.1919(.100)[.093]	.1805(.112)	.1850(.108)[.110]	.1844(.109)	.1887(.106)[.094]
	.2	.2065(.109)	.1928(.099)[.096]	.2098(.114)	.1960(.105)[.109]	.2086(.113)	.1948(.105)[.096]
	.2	.1008(.209)	.0964(.204)[.178]	.1058(.211)	.1014(.205)[.182]	.1049(.206)	.1001(.201)[.175]
100	1	1.0003(.259)	1.0352(.258)[.247]	1.0010(.260)	1.0330(.258)[.545]	1.0095(.262)	1.0425(.260)[.255]
	-.47	-.5729(.280)	-.4722(.255)[.245]	-.5938(.299)	-.4893(.269)[.798]	-.5972(.278)	-.4935(.253)[.262]
	-.42	-.4792(.236)	-.4321(.216)[.202]	-.4876(.237)	-.4387(.215)[.492]	-.4896(.234)	-.4412(.214)[.210]
	1	.9705(.035)	.9977(.034)[.033]	.9707(.036)	.9979(.034)[.077]	.9701(.036)	.9974(.034)[.034]
	1	.9625(.223)	.9978(.232)[.232]	.9672(.225)	1.0034(.235)[.266]	.9587(.223)	.9940(.233)[.232]
	1	1.0020(.106)	.9698(.100)[.099]	1.0045(.173)	.9723(.166)[.223]	1.0043(.157)	.9721(.150)[.143]
	1	.8488(.243)	1.0045(.282)[.258]	.8674(.270)	1.0263(.312)[.305]	.8531(.263)	1.0088(.301)[.280]
	.3	.3377(.028)	.3011(.027)[.026]	.3363(.029)	.2996(.028)[.056]	.3386(.029)	.3021(.028)[.027]
	.2	.1482(.125)	.1898(.113)[.113]	.1394(.134)	.1828(.120)[.415]	.1430(.128)	.1856(.115)[.120]
	.2	.2415(.121)	.1996(.107)[.109]	.2519(.129)	.2082(.114)[.378]	.2462(.124)	.2032(.109)[.116]
	.2	.1727(.185)	.1264(.182)[.178]	.1762(.192)	.1284(.189)[.294]	.1800(.184)	.1332(.180)[.181]
200	1	.9434(.184)	1.0206(.187)[.182]	.9371(.184)	1.0147(.187)[.182]	.9360(.180)	1.0136(.183)[.182]
	.14	.1501(.134)	.1413(.130)[.121]	.1501(.130)	.1414(.127)[.121]	.1555(.132)	.1468(.129)[.121]
	.35	.3582(.135)	.3494(.131)[.123]	.3598(.133)	.3509(.128)[.123]	.3608(.136)	.3519(.131)[.124]
	1	.9734(.024)	.9991(.023)[.023]	.9726(.024)	.9982(.023)[.023]	.9728(.024)	.9984(.023)[.023]
	1	.9387(.159)	.9956(.165)[.165]	.9393(.161)	.9961(.168)[.165]	.9482(.158)	1.0054(.164)[.165]
	1	1.0191(.078)	.9881(.074)[.071]	1.0163(.128)	.9856(.122)[.113]	1.0196(.113)	.9887(.107)[.104]
	1	.8562(.171)	.9940(.193)[.180]	.8719(.192)	1.0098(.214)[.200]	.8632(.184)	1.0011(.205)[.195]
	.3	.3335(.019)	.3009(.018)[.018]	.3338(.019)	.3013(.019)[.018]	.3333(.019)	.3007(.018)[.018]
	.2	.1831(.071)	.1970(.068)[.066]	.1817(.072)	.1960(.069)[.066]	.1791(.070)	.1933(.066)[.067]
	.2	.1982(.071)	.1987(.067)[.066]	.2002(.072)	.2004(.068)[.066]	.2024(.071)	.2026(.066)[.067]
	.2	.1574(.136)	.1445(.132)[.127]	.1584(.134)	.1452(.131)[.126]	.1605(.133)	.1477(.129)[.126]
400	1	.9062(.153)	1.0183(.155)[.152]	.9064(.157)	1.0186(.159)[.152]	.9048(.154)	1.0172(.157)[.152]
	.32	.3478(.100)	.3233(.097)[.095]	.3485(.104)	.3237(.100)[.096]	.3510(.101)	.3261(.097)[.095]
	-.09	-.0873(.094)	-.0850(.090)[.086]	-.0887(.096)	-.0863(.092)[.087]	-.0888(.094)	-.0864(.090)[.086]
	1	.9781(.017)	.9999(.016)[.016]	.9772(.016)	.9991(.015)[.016]	.9782(.016)	1.0000(.016)[.016]
	1	.9628(.113)	1.0026(.117)[.116]	.9626(.114)	1.0023(.118)[.116]	.9613(.110)	1.0010(.115)[.116]
	1	1.0209(.053)	.9929(.051)[.050]	1.0209(.088)	.9930(.085)[.082]	1.0219(.081)	.9939(.078)[.075]
	1	.8742(.117)	1.0004(.131)[.127]	.8767(.137)	1.0029(.152)[.141]	.8754(.129)	1.0013(.142)[.138]
	.3	.3294(.013)	.2998(.012)[.012]	.3299(.013)	.3004(.012)[.012]	.3296(.013)	.3001(.012)[.012]
	.2	.1790(.063)	.1954(.060)[.061]	.1774(.065)	.1944(.062)[.062]	.1788(.063)	.1957(.059)[.061]
	.2	.2124(.061)	.2008(.057)[.058]	.2136(.063)	.2013(.058)[.059]	.2126(.061)	.2004(.057)[.058]
	.2	.1796(.111)	.1619(.109)[.108]	.1858(.110)	.1674(.108)[.108]	.1810(.112)	.1625(.110)[.108]

Note: $\psi = (\breve{\alpha}', \beta', \gamma', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 12. Empirical sd and average of estimated standard errors of M-Estimator
DGP2, $T = 3, m = 10$, Parameter configurations as in Table 11.

		Normal Error				Normal Mixture				Chi-Square			
n	ψ	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$
50	1	.424	.366	.373	.441	.423	.385	.376	.451	.416	.370	.375	.464
	.30	.285	.251	.251	.286	.290	.260	.255	.304	.282	.249	.253	.305
	.24	.255	.235	.233	.260	.261	.239	.236	.279	.258	.234	.234	.278
	1	.049	.045	.046	.055	.049	.049	.046	.060	.050	.045	.046	.058
	1	.338	.326	.326	.382	.331	.328	.325	.394	.347	.325	.326	.391
	1	.142	.135	.137	.166	.237	.207	.140	.140	.214	.186	.139	.149
	1	.440	.385	.392	.484	.492	.411	.392	.504	.498	.408	.396	.503
	.3	.045	.039	.039	.047	.044	.041	.039	.050	.044	.039	.039	.049
	.2	.100	.093	.090	.105	.108	.110	.092	.113	.106	.094	.091	.109
	.2	.099	.096	.092	.107	.105	.109	.094	.116	.105	.096	.093	.112
	.2	.204	.178	.176	.207	.205	.182	.176	.221	.201	.175	.176	.224
100	1	.258	.247	.245	.262	.258	.545	.247	.268	.260	.255	.247	.270
	-.47	.255	.245	.234	.241	.269	.798	.238	.252	.253	.262	.239	.247
	-.42	.216	.202	.196	.204	.215	.492	.198	.213	.214	.210	.199	.210
	1	.034	.033	.033	.035	.034	.077	.033	.037	.034	.034	.033	.036
	1	.232	.232	.231	.249	.235	.266	.232	.254	.233	.232	.231	.250
	1	.100	.099	.100	.111	.166	.223	.101	.084	.150	.143	.101	.090
	1	.282	.258	.261	.290	.312	.305	.265	.295	.301	.280	.262	.288
	.3	.027	.026	.026	.028	.028	.056	.026	.029	.028	.027	.026	.028
	.2	.113	.113	.105	.107	.120	.415	.107	.111	.115	.120	.107	.109
	.2	.107	.109	.100	.101	.114	.378	.102	.105	.109	.116	.102	.104
	.2	.182	.178	.168	.176	.189	.294	.169	.184	.180	.181	.169	.180
200	1	.187	.182	.182	.188	.187	.182	.182	.191	.183	.182	.183	.194
	.14	.130	.121	.121	.124	.127	.121	.121	.126	.129	.121	.121	.126
	.35	.131	.123	.123	.125	.128	.123	.123	.128	.131	.124	.123	.128
	1	.023	.023	.023	.024	.023	.023	.023	.024	.023	.023	.023	.024
	1	.165	.165	.165	.171	.168	.165	.165	.172	.164	.165	.165	.172
	1	.074	.071	.072	.075	.122	.113	.071	.052	.107	.104	.072	.058
	1	.193	.180	.181	.190	.214	.200	.183	.188	.205	.195	.182	.188
	.3	.018	.018	.018	.018	.019	.018	.018	.019	.018	.018	.018	.019
	.2	.068	.066	.064	.064	.069	.066	.064	.065	.066	.067	.064	.065
	.2	.067	.066	.064	.064	.068	.066	.064	.065	.066	.067	.064	.065
	.2	.132	.127	.123	.125	.131	.126	.123	.128	.129	.126	.123	.127
400	1	.155	.152	.153	.156	.159	.152	.153	.158	.157	.152	.153	.159
	.32	.097	.095	.095	.095	.100	.096	.095	.097	.097	.095	.095	.097
	-.09	.090	.086	.086	.087	.092	.087	.086	.088	.090	.086	.086	.088
	1	.016	.016	.016	.016	.015	.016	.016	.016	.016	.016	.016	.016
	1	.117	.116	.116	.118	.118	.116	.116	.119	.115	.116	.116	.119
	1	.051	.050	.051	.052	.085	.082	.051	.034	.078	.075	.051	.039
	1	.131	.127	.127	.130	.152	.141	.127	.126	.142	.138	.127	.127
	.3	.012	.012	.012	.012	.012	.012	.012	.012	.012	.012	.012	.012
	.2	.060	.061	.059	.058	.062	.062	.059	.059	.059	.061	.059	.058
	.2	.057	.058	.056	.054	.058	.059	.056	.055	.057	.058	.056	.055
	.2	.109	.108	.104	.103	.108	.108	.104	.104	.110	.108	.104	.104

Table 13. Empirical Mean (sd) [se] of CQMLE and M-Estimator, **DGP2**, $T = 3$, $m = 10$
$W_1 = W_3$: **Group Interaction;** W_2: **Queen Contiguity**

		Normal Error		Normal Mixture		Chi-Square	
n	ψ	CQMLE	M-Est	CQMLE	M-Est	CQMLE	M-Est
50	1	.872 (.507)	1.087 (.499) [.459]	.900 (.513)	1.111 (.508) [.459]	.875 (.500)	1.089 (.495) [.461]
	-.16	-.114 (.272)	-.167 (.265) [.240]	-.130 (.274)	-.182 (.268) [.241]	-.110 (.269)	-.163 (.262) [.238]
	.22	.263 (.267)	.224 (.258) [.239]	.251 (.269)	.211 (.262) [.240]	.265 (.268)	.225 (.260) [.238]
	1	.978 (.046)	1.000 (.045) [.042]	.976 (.047)	.998 (.046) [.041]	.977 (.046)	.999 (.045) [.042]
	1	.954 (.328)	1.019 (.341) [.324]	.938 (.338)	1.001 (.351) [.323]	.945 (.326)	1.009 (.339) [.325]
	1	.971 (.150)	.939 (.143) [.134]	.982 (.247)	.950 (.237) [.204]	.977 (.223)	.945 (.213) [.187]
	1	.869 (.373)	1.022 (.426) [.371]	.877 (.420)	1.029 (.470) [.399]	.888 (.417)	1.043 (.473) [.397]
	.3	.332 (.040)	.298 (.039) [.035]	.333 (.041)	.300 (.040) [.035]	.331 (.040)	.297 (.039) [.035]
	.2	.186 (.051)	.187 (.051) [.047]	.187 (.053)	.187 (.053) [.047]	.187 (.053)	.188 (.053) [.047]
	.2	.209 (.075)	.196 (.070) [.067]	.205 (.076)	.192 (.071) [.067]	.207 (.073)	.194 (.069) [.068]
	.2	.130 (.156)	.127 (.158) [.134]	.133 (.154)	.130 (.155) [.130]	.131 (.153)	.130 (.154) [.129]
100	1	.933 (.377)	1.075 (.376) [.357]	.937 (.374)	1.079 (.373) [.356]	.924 (.363)	1.066 (.363) [.356]
	.49	.529 (.203)	.475 (.201) [.189]	.528 (.205)	.474 (.203) [.188]	.532 (.202)	.479 (.199) [.187]
	.17	.185 (.186)	.170 (.183) [.170]	.189 (.186)	.174 (.183) [.170]	.183 (.182)	.169 (.179) [.170]
	1	.980 (.032)	.999 (.031) [.030]	.981 (.032)	1.000 (.031) [.030]	.981 (.033)	1.000 (.032) [.030]
	1	.988 (.223)	1.004 (.231) [.228]	.996 (.226)	1.012 (.234) [.228]	.982 (.232)	.997 (.240) [.228]
	1	1.003 (.108)	.977 (.104) [.099]	1.004 (.172)	.978 (.165) [.154]	1.002 (.156)	.976 (.150) [.142]
	1	.875 (.246)	.994 (.275) [.251]	.892 (.266)	1.012 (.293) [.278]	.890 (.267)	1.009 (.293) [.273]
	.3	.327 (.025)	.300 (.025) [.023]	.327 (.025)	.300 (.025) [.023]	.327 (.025)	.299 (.024) [.023]
	.2	.193 (.041)	.191 (.041) [.039]	.191 (.040)	.189 (.041) [.039]	.193 (.040)	.191 (.041) [.039]
	.2	.192 (.047)	.194 (.045) [.044]	.193 (.046)	.194 (.045) [.044]	.196 (.046)	.197 (.045) [.043]
	.2	.145 (.120)	.147 (.121) [.111]	.148 (.120)	.150 (.120) [.110]	.153 (.117)	.155 (.117) [.107]
200	1	.939 (.265)	1.033 (.265) [.260]	.942 (.266)	1.035 (.268) [.260]	.953 (.261)	1.046 (.262) [.259]
	.22	.209 (.130)	.214 (.127) [.123]	.206 (.132)	.212 (.129) [.122]	.209 (.130)	.214 (.127) [.123]
	-.11	-.115 (.134)	-.112 (.130) [.121]	-.116 (.132)	-.113 (.128) [.121]	-.119 (.130)	-.116 (.126) [.121]
	1	.977 (.022)	1.000 (.021) [.021]	.977 (.023)	1.000 (.022) [.022]	.978 (.023)	1.000 (.022) [.021]
	1	.933 (.161)	1.001 (.168) [.168]	.930 (.162)	.998 (.169) [.168]	.931 (.166)	.998 (.173) [.167]
	1	1.018 (.077)	.988 (.074) [.071]	1.014 (.122)	.984 (.117) [.113]	1.019 (.114)	.989 (.109) [.104]
	1	.867 (.171)	1.001 (.194) [.180]	.877 (.193)	1.011 (.216) [.201]	.865 (.187)	.997 (.209) [.194]
	.3	.332 (.018)	.300 (.018) [.017]	.331 (.019)	.300 (.018) [.017]	.331 (.019)	.300 (.019) [.017]
	.2	.186 (.031)	.195 (.031) [.030]	.187 (.030)	.196 (.030) [.030]	.186 (.031)	.195 (.031) [.029]
	.2	.204 (.033)	.198 (.033) [.032]	.203 (.033)	.198 (.033) [.032]	.202 (.033)	.197 (.032) [.032]
	.2	.167 (.101)	.161 (.103) [.095]	.171 (.096)	.165 (.097) [.094]	.165 (.097)	.159 (.099) [.094]
400	1	.867 (.232)	1.032 (.234) [.234]	.879 (.243)	1.044 (.246) [.233]	.869 (.239)	1.034 (.243) [.233]
	.29	.312 (.089)	.290 (.088) [.088]	.309 (.095)	.288 (.094) [.088]	.313 (.088)	.291 (.087) [.087]
	.42	.432 (.090)	.423 (.089) [.087]	.433 (.093)	.423 (.092) [.088]	.433 (.091)	.423 (.089) [.087]
	1	.977 (.017)	1.000 (.016) [.016]	.977 (.017)	1.000 (.016) [.016]	.977 (.017)	1.000 (.016) [.016]
	1	.958 (.113)	1.001 (.117) [.116]	.956 (.109)	.998 (.114) [.116]	.958 (.112)	1.000 (.117) [.116]
	1	1.026 (.056)	.995 (.053) [.051]	1.025 (.087)	.994 (.083) [.081]	1.025 (.081)	.994 (.078) [.075]
	1	.861 (.118)	.999 (.134) [.127]	.867 (.135)	1.005 (.151) [.142]	.862 (.133)	1.001 (.148) [.139]
	.3	.332 (.013)	.300 (.013) [.012]	.332 (.013)	.299 (.013) [.012]	.332 (.013)	.300 (.013) [.012]
	.2	.197 (.032)	.196 (.033) [.032]	.196 (.033)	.195 (.034) [.032]	.196 (.033)	.195 (.033) [.032]
	.2	.198 (.023)	.199 (.023) [.023]	.198 (.023)	.199 (.023) [.023]	.198 (.024)	.199 (.024) [.023]
	.2	.172 (.082)	.174 (.083) [.078]	.176 (.081)	.177 (.081) [.078]	.170 (.081)	.172 (.082) [.078]

Note: $\psi = (\breve{\alpha}', \beta', \gamma', \sigma_v^2, \phi, \rho, \lambda')'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 14. Empirical sd and average of estimated standard errors of M-Estimator
DGP2, $T = 3, m = 10$, Parameter configurations as in Table 13.

		Normal Error				Normal Mixture				Chi-Square			
n	ψ	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$	sd	$\widehat{rse}$	$\widehat{se}$	$\widetilde{se}$
50	1	.499	.459	.468	.562	.508	.459	.467	.574	.495	.461	.468	.575
	-.16	.265	.240	.242	.275	.268	.241	.243	.290	.262	.238	.242	.288
	.22	.258	.239	.240	.270	.262	.240	.240	.283	.260	.238	.240	.280
	1	.045	.042	.042	.049	.046	.041	.042	.053	.045	.042	.042	.052
	1	.341	.324	.325	.383	.351	.323	.323	.386	.339	.325	.325	.386
	1	.143	.134	.138	.169	.237	.204	.139	.136	.213	.187	.139	.147
	1	.426	.371	.379	.469	.470	.399	.380	.475	.473	.397	.384	.480
	.3	.039	.035	.035	.042	.040	.035	.035	.044	.039	.035	.035	.044
	.2	.051	.047	.048	.059	.053	.047	.048	.061	.053	.047	.048	.059
	.2	.070	.067	.069	.084	.071	.067	.069	.088	.069	.068	.069	.086
	.2	.158	.134	.137	.166	.155	.130	.136	.175	.154	.129	.136	.174
100	1	.376	.357	.358	.387	.373	.356	.359	.396	.363	.356	.359	.397
	.49	.201	.189	.189	.200	.203	.188	.189	.206	.199	.187	.189	.207
	.17	.183	.170	.169	.177	.183	.170	.170	.184	.179	.170	.170	.183
	1	.031	.030	.030	.033	.031	.030	.030	.034	.032	.030	.030	.034
	1	.231	.228	.228	.245	.234	.228	.228	.248	.240	.228	.228	.247
	1	.104	.099	.100	.110	.165	.154	.100	.081	.150	.142	.100	.089
	1	.275	.251	.254	.283	.293	.278	.258	.284	.293	.273	.257	.283
	.3	.025	.023	.023	.025	.025	.023	.023	.026	.024	.023	.023	.026
	.2	.041	.039	.039	.043	.041	.039	.040	.044	.041	.039	.039	.044
	.2	.045	.044	.044	.048	.045	.044	.044	.049	.045	.043	.044	.049
	.2	.121	.111	.112	.123	.120	.110	.112	.127	.117	.107	.111	.128
200	1	.265	.260	.261	.271	.268	.260	.261	.274	.262	.259	.261	.276
	.22	.127	.123	.123	.126	.129	.122	.123	.128	.127	.123	.122	.127
	-.11	.130	.121	.121	.124	.128	.121	.121	.126	.126	.121	.121	.125
	1	.021	.021	.022	.022	.022	.022	.021	.023	.022	.021	.022	.023
	1	.168	.168	.168	.174	.169	.168	.168	.175	.173	.167	.167	.175
	1	.074	.071	.071	.075	.117	.113	.071	.051	.109	.104	.072	.058
	1	.194	.180	.181	.191	.216	.201	.183	.187	.209	.194	.181	.187
	.3	.018	.017	.017	.018	.018	.017	.017	.018	.019	.017	.017	.018
	.2	.031	.030	.030	.031	.030	.030	.030	.031	.031	.029	.030	.031
	.2	.033	.032	.032	.034	.033	.032	.032	.034	.032	.032	.032	.034
	.2	.103	.095	.095	.099	.097	.094	.095	.101	.099	.094	.095	.101
400	1	.234	.234	.234	.239	.246	.233	.235	.242	.243	.233	.234	.242
	.29	.088	.088	.088	.089	.094	.088	.088	.090	.087	.087	.087	.090
	.42	.089	.087	.087	.088	.092	.088	.088	.089	.089	.087	.087	.089
	1	.016	.016	.016	.016	.016	.016	.016	.017	.016	.016	.016	.017
	1	.117	.116	.116	.118	.114	.116	.116	.119	.117	.116	.116	.119
	1	.053	.051	.051	.052	.083	.081	.051	.035	.078	.075	.051	.039
	1	.134	.127	.128	.131	.151	.142	.128	.127	.148	.139	.128	.128
	.3	.013	.012	.012	.013	.013	.012	.012	.013	.013	.012	.012	.013
	.2	.033	.032	.032	.033	.034	.032	.032	.033	.033	.032	.032	.033
	.2	.023	.023	.023	.024	.023	.023	.023	.024	.024	.023	.023	.024
	.2	.083	.078	.079	.080	.081	.078	.078	.081	.082	.078	.079	.082

Table 15. Empirical Mean(sd) of CQMLE, M-Estimator, and FQMLE, **DGP3**, $T = 3$, $m = 10$; W_3: **Rook Contiguity**

		Normal Error			Normal Mixture			Chi-Square		
n	ψ	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE
50	1	.9539(.224)	1.0005(.238)	1.0101(.239)	.9525(.227)	.9980(.241)	1.0078(.241)	.9546(.225)	1.0017(.241)	1.0116(.242)
	1	.9715(.045)	.9979(.045)	.9993(.044)	.9726(.044)	.9988(.043)	1.0003(.043)	.9701(.045)	.9965(.044)	.9983(.044)
	1	.8896(.318)	1.0008(.339)	.9926(.339)	.8853(.333)	.9951(.354)	.9873(.355)	.8813(.321)	.9918(.342)	.9851(.341)
	1	1.0053(.076)	.9837(.074)	.9820(.071)	1.0068(.124)	.9855(.120)	.9832(.119)	1.0052(.113)	.9835(.109)	.9806(.107)
	1	.7896(.332)	.9931(.407)	1.0009(.373)	.8093(.388)	1.0091(.459)	1.0240(.436)	.8108(.376)	1.0168(.451)	1.0364(.426)
	.5	.5384(.031)	.5004(.032)	.4978(.031)	.5390(.032)	.5015(.032)	.4987(.032)	.5394(.032)	.5015(.033)	.4985(.032)
	.3	.2819(.114)	.2865(.114)	.2528(.114)	.2864(.115)	.2917(.115)	.2561(.115)	.2829(.116)	.2874(.116)	.2516(.118)
100	1	.9521(.169)	.9971(.177)	.9954(.177)	.9517(.179)	.9960(.188)	.9945(.188)	.9571(.177)	1.0019(.187)	1.0001(.186)
	1	.9760(.032)	1.0001(.031)	1.0004(.031)	.9757(.031)	.9995(.031)	.9998(.030)	.9751(.031)	.9989(.030)	.9993(.030)
	1	.9218(.217)	1.0012(.229)	1.0071(.229)	.9197(.227)	.9988(.239)	1.0047(.239)	.9203(.233)	.9997(.245)	1.0058(.245)
	1	1.0100(.054)	.9933(.052)	.9923(.051)	1.0055(.084)	.9889(.082)	.9881(.081)	1.0074(.078)	.9907(.076)	.9901(.076)
	1	.8334(.232)	.9940(.275)	.9977(.252)	.8586(.266)	1.0218(.313)	1.0245(.292)	.8485(.251)	1.0094(.293)	1.0121(.274)
	.5	.5311(.019)	.5004(.020)	.4997(.019)	.5305(.019)	.5000(.019)	.4993(.019)	.5310(.019)	.5004(.020)	.4997(.019)
	.3	.2879(.085)	.2902(.086)	.2836(.081)	.2904(.084)	.2935(.084)	.2858(.079)	.2940(.084)	.2969(.085)	.2889(.081)
200	1	.9518(.118)	1.0042(.124)	1.0096(.124)	.9504(.119)	1.0027(.125)	1.0081(.125)	.9413(.120)	.9934(.128)	.9986(.128)
	1	.9718(.023)	.9995(.022)	1.0007(.022)	.9726(.023)	1.0002(.022)	1.0014(.022)	.9722(.023)	.9999(.022)	1.0010(.022)
	1	.9439(.160)	1.0029(.169)	.9944(.169)	.9421(.158)	1.0013(.167)	.9927(.166)	.9419(.159)	1.0010(.169)	.9924(.168)
	1	1.0140(.039)	.9962(.037)	.9954(.036)	1.0130(.061)	.9951(.059)	.9943(.058)	1.0134(.056)	.9955(.055)	.9950(.054)
	1	.8264(.164)	.9947(.197)	.9982(.181)	.8380(.181)	1.0087(.213)	1.0118(.199)	.8326(.177)	1.0027(.211)	1.0034(.199)
	.5	.5320(.014)	.4996(.014)	.4990(.014)	.5320(.014)	.4996(.014)	.4991(.014)	.5330(.014)	.5006(.014)	.5001(.014)
	.3	.2947(.060)	.2981(.060)	.2833(.057)	.2929(.062)	.2962(.062)	.2815(.059)	.2921(.061)	.2958(.061)	.2814(.058)
400	1	.9300(.088)	1.0000(.093)	1.0103(.096)	.9305(.085)	1.0003(.090)	1.0092(.093)	.9295(.088)	.9997(.094)	1.0093(.094)
	1	.9727(.015)	1.0003(.014)	1.0062(.017)	.9729(.016)	1.0004(.015)	1.0058(.017)	.9719(.015)	.9995(.015)	1.0051(.016)
	1	.9345(.113)	1.0013(.120)	1.0189(.123)	.9374(.110)	1.0043(.117)	1.0206(.120)	.9372(.115)	1.0041(.122)	1.0209(.125)
	1	1.0178(.026)	.9982(.025)	1.0220(.030)	1.0161(.044)	.9966(.042)	1.0222(.040)	1.0178(.041)	.9982(.040)	1.0226(.038)
	1	.8173(.112)	1.0005(.136)	1.0560(.252)	.8268(.132)	1.0107(.156)	1.0489(.250)	.8211(.129)	1.0050(.154)	1.0517(.249)
	.5	.5349(.010)	.4998(.010)	.4931(.014)	.5343(.011)	.4992(.011)	.4932(.014)	.5351(.010)	.5000(.010)	.4936(.013)
	.3	.2956(.042)	.2991(.042)	.2858(.040)	.2943(.042)	.2978(.042)	.2842(.040)	.2936(.042)	.2970(.043)	.2834(.040)

Note: $\psi = (\breve{\alpha}', \beta', \gamma', \sigma_v^2, \phi, \rho, \lambda_3)'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 16. Empirical Mean(sd) of CQMLE, M-Estimator, and FQMLE, **DGP3**, $T = 3$, $m = 10$; W_3**: Queen Contiguity**

		Normal Error			Normal Mixture			Chi-Square		
n	ψ	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE
50	1	.9522(.223)	1.0007(.237)	1.0068(.238)	.9565(.230)	1.0050(.244)	1.0113(.244)	.9542(.226)	1.0024(.241)	1.0089(.242)
	1	.9712(.045)	.9983(.044)	.9992(.044)	.9736(.046)	1.0005(.045)	1.0015(.044)	.9730(.045)	.9998(.043)	1.0010(.043)
	1	.8877(.317)	.9999(.338)	.9910(.337)	.8888(.335)	1.0011(.359)	.9926(.357)	.8786(.316)	.9897(.338)	.9822(.338)
	1	1.0063(.076)	.9842(.073)	.9834(.071)	1.0033(.124)	.9815(.120)	.9800(.117)	1.0064(.117)	.9847(.113)	.9827(.111)
	1	.7847(.325)	.9922(.400)	.9942(.366)	.8392(.400)	1.0520(.474)	1.0579(.446)	.8146(.391)	1.0195(.458)	1.0331(.435)
	.5	.5394(.032)	.5004(.032)	.4989(.031)	.5383(.032)	.4994(.032)	.4977(.030)	.5399(.032)	.5012(.032)	.4993(.031)
	.3	.2654(.142)	.2693(.142)	.2353(.139)	.2660(.151)	.2712(.151)	.2356(.146)	.2624(.146)	.2668(.147)	.2327(.141)
100	1	.9549(.173)	1.0014(.181)	1.0002(.181)	.9500(.175)	.9963(.184)	.9952(.184)	.9511(.172)	.9978(.181)	.9967(.182)
	1	.9747(.032)	.9993(.031)	.9999(.031)	.9742(.032)	.9987(.032)	.9994(.031)	.9747(.032)	.9994(.032)	1.0001(.031)
	1	.9183(.224)	.9961(.237)	.9999(.236)	.9237(.227)	1.0018(.240)	1.0051(.239)	.9223(.228)	1.0004(.241)	1.0045(.240)
	1	1.0099(.052)	.9931(.051)	.9930(.050)	1.0091(.086)	.9923(.084)	.9925(.083)	1.0110(.080)	.9940(.078)	.9937(.075)
	1	.8304(.229)	.9919(.274)	.9963(.247)	.8490(.259)	1.0116(.302)	1.0141(.277)	.8411(.252)	1.0044(.295)	1.0101(.273)
	.5	.5309(.020)	.5000(.021)	.4993(.020)	.5309(.020)	.4999(.020)	.4993(.019)	.5308(.020)	.4998(.020)	.4990(.020)
	.3	.2828(.106)	.2851(.107)	.2890(.098)	.2834(.104)	.2854(.104)	.2897(.096)	.2814(.106)	.2840(.106)	.2875(.096)
200	1	.9428(.117)	1.0044(.123)	1.0064(.123)	.9435(.118)	1.0051(.124)	1.0071(.124)	.9376(.118)	.9992(.125)	1.0013(.125)
	1	.9679(.023)	.9992(.023)	1.0002(.022)	.9688(.023)	1.0000(.023)	1.0010(.022)	.9688(.023)	1.0001(.023)	1.0011(.023)
	1	.9541(.157)	.9977(.166)	.9962(.166)	.9497(.157)	.9928(.166)	.9914(.166)	.9507(.158)	.9937(.167)	.9923(.167)
	1	1.0135(.037)	.9956(.036)	.9948(.035)	1.0118(.061)	.9940(.060)	.9934(.059)	1.0122(.056)	.9943(.054)	.9934(.053)
	1	.8207(.156)	.9903(.189)	.9929(.176)	.8368(.181)	1.0073(.213)	1.0092(.204)	.8334(.175)	1.0042(.208)	1.0069(.193)
	.5	.5331(.014)	.5004(.014)	.4998(.014)	.5325(.014)	.4998(.014)	.4992(.014)	.5331(.014)	.5003(.014)	.4998(.014)
	.3	.2861(.073)	.2887(.074)	.2663(.072)	.2918(.075)	.2950(.076)	.2710(.074)	.2884(.073)	.2911(.074)	.2682(.071)
400	1	.9293(.086)	.9998(.091)	1.0001(.092)	.9284(.084)	.9988(.089)	.9999(.089)	.9289(.085)	.9992(.090)	1.0002(.092)
	1	.9723(.015)	1.0000(.015)	1.0018(.016)	.9725(.016)	1.0001(.015)	1.0023(.016)	.9725(.015)	1.0001(.015)	1.0022(.016)
	1	.9333(.112)	1.0001(.120)	1.0096(.120)	.9353(.110)	1.0022(.117)	1.0126(.118)	.9318(.114)	.9982(.121)	1.0080(.122)
	1	1.0179(.027)	.9983(.026)	1.0288(.027)	1.0164(.043)	.9969(.042)	1.0271(.036)	1.0165(.041)	.9971(.040)	1.0279(.035)
	1	.8129(.113)	.9955(.137)	.9548(.211)	.8230(.127)	1.0069(.151)	.9708(.212)	.8188(.126)	1.0011(.150)	.9612(.213)
	.5	.5353(.010)	.5000(.010)	.4978(.012)	.5350(.011)	.4998(.011)	.4971(.012)	.5352(.011)	.5000(.011)	.4975(.012)
	.3	.2927(.051)	.2965(.051)	.2853(.053)	.2912(.053)	.2948(.053)	.2832(.053)	.2925(.052)	.2961(.053)	.2839(.053)

Note: $\psi = (\breve{\alpha}', \beta', \gamma', \sigma_v^2, \phi, \rho, \lambda_3)'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

Table 17. Empirical Mean(sd) of CQMLE, M-Estimator, and FQMLE, **DGP3**, $T = 3$, $m = 10$; W_3**: Group Interaction**

		Normal Error			Normal Mixture			Chi-Square		
n	ψ	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE	CQMLE	M-Est	FQMLE
50	1	.9349(.226)	1.0001(.243)	1.0009(.242)	.9316(.223)	.9961(.238)	.9979(.239)	.9406(.222)	1.0064(.241)	1.0083(.242)
	1	.9658(.046)	.9982(.045)	.9994(.045)	.9665(.049)	.9986(.048)	1.0004(.047)	.9645(.049)	.9969(.047)	.9988(.046)
	1	.8877(.311)	.9950(.337)	.9997(.336)	.8887(.312)	.9953(.338)	1.0016(.338)	.8783(.321)	.9854(.347)	.9922(.348)
	1	1.0108(.078)	.9843(.074)	.9839(.072)	1.0103(.127)	.9837(.122)	.9818(.120)	1.0142(.119)	.9873(.113)	.9850(.112)
	1	.7506(.332)	.9934(.419)	.9940(.392)	.7697(.385)	1.0151(.473)	1.0252(.447)	.7697(.386)	1.0153(.469)	1.0305(.448)
	.5	.5473(.035)	.5011(.035)	.4999(.034)	.5474(.036)	.5015(.036)	.4996(.035)	.5480(.036)	.5016(.036)	.4996(.035)
	.3	.2611(.139)	.2675(.139)	.2432(.136)	.2607(.141)	.2659(.143)	.2415(.139)	.2631(.132)	.2687(.132)	.2455(.125)
100	1	.9566(.162)	.9985(.170)	.9962(.170)	.9622(.162)	1.0046(.171)	1.0022(.171)	.9659(.164)	1.0080(.174)	1.0058(.174)
	1	.9712(.032)	.9990(.031)	.9990(.030)	.9710(.032)	.9989(.031)	.9990(.031)	.9725(.031)	1.0002(.030)	1.0003(.030)
	1	.9104(.230)	.9945(.245)	1.0020(.245)	.9095(.224)	.9940(.237)	1.0014(.238)	.9105(.221)	.9940(.234)	1.0017(.234)
	1	1.0127(.055)	.9940(.053)	.9935(.052)	1.0132(.088)	.9940(.085)	.9934(.084)	1.0135(.080)	.9947(.078)	.9940(.077)
	1	.8191(.233)	.9969(.280)	.9998(.261)	.8323(.258)	1.0146(.304)	1.0173(.286)	.8183(.250)	.9952(.297)	1.0008(.283)
	.5	.5349(.020)	.5008(.021)	.5004(.020)	.5340(.020)	.4997(.021)	.4993(.020)	.5338(.021)	.5000(.021)	.4994(.021)
	.3	.2669(.113)	.2684(.115)	.2632(.109)	.2703(.113)	.2721(.114)	.2674(.108)	.2690(.111)	.2706(.113)	.2650(.107)
200	1	.9344(.130)	.9984(.136)	1.0006(.136)	.9372(.127)	1.0014(.134)	1.0035(.134)	.9342(.126)	.9984(.134)	1.0009(.134)
	1	.9708(.022)	.9997(.021)	1.0000(.021)	.9710(.023)	1.0000(.022)	1.0001(.022)	.9705(.023)	.9994(.022)	.9997(.022)
	1	.9596(.160)	1.0016(.169)	.9971(.169)	.9590(.157)	1.0011(.166)	.9964(.166)	.9588(.160)	1.0007(.169)	.9958(.169)
	1	1.0145(.037)	.9966(.036)	.9965(.035)	1.0144(.061)	.9963(.059)	.9966(.058)	1.0140(.057)	.9960(.055)	.9957(.054)
	1	.8267(.159)	.9958(.189)	.9971(.174)	.8360(.181)	1.0079(.213)	1.0068(.201)	.8333(.180)	1.0042(.211)	1.0062(.195)
	.5	.5328(.014)	.5003(.014)	.5005(.013)	.5323(.014)	.4997(.014)	.5000(.014)	.5324(.014)	.4999(.014)	.5000(.014)
	.3	.2803(.092)	.2831(.093)	.2964(.086)	.2793(.094)	.2818(.094)	.2964(.086)	.2796(.096)	.2823(.096)	.2961(.089)
400	1	.9347(.085)	.9961(.089)	1.0061(.090)	.9634(.084)	.9961(.089)	1.0011(.090)	.9534(.085)	.9991(.086)	1.0010(.084)
	1	.9720(.016)	1.0003(.016)	1.0045(.016)	.9750(.017)	1.0005(.016)	1.0005(.015)	.9753(.016)	1.0001(.016)	1.0005(.015)
	1	.9150(.111)	1.0025(.117)	1.0170(.119)	.9380(.111)	1.0015(.112)	1.0017(.111)	.9273(.110)	1.0005(.112)	.9971(.111)
	1	1.0162(.026)	.9975(.025)	1.0088(.023)	1.0169(.043)	.9974(.042)	1.0005(.041)	1.0259(.043)	.9984(.042)	.9965(.041)
	1	.8288(.114)	1.0052(.137)	1.0679(.131)	.8389(.115)	1.0032(.128)	1.0022(.109)	.8385(.115)	1.0002(.123)	1.0023(.112)
	.5	.5339(.010)	.5001(.010)	.4945(.012)	.5333(.011)	.5001(.010)	.4975(.010)	.5342(.010)	.5000(.011)	.4995(.011)
	.3	.2811(.076)	.2966(.077)	.2902(.075)	.2801(.075)	.2919(.076)	.2899(.074)	.2798(.076)	.2917(.077)	.2901(.074)

Note: $\psi = (\breve{\alpha}', \beta', \gamma', \sigma_v^2, \phi, \rho, \lambda_3)'$; X_t values are generated with $\theta_x = (g, \phi_1, \phi_2, \sigma_1, \sigma_2) = (.01, .5, .5, 2, 1)$.

IV. More details on Empirical Application

Table E1. Spatial correlation in investment level between cities

	$W_{province}$		$W_{geography}$	
Population	0.266**	[0.107]	0.278***	[0.107]
GDP	0.505***	[0.063]	0.504***	[0.065]
GDP per capita	−12.119	[10.517]	−12.963	[12.496]
Fiscal account balance	0.230*	[0.138]	0.195	[1.049]
Fiscal expenditure	0.726**	[0.362]	0.692*	[0.395]
Provincial fiscal revenue	−0.250***	[0.057]	−0.244***	[0.057]
Provincial fiscal expenditure	0.143***	[0.038]	0.189***	[0.037]
Public capital investment	−0.103***	[0.038]	−0.150***	[0.035]
Autonomous city	−87.107**	[38.930]	−77.327**	[36.010]
Year 2011	−42.468**	[17.630]	−66.059***	[14.981]
Year 2012	−16.305*	[9.105]	−26.101***	[8.533]
Fujian	167.387*	[100.055]	346.286**	[138.638]
Heilongjiang	−216.518**	[104.343]	−206.837**	[94.651]
Henan	−188.019**	[87.900]	−196.181**	[87.113]
Hunan	−167.008**	[81.538]	−139.165*	[79.944]
Jiangsu	293.630**	[148.565]	413.592*	[248.087]
Liaoning	327.488*	[181.071]	436.245***	[168.468]
Shandong	160.858	[132.777]	231.139*	[132.878]
Tibet	−97.208**	[38.669]	−88.272**	[36.256]
Xinjiang	−93.910*	[52.352]	−110.864**	[51.765]
Yunnan	−154.002**	[67.028]	−207.147***	[65.219]
Anhui	2.041	[102.994]	10.073	[105.556]
Gansu	−51.650	[57.839]	−41.154	[54.623]
Guangdong	−0.905	[137.840]	−149.033	[145.038]
Guangxi	−54.642	[73.815]	−25.222	[71.832]
Guizhou	−130.958	[90.312]	−113.002	[87.384]
Hainan	30.291	[63.905]	91.969	[62.443]
Hebei	−15.927	[119.205]	115.235	[117.940]
Inner Mongolia	4.324	[96.436]	56.223	[92.472]
Jiangxi	−54.390	[79.653]	−7.885	[76.056]
Jilin	−100.333	[81.584]	−41.585	[74.297]
Ningxia	−7.582	[62.648]	37.187	[54.897]
Qinghai	−6.420	[44.715]	−3.215	[43.236]
Shaanxi	72.104	[153.234]	159.510	[141.590]
Shanxi	0.336	[44.828]	22.009	[46.748]
Sichuan	−82.504	[99.736]	−117.881	[97.353]
Zhejiang	82.416	[184.019]	209.249	[180.198]
Average GDP	−0.242***	[0.034]	−0.248***	[0.033]
Y_{-1}	0.216***	[0.063]	0.237***	[0.062]
WY	0.183***	[0.069]	0.008	[0.008]

Note: 1. Population and GDP per capita are measured in 10^4, and other variables are measured in 10^8. 2. ***, **, and * represent significance at 1 %, 5% and 10% level.

Table E2. Spatial correlation in investment level between cities

Population	0.207	[0.104]
GDP	0.488	[0.061]
Fiscal revenue	0.549	[0.503]
Fiscal expenditure	0.628	[0.256]
Fiscal account balance	-0.336	[0.295]
Provincial fiscal revenue	-0.093	[0.036]
Provincial fiscal expenditure	0.058	[0.031]
Public capital investment	-0.085	[0.045]
Autonomous city	-91.814	[31.775]
Pearl delta economic zone	-854.630	[336.878]
Yangzi delta economic zone	-32.170	[132.139]
Guantian economic zone	159.946	[148.943]
Beibu gulf economic zone	68.501	[98.744]
2001	-46.731	[25.332]
2012	-17.776	[16.030]
Constant	58.823	[60.119]
Average GDP	-0.119	[0.109]
Average fiscal revenue	-0.386	[0.745]
Average fiscal expenditure	-0.608	[1.034]
Average fiscal account balance	-0.079	[0.435]
Y_{-1}	0.212	[0.060]
WY	0.089	[0.079]
WY_{-1}	0.016	[0.105]
Wu	0.429	[0.076]

Note: Results are based on $W_{province}$

Table E3. Spatial correlation in investment level between cities

Population	0.266	[0.120]
GDP	0.489	[0.058]
Fiscal revenue	0.577	[0.486]
Fiscal expenditure	0.632	[0.246]
Fiscal account balance	-0.336	[0.294]
Provincial fiscal revenue	-0.170	[0.051]
Provincial fiscal expenditure	0.079	[0.034]
Public capital investment	-0.061	[0.037]
Autonomous city	-97.176	[35.378]
Northeast	63.839	[81.568]
East	41.646	[82.046]
Middle	-37.720	[43.627]
2001	-31.235	[26.619]
2012	-12.007	[15.220]
Constant	5.109	[62.984]
Average GDP	-0.171	[0.113]
Average fiscal revenue	-0.155	[0.781]
Average fiscal expenditure	-0.593	[1.145]
Average fiscal account balance	-0.055	[0.423]
Y_{-1}	0.213	[0.061]
WY	0.231	[0.073]
WY_{-1}	-0.037	[0.076]
Wu	0.304	[0.113]

Note: Results are based on $W_{province}$

Table E4. List of Cities

Province	**Table S1.** Cities							
Anhui	Hefei City	Wuhu City	Bengbu City	Huainan City	Maanshan City	Huaibei City	Tongling City	Anqing City
	Huangshan City	Chuzhou City	Fuyang City	Suzhou City	Liuan City	Bozhou City	Chizhou City	Xuancheng City
Fujian	Fuzhou City	Xiamen City	Putian City	Sanming City	Quanzhou City	Zhangzhou City	Nanping City	Longyan City
	Ningde City							
Gansu	Lanzhou City	Jiayuguan City	Jinchang City	Baiyin City	Tianshui City	Wuwei City	Zhangye City	Pingliang City
	Jiuquan City	Qingyang City	Dingxi City	Longnan City	Linxia Hui A.P	Cannan Zang A.P		
Guangdong	Guangzhou City	Shaoguan City	Shenzhen City	Zhuhai City	Shantou City	Foshan City	Jiangmen City	Zhanjiang City
	Maoming City	Zhaoqing City	Huizhou City	Meizhou City	Shanwei City	Heyuan City	Yangjiang City	Qingyuan City
	Dongguan City	ZhongShan City	Chaozhou City	Jieyang City	Yunfu City			
Guangxi	Nanning City	Liuzhou City	Guilin City	Wuzhou City	Beihai City	Fangchenggang City	Qinzhou City	Guigang City
	Yulin City	Baise City	Hezhou City	Hechi City	Laibin City	Chongzuo City		
Guizhou	Guiyang City	Liupanshui City	Zunyi City	Anshun City	South Guizhou Buyi & Miao A.P	Southwest Guizhou Buyi & Miao A.P	Southeast Guizhou Miao & Dong A.P	Tongren Prefecture
	Bijie Prefecture							
Hainan	Haikou City	Sanya City						
Hebei	Shijiazhuang City	Tangshan City	Qinhuangdao City	Handan City	Xingtai City	Baoding City	Zhangjiakou City	Chengde City
	Cangzhou City	Langfang City	Hengshui City					
Heilongjiang	Harbin City	Qiqihar City	Jixi City	Hegang City	Shuangyashan City	Daqing City	Yichun City	Jiamusi City
	Qitaihe City	Mudanjiang City	Heihe City	Suihua City				
Henan	Zhengzhou City	Kaifeng City	Luoyang City	Pingdingshan City	Anyang City	Hebi City	Xinxiang City	Jiaozuo City
	Puyang City	Xuchang City	LuoheCity	Sanmenxia City	Nanyang City	Shangqiu City	Xinyang City	Zhoukou City
	Zhumadian City	Jiyuan City						
Hubei	Wuhan City	Huangshi City	Shiyan City	Yichang City	Xiangyang City	EzhouCity	Jingmen City	Xiaogan City
	Jingzhou City	Huanggang City	Xianning City	Suizhou City	Enshi Tujia & Miao A.P	Xiantao City	Qianjiang City	Tianmen City
Hunan	ChangshaCity	Zhuzhou City	Xiangtan City	Hengyang City	Shaoyang City	Yueyang City	Changde City	Zhangjiajie City
	Yiyang City	Chenzhou City	Yongzhou City	Huaihua City	Loudi City	West Hunan Tujia & Miao A.P		
Inner Mongolia	Hohhot City	Baotou City	Hulunbuir City	Xingan League	Tongliao City	Chifeng City	Xilingol League	Ulanqab City
	Erdos City	Bayannur City	Wuhai City	Alxa League				

Table E5. List of Cities - cont.

Province	**Table S1.** Cities							
Jiangsu	Nanjing City	Wuxi City	Xuzhou City	Changzhou City	Suzhou City	Nantong City	Lianyungang City	Huaian City
	Yancheng City	Yangzhou City	Zhenjiang City	Taizhou City	Suqian City			
Jiangxi	Nanchang City	Jingdezhen City	Pingxiang City	Jiujiang City	Xinyu City	Yingtan City	Ganzhou City	Jian City
	Yichun City	Fuzhou City	Shangrao City					
Jilin	Changchun City	Jilin City	Siping City	Liaoyuan City	Tonghua City	Baishan City	Songyuan City	Baicheng City
	Yanbian Korean A.P							
Liaoning	Shenyang City	Dalian City	Anshan City	Fushun City	Benxi City	Dandong City	Jinzhou City	Yingkou City
	Fuxin City	Liaoyang City	Panjin City	Tieling City	Chaoyang City	Huludao City		
Ningxia	Yinchuan City	Shizuishan City	Wuzhong City	Guyuan City	Zhongwei City			
Qinghai	Xining City	Haidong Prefecture	Haibei Zang A.P	Huangnan Zang AP	Hainan Zang A.P	Golog Zang A.P	Yushu Zang A.P	Haixi Mongolian & Zang A.P
Shaanxi	Xi'an City	Tongchuan City	Baoji City	Xianyang City	Weinan City	Yan'an City	Hanzhong City	YulinCity
	AnkangCity	Shangluo City						
Shandong	Jinan City	Qingdao City	Zibo City	Zaozhuang City	Dongying City	Yantai City	Weifang City	Jining City
	Taian City	Weihai City	Rizhao City	Laiwu City	Linyi City	Dezhou City	Liaocheng City	Binzhou City
	Heze City							
Shanxi	Taiyuan City	Datong City	Yangquan City	Changzhi City	Jincheng City	Shuozhou City	Jinzhong City	Yuncheng City
	Xinzhou City	Linfen City	Luliang City					
Sichuan	Chengdu City	Zigong City	Panzhihua City	Luzhou City	Deyang City	Mianyang City	Guangyuan City	Suining City
	Neijiang City	Leshan City	Nanchong City	Meishan City	Yibin City	Guangan City	Dazhou City	YaanCity
	Bazhong City	Ziyang City	Aba Zang	Qiang A.P	Ganzi Zang A.P	Liangshan Yi A.P		
Tibet	Lhasa City	Qamdu Prefecture	Lhokha Prefecture	Xigaze Prefecture	Narqu Prefecture	Ngri Prefecture	Nyingchi Prefecture	
Xinjiang	Urumqi City	Karamay City	Turpan Prefecture	Hami Prefecture	Changji Hui AP	Altay Prefecture	ShiheziCity	Aksu Prefecture
	Kizilsu Kirgiz A.P	Kashi Prefecture	Hotan Prefecture	Ili Kazak A.P	Tacheng Prefecture	Bortala Mongolian A.P	Bayingolin Mongolian A.P	Alar City
	Tumxuk City							
Yunnan	Kunming City	Qujing City	Yuxi City	Baoshan City	Zhaotong City	Lijiang City	Puer City	Lincang City
	Chuxiong Yi A.P	Honghe Hani & Yi A.P	Wenshan Zhuang & Miao A.P	Xishuangbanna Dai A.P	Dali Bai A.P	Dehong Dai & Jingpo A.P	Nujiang Lisu A.P	Diqing Zang A.P
Zhejiang	Hangzhou City	Ningbo City	Wenzhou City	Jiaxing City	Huzhou City	Shaoxing City	Jinhua City	Quzhou City
	Zhoushan City	Taizhou City	Lishui City					

V. Appendices: Detailed Proofs

Appendix V1: Some Basic Lemmas

Lemma V1.1. *(Kelejian and Prucha, 1999; Lee, 2002): Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ matrices that are uniformly bounded in both row and column sums. Let C_n be a sequence of conformable matrices whose elements are uniformly $O(h_n^{-1})$. Then*

(i) the sequence $\{A_n B_n\}$ are uniformly bounded in both row and column sums,

(ii) the elements of A_n are uniformly bounded and $tr(A_n) = O(n)$, and

(iii) the elements of $A_n C_n$ and $C_n A_n$ are uniformly $O(h_n^{-1})$.

Lemma V1.2. *(Lee, 2004, p.1918): For W_1 and B_1 defined in Model (1.1) in the main paper, if $\|W_1\|$ and $\|B_{10}^{-1}\|$ are uniformly bounded, where $\|\cdot\|$ is a matrix norm, then $\|B_1^{-1}\|$ is uniformly bounded in a neighbourhood of λ_{10}.*

Lemma V1.3. *(Lee, 2004, p.1918): Let X_n be an $n \times p$ matrix. If the elements X_n are uniformly bounded and $\lim_{n\to\infty} \frac{1}{n} X_n' X_n$ exists and is nonsingular, then $P_n = X_n(X_n'X_n)^{-1}X_n'$ and $M_n = I_n - P_n$ are uniformly bounded in both row and column sums.*

Lemma V1.4. *(Lemma A.4, Yang, 2018): Let $\{A_n\}$ be a sequence of $n \times n$ matrices that are uniformly bounded in either row or column sums. Suppose that the elements $a_{n,ij}$ of A_n are $O(h_n^{-1})$ uniformly in all i and j. Let v_n be a random n-vector of iid elements with mean zero, variance σ^2 and finite 4th moment, and b_n a constant n-vector of elements of uniform order $O(h_n^{-1/2})$. Then*

(i) $\mathrm{E}(v_n' A_n v_n) = O(\frac{n}{h_n})$,	*(ii)* $\mathrm{Var}(v_n' A_n v_n) = O(\frac{n}{h_n})$,
(iii) $\mathrm{Var}(v_n' A_n v_n + b_n' v_n) = O(\frac{n}{h_n})$,	*(iv)* $v_n' A_n v_n = O_p(\frac{n}{h_n})$,
(v) $v_n' A_n v_n - \mathrm{E}(v_n' A_n v_n) = O_p((\frac{n}{h_n})^{\frac{1}{2}})$,	*(vi)* $v_n' A_n b_n = O_p((\frac{n}{h_n})^{\frac{1}{2}})$,

and (vii), the results (iii) and (vi) remain valid if b_n is a random n-vector independent of v_n such that $\{\mathrm{E}(b_{ni}^2)\}$ are of uniform order $O(h_n^{-1})$.

Lemma V1.5. *(Lemma A.5, Yang, 2018): Let $\{\Phi_n\}$ be a sequence of $n \times n$ matrices with row and column sums uniformly bounded, and elements of uniform order $O(h_n^{-1})$. Let $v_n = (v_1, \cdots, v_n)'$ be a random vector of iid elements with mean zero, variance σ_v^2, and finite $(4 + 2\epsilon_0)$th moment for some $\epsilon_0 > 0$. Let $b_n = \{b_{ni}\}$ be an $n \times 1$ random vector, independent of v_n, such that (i) $\{\mathrm{E}(b_{ni}^2)\}$ are of uniform order $O(h_n^{-1})$, (ii) $sup_i E|b_{ni}|^{2+\epsilon_0} < \infty$, (iii) $\frac{h_n}{n}\sum_{i=1}^n [\phi_{n,ii}(b_{ni} - \mathrm{E}b_{ni})] = o_p(1)$ where $\{\phi_{n,ii}\}$ are the diagonal elements of Φ_n, and (iv) $\frac{h_n}{n}\sum_{i=1}^n [b_{ni}^2 - \mathrm{E}(b_{ni}^2)] = o_p(1)$. Define the bilinear-quadratic form:*

$$Q_n = b_n' v_n + v_n' \Phi_n v_n - \sigma_v^2 tr(\Phi_n),$$

and let $\sigma_{Q_n}^2$ be the variance of Q_n. If $lim_{n\to\infty} h_n^{1+2/\epsilon_0}/n = 0$ and $\{\frac{h_n}{n}\sigma_{Q_n}^2\}$ are bounded away from zero, then $Q_n/\sigma_{Q_n} \xrightarrow{d} N(0, 1)$.

Lemma V1.6. *Under Assumption F, for an $n \times n$ matrix Φ uniformly bounded in either row or column sums, with elements of uniform order h_n^{-1}, and an $n \times 1$ vector ϕ with elements of uniform order $h_n^{-1/2}$, we have:*
(i) $\frac{h_n}{n} y_0' \Phi y_0 = O_p(1)$; *(ii)* $\frac{h_n}{n}[y_0 - \mathrm{E}(y_0)]'\phi = o_p(1)$; *(iii)* $\frac{h_n}{n}[y_0'\Phi y_0 - \mathrm{E}(y_0'\Phi y_0)] = o_p(1)$.

Also in the proofs, the following results are useful: (i) eigenvalues of a projection matrix are either 0 or 1; (ii) eigenvalues of a positive definite matrix are strictly positive; (iii) for symmetric matrix A and positive semidefinite (p.s.d.) matrix B, $\gamma_{\min}(A)\mathrm{tr}(B) \le \mathrm{tr}(AB) \le \gamma_{\max}(A)\mathrm{tr}(B)$; (iv) for symmetric matrices A and B, $\gamma_{\max}(A+B) \le \gamma_{\max}(A) + \gamma_{\max}(B)$; and (v) for p.s.d. matrices A and B, $\gamma_{\max}(AB) \le \gamma_{\max}(A)\gamma_{\max}(B)$. See, e.g, Bernstein (2009).

Appendix V2: Proofs for Section 2

Note: all the equations, lemmas and theorems referred to in the following proofs correspond to those appeared in the main paper or Section V of this `Supplementary Appendix`.

Proof of Lemma 2.1: By (2.1), backward substitution leads to, for $t = -m+1, \dots, T$,

$$\begin{aligned} \mathrm{E}(y_t\varepsilon') &= B_1^{-1}B_2E(y_{t-1}\varepsilon') + B_1^{-1}E(\varepsilon\varepsilon') + B_1^{-1}B_3^{-1}\mathrm{E}(v_t\varepsilon') \\ &= (B_1^{-1}B_2)^2E(y_{t-2}\varepsilon') + (B_1^{-1}B_2 + I_n)B_1^{-1}\mathrm{E}(\varepsilon\varepsilon') \\ &= \mathcal{B}^t\mathrm{E}(y_0\varepsilon') + (\textstyle\sum_{i=0}^{t-1}\mathcal{B}^i)B_1^{-1}\mathrm{E}(\varepsilon\varepsilon') \\ &= \mathcal{B}^{t+m}\mathrm{E}(y_{-m}\varepsilon') + \mathcal{B}^{t+m-1}B_1^{-1}\mathrm{E}(\varepsilon\varepsilon') + (\textstyle\sum_{i=0}^{t+m-2}\mathcal{B}^i)B_1^{-1}\mathrm{E}(\varepsilon\varepsilon') \\ &= (\textstyle\sum_{i=0}^{t+m-1}\mathcal{B}^i)B_1^{-1}\sigma_{\varepsilon 0}^2. \end{aligned}$$

Therefore, $\mathrm{E}(Y_{-1}\boldsymbol{\varepsilon}') = \sigma_{\varepsilon 0}^2\mathbf{C}_{-1}$ and $\mathrm{E}(Y\boldsymbol{\varepsilon}') = \sigma_{\varepsilon 0}^2\mathbf{C}$.

For $t, s = 1, \dots, T$, we have $\mathrm{E}(y_tv_t') = B_1^{-1}B_2E(y_{t-1}v_t') + B_1^{-1}B_3^{-1}\mathrm{E}(v_tv_t') = \sigma_{v0}^2B_1^{-1}B_3^{-1}$; $\mathrm{E}(y_tv_s') = 0$ when $t < s$; and

$$\begin{aligned} \mathrm{E}(y_tv_s') &= B_1^{-1}B_2E(y_{t-1}v_s') + B_1^{-1}B_3^{-1}E(v_tv_s') = \mathcal{B}^2E(y_{t-2}v_s') = \cdots \\ &= \mathcal{B}^{t-s}E(y_sv_s') = \mathcal{B}^{t-s}E(B_1^{-1}B_3^{-1}v_sv_s') = \mathcal{B}^{t-s}B_1^{-1}B_3^{-1}\sigma_{v0}^2, \end{aligned}$$

when $t > s$. Therefore, $\mathrm{E}(Y_{-1}v')(\mathbf{B}_3^{-1})' = \sigma_{v0}^2\mathbf{D}_{-1}$ and $\mathrm{E}(Yv')(\mathbf{B}_3^{-1})' = \sigma_{v0}^2\mathbf{D}$. Combining these results, we obtain the results of Lemma 2.1. ■

Proof of (2.14): Backward substitution on (2.1) gives, for $t = 1, \dots, T$,

$$\begin{aligned} y_t &= \mathcal{B}y_{t-1} + B_1^{-1}\mathbf{X}_t\beta_0 + B_1^{-1}\varepsilon + B_1^{-1}B_3^{-1}v_t \\ &= \mathcal{B}^2y_{t-2} + \mathcal{B}B_1^{-1}\mathbf{X}_t\beta_0 + \mathcal{B}B_1^{-1}\varepsilon + \mathcal{B}B_1^{-1}B_3^{-1}v_t + B_1^{-1}\mathbf{X}_t\beta_0 + B_1^{-1}\varepsilon + B_1^{-1}B_3^{-1}v_t \\ &\;\;\vdots \\ &= \mathcal{B}^ty_0 + \textstyle\sum_{k=1}^t\mathcal{B}^{t-k}B_1^{-1}\mathbf{X}_k\beta_0 + \sum_{k=1}^t\mathcal{B}^{t-k}B_1^{-1}\varepsilon + \sum_{k=1}^t\mathcal{B}^{t-k}B_1^{-1}B_3^{-1}v_k. \end{aligned}$$

The results of Lemma 2.2 follows. ■

Proof of Theorem 2.1: The concentrated AQS function $S^{*c}_{\texttt{SDPD}}(\delta)$ with $\boldsymbol{\beta}$ and σ_v^2 being concentrated out, and its population counterpart $\bar{S}^{*c}_{\texttt{SDPD}}(\delta)$ are given by

$$S^{*c}_{\texttt{SDPD}}(\delta) = \begin{cases} \frac{1}{2\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\hat{e}(\delta) - \frac{1}{2}\text{tr}[(\Omega^{-1}(J_T\otimes I_n)], \\ \frac{1}{\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}Y_{-1} - \text{tr}[(\phi\mathbf{C}_{-1}+\mathbf{D}_{-1})\Omega^{-1}], \\ \frac{1}{\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_1 Y - \text{tr}[(\phi\mathbf{C}+\mathbf{D})\Omega^{-1}\mathbf{W}_1], \\ \frac{1}{\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1} - \text{tr}[(\phi\mathbf{C}_{-1}+\mathbf{D}_{-1})\Omega^{-1}\mathbf{W}_2], \\ \frac{1}{2\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta) - \frac{1}{2}\text{tr}(\Omega^{-1}\dot{\Omega}_{\lambda_3}), \end{cases} \quad \text{(V2.1)}$$

$$\bar{S}^{*c}_{\texttt{SDPD}}(\delta) = \begin{cases} \frac{1}{2\bar{\sigma}_v^2(\delta)}\text{E}[\bar{\mathbf{e}}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\bar{\mathbf{e}}(\delta)] - \frac{1}{2}\text{tr}[(\Omega^{-1}(J_T\otimes I_n)], \\ \frac{1}{\bar{\sigma}_v^2(\delta)}\text{E}[\bar{\mathbf{e}}'(\delta)\Omega^{-1}Y_{-1}] - \text{tr}[(\phi\mathbf{C}_{-1}+\mathbf{D}_{-1})\Omega^{-1}], \\ \frac{1}{\bar{\sigma}_v^2(\delta)}\text{E}[\bar{\mathbf{e}}'(\delta)\Omega^{-1}\mathbf{W}_1 Y] - \text{tr}[(\phi\mathbf{C}+\mathbf{D})\Omega^{-1}\mathbf{W}_1], \\ \frac{1}{\bar{\sigma}_v^2(\delta)}\text{E}[\bar{\mathbf{e}}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1}] - \text{tr}[(\phi\mathbf{C}_{-1}+\mathbf{D}_{-1})\Omega^{-1}\mathbf{W}_2], \\ \frac{1}{2\bar{\sigma}_v^2(\delta)}\text{E}[\bar{\mathbf{e}}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\bar{\mathbf{e}}(\delta)] - \frac{1}{2}\text{tr}(\Omega^{-1}\dot{\Omega}_{\lambda_3}), \end{cases} \quad \text{(V2.2)}$$

where $\hat{\sigma}_v^2(\delta)$ is defined in (2.13), and $\bar{\sigma}_v^2(\delta)$ is defined above Theorem 2.1.

From (V2.1) and (V2.2), we have

$$S^{*c}_{\texttt{SDPD}}(\delta)-\bar{S}^{*c}_{\texttt{SDPD}}(\delta) = \begin{cases} \frac{1}{2\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\hat{e}(\delta) - \frac{1}{2\bar{\sigma}_v^2(\delta)}\text{E}[\bar{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\bar{e}(\delta)], \\ \frac{1}{\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}Y_{-1} - \frac{1}{\bar{\sigma}_v^2(\delta)}\text{E}[\bar{e}'(\delta)\Omega^{-1}Y_{-1}], \\ \frac{1}{\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_1 Y - \frac{1}{\bar{\sigma}_v^2(\delta)}\text{E}[\bar{e}'(\delta)\Omega^{-1}\mathbf{W}_1 Y], \\ \frac{1}{\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1} - \frac{1}{\bar{\sigma}_v^2(\delta)}\text{E}[\bar{e}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1}], \\ \frac{1}{2\hat{\sigma}_v^2(\delta)}\hat{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta) - \frac{1}{2\bar{\sigma}_v^2(\delta)}\text{E}[\bar{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta)]. \end{cases}$$

Under Assumption G, the consistency of $\hat{\delta}_{\texttt{M}}$ follows if $\sup_{\delta\in\Delta}\frac{1}{nT}\|S^{*c}_{\texttt{SDPD}}(\delta)-\bar{S}^{*c}_{\texttt{SDPD}}(\delta)\| \xrightarrow{p} 0$ as $n\to\infty$, by Theorem 5.9 of van der Vaart (1998), boils down to the proofs of the following:

(a) $\inf_{\delta\in\Delta}\bar{\sigma}^2_{v,\texttt{M}}(\delta)$ is bounded away from zero,

(b) $\sup_{\delta\in\Delta}\big|\hat{\sigma}^2_{v,\texttt{M}}(\delta)-\bar{\sigma}^2_{v,\texttt{M}}(\delta)\big| = o_p(1)$,

(c) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{2\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\hat{e}(\delta) - \frac{1}{2\bar{\sigma}_v^2}\text{E}[\bar{e}'(\delta)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}\bar{e}(\delta)]\big| = o_p(1)$,

(d) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}Y_{-1} - \frac{1}{\bar{\sigma}_v^2}\text{E}[\bar{e}'(\delta)\Omega^{-1}Y_{-1}]\big| = o_p(1)$,

(e) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_1 Y - \frac{1}{\bar{\sigma}_v^2}\text{E}[\bar{e}'(\delta)\Omega^{-1}\mathbf{W}_1 Y]\big| = o_p(1)$,

(f) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1} - \frac{1}{\bar{\sigma}_v^2}\text{E}[\bar{e}'(\delta)\Omega^{-1}\mathbf{W}_2 Y_{-1}]\big| = o_p(1)$,

(g) $\sup_{\delta\in\Delta}\frac{1}{nT}\big|\frac{1}{\hat{\sigma}_v^2}\hat{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta) - \frac{1}{\bar{\sigma}_v^2}\text{E}[\bar{e}'(\delta)\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}\hat{e}(\delta)]\big| = o_p(1)$.

Define $\bar{\mathbf{e}}^*(\delta) = \Omega^{-\frac{1}{2}}\bar{\mathbf{e}}(\delta)$ and $\mathbf{B}^*_r = \Omega^{-\frac{1}{2}}\mathbf{B}_r, r=1,2$, where $\Omega^{\frac{1}{2}}$ is the square-root matrix of Ω. Let $Y^\circ = Y - \mathrm{E}(Y)$ and $Y^\circ_{-1} = Y_{-1} - \mathrm{E}(Y_{-1})$. We first present a useful identity:

$$\bar{\mathbf{e}}^*(\delta) = \mathbf{M}(\mathbf{B}_1^*\Delta Y - \mathbf{B}_2^* Y_{-1}) + \mathbf{P}(\mathbf{B}_1^* Y^\circ - \mathbf{B}_2^* Y^\circ_{-1}), \tag{V2.3}$$

where $\mathbf{M} = I_{nT} - \Omega^{-\frac{1}{2}}\mathbf{X}(\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-\frac{1}{2}}$ and $\mathbf{P} = I_{nT} - \mathbf{M}$.

Proof of (a). By (V2.3) and the orthogonality between $\mathbf{M}$ and $\mathbf{P}$, we have

$$\begin{aligned}
\bar{\sigma}^2_{v,\mathtt{M}}(\delta) &= \tfrac{1}{nT}\mathrm{E}[\bar{\mathbf{e}}^{*\prime}(\delta)\bar{\mathbf{e}}^*(\delta)] \\
&= \tfrac{1}{nT}\mathrm{E}[(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1})'\mathbf{M}(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1})] + \tfrac{1}{nT}\mathrm{E}[(\mathbf{B}_1^*Y^\circ - \mathbf{B}_2^*Y^\circ_{-1})'\mathbf{P}(\mathbf{B}_1^*Y^\circ - \mathbf{B}_2^*Y^\circ_{-1})] \\
&= \tfrac{1}{nT}\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1})] + \tfrac{1}{nT}(\mathbf{B}_1^*\mathrm{E}Y - \mathbf{B}_2^*\mathrm{E}Y_{-1})'\mathbf{M}(\mathbf{B}_1^*\mathrm{E}Y - \mathbf{B}_2^*\mathrm{E}Y_{-1}).
\end{aligned}$$

As $\mathbf{M}$ is p.s.d, the second term is nonnegative for every in $\delta \in \Delta$. By Assumption E(iv) and the assumptions given in the theorem, the first term is such that for every $\delta \in \Delta$,

$$\begin{aligned}
\tfrac{1}{nT}\mathrm{tr}[\Omega^{-1}\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})] &\geq \tfrac{1}{nT}\gamma_{\min}(\Omega^{-1})\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})] \\
&= \tfrac{1}{nT}\gamma^{-1}_{\max}(\Omega)\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})] \\
&\geq \tfrac{1}{nT}[\phi\gamma_{\max}(J_T \otimes I_n) + \gamma_{\max}\big((B_3'B_3)^{-1}\big)]^{-1}\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})] \\
&= \tfrac{1}{nT}[\phi + \gamma^{-1}_{\min}(B_3'B_3)]^{-1}\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})] \\
&\geq \tfrac{1}{nT}\tfrac{\underline{c}_3}{1+\underline{c}_3\phi}\mathrm{tr}[\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})] \geq c > 0.
\end{aligned}$$

It follows that $\inf_{\delta\in\Delta}\bar{\sigma}^2_{v,\mathtt{M}}(\delta) > c > 0$.

Proof of (b). Let $\hat{\mathbf{e}}^*(\delta) = \Omega^{-\frac{1}{2}}\hat{\mathbf{e}}(\delta) = \mathbf{M}(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1})$, we have,

$$\hat{\sigma}^2_{v,\mathtt{M}}(\delta) = \tfrac{1}{nT}\hat{\mathbf{e}}^{*\prime}(\delta)\hat{\mathbf{e}}^*(\delta) = \tfrac{1}{nT}(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1})'\mathbf{M}(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1}).$$

It follows that,

$$\begin{aligned}
\hat{\sigma}^2_{v,\mathtt{M}}(\delta) - \bar{\sigma}^2_{v,\mathtt{M}}(\delta) &= \frac{1}{nT}[(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1})'\mathbf{M}(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1})] \qquad \text{(V2.4)} \\
&- \frac{1}{nT}\mathrm{E}[(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1})'\mathbf{M}(\mathbf{B}_1^*Y - \mathbf{B}_2^*Y_{-1})] - \frac{1}{nT}\mathrm{E}[(\mathbf{B}_1^*Y^\circ - \mathbf{B}_2^*Y^\circ_{-1})'\mathbf{P}(\mathbf{B}_1^*Y^\circ - \mathbf{B}_2^*Y^\circ_{-1})] \\
&= \frac{1}{nT}\left[Y'\mathbf{B}_1^{*\prime}\mathbf{M}\mathbf{B}_1^*Y - \mathrm{E}(Y'\mathbf{B}_1^{*\prime}\mathbf{M}\mathbf{B}_1^*Y)\right] + \frac{1}{nT}[Y_{-1}'\mathbf{B}_2^{*\prime}\mathbf{M}\mathbf{B}_2^*Y_{-1} - \mathrm{E}(Y_{-1}'\mathbf{B}_2^{*\prime}\mathbf{M}\mathbf{B}_2^*Y_{-1})] \\
&- \frac{2}{nT}[Y'\mathbf{B}_1^{*\prime}\mathbf{M}\mathbf{B}_2^*Y_{-1} - \mathrm{E}(Y'\mathbf{B}_1^{*\prime}\mathbf{M}\mathbf{B}_2^*Y_{-1})] - \frac{1}{nT}\mathrm{E}[(\mathbf{B}_1^*Y^\circ - \mathbf{B}_2^*Y^\circ_{-1})'\mathbf{P}(\mathbf{B}_1^*Y^\circ - \mathbf{B}_2^*Y^\circ_{-1})] \\
&\equiv (Q_1 - \mathrm{E}Q_1) + (Q_2 - \mathrm{E}Q_2) - 2(Q_3 - \mathrm{E}Q_3) - \mathrm{E}Q_4.
\end{aligned}$$

The results follows if $Q_j - \mathrm{E}Q_j \xrightarrow{p} 0,\ j = 1, 2, 3$, and $\mathrm{E}Q_4 \longrightarrow 0$, uniformly in $\delta \in \Delta$.

By (2.16) and letting $\mathbf{M}^* = \Omega^{-\frac{1}{2}}\mathbf{M}\Omega^{-\frac{1}{2}}$, we have,

$$\begin{aligned}
Q_1 = \tfrac{1}{nT}\big(&\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{Q}\mathbf{y}_0 + \boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\boldsymbol{\eta} + \boldsymbol{\varepsilon}'\mathbb{S}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{S}\boldsymbol{\varepsilon} + \mathbf{v}'\mathbb{B}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{B}\mathbf{v} \\
&+ 2\boldsymbol{\varepsilon}'\mathbb{S}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{B}\mathbf{v} + 2\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\boldsymbol{\eta} + 2\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{S}\boldsymbol{\varepsilon} + 2\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{B}\mathbf{v} \\
&+ 2\boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{S}\boldsymbol{\varepsilon} + 2\boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{B}\mathbf{v}\big),
\end{aligned}$$

which leads to $Q_1 - \mathrm{E}Q_1 = \sum_{\ell=1}^{9}(Q_{1,\ell} - \mathrm{E}Q_{1,\ell})$, where $Q_{1,\ell}, \ell = 1, \ldots, 9$, denote the nine stochastic terms of Q_1, and the expectations of the forth and the last two terms are zero;

$$Q_2 = \frac{1}{nT}\big(\mathbf{y}_0'\mathbb{Q}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0 + \boldsymbol{\eta}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1} + \boldsymbol{\varepsilon}'\mathbb{S}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon}$$
$$+\mathbf{v}'\mathbb{B}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + 2\boldsymbol{\varepsilon}'\mathbb{S}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + 2\mathbf{y}_0'\mathbb{Q}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1}$$
$$+2\mathbf{y}_0'\mathbb{Q}_{-1}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + 2\mathbf{y}_0'\mathbb{Q}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + 2\boldsymbol{\eta}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + 2\boldsymbol{\eta}_{-1}'\mathbf{B}_2'\mathbf{M}^*\mathbf{B}_2\mathbb{B}\mathbf{v}\big),$$

which leads to $Q_2 - \mathrm{E}Q_2 = \sum_{\ell=1}^{9}(Q_{2,\ell} - \mathrm{E}Q_{2,\ell})$, where $Q_{2,\ell}, \ell = 1, \ldots, 9$, denote the nine stochastic terms of Q_2, and the expectations of the forth and the last two terms are zero.

$$Q_3 = \frac{1}{nT}\big(\mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0 + \boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1} + \boldsymbol{\varepsilon}'\mathbb{S}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + \mathbf{v}'\mathbb{B}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v}$$
$$+2\boldsymbol{\varepsilon}'\mathbb{S}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + \mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1} + \mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + \mathbf{y}_0'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v}$$
$$+\boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0 + \boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{S}_{-1}\boldsymbol{\varepsilon} + \boldsymbol{\eta}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{B}_{-1}\mathbf{v} + \boldsymbol{\varepsilon}'\mathbb{S}_{-1}\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0$$
$$+\mathbf{v}'\mathbb{B}_{-1}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\mathbb{Q}_{-1}\mathbf{y}_0 + \boldsymbol{\varepsilon}'\mathbb{S}_{-1}\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1} + \mathbf{v}'\mathbb{B}_{-1}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_2\boldsymbol{\eta}_{-1}\big),$$

which leads to $Q_3 - \mathrm{E}Q_3 = \sum_{\ell=1}^{14}(Q_{3,\ell} - \mathrm{E}Q_{3,\ell})$, where $Q_{3,\ell}, \ell = 1, \ldots, 14$, denote the fourteen stochastic terms of Q_3. The forth,ninth, tenth and the last two terms have expectations zero.

Thus, $Q_k, k = 1, 2, 3$, are decomposed into sums of terms of the forms: $\frac{1}{nT}\mathbf{y}_0'\Phi\mathbf{y}_0$, $\frac{1}{nT}\mathbf{v}'\Pi\mathbf{v}$, $\frac{1}{nT}\boldsymbol{\varepsilon}'\Pi_\varepsilon\boldsymbol{\varepsilon}$, $\frac{1}{nT}\boldsymbol{\varepsilon}'\Theta\mathbf{v}$,$\frac{1}{nT}\mathbf{y}_0'\Psi\mathbf{v}$, $\frac{1}{nT}\mathbf{y}_0'\Psi_\varepsilon\boldsymbol{\varepsilon}$, $\frac{1}{nT}\mathbf{y}_0'\phi$, $\frac{1}{nT}\mathbf{v}'\psi$, and $\frac{1}{nT}\boldsymbol{\varepsilon}'\psi_\varepsilon$. The matrices Φ, Π, Π_ε, Θ, and Ψ, and the vectors ϕ, ψ and ψ_ε are defined in terms of $\mathbb{Q}$, $\mathbb{Q}_{-1}$, $\mathbb{S}$, $\mathbb{S}_{-1}$, $\mathbb{B}$, $\mathbb{B}_{-1}$, $\boldsymbol{\eta}$, and $\boldsymbol{\eta}_{-1}$, which depend on true parameter values and $\mathbf{B}_1$, which depends on λ_1, $\mathbf{B}_2$, which depends on ρ and λ_2 and $\mathbf{M}^*$, which depends on λ_3 and ϕ. By Lemma V1.1, Assumption E and the expressions given under the AQS function, the $nT \times N$ matrices $\mathbb{R}$, $\mathbb{R}_{-1}$, $\mathbb{S}$, $\mathbb{S}_{-1}$, $\mathbb{B}$ and $\mathbb{B}_{-1}$ are uniformly bounded in both row and colum sums, elements of the $nT \times 1$ vectors $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_{-1}$ uniformly bounded. By Assuption E(iv), $\mathbf{B}_1$ and $\mathbf{B}_2$ are uniformly bounded in either row and colum sums. By Assumption E and the expression of Ω in eqation (2.3), we know $0 < \underline{c} \leq \inf_{\lambda_3,\phi\in\boldsymbol{\Lambda}}\gamma_{\min}(\Omega) \leq \sup_{\lambda_3,\phi\in\boldsymbol{\Lambda}}\gamma_{\max}(\Omega) \leq \bar{c} < \infty$. Therefore, $0 < \frac{1}{\bar{c}} \leq \inf_{\lambda_3,\phi\in\boldsymbol{\Lambda}}\gamma_{\min}(\Omega^{-1}) \leq \sup_{\lambda_3,\phi\in\boldsymbol{\Lambda}}\gamma_{\max}(\Omega^{-1}) \leq \frac{1}{\underline{c}} < \infty$. For $nT \times 1$ vector e_k whose kth element is one and all other elements are zeros. $\|\Omega^{-1}e_k\| \leq \|\Omega^{-1}\|\|e_k\| \leq \gamma_{\max}(\Omega^{-1}) \leq \frac{1}{\underline{c}}$. It follows that Ω^{-1} and therefore $\mathbf{M}^*$ is bounded either row and colum sums.

The quadratic terms of $\mathbf{y}_0$ can be written as $\frac{1}{nT}y_0'\Phi_{++}(\delta)y_0$ where $\Phi_{++}(\delta) = \sum_t\sum_s\Phi_{t,s}(\delta)$. Each $\delta \in \Delta$, $\Phi_{t,s}(\delta)$ are uniformly bounded in either row or column sums. The pointwise convergence of $\frac{1}{n}[y_0'\Phi_{++}(\delta)y_0 - \mathrm{E}(y_0'\Phi_{++}(\delta)y_0)]$ thus follows from Assumption F(iii). The quadratic terms of $\mathbf{v}$ can be written as $\frac{1}{nT}\sum_{t=1}^{T}\sum_{s=1}^{T}v_t'\Pi_{ts}v_s$. The quadratic terms of ε can be written as $\frac{1}{nT}\varepsilon'\Pi_{\varepsilon,++}\varepsilon$, where $\Pi_{\varepsilon,++} = \sum_t\sum_s\Pi_{\varepsilon,ts}$. The pointwise convergence of $\frac{1}{n}[v_t'\Pi_{ts}v_s - \mathrm{E}(v_t'\Pi_{ts}v_s)]$ follows from Lemma V1.4 (v), for each $t, s = 1, \ldots, T$, and the pointwise convergence of $\frac{1}{n}[\varepsilon'\Pi_{\varepsilon,++}\varepsilon - \mathrm{E}(\varepsilon'\Pi_{\varepsilon,++}\varepsilon)]$ also follows from Lemma V1.4 (v). The pointwise convergence of $\frac{1}{n}[\boldsymbol{\varepsilon}'\Theta\mathbf{v} - \mathrm{E}(\boldsymbol{\varepsilon}'\Theta\mathbf{v})]$ and $\frac{1}{nT}[\mathbf{y}_0'\Psi\mathbf{v} - \mathrm{E}(\mathbf{y}_0'\Psi\mathbf{v})]$ follows by writing $\boldsymbol{\varepsilon}'\Theta\mathbf{v} = \sum_s\varepsilon'\Theta_{+s}v_s$, $\mathbf{y}_0'\Psi\mathbf{v} = \sum_s y_0'\Psi_{+s}v_s$ and then applying Lemma V1.4 (vii). Similarly, the pointwise convergence of $\frac{1}{nT}[\mathbf{y}_0'\Psi_\varepsilon\boldsymbol{\varepsilon} - \mathrm{E}(\mathbf{y}_0'\Psi_\varepsilon\boldsymbol{\varepsilon})]$ follows by writing,

$\mathbf{y}_0'\Psi_\varepsilon\boldsymbol{\varepsilon} = y_0'\Psi_{++}\varepsilon = \varepsilon'\Psi_{++}\varepsilon + (\eta_m + V_m^*)'\Psi_{++}\varepsilon$, and then applying Lemma V1.4 (i) to the quadratic term and Chebyshev inequality for the linear term. The pointwise convergence of $\frac{1}{nT}[\mathbf{y}_0'\phi - \mathrm{E}(\mathbf{y}_0'\phi)]$ follows from Assumption F(ii), and of $\frac{1}{nT}\mathbf{v}'\psi$ and $\frac{1}{nT}\boldsymbol{\varepsilon}'\psi_\varepsilon$ from Chebyshev inequality. Thus, $Q_{k,\ell}(\delta) - \mathrm{E}Q_{k,\ell}(\delta) \xrightarrow{p} 0$, for each $\delta \in \Delta$, and all k and ℓ.

Now, for all the $Q_{k,\ell}(\delta)$ terms, let δ_1 and δ_2 be in $\boldsymbol{\Delta}$. We have by the mean value theorem (MVT):

$$Q_{k,\ell}(\delta_2) - Q_{k,\ell}(\delta_1) = \tfrac{\partial}{\partial\delta'}Q_{k,\ell}(\bar{\delta})(\delta_2 - \delta_1),$$

where $\bar{\delta}$ lies between δ_1 and δ_2 elementwise. Note that $Q_{k,\ell}(\delta)$ is linear or quadratic in ρ, λ_1 and λ_2, and thus the corresponding partial derivatives takes simple form. It is easy to show that $\sup_{\delta\in\boldsymbol{\Delta}} |\frac{\partial}{\partial\omega}Q_{k,\ell}(\delta)| = O_p(1)$, for $\omega = \rho, \lambda_1, \lambda_2$. For $\frac{\partial}{\partial\lambda_3}Q_{k,\ell}(\delta)$, note that only the matrix M^* involves λ_3. Some algebra leads to the following simple expression for its derivative:

$$\tfrac{d}{d\lambda_3}M^* = \mathbf{M}^*\dot{\Omega}_{\lambda 3}\mathbf{M}^*,$$

where $\dot{\Omega}_{\lambda 3} = \frac{d}{d\lambda_3}\Omega = I_T \otimes (B_3'B_3)(B_3'W_3 + W_3'B_3)(B_3'B_3)$. Thus, it is easy to show that for all k and l $\sup_{\delta\in\boldsymbol{\Delta}} |\frac{\partial}{\partial\lambda_3}Q_{k,\ell}(\delta)| = O_p(1)$. For example, for $Q_{1,1}(\delta)$, noting that $\gamma_{\max}(\mathbf{M}) = 1$,

$$\begin{aligned}
\sup_{\delta\in\boldsymbol{\Delta}} |\tfrac{\partial}{\partial\lambda_3}Q_{1,1}(\delta)| &= \sup_{\delta\in\boldsymbol{\Delta}} |\tfrac{1}{n(T-1)}\tfrac{\partial}{\partial\lambda_3}\Delta\mathbf{y}_1'\mathbb{Q}'\mathbf{B}_1'\mathbf{M}^*\mathbf{B}_1\mathbb{Q}\Delta\mathbf{y}_1| \\
&= \sup_{\delta\in\boldsymbol{\Delta}} \tfrac{1}{n(T-1)}|\Delta\mathbf{y}_1'\mathbb{R}'\mathbf{B}_1'\mathbf{M}^*\dot{\Omega}_{\lambda 3}\mathbf{M}^*\mathbf{B}_1\mathbb{R}\Delta\mathbf{y}_1| \\
&\leq \sup_{\delta\in\boldsymbol{\Delta}} \tfrac{1}{n(T-1)}|\Delta\mathbf{y}_1'\mathbb{R}'\mathbf{B}_1'\dot{\Omega}_{\lambda 3}\mathbf{B}_1\mathbb{R}\Delta\mathbf{y}_1| \\
&\leq \gamma_{\max}(\dot{\Omega}_{\lambda 3})\gamma_{\max}(\mathbf{B}_1'\mathbf{B}_1)\tfrac{1}{nT}|\Delta\mathbf{y}_1'\mathbb{R}'\mathbb{R}\Delta\mathbf{y}_1| \\
&= O(1) \times O(1) \times O_p(1) = O_p(1), \text{ by Assumption F(i).}
\end{aligned}$$

It follows that $Q_{k,\ell}(\delta)$ are stochastic equicontinuous, and by Theorem 1 of Andrews (1992) $Q_{k,\ell}(\delta) - \mathrm{E}Q_{k,\ell}(\delta) \xrightarrow{p} 0$, uniformly in $\delta \in \boldsymbol{\Delta}$. Thus, $Q_k(\delta) - \mathrm{E}Q_k(\delta) \xrightarrow{p} 0$, uniformly in $\delta \in \Delta$, $k = 1, 2, 3$. It left to show that $\mathrm{E}Q_4(\delta) \to 0$, uniformly in $\delta \in \Delta$. We have

$$\begin{aligned}
\mathrm{E}Q_4 &= \tfrac{1}{nT}\mathrm{tr}[\Omega^{-1}\mathbf{X}(\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})] \\
&\leq \tfrac{1}{nT}\gamma^2_{\max}(\Omega^{-1})\gamma_{\max}[(\mathbf{X}'\Omega^{-1}\Delta X)^{-1}]\ \mathrm{tr}[\mathbf{X}'\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})\mathbf{X}] \\
&= \tfrac{1}{nT}\gamma^{-2}_{\min}(\Omega)\gamma^{-1}_{\min}\big(\tfrac{\mathbf{X}'\Omega^{-1}\mathbf{X}}{nT}\big)\ \tfrac{1}{nT}\mathrm{tr}[\mathbf{X}'\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})\mathbf{X}].
\end{aligned}$$

By Assumption E(iv), $\gamma_{\min}(\Omega) \geq \phi\gamma_{\min}(J_T \otimes I_n) + \gamma^{-1}_{\max}(B_3'B_3) \geq \frac{1}{\inf_{\lambda_3\in\Lambda_3}\gamma_{\max}(B_3'B_3)} \geq \frac{1}{\bar{c}_3}$. By Assumption D, $\gamma_{\min}(\frac{\mathbf{X}'\Omega^{-1}\mathbf{X}}{nT}) \geq \inf_{\lambda_3\in\Lambda_3}\gamma_{\min}(\Omega^{-1})\gamma_{\min}\big(\frac{\mathbf{X}'\mathbf{X}}{nT}\big) \geq \underline{c}_x \geq 0$. It follows that,

$$\begin{aligned}
\mathrm{E}Q_4 &\leq \tfrac{1}{nT}\bar{c}_3^2\tfrac{1}{\underline{c}_x}\tfrac{1}{nT}\mathrm{tr}[\mathbf{X}'\mathrm{Var}(\mathbf{B}_1Y - \mathbf{B}_2Y_{-1})\mathbf{X}] \\
&\leq \tfrac{1}{n(T-1)}\bar{c}_c^2\tfrac{1}{\underline{c}_x}\bar{c}_y\ \tfrac{1}{nT}\mathrm{tr}[\mathbf{X}'\mathbf{X}], \text{ by the assumption in Theorem 2.1} \\
&= O(n^{-1}), \text{ by Assumption D.}
\end{aligned}$$

Hence, $\hat{\sigma}^2_{v,\mathtt{M}}(\delta) - \bar{\sigma}^2_{v,\mathtt{M}}(\delta) \xrightarrow{p} 0$, uniformly in $\delta \in \Delta$, completing the proof of **(b)**.

Proofs of (c)-(g). By the expressions of $\hat{\mathbf{e}}(\delta)$ and $\bar{\mathbf{e}}(\delta)$ given earlier and (2.16), all the quantities inside $|\cdot|$ in (c)-(g) can all be expressed in the forms similar to (V2.4). Thus, the proofs of (c)-(g) follow the proof of (b). ∎

Proof of Theorem 2.2: We have by the mean value theorem,

$$0 = \tfrac{1}{\sqrt{nT}} S^*_{\texttt{SDPD}}(\hat{\psi}_{\texttt{SDPD}}) = \tfrac{1}{\sqrt{nT}} S^*_{\texttt{SDPD}}(\psi_0) + \big[\tfrac{1}{nT}\tfrac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\bar{\psi})\big]\sqrt{nT}(\hat{\psi}_{\texttt{M}} - \psi_0),$$

where $\bar{\psi}$ lies elementwise between $\hat{\psi}_{\texttt{M}}$ and ψ_0. The result of the theorem follows if

(a) $\frac{1}{\sqrt{nT}} S^*_{\texttt{SDPD}}(\psi_0) \xrightarrow{D} N\big[0, \lim_{n\to\infty} \Gamma^*_{\texttt{SDPD}}(\psi_0)\big]$,
(b) $\frac{1}{nT}\big[\frac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\bar{\psi}) - \frac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\psi_0)\big] \xrightarrow{p} 0$, and
(c) $\frac{1}{nT}\big[\frac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\psi_0) - \mathrm{E}\big(\frac{\partial}{\partial \psi'} S^*_{\texttt{SDPD}}(\psi_0)\big)\big] \xrightarrow{p} 0$.

Proof of (a). By $\mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}_{30}^{-1}\mathbf{v}$ and letting $\Pi_r^\circ = \mathbf{B}_{30}'^{-1}\Pi_r, r = 1, \ldots, 4$, $\Psi_r^\circ = \mathbf{B}_{30}'^{-1}\Psi_r, r = 1, 2, 3$, and $\Phi_r^\circ = \mathbf{B}_{30}'^{-1}\Phi_r\mathbf{B}_{30}^{-1}, r = 1, \ldots, 6$, and $\Phi_r^\circ = \mathbf{B}_{30}'^{-1}\Phi_r$ and $\Phi_r^\diamond = \mathbf{B}_{30}'^{-1}\Phi_r\mathbf{B}_{30}^{-1}, r = 1, \ldots, 6$, the AQS functions (2.15) can be further expressed as follows,

$$S^*_{\texttt{SDPD}}(\psi_0) = \begin{cases} \boldsymbol{\Pi}_1'\varepsilon + \Pi_1^{\circ\prime}\mathbf{v}, \\ \boldsymbol{\varepsilon}'\Phi_1\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_1^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_1^\circ\boldsymbol{\varepsilon} - \mu_{\sigma^2}, \\ \boldsymbol{\varepsilon}'\Phi_2\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_2^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_2^\circ\boldsymbol{\varepsilon} - \mu_{\phi}, \\ \boldsymbol{\varepsilon}'\Psi_1\mathbf{y}_0 + \mathbf{v}'\Psi_1^\circ\mathbf{y}_0 + \Pi_2'\boldsymbol{\varepsilon} + \Pi_2^{\circ\prime}\mathbf{v} + \boldsymbol{\varepsilon}'\Phi_3\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_3^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_3^\circ\boldsymbol{\varepsilon} - \mu_{\rho}, \\ \boldsymbol{\varepsilon}'\Psi_2\mathbf{y}_0 + \mathbf{v}'\Psi_2^\circ\mathbf{y}_0 + \Pi_3'\boldsymbol{\varepsilon} + \Pi_3^{\circ\prime}\mathbf{v} + \boldsymbol{\varepsilon}'\Phi_4\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_4^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_4^\circ\boldsymbol{\varepsilon} - \mu_{\lambda_1}, \\ \boldsymbol{\varepsilon}'\Psi_3\mathbf{y}_0 + \mathbf{v}'\Psi_3^\circ\mathbf{y}_0 + \Pi_4'\boldsymbol{\varepsilon} + \Pi_4^{\circ\prime}\mathbf{v} + \boldsymbol{\varepsilon}'\Phi_5\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_5^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_5^\circ\boldsymbol{\varepsilon} - \mu_{\lambda_{12}}, \\ \boldsymbol{\varepsilon}'\Phi_6\boldsymbol{\varepsilon} + \mathbf{v}'\Phi_6^\diamond\mathbf{v} + 2\mathbf{v}'\Phi_6^\circ\boldsymbol{\varepsilon} - \mu_{\lambda_3}, \end{cases} \tag{V2.5}$$

where $\mu_{\sigma^2} = \frac{nT}{2\sigma_{v0}^2}$, $\mu_\phi = \frac{1}{2}\mathrm{tr}[\Omega_0^{-1}(J_T \otimes I_n)]$, $\mu_\rho = \mathrm{tr}[(\phi_0\mathbf{C}_{-10} + \mathbf{D}_{-10})\Omega_0^{-1}]$, $\mu_{\lambda_1} = \mathrm{tr}[(\phi_0\mathbf{C}_0 + \mathbf{D}_0)\Omega_0^{-1}\mathbf{W}_1]$, $\mu_{\lambda_2} = \mathrm{tr}[(\phi_0\mathbf{C}_{-10} + \mathbf{D}_{-10})\Omega_0^{-1}\mathbf{W}_2]$, and $\mu_{\lambda_3} = \mathrm{tr}(\Omega_0^{-1}\dot{\Omega}_{\lambda_3 0})$.

Partition the vectors or matrices Π_r and Π_r° according to $t = 1, \ldots, T$, and denote the partitioned vectors or matrices, respectively, by $\{\Pi_{rt}\}$ and $\{\Pi_{rt}^\circ\}$; partition the matrices Φ_r, Φ_r°, $\Phi_r^\diamond$, Ψ_r, and Ψ_r° according to $t, s = 1, \ldots, T$, and denote the partitioned matrices, respectively, by $\{\Phi_{rts}\}$, $\{\Phi_{rts}^\circ\}$, $\{\Phi_{rts}^\diamond\}$, $\{\Psi_{rts}\}$, and $\{\Phi_{rts}^\circ\}$. As $\boldsymbol{\varepsilon} = 1_T \otimes \varepsilon$ and $\mathbf{y}_0 = 1_T \otimes y_0$, denoting $\Pi_{r+} = \sum_{t=1}^T \Pi_{rt}$, $\Phi_{rt+}^\circ = \sum_{s=1}^T \Phi_{rts}^\circ$, $\Phi_{r++} = \sum_{s=1}^T\sum_{s=1}^T \Phi_{rts}$, we have

$$\begin{aligned} &\Pi_r'\boldsymbol{\varepsilon} = \Pi_{r+}\varepsilon, \quad \boldsymbol{\varepsilon}'\Phi\boldsymbol{\varepsilon} = \varepsilon'\Phi_{r++}\varepsilon, \quad \boldsymbol{\varepsilon}'\Psi\mathbf{y}_0 = \varepsilon'\Psi_{r++}y_0, \\ &\mathbf{v}'\Psi_r^\circ\mathbf{y}_0 = \mathbf{v}'\Psi_{r+}^\circ y_0, \quad \mathbf{v}'\Phi_r^\circ\boldsymbol{\varepsilon} = \mathbf{v}'\Phi_{r+}^\circ\varepsilon. \end{aligned}$$

where $\Psi_{r+}^\circ = \Psi_r^\circ(1_T \otimes I_n)$ and $\Phi_{r+}^\circ = \Phi_r^\circ(1_T \otimes I_n)$. Now, by (3.2), the terms bilinear in ε and y_0, and the terms bilinear in $\mathbf{v}$ and y_0 can be expressed as

$$\begin{aligned} &\varepsilon'\Psi_{r++}y_0 = \varepsilon'\Psi_{r++}K_m\varepsilon + \varepsilon'\Psi_{r++}K_m(\eta_m^* + V_m^*), \text{ and} \\ &\mathbf{v}'\Psi_{r+}^\circ y_0 = \mathbf{v}'\Psi_{r+}^\circ K_m\varepsilon + \mathbf{v}'\Psi_{r+}^\circ K_m(\eta_m^* + V_m^*). \end{aligned}$$

Therefore, the AQS vector at the true parameters consists of terms linear-quadratic in $\mathbf{v}$, linear-quadratic in ε, and bilinear in ε and $\mathbf{v}$. Thus, for every non-zero $\dim(\psi) \times 1$ vector of constants c, $c'S^*_{\texttt{SDPD}}(\psi_0)$ can be expressed as

$$c'S^*_{\texttt{SDPD}}(\psi_0) = \mathbf{v}'A\mathbf{v} + \mathbf{v}'\pi + \varepsilon'B\varepsilon + \varepsilon'\varphi + \mathbf{v}D\varepsilon - c'\mu_\psi,$$

for suitably defined non-stochastic matrices A, B and D, and (random) vectors π and φ, where $\mu_\psi = \{0'_p, \mu_{\sigma^2}, \mu_\rho, \mu_{\lambda_1}, \mu_{\lambda_2}, \mu_{\lambda_3}, \}'$. Both π and φ are measurable functions of V_m, and hence are independent of ε and $\mathbf{v}$. Putting $c'S^*_{\texttt{SDPD}}(\psi_0)$ in a more compact form: $\mathbb{V}'\mathbb{A}\mathbb{V} + \mathbb{V}'\varpi - c'\mu_\psi$, where $\mathbb{V} = (\mathbf{v}', \varepsilon')'$, $\mathbb{A} = \{A, D; \mathbf{0}, B\}$, $\varpi = (\pi', \varphi)'$, and $\mathbf{0}$ denotes a matrix of zeros, the asymptotic normality of $\frac{1}{\sqrt{nT}}c'S^*_{\texttt{SDPD}}(\psi_0)$ follows from Lemma V1.5. Finally, the Cramér-Wold devise leads to the joint asymptotic normality of $\frac{1}{\sqrt{nT}}S^*_{\texttt{SDPD}}(\psi_0)$.

Proof of (b). The Hessian matrix,$H^*_{\texttt{SDPD}}(\psi) = \frac{\partial}{\partial\psi'}S^*_{\texttt{SDPD}}(\psi)$, has the elements:

$$\begin{aligned}
&H^*_{\beta\beta} = -\tfrac{1}{\sigma_v^2}X'\Omega^{-1}X, && H^*_{\beta\sigma_v^2} = -\tfrac{1}{\sigma_v^4}X'\Omega^{-1}\mathbf{e}(\theta), && H^*_{\beta\phi} = \tfrac{1}{\sigma_v^2}X'\dot{\Omega}^-_\phi\mathbf{e}(\theta),\\
&H^*_{\beta\rho} = -\tfrac{1}{\sigma_v^2}X'\Omega^{-1}Y_{-1}, && H^*_{\beta\lambda_1} = -\tfrac{1}{\sigma_v^2}X'\Omega^{-1}W_1Y, && H^*_{\beta\lambda_2} = -\tfrac{1}{\sigma_v^2}X'\Omega^{-1}W_2Y_{-1}\\
&H^*_{\beta\lambda_3} = \tfrac{1}{\sigma_v^2}X'\dot{\Omega}^-_{\lambda_3}\mathbf{e}(\theta), && H^*_{\sigma_v^2\sigma_v^2} = -\tfrac{1}{\sigma_v^6}\mathbf{e}'(\theta)\Omega^{-1}\mathbf{e}(\theta) + \tfrac{nT}{2\sigma_v^4}, && H^*_{\sigma_v^2\phi} = \tfrac{1}{2\sigma_v^4}\mathbf{e}'(\theta)\dot{\Omega}^-_\phi\mathbf{e}(\theta),\\
&H^*_{\sigma_v^2\rho} = -\tfrac{1}{\sigma_v^4}\mathbf{e}'(\theta)\Omega^{-1}Y_{-1}, && H^*_{\sigma_v^2\lambda_1} = -\tfrac{1}{\sigma_v^4}\mathbf{e}'(\theta)\Omega^{-1}W_1Y, && H^*_{\sigma_v^2\lambda_2} = -\tfrac{1}{\sigma_v^4}\mathbf{e}'(\theta)\Omega^{-1}W_2Y_{-1},\\
&H^*_{\sigma_v^2\lambda_3} = \tfrac{1}{2\sigma_v^4}\mathbf{e}'(\theta)\dot{\Omega}^-_{\lambda_3}\mathbf{e}(\theta), && H^*_{\phi\rho} = \tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_\phi Y_{-1}, && H^*_{\phi\lambda_1} = \tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_\phi W_1Y,\\
&H^*_{\phi\lambda_2} = \tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_\phi W_2Y_{-1}, && H^*_{\phi\phi} = -\tfrac{1}{2\sigma_v^2}\mathbf{e}'(\theta)\ddot{\Omega}^-_\phi\mathbf{e}(\theta) - \tfrac{1}{2}\mathrm{tr}[\dot{\Omega}^-_\phi(J_T\otimes I_n)],
\end{aligned}$$

$$\begin{aligned}
H^*_{\phi\lambda_3} &= -\tfrac{1}{2\sigma_v^2}\mathbf{e}'(\theta)\ddot{\Omega}^-_{\phi,\lambda_3}\mathbf{e}(\theta) - \tfrac{1}{2}\mathrm{tr}[\dot{\Omega}^-_{\lambda_3}(J_T\otimes I_n)],\\
H^*_{\rho\rho} &= -\tfrac{1}{\sigma_v^2}Y'_{-1}\Omega^{-1}Y_{-1} - \mathrm{tr}[(\phi\dot{\mathbf{C}}_{-1,\rho} + \dot{\mathbf{D}}_{-1,\rho})\Omega^{-1}],\\
H^*_{\rho\lambda_1} &= -\tfrac{1}{\sigma_v^2}Y'W_1'\Omega^{-1}Y_{-1} - \mathrm{tr}[(\phi\dot{\mathbf{C}}_{-1,\lambda_1} + \dot{\mathbf{D}}_{-1,\lambda_1})\Omega^{-1}],\\
H^*_{\rho\lambda_2} &= -\tfrac{1}{\sigma_v^2}Y_1'W_2'\Omega^{-1}Y_{-1} - \mathrm{tr}[(\phi\dot{\mathbf{C}}_{-1,\lambda_2} + \dot{\mathbf{D}}_{-1,\lambda_2})\Omega^{-1}],\\
H^*_{\rho\lambda_3} &= \tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_{\lambda_3}Y_{-1} - \mathrm{tr}[(\dot{D}_{-1,\lambda_3}\Omega^{-1}) + (\phi\mathbf{C}_1 + \mathbf{D}_1)\dot{\Omega}^-_{\lambda_3}],\\
H^*_{\lambda_1\lambda_1} &= -\tfrac{1}{\sigma_v^2}Y'W_1'\Omega^{-1}W_1Y - \mathrm{tr}[(\phi\dot{\mathbf{C}}_{\lambda_1} + \dot{\mathbf{D}}_{\lambda_1})\Omega^{-1}W_1],\\
H^*_{\lambda_1\lambda_2} &= -\tfrac{1}{\sigma_v^2}Y'_{-1}W_2'\Omega^{-1}W_1Y - \mathrm{tr}[(\phi\dot{\mathbf{C}}_{\lambda_2} + \dot{\mathbf{D}}_{\lambda_2})\Omega^{-1}W_2],\\
H^*_{\lambda_1\lambda_3} &= \tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_{\lambda_3}W_1Y - \mathrm{tr}\{[\dot{D}_{\lambda_3}\Omega^{-1} + (\phi\mathbf{C} + \mathbf{D})\dot{\Omega}^-_{\lambda_3}]W_1\},\\
H^*_{\lambda_2\lambda_2} &= -\tfrac{1}{\sigma_v^2}Y'_{-1}W_2'\Omega^{-1}W_2Y_1 - \mathrm{tr}[(\phi\dot{\mathbf{C}}_{-1,\lambda_2} + \dot{\mathbf{D}}_{-1,\lambda_2})\Omega^{-1}W_2],\\
H^*_{\lambda_2\lambda_3} &= \tfrac{1}{\sigma_v^2}\mathbf{e}'(\theta)\dot{\Omega}^-_{\lambda_3}W_2Y_{-1} - \mathrm{tr}\{[\dot{D}_{-1\lambda_3}\Omega^{-1} + (\phi\mathbf{C}_1 + \mathbf{D}_1)\dot{\Omega}^-_{\lambda_3}]W_2\},\\
H^*_{\lambda_3\lambda_3} &= -\tfrac{1}{2\sigma_v^2}\mathbf{e}'(\theta)\ddot{\Omega}^-_{\lambda_3}\mathbf{e}(\theta) - \tfrac{1}{2}\mathrm{tr}(\dot{\Omega}^-_{\lambda_3}\dot{\Omega}_{\lambda_3} + \Omega^{-1}\ddot{\Omega}_{\lambda_3}),
\end{aligned}$$

where $\dot{\mathbf{C}}_\omega = \frac{\partial\mathbf{C}}{\partial\omega}$, $\dot{\mathbf{D}}_\omega = \frac{\partial\mathbf{D}}{\partial\omega}$, $\dot{\mathbf{C}}_{-1,\omega} = \frac{\partial\mathbf{C}_{-1}}{\partial\omega}$, $\dot{\mathbf{D}}_{-1,\omega} = \frac{\partial\mathbf{D}_{-1}}{\partial\omega}$, for $\omega = \rho, \lambda_1, \lambda_2, \lambda_3$, and these expressions can easily be obtained from the expressions of $\mathbf{C}$, $\mathbf{C}_{-1}$, $\mathbf{D}$, and $\mathbf{D}_{-1}$ given in Lemma 2.1; and further,

$$\begin{aligned}
&\dot{\Omega}_{\lambda_3} = \tfrac{\partial\Omega_{\lambda_3}}{\partial\lambda_3} = (\mathbf{B}_3'\mathbf{B}_3)^{-1}(\mathbf{B}_3'\mathbf{W}_3 + \mathbf{W}_3'\mathbf{B}_3)(\mathbf{B}_3'\mathbf{B}_3)^{-1},\\
&\ddot{\Omega}_{\lambda_3} = \tfrac{\partial\dot{\Omega}_{\lambda_3}}{\partial\lambda_3} = 2[\dot{\Omega}_{\lambda_3}(\mathbf{B}_3'\mathbf{W}_3 + \mathbf{W}_3'\mathbf{B}_3)(\mathbf{B}_3'\mathbf{B}_3)^{-1} - (\mathbf{B}_3'\mathbf{B}_3)^{-1}(\mathbf{W}_3'\mathbf{W}_3)(\mathbf{B}_3'\mathbf{B}_3)^{-1}],\\
&\dot{\Omega}^-_{\lambda_3} = \tfrac{\partial\Omega^{-1}}{\partial\lambda_3} = -\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}, \qquad \ddot{\Omega}^-_{\lambda_3} = \tfrac{\partial\dot{\Omega}^-_{\lambda_3}}{\partial\lambda_3} = -2\Omega^{-1}\dot{\Omega}_{\lambda_3}\dot{\Omega}^-_{\lambda_3} - \Omega^{-1}\ddot{\Omega}_{\lambda_3}\Omega^{-1},\\
&\dot{\Omega}^-_\phi = \tfrac{\partial\Omega^{-1}}{\partial\phi} = \Omega^{-1}(J_T\otimes I_n)\Omega^{-1}, \quad \ddot{\Omega}^-_\phi = \tfrac{\partial\dot{\Omega}^{-1}_\phi}{\partial\phi} = 2\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}(J_T\otimes I_n)\Omega^{-1},\\
&\ddot{\Omega}^-_{\phi,\lambda_3} = \tfrac{\partial\dot{\Omega}^-_\phi}{\partial\lambda_3} = 2\Omega^{-1}\dot{\Omega}_{\lambda_3}\Omega^{-1}(J_T\otimes I_n)\Omega^{-1}.
\end{aligned}$$

It is easy to show that $\frac{1}{nT}H^*_{\texttt{SDPD}}(\psi_0) = O_p(1)$ by Lemma V1.1 and the model assumptions.

Thus, $\frac{1}{nT}H^*_{\texttt{SDPD}}(\bar{\psi}) = O_p(1)$ because $\bar{\psi} - \psi_0 = o_p(1)$ which is implied by $\hat{\psi}_{\texttt{M}} \xrightarrow{p} \psi_0$. As $\bar{\sigma}^2 \xrightarrow{p} \sigma^2_{v0}$, $\bar{\sigma}^{-r} = \sigma^{-r}_{v0} + o_p(1), r = 2, 4, 6$. As σ^r appears in $H^*_{\texttt{SDPD}}(\psi)$ multiplicatively,

$$\tfrac{1}{n(T-1)}H^*_{\texttt{SDPD}}(\bar{\psi}) = \tfrac{1}{n(T-1)}H^*_{\texttt{SDPD}}(\bar{\beta}, \sigma^2_{v0}, \bar{\rho}, \bar{\lambda}, \phi) + o_p(1).$$

The proof of (**b**) is thus equivalent to the proof of

$$\tfrac{1}{n(T-1)}\left[H^*_{\texttt{SDPD}}(\bar{\beta}, \sigma^2_{v0}, \bar{\rho}, \bar{\lambda}, \bar{\phi}) - H^*_{\texttt{SDPD}}(\psi_0)\right] \xrightarrow{p} 0.$$

Writing $\mathbf{e}(\theta) = \mathbf{e} - (\lambda_1 - \lambda_{10})\mathbf{W}_1 Y - (\rho - \rho_0)Y_{-1} - (\lambda_2 - \lambda_{20})\mathbf{W}_2 Y_{-1} - X(\beta - \beta_0)$, and by the expressions for Y and Y_{-1} given in (2.14), we see that all the random elements of $H^*_{\texttt{SDPD}}(\psi)$ can be written as linear combinations of terms:

quadratic in $\mathbf{e}$:	$(\varpi - \varpi_0)^j(\omega - \omega_0)^k\mathbf{e}'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbf{e}$,
quadratic in $\mathbf{y}_0$:	$(\varpi - \varpi_0)^j(\omega - \omega_0)^k\mathbf{y}_0'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbf{y}_0$,
linear in $\mathbf{e}$:	$(\varpi - \varpi_0)^j\mathbf{e}'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbb{Z}$,
linear in $\mathbf{y}_0$:	$(\varpi - \varpi_0)^j\mathbf{y}_0'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbb{Z}$,
bylinear in $\mathbf{e}$ *and* $\mathbf{y}_0$:	$(\varpi - \varpi_0)^j(\omega - \omega_0)^k\mathbf{e}'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbf{y}_0$,

for $j, k = 0, 1$, $\varpi, \omega = \rho, \lambda_1, \lambda_2$, where $\mathbb{A}$ and $\mathbb{B}$ denote generically $nT \times nT$ nonstochastic matrices, and $\mathbb{Z}$ generically $nT \times d$ nonstochastic vector or matrices, free from parameters; and $\mathbf{G}(\phi, \lambda_3)$ can be Ω^{-1}, $\dot{\Omega}^-_{\lambda_3}$, $\ddot{\Omega}^-_{\lambda_3}$, $\dot{\Omega}^-_{\phi}$, $\ddot{\Omega}^-_{\phi}$, and $\ddot{\Omega}^-_{\phi,\lambda_3}$.

Take a typical quadratic term of $\mathbf{e}$, $\mathbf{e}'\mathbb{A}\mathbf{G}(\phi, \lambda_3)\mathbb{B}\mathbf{e}$, for example. Letting (λ_3^*, ϕ^*) be between $(\bar{\lambda}_3, \bar{\phi})$ and (λ_{30}, ϕ_0), we have by MVT,

$$\tfrac{1}{nT}[\mathbf{e}'\mathbb{A}\mathbf{G}(\bar{\lambda}_3, \bar{\phi})\mathbb{B}\mathbf{e} - \mathbf{e}'\mathbb{A}\mathbf{G}(\lambda_{30}, \phi_0)\mathbb{B}\mathbf{e}] = \tfrac{\bar{\phi} - \phi_0}{nT}\mathbf{e}'\mathbb{A}\dot{\mathbf{G}}_{\phi^*}\mathbb{B}\mathbf{e} + \tfrac{\bar{\lambda}_3 - \lambda_0}{nT}\mathbf{e}'\mathbb{A}\dot{\mathbf{G}}_{\lambda_3^*}\mathbb{B}\mathbf{e},$$

where $\dot{\mathbf{G}}_\phi$ and $\dot{\mathbf{G}}_{\lambda_3}$ are the partial derivatives of $\mathbf{G}$ evaluated at (λ_3^*, ϕ^*). From the expression of the Hessian matrix given earlier, we see that $\mathbf{G}$ depends on λ_3 and ϕ, and is the multiplications and linear combinations of matrices Ω^{-1}, $\mathbf{B}_3^{-1}$ and $\mathbf{W}_3$. Therefore, its partial derivatives evaluated at (λ_3, ϕ) are the multiplications and linear combinations of $\Omega^{-1}(\lambda_3, \phi)$, $\mathbf{B}_3^{-1}(\lambda_3)$ and $\mathbf{W}_3$, and hence are uniformly bounded in both row and column sums for (λ_3, ϕ) in a neighborhood of (λ_{30}, ϕ_0). Recall that $\mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{B}_0^{-1}\mathbf{v}$. By applying Lemma V1.4 (i) and using the consistency of $\hat{\psi}_{\texttt{M}}$, we have $\frac{1}{nT}[\mathbf{e}'\mathbb{A}\mathbf{G}(\bar{\lambda}_3, \bar{\phi})\mathbb{B}\mathbf{e} - \mathbf{e}'\mathbb{A}\mathbf{G}(\lambda_{30}, \phi_0)\mathbb{B}\mathbf{e}] \xrightarrow{p} 0$. The convergence of all other terms can be shown similarly by using Lemma V1.4, Assumption F, and the consistency of the M-estimator.

It left to show that all the 'trace' terms in $\frac{1}{nT}\left[H^*_{\texttt{SDPD}}(\bar{\beta}, \sigma^2_{v0}, \bar{\rho}, \bar{\lambda}, \bar{\phi}) - H^*_{\texttt{SDPD}}(\psi_0)\right]$ are $o_p(1)$. For example, let $(\phi^*, \rho^*, \lambda^*)$ be between $(\bar{\phi}, \bar{\rho}, \bar{\lambda})$ and $(\phi_0, \rho_0, \lambda_0)$. By MVT,

$$\begin{aligned}
&\tfrac{1}{nT}\{\mathrm{tr}[\dot{\mathbf{E}}_{-1,\rho}(\bar{\phi}, \bar{\rho}, \bar{\lambda})\Omega^{-1}(\bar{\phi}, \bar{\lambda}_3)] - \mathrm{tr}[\dot{\mathbf{E}}_{-1,\rho}(\phi_0, \rho_0, \lambda_0)\Omega^{-1}(\phi_0, \lambda_{30})]\} \\
&= \tfrac{\bar{\phi} - \phi_0}{nT}\mathrm{tr}[\phi^*\Omega^{-1}(\phi^*, \lambda_3^*) + \dot{\mathbf{E}}_{-1,\rho^*}\dot{\Omega}^{-1}_{\phi^*}] + \tfrac{\bar{\rho} - \rho_0}{nT}\mathrm{tr}[\ddot{\mathbf{E}}^{\rho^*}_{-1,\rho}\Omega^{-1}(\phi^*, \lambda_3^*)] \\
&+ \tfrac{\bar{\lambda}_1 - \lambda_{10}}{nT}\mathrm{tr}[\ddot{\mathbf{E}}^{\lambda_1^*}_{-1,\rho}\Omega^{-1}(\phi^*, \lambda_3^*)] + \tfrac{\bar{\lambda}_2 - \lambda_{20}}{nT}\mathrm{tr}[\ddot{\mathbf{E}}^{\lambda_2^*}_{-1,\rho}\Omega^{-1}(\phi^*, \lambda_3^*)] \\
&+ \tfrac{\bar{\lambda}_3 - \lambda_{30}}{nT}\mathrm{tr}[\ddot{\mathbf{E}}^{\lambda_3^*}_{-1,\rho}\Omega^{-1}(\phi^*, \lambda_3^*) + \dot{\mathbf{E}}_{-1,\rho}\dot{\Omega}^{-1}(\lambda_3^*)],
\end{aligned}$$

where $\ddot{\mathbf{E}}^{r^*}_{-1,\rho}, r = \phi, \rho, \lambda_1, \lambda_2, \lambda_3$,are the partial derivatives of $\dot{\mathbf{E}}_\rho$ evaluated at $(\phi^*, \rho^*, \lambda^*)$. Consider w.l.o.g. $T = 2$. Recall the expression of $\mathbf{E}$ and the definitions of C and D, we have,

$$\mathbf{D}(\rho, \lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} B_1^{-1}(B_3' B_3)^{-1}, & B_1^{-1}(B_3' B_3)^{-1} \\ \mathcal{B} B_1^{-1}(B_3' B_3)^{-1}, & \mathcal{B} B_1^{-1}(B_3' B_3)^{-1} \end{pmatrix},$$

$$\mathbf{C}(\rho, \lambda_1, \lambda_2) = \begin{pmatrix} (\sum_{i=0}^{m} \mathcal{B}^i) B_1^{-1}, & (\sum_{i=0}^{m} \mathcal{B}^i) B_1^{-1} \\ (\sum_{i=0}^{m+1} \mathcal{B}^i) B_1^{-1}, & (\sum_{i=0}^{m+1} \mathcal{B}^i) B_1^{-1} \end{pmatrix}.$$

This shows that elements of $\mathbf{E}$ and $\mathbf{E}_\rho$ are linear combinations of multiplications of the matricies W_1, B_1^{-1}, B_2 and B_3^{-1}. Therefore, $\ddot{\mathbf{E}}^{r}_{-1,\rho}$ have elements being the linear combinations of W_1, W_2,W_3, $B_1^{-1}(\lambda_1)$, $B_2(\rho, \lambda_2)$,and $B_3^{-1}(\lambda_3)$ and hence are uniformly bounded in both row and column sums for (ρ, λ) in the neighborhood of (ρ_0, λ_0) by Lemmas V1.1 and V1.2. Therefore, each trace term in the equation above divided by nT such as $\frac{1}{nT}\mathrm{tr}[\phi^*\Omega^{-1}(\phi^*, \lambda_3^*) + \dot{\mathbf{E}}_{-1,\rho^*}\dot{\Omega}^{-1}_{\phi^*}]$ is $O_p(1)$. So, (b) is proved.

Proof of (c). By (2.14) and the definition of $\mathbf{e}$, elements of Hessian matrix can be written as linear combinations of quadratic and linear terms of $\mathbf{v}$ and $\boldsymbol{\varepsilon}$, quadratic and linear terms of $\mathbf{y}_0$, bilinear terms of $\mathbf{v}$ and $\mathbf{y}_0$, $\boldsymbol{\varepsilon}$ and $\mathbf{y}_0$, $\mathbf{v}$ and $\boldsymbol{\varepsilon}$. Thus, the results follow by repeatedly applying Lemma V1.1, Lemma V1.4, and Assumption F. ■

Appendix V3: Proofs for Section 3

Proof of Lemma 3.1. The result (3.7) is obvious. To show (3.9), write $g_{\Psi i}$ defined in (3.3) as $g_{\Psi i} = Q_{1i} + Q_{2i}$, where $Q_{1i} = \sum_{t=1}^{T}(e_{it}\Psi^*_{ii,t+}y^*_{0i} - d_{\Psi it})$ and $Q_{2i} = \sum_{t=1}^{T} e'_{it}\xi_{it}$. Then, $\mathrm{E}(g_{\Psi i}\ g_{\Psi j}) = \mathrm{E}[(Q_{1i} + Q_{2i})(Q_{1j} + Q_{2j})]$. Defined above (3.3), b'_i and w'_{it} are the ith row of B_3^{-1} and $(\Psi^{*l}_{t+} + \Psi^{*u}_{t+})$. Thus, $\xi_{it} = w'_{it}y^*_0$ and $e_{it} = \varepsilon_i + b'_i v_t$. It is easy to show that

$$\begin{aligned}\mathrm{E}(Q_{1i}Q_{1j}) &= \textstyle\sum_{t=1}^{T} \mathrm{Cov}(e'_{it}\Psi^*_{ii,t+}y^*_{0i}, e'_{jt}\Psi^*_{jj,t+}y^*_{0j}) + \sum_{t=1}^{T}\sum_{s(\neq t)} \mathrm{Cov}(e'_{it}\Psi^*_{ii,t+}y^*_{0i}, e'_{js}\Psi^*_{jj,s+}y^*_{0j}) \\ &= \sigma^2_{v_0}(b'_i b_j)\textstyle\sum_{t=1}^{T}(\Psi^*_{ii,t+}\Psi^*_{jj,t+})\mathrm{E}(y^*_{0i}y^*_{0j}),\end{aligned}$$

where the double summation part vanishes, as for $i \neq j$ and $t \neq s$, $e'_{it}\Psi^*_{ii,t+}y^*_{0i}$ and $e'_{js}\Psi^*_{jj,s+}y^*_{0j}$ (respectively measurable-$(\varepsilon_i, v_t, V_m)$ and $(\varepsilon_j, v_s, V_m)$) are conditionally independent given V_m. Similarly, $\mathrm{E}(Q_{ri}Q_{\nu j}), r, \nu = 1, 2$, are

$$\begin{aligned}\mathrm{E}(Q_{1i}Q_{2j}) &= \sigma^2_{v_0}(b'_i b_j)\textstyle\sum_{t=1}^{T}\Psi^*_{ii,t+}\mathrm{E}(y^*_{0i}\xi_{jt}), \\ \mathrm{E}(Q_{2i}Q_{1j}) &= \sigma^2_{v_0}(b'_i b_j)\textstyle\sum_{t=1}^{T}\Psi^*_{jj,t+}\mathrm{E}(y^*_{0j}\xi_{it}), \\ \mathrm{E}(Q_{2i}Q_{2j}) &= \sigma^2_{v_0}(b'_i b_j)\textstyle\sum_{t=1}^{T}\mathrm{E}(\xi_{it}\xi_{jt}) + \sigma^4_{\epsilon_0}(1'_j w_{i+})(1'_i w_{j+}),\end{aligned}$$

where 1_i denotes an $n\times 1$ vector of element 1 at the ith position and zero elsewhere. Summarizing and simplifying, we have $\mathrm{E}(g_{\Psi i}\ g_{\Psi j}) = \sigma^4_{\varepsilon_0}(w_{ij,+}w_{ji,+}) + \sigma^2_{v_0}\sum_{t=1}^{T}(b'_i b_j)\mathrm{E}[(w^{*\prime}_{ti}y^*_0)(w^{*\prime}_{tj}y^*_0)]$.

More generally, for Ψ_r and Ψ_ν, $r, \nu = 1, 2, 3$, we have,

$$\mathrm{E}(g_{\Psi_r i}\ g_{\Psi_\nu j}) = \sigma_{\varepsilon 0}^4 (w_{rij,+} + w_{\nu ji,+}) + \sigma_{v0}^2 \sum_{t=1}^T (b_i' b_j) \mathrm{E}[(w_{ri,t}^{*\prime} y_0^*)(w_{\nu j,t}^{*\prime} y_0^*)], \tag{V3.1}$$

which is the result (3.9) in Lemma 3.1.

To show (3.8), first note that, for $n \times 1$ vectors a, b, c, and d:

$$\mathrm{E}[(a'v_t)(b'v_t)(c'v_t)(d'v_t)] = (\mu_{v0}^{(4)} - 3\sigma_{v0}^4)(a \odot b)'(c \odot d) + \sigma_{v0}^4[(a'b)(c'd) + (a'c)(b'd) + (a'd)(b'c)],$$

where $\odot$ denotes the Hadamard product, and $\mu_{v0}^{(4)}$ is the 4th moment of v_{it}. Immediately following this result we have:

$$\mathrm{E}[(a'v_t)^2 (b'v_t)^2] = (\mu_{v0}^{(4)} - 3\sigma_{v0}^4)(a \odot a)'(b \odot b) + \sigma_{v0}^4[(a'a)(b'b) + 2(a'b)^2].$$

Writing $g_{\Phi i} = Q_{1i} + Q_{2i}$, where $Q_{1i} = \sum_{t=1}^T (e_{it} e_{it}^* - d_{1\Phi it})$ and $Q_{2i} = \sum_{t=1}^T (e_{it}\varphi_{it} - d_{2\Phi it})$ by (3.4), we have $\mathrm{E}(g_{\Phi i} g_{\Phi j}) = \mathrm{E}[(Q_{1i} + Q_{2i})(Q_{1j} + Q_{2j})]$. Some tedious algebra leads to

$$\begin{aligned}
\mathrm{E}(Q_{1i}Q_{1j}) &= \textstyle\sum_{t=1}^T \mathrm{Cov}(e_{it}e_{it}^*, e_{jt}e_{jt}^*) + \sum_{t=1}^T \sum_{s=1, s\neq t}^T \mathrm{Cov}(e_{it}e_{it}^*, e_{js}e_{js}^*) \\
&= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T \left[\Phi_{ii,ts}\Phi_{jj,st}(b_i'b_j)^2 + \Phi_{ii,ts}\Phi_{jj,ts}(b_i'b_j)^2\right] \\
&\quad + (\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T \Phi_{ii,tt}\Phi_{jj,tt}(b_i \odot b_i)'(b_j \odot b_j), \\
\mathrm{E}(Q_{1i}Q_{2j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [\Phi_{ii,ts}(b_i'c_{j,st})(b_i'b_j) + \Phi_{ii,ts}(b_i'c_{j,ts})(b_i'b_j)] \\
&\quad + \sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T (\Phi_{ii,t+} + \Phi_{ii,+t})(1_i'a_{j,t+})(b_i'b_j) \\
&\quad + (\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T \Phi_{ii,tt}(b_i \odot b_i)'(b_j \odot c_{j,tt}), \\
\mathrm{E}(Q_{2i}Q_{1j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [\Phi_{jj,ts}(b_j'c_{i,st})(b_i'b_j) + \Phi_{jj,ts}(b_j'c_{i,ts})(b_i'b_j)] \\
&\quad + \sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T (\Phi_{jj,t+} + \Phi_{jj,+t})(1_j'a_{i,t+})(b_i'b_j) \\
&\quad + (\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T \Phi_{jj,tt}(b_j \odot b_j)'(b_i \odot c_{i,tt}), \\
\mathrm{E}(Q_{2i}Q_{2j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [(b_i'c_{j,st})(b_j'c_{i,ts}) + (b_i'b_j)(c_{i,ts}'c_{j,ts})] \\
&\quad + \sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T [(1_j'a_{i,t+})(b_i'c_{j,+t}) + (1_i'a_{j,t+})(b_j'c_{i,+t}) + (a_{i,t+}'a_{j,t+})(b_i'b_j)] \\
&\quad + (\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T (b_i \odot c_{i,tt})'(b_j \odot c_{j,tt}),
\end{aligned}$$

Denote $\Phi_{ts}^* = \Phi_{ts}^l + \Phi_{st}^{u\prime} + \Phi_{ts}^d$, and let $c_{i,ts}^*$ be ith row of $\Phi_{ts}^* B_3^{-1}$. Further let $a_{ji,ts}$ be the ith element of $a_{j,ts}$. Then we have,

$$\begin{aligned}
\mathrm{E}(g_{\Phi i} g_{\Phi j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [(b_j'c_{i,ts}^*)(b_i'c_{j,st}^*) + (b_i'b_j)(c_{i,ts}^{*\prime}c_{j,ts}^*)] \\
&\quad + \sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T [a_{ji,t+}(b_j'c_{i,+t}^*) + a_{ij,t+}(b_i'c_{j,+t}^*) + (b_i'b_j)(a_{i,t+}^{*\prime}a_{j,t+}^*)] \\
&\quad + (\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T [(b_i \odot c_{i,tt}^*)'(b_j \odot c_{j,tt}^*)].
\end{aligned}$$

More generally, for Φ_r and Φ_ν, $r, \nu = 1, \ldots, 6$, we have

$$\begin{aligned}
\mathrm{E}(g_{\Phi_r i} g_{\Phi_\nu j}) &= \sigma_{v0}^4 \textstyle\sum_{t=1}^T \sum_{s=1}^T [(b_j'c_{ri,ts}^*)(b_i'c_{\nu j,st}^*) + (b_i'b_j)(c_{ri,ts}^{*\prime}c_{\nu j,ts}^*)] \\
&\quad + \sigma_{v0}^2\sigma_{\varepsilon 0}^2 \textstyle\sum_{t=1}^T a_{rji,t+}(b_j'c_{\nu i,+t}^*) + a_{\nu ij,t+}(b_i'c_{rj,+t}^*) + (b_i'b_j)(a_{ri,t+}^{*\prime}a_{\nu j,t+}^*)] \\
&\quad + (\mu_{v0}^{(4)} - 3\sigma_{v0}^4) \textstyle\sum_{t=1}^T [(b_i \odot c_{ri,tt}^*)'(b_j \odot c_{\nu j,tt}^*)].
\end{aligned}$$

To show (3.8), write $g_{\Pi i} = \sum_{t=1}^T \Pi'_{jt} e_{jt} = P_i$ and write $g_{\Phi i} = Q_{1i} + Q_{2i}$, where $Q_r, r = 1, 2$, are given above. Then, $\mathrm{E}(g_{\Phi i} g_{\Pi j}) = \mathrm{E}[(Q_{1i} + Q_{2i})P_j]$. Some algebra leads to

$$\mathrm{E}(Q_{1i}P_j) = \mu_{v0}^{(3)} \sum_{t=1}^T \Phi_{ii,tt}(b_i \odot b_i)' b_j \Pi'_{jt}, \quad \mathrm{E}(Q_{2i}P_j) = \mu_{v0}^{(3)} \sum_{t=1}^T (b_i \odot c_{i,tt})' b_j \Pi'_{jt}.$$

Combining the two terms we have, $\mathrm{E}(g_{\Phi i} g_{\Pi j}) = \mu_{v0}^{(3)} \sum_{t=1}^T (b_i \odot c^*_{i,tt})' b_j \Pi'_{jt}$.

Proof of (3.11) is similar. Write $g_{\Psi i} = Q_{1i} + Q_{2i}$ and $g_{\Pi i} = P_i$. Then $\mathrm{E}(g_{\Psi i} g'_{\Pi j}) = \mathrm{E}[(Q_{1i} + Q_{2i})P_j]$. After some algebra, we obtain

$$\mathrm{E}(Q_{1i}P_j) = \sigma_{v0}^2 \sum_{t=1}^T (b'_i b_j) \Pi'_{jt} \Psi^*_{ii,t} \mathrm{E}(y^*_{0i}) \text{ and } \mathrm{E}(Q_{2i}P_j) = \sigma_{v0}^2 \sum_{t=1}^T (b'_i b_j) \Pi'_{jt} \mathrm{E}(\xi_{it}),$$

leading to $\mathrm{E}(g_{\Psi i} g'_{\Pi j}) = \sigma_{v0}^2 \sum_{t=1}^T \Pi'_{jt} \mathrm{E}(w^{*\prime}_{it} y^*_0)(b'_i b_j)$.

To show the result (3.12) in Lemma 3.1, Write $g_{\Phi i} = Q_{1i} + Q_{2i}$ and $g_{\Psi i} = P_{1i} + P_{2i}$ where $P_{1i} = \sum_{t=1}^T (e_{it} \Psi^*_{ii,t} y^*_{0i} - d_{\Psi it})$ and $P_{2i} = \sum_{t=1}^T e'_{it} \xi_{it}$. Then $\mathrm{E}(g_{\Phi i} g_{\Psi j}) = \mathrm{E}[(Q_{1i} + Q_{2i})(P_{1j} + P_{2j})]$. Some tedious algebra leads to

$$\begin{aligned}
\mathrm{E}(Q_{1i}P_{1j}) &= \mu_{v0}^{(3)} \sum_{t=1}^T \Phi_{ii,tt} \Psi^*_{jj,t} (b_i \odot b_i)' b_j \mathrm{E}(y^*_{0j}), \\
\mathrm{E}(Q_{1i}P_{2j}) &= \sigma_{\varepsilon 0}^2 \sigma_{v0}^2 \sum_{t=1}^T \left[\Phi_{ii,+t}(1'_i w_{jt})(b'_i b_j) + \Phi_{ii,t+}(1'_i w_{jt})(b'_i b_j)\right] \\
&\quad + \mu_{v0}^{(3)} \sum_{t=1}^T \Phi_{ii,tt}(b_i \odot b_i)' b_j \mathrm{E}(\xi_{jt}), \\
\mathrm{E}(Q_{2i}P_{1j}) &= \sigma_{\varepsilon 0}^2 \sigma_{v0}^2 \sum_{t=1}^T \Psi^*_{jj,t}(1'_j a_{i,t+})(b'_i b_j) + \mu_{v0}^{(3)} \sum_{t=1}^T \Psi^*_{jj,t}(b_i \odot b_j)' c_{i,tt} \mathrm{E}(y^*_{0j}), \\
\mathrm{E}(Q_{2i}P_{2j}) &= \sigma_{\varepsilon 0}^2 \sigma_{v0}^2 \sum_{t=1}^T \left[(1'_i w_{jt})(b'_j c_{i,+t}) + (b'_i b_j)(a'_{i,t+} w_{jt})\right] + \sigma_{\varepsilon 0}^4 (1'_i w_{j+})(1'_j a_{i,++}) \\
&\quad + \mu_{v0}^{(3)} \sum_{t=1}^T (b_i \odot b_j)' c_{i,tt} \mathrm{E}(\xi_{jt}).
\end{aligned}$$

Summarizing the above and simplifying give the result (3.12). ∎

Proof of Theorem 3.1. First, the result $\Sigma^*_{\mathtt{SDPD}}(\hat{\psi}_{\mathtt{M}}) - \Sigma^*_{\mathtt{SDPD}}(\psi_0) \xrightarrow{p} 0$ is implied by the result **(b)** in the proof of Theorem 2.2. To show $\widehat{\Gamma}^*_{\mathtt{SDPD}} - \Gamma^*_{\mathtt{SDPD}}(\psi_0) \xrightarrow{p} 0$, for the single summation part, the result $\frac{1}{nT} \sum_{i=1}^n [\hat{\mathbf{g}}_i \hat{\mathbf{g}}'_i - \mathrm{E}(\mathbf{g}_i \mathbf{g}'_i)] \xrightarrow{p} 0$ follows from $\frac{1}{nT} \sum_{i=1}^n [\hat{\mathbf{g}}_i \hat{\mathbf{g}}'_i - \mathbf{g}_i \mathbf{g}'_i] \xrightarrow{p} 0$ and $\frac{1}{nT} \sum_{i=1}^n [\mathbf{g}_i \mathbf{g}'_i - \mathrm{E}(\mathbf{g}_i \mathbf{g}'_i)] \xrightarrow{p} 0$. The proof of the former is straightforward by MVT. We focus on the proof of the latter result. As the elements of $S^*_{\mathtt{SDPD}}(\psi_0)$ are mixtures of terms of the forms $\Pi' \mathbf{e} = \sum_{i=1}^n g_{\Pi i}$, $\mathbf{e}' \Psi \mathbf{y}_0 - \mathrm{E}(\mathbf{e}' \Psi \mathbf{y}_0) = \sum_{i=1}^n g_{\Psi i}$, and $\mathbf{e}' \Phi \mathbf{e} - \mathrm{E}(\mathbf{e}' \Phi \mathbf{e}) = \sum_{i=1}^n g_{\Phi i}$, it suffices to show that

$$\tfrac{1}{nT} \sum_{i=1}^n [g_{ki} g'_{ri} - \mathrm{E}(g_{ki} g'_{ri})] = o_p(1), \quad \text{for } g_{ki}, g_{ri} = g_{\Pi i}, g_{\Psi i}, g_{\Phi i}. \tag{V3.2}$$

First, assuming, W.L.O.G, $\{\Pi_{it}\}$ are scalars, we have

$$g_{\Pi i} = \sum_{t=1}^T \Pi_{it} \mathbf{e}_{it} = \sum_{t=1}^T \Pi_{it}(\varepsilon_i + b'_i v_t) = \Pi_{i+} \varepsilon_i + b'_i \mathbf{v}_i, \tag{V3.3}$$

where $\Pi_{i+} = \sum_{t=1}^T \Pi_{it}$ and $\mathbf{v}_i = \sum_{t=1}^T \Pi_{it} v_t$. It follows that

$$\tfrac{1}{nT} \sum_{i=1}^n \left[g^2_{\Pi i} - \mathrm{E}(g^2_{\Pi i})\right] \equiv U_1 + U_2 + U_3,$$

where $U_1 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}^2(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)$, $U_2 = \frac{2}{nT}\sum_{i=1}^n(\Pi_{i+}\varepsilon_i)(b_i'\mathbf{v}_i)$ and $U_3 = \frac{1}{nT}\sum_{i=1}^n \big[(b_i'\mathbf{v}_i)^2 - \sigma_{v0}^2(\sum_{t=1}^T \Pi_{it}^2)(b_i'b_i)\big]$. It is obvious that $\mathrm{E}(U_1) = 0$ and $\mathrm{Var}(U_1) = \frac{1}{n^2T^2}\sum_{i=1}^n \Pi_{i+}^4 \mathrm{E}[(\varepsilon_i^2 - \sigma_{\varepsilon_0}^2)^2]$ $= \frac{1}{n^2T^2}(\mu_\varepsilon^{(4)} - \sigma_{\varepsilon_0}^4)\sum_{i=1}^n \Pi_{i+}^4$. Given Assumption B and that the elements of Π_{it} are uniformly bounded for each t, $\mathrm{Var}(U_1) = o(1)$. Hence $U_1 \xrightarrow{p} 0$ by Chebyshev's inequality.

Write $U_2 = \frac{2}{nT}\sum_{i=1}^n(\Pi_{i+}\varepsilon_i)(b_i'\mathbf{v}_i) = \frac{2}{nT}\sum_{t=1}^T \varepsilon\prime\Theta_t v_t$, where $\Theta_t = \mathtt{diag}(\Pi_+)\mathtt{diag}(\Pi_t)B_3^{-1}$, and is uniformly bounded in either row or column sum. Then $U_2 \xrightarrow{p} 0$, by Lemma A.4 (vii) and independence of ε and v_t. For U_3, we have

$$\begin{aligned}
\tfrac{1}{nT}\textstyle\sum_{i=1}^n(b_i'\mathbf{v}_i)^2 &= \tfrac{1}{nT}\textstyle\sum_{i=1}^n \big[(\sum_{t=1}^T \Pi_{it}v_t')(b_ib_i')(\sum_{t=1}^T \Pi_{it}v_t)\big] \\
&= \tfrac{1}{nT}\textstyle\sum_{t=1}^T\sum_{s=1}^T v_t'\big(\sum_{i=1}^n \Pi_{it}b_ib_i'\Pi_{is}\big)v_s \\
&= \tfrac{1}{nT}\textstyle\sum_{t=1}^T\sum_{s=1}^T v_t'\big[B_3^{-1}\mathtt{diag}(\Pi_t)\mathtt{diag}(\Pi_s)B_3'^{-1}\big]v_s.
\end{aligned}$$

Denote $A_{ts} = B_3^{-1}\mathtt{diag}(\Pi_t)\mathtt{diag}(\Pi_s)B_3'^{-1}$. As t is fixed and the elements of Π_t are uniformly bounded, one can easily show that $\frac{1}{n}[v_s'A_{ts}v_s - \mathrm{E}(v_t'A_{ts}v_s)] \xrightarrow{p} 0$ by applying Lemma A.4 (v) for the case of $t = s$, and Lemma A.4 (vii) for the case of $t \neq s$. Thus, $U_3 \xrightarrow{p} 0$. Therefore, $\frac{1}{nT}\sum_{i=1}^n \big[g_{\Pi i}^2 - \mathrm{E}(g_{\Pi i}^2)\big] = o_p(1)$.

Second, for $g_{\Phi i} = \sum_{t=1}^T e_{it}e_{it}^* + \sum_{t=1}^T e_{it}\varphi_{it} - d_{\Phi i}$, recall $e_{it} = \varepsilon_i + b_i'v_t$, $e_{it}^* = \Phi_{ii,t+}\varepsilon_i + b_i'\mathbf{v}_{it}^*$ where $\mathbf{v}_{it}^* = \sum_{s=1}^T \Phi_{ii,ts}v_s$, and $\varphi_{it} = a_{i,t+}'\varepsilon + \sum_{s=1}^T c_{i,ts}'v_s$. Some algebra leads to

$$g_{\Phi i} = k_i(\varepsilon_i^2 - \sigma_\varepsilon^2) + \varepsilon_i z_{1i} + \varepsilon_i(r_i'\varepsilon) + (u_i - \mu_{u_i}) + \textstyle\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t), \qquad \text{(V3.4)}$$

where $z_{1i} = \sum_{t=1}^T p_{it}'v_t$ with p_{it}' being the ith row of some non-stochastic matrix uniformly bounded in row or column sums; $u_i = \sum_{t=1}^T\sum_{s=1}^T v_t'A_{i,ts}v_s$ with mean $\mu_{u_i} = \sigma_v^2\sum_{t=1}^T \mathtt{tr}(A_{i,tt})$, where $A_{i,ts} = \Phi_{ii,ts}(b_ib_i') + (b_ic_{i,ts}')$; k_i are uniformly bounded scalar constants, b_i is defined as before, and r_i' and q_{it}' represent ith row of some non-stochastic lower triangular matrices which are uniformly bounded in either row or column sums. Noticing that the terms in (V3.4) are uncorrelated, it follows that

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^n \big(g_{\Phi i}^2 - \mathrm{E}(g_{\Phi i}^2)\big) = \sum_{r=1}^{15} U_r, \quad \text{where} \qquad \text{(V3.5)}$$

$U_1 = \frac{1}{nT}\sum_{i=1}^n k_i^2\{(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)^2 - \mathrm{E}[(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)^2]\}$, $U_2 = \frac{1}{nT}\sum_{i=1}^n[\varepsilon_i^2 z_{1i}^2 - (\sum_{t=1}^T p_{it}'p_{it})\sigma_{v_0}^2\sigma_{\varepsilon_0}^2]$,
$U_3 = \frac{1}{nT}\sum_{i=1}^n[\varepsilon_i^2(r_i'\varepsilon)^2 - \sigma_{\varepsilon 0}^4\sum_{j=1}^n r_{ij}^2]$, $U_4 = \frac{1}{nT}\sum_{i=1}^n\{(u_i - \mu_{u_i})^2 - \mathrm{E}[(u_i - \mu_{u_i})^2]\}$,
$U_5 = \frac{1}{nT}\sum_{i=1}^n\{[\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t)]^2 - \sigma_{v_0}^2\sigma_{\varepsilon_0}^2(\sum_{t=1}^T q_{it}'q_{it})(b_i'b_i)\}$,
$U_6 = \frac{2}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2 - \sigma_{\varepsilon_0}^2)\varepsilon_i z_{1i}$, $U_7 = \frac{2}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2 - \sigma_{\varepsilon_0}^2)\varepsilon_i(r_i'\varepsilon)$,
$U_8 = \frac{2}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)(u_i - \mu_{u_i})$, $U_9 = \frac{2}{nT}\sum_{i=1}^n \varepsilon_i^2(r_i'\varepsilon)z_{1i}$,
$U_{10} = \frac{2}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2 - \sigma_{\varepsilon_0}^2)\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t)$, $U_{11} = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i z_{1i}(u_i - \mu_{u_i})$,
$U_{12} = \frac{2}{nT}\sum_{i=1}^n[\varepsilon_i z_{1i}\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t)]$, $U_{13} = \frac{2}{nT}\sum_{i=1}^n \varepsilon_i(r_i'\varepsilon)(u_i - \mu_{u_i})$,
$U_{14} = \frac{2}{nT}\sum_{i=1}^n \varepsilon_i(r_i'\varepsilon)\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t)$, $U_{15} = \frac{2}{nT}\sum_{i=1}^n(u_i - \mu_{u_i})\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t)$.

Each of the fifteen terms above is or can be written as the sum of a MD array, and thus the weak law of large numbers (WLLN) for a MD array, i.e., Theorem 19.7 of Davidson (1994, p.299), can be applied to prove its convergence in probability to zero. Details are as follows.

For the first term, writing $U_1 = \frac{1}{nT}\sum_{i=1}^n k_i^2[(\varepsilon_i^4 - \mu_{\varepsilon 0}^{(4)}) + 2\sigma_\varepsilon^2(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)] = \sum_{i=1}^n H_{ni}$. As H_{ni} are independent across i and $\{k_i\}$ are uniformly bounded, the conditions for WLLN for a MD array of Davidson can easily be verified and thus $U_1 \xrightarrow{p} 0$.

For the 14th term, denote $U_{14} = \frac{2}{nT}\sum_{i=1}^n \varepsilon_i(r_i'\varepsilon)\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t) = \sum_{i=1}^n H_{ni}$. Let $\mathcal{G}_{ni}$ be the increasing σ-field generated by $(\mathbf{v}, \varepsilon_1, \ldots, \varepsilon_i)$. Notice that $\mathrm{E}(H_{ni}|\mathcal{G}_{n,i-1}) = 0$. Thus $\{H_{ni}, \mathcal{G}_{ni}\}$ form a MD array. It is easy to show that $\mathrm{E}|H_{n,i}^{1+\epsilon}| \leq K_h < \infty$, for some $\epsilon > 0$. In particular, $\mathrm{E}(H_{ni}^2) = \sum_{t=1}^T(b_i'b_i)\sigma_{\varepsilon 0}^2\sigma_{v0}^2\{(\mu_{\varepsilon 0}^{(4)} - 3\sigma_{\varepsilon 0}^4)(r_i \odot r_i)'(q_{it} \odot q_{it}) + \sigma_{\varepsilon 0}^4[(r_i'r_i)(q_{it}'q_{it}) + 2(r_i'q_{it})^2]\}$, which is bounded by Lemma V1.1. Thus, $\{H_{ni}\}$ is uniformly integrable. The other two conditions of the WLLN for MD arrays of Davidson are satisfied. So we have $U_{14} = \sum_{i=1}^n H_{ni} \xrightarrow{p} 0$. The proofs of the terms U_{11}, U_{12}, and U_{13} proceed similarly.

The 8th term can be written as $U_8 = \frac{2}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)(u_i - \mu_{u_i}) = \frac{2}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)u_i - \frac{2}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2 - \sigma_{\varepsilon 0}^2)\mu_{ui} = \frac{2}{nT}\sum_{i=1}^n V_{1n,i} + \frac{2}{nT}\sum_{i=1}^n V_{2n,i}$. As k_i and μ_{ui} are uniformly bounded, we immediately have $\frac{2}{nT}\sum_{i=1}^n V_{2n,i} \xrightarrow{p} 0$ by invoking Kolmogorov's law of large numbers (LLN). For $V_{1n,i}$, first we notice that u_i is independent of ε_i for all i. Let $\mathcal{F}_{ni}$ be sigma field generated by $(u_i, \varepsilon_1, \ldots, \varepsilon_i)$, then $\mathrm{E}(V_{1n,i}|\mathcal{F}_{n,i-1}) = 0$. So, $\{V_{1n,i}, \mathcal{F}_{n,i}\}$ form a MD array. Now, $\mathrm{E}(V_{1n,i}^2) = \mathrm{E}(\varepsilon_i^2 - \sigma_{\varepsilon_0}^2)^2\,\mathrm{E}(u_i^2)$, and

$$\begin{aligned}
\mathrm{E}(u_i^2) &= \textstyle\sum_{t=1}^T \mathrm{E}(v_t'A_{i,tt}v_tv_t'A_{i,tt}v_t) + \sum_{t=1}^T\sum_{s\neq t}[\mathrm{E}(v_t'A_{i,tt}v_t)\mathrm{E}(v_s'A_{i,ss}v_s) \\
&\quad +\mathrm{E}(v_t'A_{i,ts}v_sv_t'A_{i,ts}v_s) + \mathrm{E}(v_t'A_{i,ts}v_sv_s'A_{i,st}v_t)] \\
&= \sigma_{v_0}^4\textstyle\sum_{t=1}^T\sum_{s\neq t}[\mathrm{tr}(A_{i,tt})\mathrm{tr}(A_{i,ss}) + \mathrm{tr}(A_{i,ts}A_{i,ts}') + \mathrm{tr}(A_{i,ts}A_{i,st})] \\
&\quad +(\mu_{v_0}^{(4)} - 3\sigma_{v_0}^4)\textstyle\sum_{t=1}^T\sum_{j=1}^n a_{itt,jj}^2,
\end{aligned}$$

where $a_{its,ij}$ is the (i,j) element of $A_{i,ts}$. $\sum_{j=1}^n a_{itt,jj}^2 \leq \mathrm{tr}(A_{i,tt}A_{i,tt}')$. Recall that $A_{i,ts} = \Phi_{ii,ts}(b_ib_i') + (b_ic_{i,ts}')$. So we have $\mathrm{tr}(A_{i,ts}) = \Phi_{ii,ts}\sum_{j=1}^n(b_{ij}c_{ij}^*) = O(1)$ as $(B_3'B_3')^{-1}$ and Φ_{ts}^* are uniformly bounded in both row and column sums at true parameter values. Similarly we have $\mathrm{tr}(A_{i,ts}A_{i,ts}') = O(1)$ and $\mathrm{tr}(A_{i,ts}A_{i,st}) = O(1)$. Therefore, the condition, $\mathrm{E}(|V_{1n,i}|^{1+\epsilon}) < K_v < \infty$ for some $\epsilon > 0$, is satisfied. With constant coefficients $\frac{1}{nT}$, the other two conditions of WLLN for MD array of Davidson are satisfied. So we have $\frac{2}{nT}\sum_{i=1}^n V_{1n,i} \xrightarrow{p} 0$ and thus, $U_8 \xrightarrow{p} 0$. The proofs of convergence of U_2, U_5 and U_{15} are similar as that of U_8.

The third term can be written as: $U_3 = \frac{1}{nT}\sum_{i=1}^n(\varepsilon_i^2 - \sigma_{v_0}^2)(r_i'\varepsilon)^2 + \frac{1}{nT}\sum_{i=1}^n \sigma_{v_0}^2[(r_i'\varepsilon)^2 - \sigma_{v_0}^2\sum_{j=1}^n r_{ij}^2]$. Similarly, the first term is the average of a MD array and its convergence follows from Davidson's WLLN for MD array. Letting $n \times n$ matrix $\mathbf{r} = (r_1, \ldots, r_n)(r_1, \ldots, r_n)'$, the second term becomes $\frac{1}{nT}\sigma_{v_0}^2[\varepsilon'\mathbf{r}\varepsilon - \mathrm{E}(\varepsilon'\mathbf{r}\varepsilon)]$. Then, by Lemma A.1 and Lemma A.4 (v), we have $\frac{1}{nT}\sigma_{v_0}^2[\varepsilon'\mathbf{r}\varepsilon - \mathrm{E}(\varepsilon'\mathbf{r}\varepsilon)] \xrightarrow{p} 0$. So, we have $U_3 \xrightarrow{p} 0$.

Next, define $n\times 1$ vectors $p_{it}^l = (p_{1t}, \ldots, p_{i-1,t}, 0, \ldots, 0)'$, and $p_{it}^u = (0, \ldots, 0, p_{it}, \ldots, p_{nt},)'$. Then, U_9 can be written as $\frac{1}{nT}\sum_{i=1}^n(\varepsilon_i^2 - \sigma_{\varepsilon_0}^2)(r_i'\varepsilon)(\sum_{t=1}^T p_{it}^{l\prime}v_t) + \frac{1}{nT}\sum_{i=1}^n \varepsilon_i(r_i'\varepsilon)(\sum_{t=1}^T p_{it}^{u\prime}v_t)$. It can be easily seen that the first term is the average of a MD array as $(r_i'\varepsilon)(\sum_{t=1}^T p_{it}^{u\prime}v_t)$ is

$\mathcal{G}_{n,i-1}$ measurable, and the second term is the average of n independent terms. Conditions of Theorem 19.7 of Davidson (1994) are easily verified and hence $U_9 \xrightarrow{p} 0$. Convergence of U_6 and U_{10} can be proved similarly as the first term of U_9.

For the 7th term, we have $U_7 = \frac{2}{nT}\sum_{i=1}^n k_i\varepsilon_i^3(r_i'\varepsilon) - \frac{2}{nT}\sigma_{\varepsilon 0}^2\sum_{i=1}^n k_i\varepsilon_i(r_i'\varepsilon) = \frac{2}{nT}\sum_{i=1}^n V_{1n,i} - \frac{2}{nT}\sigma_{\varepsilon 0}^2\sum_{i=1}^n V_{2n,i}$. For $V_{1n,i}$, we can write $\frac{2}{nT}\sum_{i=1}^n V_{1n,i} = \frac{2}{nT}\sum_{i=1}^n k_i\varepsilon_i^3(r_i'\varepsilon) = \frac{2}{nT}\sum_{i=1}^n k_i(\varepsilon_i^3 - \mu_{\varepsilon_0}^3)(r_i'\varepsilon) + \frac{2}{nT}\sum_{i=1}^n k_i\mu_{\varepsilon_0}^3(r_i'\varepsilon)$. The convergence of the second term follows immediately from Lemma A.4 (vi). For the first term, let $H_{n,i} = k_i(\varepsilon_i^3 - \mu_{\varepsilon_0}^3)(r_i'\varepsilon)$. As $(r_i'\varepsilon)$ is $\mathcal{G}_{n,i-1}$ measurable, $\mathrm{E}(H_{n,i}|\mathcal{G}_{n,i-1}) = 0$. Therefore $\{H_{n,i}, \mathcal{G}_{n,i}\}$ form a MD array and $\mathrm{E}|H_{n,i}^{1+\epsilon}| \le K_v < \infty$, for some $\epsilon > 0$. The other two conditions of Davidson's WLLN for MD arrays are satisfied. Thus, $\frac{1}{nT}\sum_{i=1}^n H_{n,i} \xrightarrow{p} 0$, leading to $\frac{2}{nT}\sum_{i=1}^n V_{1n,i} \xrightarrow{p} 0$. It is easy to see that $\frac{2}{nT}\sigma_\varepsilon^2\sum_{i=1}^n V_{2n,i}$ is the average of a MD array and its convergence follows from Davidson's WLLN for MD arrays, and therefore we have $U_7 \xrightarrow{p} 0$.

Lastly, for the 4th term, we have $U_4 = \frac{1}{nT}\sum_{i=1}^n\{(u_i - \mu_{u_i})^2 - \mathrm{E}[(u_i - \mu_{u_i})^2]\} = \frac{1}{nT}\sum_{i=1}^n[u_i^2 - \mathrm{E}(u_i^2)] - \frac{1}{nT}\sum_{i=1}^n \mu_{u_i}(u_i - \mu_{u_i})$. The convergence of the second term follows from Lemma V1.4. For the first term, note that $(\sum_{t=1}^T\sum_{s=1}^T v_t'A_{i,ts}v_s)^2$ can be written as a sum of four types of terms: $H_{r,ni}, r = 1, 2, 3, 4$. The first type is $H_{1,ni} = \sum_t\sum_s\sum_k\sum_{l\neq t,s,k} v_t'A_{i,ts}v_sv_k'A_{i,kl}v_l = \sum_l v_l'\varphi_{il}$, where $\varphi_{il} = \sum_{t\neq l}\sum_{s\neq l}\sum_{k\neq l} A_{i,kl}'v_kv_t'A_{i,ts}v_s$. By the independence between v_l and φ_{il}, we have $\mathrm{E}(v_l'\varphi_{il}) = 0$. As T is fixed, we ignore the sum over t and we have $\frac{1}{nT}\sum_{i=1}^n v_l'\varphi_{il} = \frac{1}{nT}v_l'\sum_{i=1}^n\varphi_{il} = \frac{1}{nT}v_l'\varphi_l = \frac{1}{nT}\sum_{j=1}^n v_{lj}\varphi_{lj}$. Therefore we have average of n uncorrelated terms. It is easy to verify that the conditions of WLLN for MD array of Davidson are satisfied, and thus $\frac{1}{nT}\sum_{i=1}^n H_{1,ni} \xrightarrow{p} 0$.

The second type of terms is $H_{2,ni} = \sum_t\sum_{s\neq t} v_t'A_{i,tt}v_tv_s'A_{i,ss}v_s = \sum_t\sum_{s\neq t} u_{it}u_{is}$. For each t and s, we can write $\frac{1}{nT}\sum_{i=1}^n[u_{it}u_{is} - \mathrm{E}(u_{it})\mathrm{E}(u_{is})] = \frac{1}{nT}\sum_{i=1}^n[u_{it} - \mathrm{E}(u_{it})]\mathrm{E}(u_{is}) + \frac{1}{nT}\sum_{i=1}^n[u_{is} - \mathrm{E}(u_{is})]u_{it} \equiv \frac{1}{nT}\sum_{i=1}^n V_{1n,i} + \frac{1}{nT}\sum_{i=1}^n V_{2n,i}$. We have,

$$\begin{aligned}
\tfrac{1}{nT}\textstyle\sum_{i=1}^n V_{1n,i} &= \tfrac{1}{nT}\textstyle\sum_{i=1}^n\left[\left(v_t'A_{i,tt}^dv_t - \sigma_{v0}^2\mathrm{tr}(A_{i,tt})\right) + v_t'(A_{i,tt}^l + A_{i,tt}^{u\prime})v_t\right]\mathrm{E}(u_{is})\\
&= \tfrac{1}{nT}v_t'[\textstyle\sum_{i=1}^n A_{i,tt}^d\sigma_{v0}^2\mathrm{tr}(A_{i,ss})]v_t - \tfrac{1}{nT}\textstyle\sum_{i=1}^n\sigma_{v0}^4\mathrm{tr}(A_{i,tt})\mathrm{tr}(A_{i,ss}) + \tfrac{1}{nT}v_t'[\textstyle\sum_{i=1}^n(A_{i,tt}^l + A_{i,tt}^{u\prime})]v_t\\
&= \tfrac{1}{nT}[v_t'v_t^* - \mathrm{E}(v_t'v_t^*)] + \tfrac{1}{nT}v_t'\xi_t = \tfrac{1}{nT}\textstyle\sum_{j=1}^n(v_{jt}v_{jt}^* - \mathrm{E}(v_{jt}v_{jt}^*)) + \tfrac{1}{nT}\textstyle\sum_{j=1}^n v_{jt}\xi_{jt},
\end{aligned}$$

where $v_t^* = [\sum_{i=1}^n A_{i,tt}^d\sigma_{v0}^2\mathrm{tr}(A_{i,ss})]v_t$ and $\xi_t = [\sum_{i=1}^n(A_{i,tt}^l + A_{i,tt}^{u\prime})]v_t$. Clearly, the first term is the average of n independent terms, and the second term is the average of an MD array as ξ_{jt} is $\mathcal{G}_{n,j-1}$-measurable and $\{v_{jt}\xi_{jt}, \mathcal{G}_{n,j}\}$ form an MD array. Conditions of Theorem 19.7 of Davidson(1994) are easily verified and hence $\frac{1}{nT}\sum_{i=1}^n V_{1n,i} \xrightarrow{p} 0$. Similarly, we have $\frac{1}{nT}\sum_{i=1}^n V_{2n,i} \xrightarrow{p} 0$, and therefore $\frac{1}{nT}\sum_{i=1}^n(H_{2,ni} - \mathrm{E}H_{2,ni}) \xrightarrow{p} 0$.

The third type of terms is $H_{3,ni} = \sum_t\sum_{s\neq t} v_t'A_{i,ts}v_sv_t'A_{i,ts}v_s = \sum_t\sum_{s\neq t}(v_t'\xi_{its})^2 = \sum_t\sum_{s\neq t} v_t'\xi_{its}\xi_{its}'v_t = \sum_t\sum_{s\neq t} v_t'\mathbb{A}_{its}v_t$. For each t and s, we have $\frac{1}{nT}\sum_{i=1}^n v_t'\mathbb{A}_{its}v_t = \frac{1}{nT}v_t'\mathbb{A}_{+ts}v_t = \frac{1}{nT}v_t'v_t^* + \frac{1}{nT}v_t'\xi_t$. Therefore, similar to the proof of the second type of terms, we have $\frac{1}{nT}\sum_{i=1}^n[H_{3,ni} - \mathrm{E}H_(3, ni)] \xrightarrow{p} 0$.

The forth type of terms: $H_{4,ni} = \sum_t v_t'A_{i,tt}v_tv_t'A_{i,tt}v_t = \sum_t(v_t'v_{it}^* + v_t'\xi_{it})^2$, where $v_{it}^* =$

$A^{d}_{i,tt}v_t$, and $\xi_{it} = (A^l_{i,tt}+A^{u\prime}_{i,tt})v_t$. For each t, we have $\frac{1}{nT}\sum_{i=1}^n(v_t'v_{it}^*+v_t'\xi_{it})^2 = \frac{1}{nT}\sum_{i=1}^n(v_t'v_{it}^*)^2+\frac{1}{nT}\sum_{i=1}^n(v_t'\xi_{it})^2+\frac{2}{nT}\sum_{i=1}^n v_t'v_{it}^*v_t'\xi_{it} \equiv \frac{1}{nT}\sum_{i=1}^n V_{1n,i}+\frac{1}{nT}\sum_{i=1}^n V_{2n,i}+\frac{1}{nT}\sum_{i=1}^n V_{3n,i}$. First, $\frac{1}{nT}\sum_{i=1}^n[V_{1n,i}-\text{E}(V_{1n,i})] = \frac{1}{nT}\sum_{j=1}^n(v_j^4-\mu_v^{(4)})a_{jj}+\frac{1}{nT}\sum_{j=1}^n\sum_{k\neq j}(v_j^2v_k^2-\sigma_{v0}^4)a_{kj} \xrightarrow{p} 0$. Similarly, $\frac{1}{nT}\sum_{i=1}^n[V_{3n,i}-\text{E}(V_{3n,i})] \xrightarrow{p} 0$. Now, $\frac{1}{nT}\sum_{i=1}^n[V_{2n,i}-\text{E}(V_{2,ni})]$ van be written as

$$\begin{aligned}&\tfrac{1}{nT}\textstyle\sum_{j=1}^n[v_j^2(\sum_{i=1}^n\xi_{i,j}^2)-\sigma_{v0}^2\text{E}(\sum_{i=1}^n\xi_{i,j}^2)]+\tfrac{1}{nT}\sum_{j=1}^n\sum_{k\neq j}v_jv_k(\sum_{i=1}^n\xi_{i,j}\xi_{i,k})\\ =&\tfrac{1}{nT}\textstyle\sum_{j=1}^n(v_j^2-\sigma_{v0}^2)(\sum_{i=1}^n\xi_{i,j}^2)+\tfrac{1}{nT}\sum_{j=1}^n[(\sum_{i=1}^n\xi_{i,j}^2)-\text{E}(\sum_{i=1}^n\xi_{i,j}^2)]\\ &+\tfrac{\sigma_{v0}^2}{nT}\textstyle\sum_{j=1}^n v_j[\sum_{k\neq j}v_k(\sum_{i=1}^n\xi_{i,j}\xi_{i,k})].\end{aligned}$$

The first term and third term can be proved by WLLN for MD arrays as $\xi_{i,j}$ is $\mathcal{G}_{n,j-1}$ measurable and the third term is average of n uncorrelated terms. Let $a_{i,j}'$ be the j th row of $A^l_{i,tt}+A^{u\prime}_{i,tt}$. Then, $\xi_{i,j}=a_{i,j}'v_t$ and the second term becomes $\frac{1}{nT}\sum_{j=1}^n\left[v_t'(\sum_{i=1}^n a_{i,j}a_{i,j}')v_t-\sigma_{v0}^2\text{tr}(\sum_{i=1}^n a_{i,j}a_{i,j}')\right]=o_p(1)$ by Lemma V1.1, V1.2 and V1.4. So we have $\frac{1}{nT}\sum_{i=1}^n H_{4,ni}\xrightarrow{p} 0$. Combining these results, we have $U_4=o_p(1)$. We have proved that each of $U_r\xrightarrow{p}0$ for $r=1,\dots,15$. Therefore, $\frac{1}{nT}\sum_{i=1}^n\left[g_{\Phi i}^2-\text{E}(g_{\Phi i}^2)\right]\xrightarrow{p}0$.

By (V3.3) and (V3.4), we have for the cross-product term,

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^n\left[g_{\Pi i}g_{\Phi i}-\text{E}(g_{\Pi i}g_{\Phi i})\right]=\sum_{r=1}^{10}U_r,\quad\text{where}\tag{V3.6}$$

$$\begin{aligned}U_1&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n k_i\Pi_{i+}[(\varepsilon_i^2-\sigma_\varepsilon^2)\varepsilon_i-\mu_{\varepsilon 0}^{(3)}], & U_2&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n k_i(\varepsilon_i^2-\sigma_\varepsilon^2)b_i'\mathbf{v}_i,\\ U_3&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n\Pi_{i+}\varepsilon_i^2z_{1i}, & U_4&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n\varepsilon_iz_{1i}b_i'\mathbf{v}_i,\\ U_5&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n\Pi_{i+}[\varepsilon_i^2(r_i'\varepsilon)+r_{ii}\mu_{\varepsilon0}^{(3)}], & U_6&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n\Pi_{i+}\varepsilon_i^2z_{1i},\\ U_7&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n\Pi_{i+}\varepsilon_i(u_i-\mu_{ui}), & U_8&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n b_i'\mathbf{v}_i(u_i-\mu_{ui}),\\ U_9&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n b_i'\mathbf{v}_i\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t), & U_{10}&=\tfrac{1}{nT}\textstyle\sum_{i=1}^n\Pi_{i+}\varepsilon_i\sum_{t=1}^T(q_{it}'\varepsilon)\end{aligned}$$

As the terms contained in (V3.6) are similar to the terms contained (V3.5), we skip the proofs.

Third, similarly, for $g_{\Psi i}=\sum_{t=1}^T e_{it}\Psi_{ii,t}^*y_{0i}^*+\sum_{t=1}^T e_{it}\xi_{it}-d_{\Psi i}$, recall $\xi_{it}=w_{it}'y_0^*$ and $y_0^*=\eta_m^*+\varepsilon+V_m^*$. Some algebra lead to

$$g_{\Psi i}=\varepsilon_ih_i+\Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2)+\varepsilon_i(w_{i+}'\varepsilon)+z_{2i}+\textstyle\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon),\tag{V3.7}$$

where $h_i=a_i'V_m^*+\sum_{t=1}^T c_{it}'v_t$, $z_{2i}=\sum_{t=1}^T s_{it}'v_t$, and a_i', s_i', and c_{it}' are non-stochastic vectors. It follows that

$$\tfrac{1}{nT}\textstyle\sum_{i=1}^n\left(g_{\Psi i}^2-\text{E}(g_{\Psi i}^2)\right)=\sum_{r=1}^{15}U_r,\quad\text{where}\tag{V3.8}$$

$U_1 = \frac{1}{nT}\sum_{i=1}^n(\varepsilon_i^2 h_i^2 - \sigma_{\varepsilon 0}^2 \mathrm{E}(h_i^2)),$ $\quad U_2 = \frac{1}{nT}\sum_{i=1}^n \Psi_{ii+}^{*2}\{(\varepsilon_i^2-\sigma_\varepsilon^2)^2 - \mathrm{E}[(\varepsilon_i^2-\sigma_\varepsilon^2)^2]\},$
$U_3 = \frac{1}{nT}\sum_{i=1}^n[\varepsilon_i^2(w_{i+}'\varepsilon)^2 - \sigma_{\varepsilon 0}^4\sum_{j=1}^n w_{ij}^2],$ $\quad U_4 = \frac{1}{nT}\sum_{i=1}^n[z_{2i}^2 - \mathrm{E}(z_{2i}^2)],$
$U_6 = \frac{2}{nT}\sum_{i=1}^n \Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2)\varepsilon_i h_i,$ $\quad U_7 = \frac{2}{nT}\sum_{i=1}^n \varepsilon_i^2(w_{i+}'\varepsilon)h_i,$
$U_8 = \frac{2}{nT}\sum_{i=1}^n \varepsilon_i h_i z_{2i},$ $\quad U_9 = \frac{2}{nT}\sum_{i=1}^n \varepsilon_i h_i\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon),$
$U_{10} = \frac{2}{nT}\sum_{i=1}^n \Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2)\varepsilon_i(w_{i+}'\varepsilon),$ $\quad U_{11} = \frac{2}{nT}\sum_{i=1}^n \Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2)\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon),$
$U_{12} = \frac{2}{nT}\sum_{i=1}^n \Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2)z_{2i},$ $\quad U_{13} = \frac{2}{nT}\sum_{i=1}^n \varepsilon_i(w_{i+}'\varepsilon)\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon),$
$U_{14} = \frac{2}{nT}\sum_{i=1}^n \varepsilon_i(w_{i+}'\varepsilon)z_{2i},$ $\quad U_{15} = \frac{2}{nT}\sum_{i=1}^n z_{2i}\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon).$
$U_5 = \frac{1}{nT}\sum_{i=1}^n[(\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon))^2 - \sigma_{v0}^2\sigma_{\varepsilon 0}^2(b_i'b_i)\sum_{t=1}^T(w_{it}'w_{it})].$

By (V3.3) and (V3.7), we have,

$$\frac{1}{nT}\sum_{i=1}^n\big(g_{\Pi i}g_{\Psi i} - \mathrm{E}(g_{\Pi i}g_{\Psi i})\big) = \sum_{r=1}^{10}U_r, \quad \text{where} \qquad \text{(V3.9)}$$

$U_1 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}[\varepsilon_i^2 h_i - \sigma_\varepsilon^2\mathrm{E}(h_i)],$ $\quad U_2 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}\Psi_{ii+}^*[(\varepsilon_i^2-\sigma_\varepsilon^2)\varepsilon_i - \mu_{\varepsilon 0}^{(3)}],$
$U_3 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}\varepsilon_i^2(w_{i+}'\varepsilon),$ $\quad U_4 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}\varepsilon_i z_{2i},$
$U_5 = \frac{1}{nT}\sum_{i=1}^n \Pi_{i+}\varepsilon_i\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon),$ $\quad U_6 = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i h_i b_i'\mathbf{v}_i,$
$U_7 = \frac{1}{nT}\sum_{i=1}^n b_i'\mathbf{v}_i\Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2),$ $\quad U_8 = \frac{1}{nT}\sum_{i=1}^n b_i'\mathbf{v}_i\varepsilon_i(w_{i+}'\varepsilon),$
$U_9 = \frac{1}{nT}\sum_{i=1}^n[b_i'\mathbf{v}_i z_{2i} - \mathrm{E}(b_i'\mathbf{v}_i z_{2i})],$ $\quad U_{10} = \frac{1}{nT}\sum_{i=1}^n b_i'\mathbf{v}_i\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon).$

By (V3.4) and (V3.7), we have,

$$\frac{1}{nT}\sum_{i=1}^n\big(g_{\Phi i}g_{\Psi i} - \mathrm{E}(g_{\Phi i}g_{\Psi i})\big) = \sum_{r=1}^{25}U_r, \quad \text{where} \qquad \text{(V3.10)}$$

$U_1 = \frac{1}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2-\sigma_\varepsilon^2)\varepsilon_i h_i,$ $\quad U_2 = \frac{1}{nT}\sum_{i=1}^n k_i\Psi_{ii+}^*\{(\varepsilon_i^2-\sigma_\varepsilon^2)^2 - \mathrm{E}[(\varepsilon_i^2-\sigma_\varepsilon^2)^2]\},$
$U_3 = \frac{1}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2-\sigma_\varepsilon^2)\varepsilon_i(w_{i+}'\varepsilon),$ $\quad U_4 = \frac{1}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2-\sigma_\varepsilon^2)z_{2i},$
$U_5 = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i^2 h_i z_{1i},$ $\quad U_6 = \frac{1}{nT}\sum_{i=1}^n k_i(\varepsilon_i^2-\sigma_\varepsilon^2)\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon),$
$U_7 = \frac{1}{nT}\sum_{i=1}^n \Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2)\varepsilon_i z_{1i},$ $\quad U_8 = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i^2(w_{i+}'\varepsilon)z_{1i},$
$U_9 = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i z_{1i}z_{2i},$ $\quad U_{10} = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i z_{1i}\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon),$
$U_{11} = \frac{1}{nT}\sum_{i=1}^n h_i\varepsilon_i^2(r_i'\varepsilon),$ $\quad U_{12} = \frac{1}{nT}\sum_{i=1}^n \Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2)\varepsilon_i(r_i'\varepsilon),$
$U_{13} = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i(r_i'\varepsilon)z_{2i},$ $\quad U_{14} = \frac{1}{nT}\sum_{i=1}^n[\varepsilon_i^2(w_{i+}'\varepsilon)(r_i'\varepsilon) - \sigma_{\varepsilon 0}^4(w_{i+}'r_i)],$
$U_{15} = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i(r_i'\varepsilon)\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon),$ $\quad U_{16} = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i h_i(u_i-\mu_{ui}),$
$U_{17} = \frac{1}{nT}\sum_{i=1}^n \Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2)(u_i-\mu_{ui}),$ $\quad U_{18} = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i(w_{i+}'\varepsilon)(u_i-\mu_{ui}),$
$U_{19} = \frac{1}{nT}\sum_{i=1}^n[(u_i-\mu_{ui})z_{2i} - \mathrm{E}(u_i z_{2i})],$ $\quad U_{20} = \frac{1}{nT}\sum_{i=1}^n(u_i-\mu_{ui})\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon),$
$U_{21} = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i h_i\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t),$ $\quad U_{22} = \frac{1}{nT}\sum_{i=1}^n \Psi_{ii+}^*(\varepsilon_i^2-\sigma_\varepsilon^2)\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t),$
$U_{23} = \frac{1}{nT}\sum_{i=1}^n \varepsilon_i(w_{i+}'\varepsilon)\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t),$ $\quad U_{25} = \frac{1}{nT}\sum_{i=1}^n z_{2i}\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t).$
$U_{24} = \frac{1}{nT}\sum_{i=1}^n[\sum_{t=1}^T(b_i'v_t)(w_{it}'\varepsilon)\sum_{t=1}^T(q_{it}'\varepsilon)(b_i'v_t) - \sigma_{\varepsilon 0}^2\sigma_{v0}\sum_{t=1}^T(b_i'b_i)(w_{it}'q_{it})],$

As V_m^* is independent of ε and v_t, and η_m^* is exogenous. The terms in (V3.8)-(V3.10) are similar to those in (V3.5), and therefore their convergence is proved similarly. These complete the prove of convergence in the single summation part of Theorem 3.1.

To prove the convergence of the **double summation** part in Theorem 3.1, the result $\frac{1}{nT}\sum_{1=1}^{n}\sum_{j=1,j\neq i}^{n}(\widehat{\Upsilon}_{ij}-\Upsilon_{ij})\xrightarrow{p}0$ follows if

(a) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}(\widehat{\Upsilon}_{ij}-\widetilde{\Upsilon}_{ij})\xrightarrow{p}0$, and

(b) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}[\widetilde{\Upsilon}_{ij}-\Upsilon_{ij}]\xrightarrow{p}0$,

where $\widetilde{\Upsilon}_{ij}$ is Υ_{ij} with $\mathrm{E}(\cdot)$ corresponding to y_0^* being removed. As each element of Υ_{ij} is a linear combination of the terms specified in Lemma 3.1. So, we only need to prove the consistency of those terms. Also, as T is fixed, the proof can be done with a fixed t and s.

Proof of (a): (i) By lemma 3.1 we have, $\mathrm{E}(g_{\boldsymbol{\pi}_r}g'_{\boldsymbol{\pi}_\nu})=\sigma_{v0}^2\sum_{t=1}^{T}(b_i'b_j)\boldsymbol{\pi}'_{r,it}\boldsymbol{\pi}_{\nu,jt}$. As σ_{v0}^2 enters linearly and $\hat{\sigma}_v^2$ is consistent, and as this term does not involve y_0^*, it suffices to prove

$$Q_0^t=\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}[(\hat{b}_i'\hat{b}_j)\hat{\boldsymbol{\pi}}_{it}\hat{\boldsymbol{\pi}}_{jt}-(b_i'b_j)\boldsymbol{\pi}_{it}\boldsymbol{\pi}_{jt}]\xrightarrow{p}0,\ \text{ for each } t=1,\ldots,T.$$

Denote $\eta_t=\Pi_t\odot\Pi_t$. Let $\mathbb{B}=(B_3'B_3)^{-1}$ and $\mathbb{B}_{ij}$ be its $(i,j)th$ element. Rewrite,

$$Q_0^t=\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(\hat{b}_i'\hat{b}_j-b_i'b_j)\hat{\Pi}_{it}\hat{\Pi}_{jt}+\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(\hat{\Pi}_{it}\hat{\Pi}_{jt}-\Pi_{it}\Pi_{jt})(b_i'b_j)\equiv Q_{0,1}^t+Q_{0,2}^t.$$

By Holder's inequality, $Q_{0,1}^t\leq\big[\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(\hat{b}_i'\hat{b}_j-b_i'b_j)^2\big]^{\frac{1}{2}}\big[\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(\hat{\Pi}_{it}\hat{\Pi}_{jt})^2\big]^{\frac{1}{2}}$. By Lemma A.6, $\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(\hat{\Pi}_{it}\hat{\Pi}_{jt})^2\leq\frac{1}{n}\eta_t'\eta_t=O(1)$. By MVT, $\hat{\mathbb{B}}-\mathbb{B}=(\hat{\lambda}_3-\lambda_{30})\dot{\mathbb{B}}_{\lambda_3^*}$, where $\dot{\mathbb{B}}_{\lambda_3^*}=\frac{d}{d\lambda_3}\mathbb{B}(\lambda_3^*)=\mathbb{B}_{\lambda_3^*}(W_3'B_3+B_3'W_3)\mathbb{B}_{\lambda_3^*}$, with λ_3^* lying between $\hat{\lambda}_3$ and λ_{30}. Then, by Lemmas V1.1 and V1.2, and the consistency of $\hat{\lambda}_3$, $\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(\hat{b}_i'\hat{b}_j-b_i'b_j)^2=\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}\big[(\hat{\lambda}_3-\lambda_{30})\dot{\mathbb{B}}_{\lambda_3^*,ij}\big]^2\leq\frac{1}{n}(\hat{\lambda}_3-\lambda_{30})^2\mathrm{tr}(\dot{\mathbb{B}}_{\lambda_3^*}\dot{\mathbb{B}}'_{\lambda_3^*})=o_p(1)$. Therefore, $Q_{0,1}^t\xrightarrow{p}0$.

By Holder's inequality, $Q_{0,2}^t\leq\big[\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(\hat{\Pi}_{it}\hat{\Pi}_{jt}-\Pi_{it}\Pi_{jt})^2\big]^{\frac{1}{2}}\big[\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(b_i'b_j)^2\big]^{\frac{1}{2}}$. Applying MVT on $\hat{\Pi}_t\hat{\Pi}_t'-\Pi_t\Pi_t'$, we have $\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(\hat{\Pi}_{it}\hat{\Pi}_{jt}-\Pi_{it}\Pi_{jt})^2=o_p(1)$ by the consistency of the estimator, and Lemmas V1.1 and V1.2. Next, by Assumption E(iii) and Lemma A.1, we have $\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}(b_i'b_j)^2=\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{B}_{ij}^2\leq\frac{1}{n}\mathrm{tr}(\mathbb{B}\mathbb{B}')=O(1)$, and thus $Q_{0,2}^t\xrightarrow{p}0$. Therefore, $Q_0^t=Q_{0,1}^t+Q_{0,2}^t\xrightarrow{p}0$.

(ii) By lemma 3.1, $\mathrm{E}(g_{\Phi_r i}\,g_{\Phi_\nu j})=\sigma_{v0}^4\sum_{t=1}^{T}\sum_{s=1}^{T}[(b_j'c^*_{ri,ts})(b_i'c^*_{\nu j,st})+(b_i'b_j)(c^{*\prime}_{ri,ts}c^*_{\nu j,ts})]+\sigma_{v0}^2\sigma_{\varepsilon0}^2\sum_{t=1}^{T}[a_{\nu ji,t+}(b_j'c^*_{ri,+t})+a_{rij,t+}(b_i'c^*_{\nu j,+t})+(b_i'b_j)(a^{*\prime}_{ri,t+}a^*_{\nu j,t+})]+(\mu_{v0}^{(4)}-3\sigma_{v0}^4)\sum_{t=1}^{T}[(b_i\odot c^*_{ri,tt})'(b_j\odot c^*_{\nu j,tt})]$, $r,\nu=1,\ldots,6$. Therefore, we need to prove:

$$\begin{aligned}
Q_1^t&=\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}[(\hat{b}_j'\hat{c}^*_{i,ts})(\hat{b}_i'\hat{c}^*_{j,st})-(b_j'c^*_{i,ts})(b_i'c^*_{j,st})]\xrightarrow{p}0,\\
Q_2^t&=\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}[(\hat{b}_i'\hat{b}_j)(\hat{c}^{*\prime}_{i,ts}\hat{c}^*_{j,ts})-(b_i'b_j)(c^{*\prime}_{i,ts}c^*_{j,ts})]\xrightarrow{p}0,\\
Q_3^t&=\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}[\hat{a}_{ji,t+}(\hat{b}_j'\hat{c}^*_{i,+t})-a_{ji,t+}(b_j'c^*_{i,+t})]\xrightarrow{p}0,\\
Q_4^t&=\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}[\hat{a}_{ij,t+}(b_i'\hat{c}^*_{j,+t})-a_{ij,t+}(b_i'c^*_{j,+t})]\xrightarrow{p}0,\\
Q_5^t&=\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}[(\hat{a}^{*\prime}_{i,t+}\hat{a}^*_{j,t+})(\hat{b}_i'\hat{b}_j)-(a^{*\prime}_{i,t+}a^*_{j,t+})(b_i'b_j)]\xrightarrow{p}0,\\
Q_6^t&=\tfrac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}[(\hat{b}_i\odot\hat{c}^*_{i,tt})'(\hat{b}_j\odot\hat{c}^*_{j,tt})-(b_i\odot c^*_{i,tt})'(b_j\odot c^*_{j,tt})]\xrightarrow{p}0.
\end{aligned}$$

To prove $Q_1^t \xrightarrow{p} 0$, rewrite

$Q_1^t = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^* - b_i' c_{j,st}^*)(b_j' c_{i,ts}^*) + \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_j'\hat{c}_{i,ts}^* - b_j' c_{i,ts}^*)(\hat{b}_i'\hat{c}_{j,st}^*) \equiv Q_{1,1}^t + Q_{1,2}^t$.

By Holder's inequality, $Q_{1,1}^t \le (\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^* - b_i' c_{j,st}^*)^2)^{\frac{1}{2}} (\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(b_j' c_{i,ts}^*)^2)^{\frac{1}{2}}$. Let $Q_{ts} = \mathbb{B}\Phi_{ts}^*$ and $Q_{ts,ij}$ be its (i,j)th element. By MVT, we have $\hat{Q}_{st} - Q_{st} = \dot{Q}_{st}(\delta^*)(\hat{\delta} - \delta)$, where $\dot{Q}_{st}(\delta^*) = \frac{\partial}{\partial \delta'} Q_{st}(\delta^*)$, and δ^* lies between $\hat{\delta}$ and δ elementwise. Then, by Lemmas V1.1 and V1.2, and the consistency of $\hat{\delta}$, we have

$$\begin{aligned}
&\tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^* - b_i' c_{j,st}^*)^2 = \tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{Q}_{st,ij} - Q_{st,ij})^2 \\
&\quad = \tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{\delta} - \delta)^2 \dot{Q}_{st,ij}^2(\delta^*) \le \tfrac{1}{n}(\hat{\delta} - \delta)^2 \operatorname{tr}(\dot{Q}_{st}(\delta^*)\dot{Q}_{st}'(\delta^*)) = o_p(1).
\end{aligned}$$

Next, $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(b_j' c_{i,ts}^*)^2 = \frac{1}{n}\sum_{i=1}^n \sum_{j\neq i} Q_{ts,ij}^2 \le \frac{1}{n}\operatorname{tr}(Q_{ts}Q_{ts}') = O(1)$ by Assumption E(iii) and Lemma A(?). Thus, $Q_{1,1}^t \xrightarrow{p} 0$. Similarly for the 2nd term, by Holder's inequality, $Q_{1,2}^t \le \big(\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_j'\hat{c}_{i,ts}^* - b_j' c_{i,ts}^*)^2\big)^{\frac{1}{2}} \big(\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^*)^2\big)^{\frac{1}{2}}$. By MVT, Assumption E$(iv)$, Lemmas V1.1 and V1.2, and the consistency of the estimator, we have the 1st part $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_j'\hat{c}_{i,ts}^* - b_j' c_{i,ts}^*)^2 \xrightarrow{p} 0$. For the 2nd part, we have $\frac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{b}_i'\hat{c}_{j,st}^*)^2 \le \frac{1}{n}\operatorname{tr}(\hat{Q}_{st}\hat{Q}_{st}') = O(1)$ by Assumption E(iv) and Lemmas V1.1 and V1.2, leading to $Q_{1,2}^t \xrightarrow{p} 0$. Therefore, $Q_1^t \xrightarrow{p} 0$. The results $Q_r^t \xrightarrow{p} 0$, $r = 2, \ldots, 5$, can be proved in a similar manner.

To prove $Q_6^t \xrightarrow{p} 0$, let $Q_{tt} = B_3^{-1} \odot \Phi_{tt}^* B_3^{-1}$, q_i' be its ith row, and q_{ij} be its (i,j)th element, then we can rewrite Q_6^t as:

$$Q_6^t = \tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{q}_j' - q_j')q_i + \tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{q}_i' - q_i')\hat{q}_j \equiv Q_{6,1}^t + Q_{6,2}^t.$$

By MVT, $\hat{Q}_{tt} - Q_{tt} = \dot{Q}_{tt}(\delta^*)(\hat{\delta} - \delta)$, where $\dot{Q}_{tt}(\delta^*) = \frac{\partial}{\partial \delta'} Q_{tt}(\delta^*)$, and δ^* lies between $\hat{\delta}$ and δ elementwise. By Lemmas V1.1 and V1.2, it can be easily seen that $\dot{Q}_{tt}(\delta^*)$ is uniformly bounded in either row or column sum. Let $\dot{q}_i'$ be the i th row of $\dot{Q}_{tt}(\delta^*)$. We have, $Q_{6,1}^t = \frac{1}{n}(\sum_{i=1}^n q_i')(\sum_{j\neq i}\hat{q}_j - q_j) = \frac{1}{n}(\hat{\delta} - \delta)(\sum_{i=1}^n q_i')(\sum_{j\neq i}\dot{q}_j) \le \frac{1}{n}(\hat{\delta} - \delta)\sum_{m=1}^n \sum_{i=1}^n |q_{im}| \sum_{j=1}^n |\dot{q}_{jm}|$ $\le \frac{1}{n}(\hat{\delta} - \delta)c_1 \sum_{m=1}^n \sum_{j=1}^n |\dot{q}_{jm}| = \frac{1}{n}(\hat{\delta} - \delta)c_1 n c_2 = o_p(1)$. The convergence of $Q_{6,2}^t$ proceeds similarly as the first term. Therefore, $Q_6^t \xrightarrow{p} 0$.

(iii) $\mathrm{E}(g_{\Psi_r,i}g_{\Psi_\nu,j}) = \sigma_{\varepsilon 0}^4(w_{rij,+}w_{\nu ji,+}) + \sigma_{v0}^2 \sum_{t=1}^T (b_i'b_j)\mathrm{E}(\xi_{ri,t}^*\xi_{\nu j,t}^*)$ from Lemma 3.1, and thus we need to show:

$$\begin{aligned}
Q_7^t &= \tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i}(\hat{w}_{ij,+}\hat{w}_{ji,+} - w_{ij,+}w_{ji,+}) \xrightarrow{p} 0, \quad \text{and} \\
Q_8^t &= \tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i}[(\hat{b}_i'\hat{b}_j)(\hat{\xi}_{i,t}^*\hat{\xi}_{j,t}^*) - (b_i'b_j)(\xi_{i,t}^*\xi_{j,t}^*)] \xrightarrow{p} 0.
\end{aligned}$$

(iv) $\mathrm{E}(g_{\Phi i}g_{\Pi' j}) = \mu_{v0}^{(3)} \sum_{t=1}^T (b_i \odot c_{i,tt}^*)' b_j \boldsymbol{\pi}_{j,t}$ from Lemma 3.1, and thus we need to show:

$$Q_9^t = \tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i}[(\hat{b}_i \odot \hat{c}_{i,tt}^*)'\hat{b}_j\hat{\boldsymbol{\pi}}_{j,t} - (b_i \odot c_{i,tt}^*)'b_j\boldsymbol{\pi}_{j,t}] \xrightarrow{p} 0.$$

(v) $\mathrm{E}(g_{\Psi i}g_{\Pi' j}) = \sigma_{v0}^2 \sum_{t=1}^T \boldsymbol{\pi}_{j,t}\mathrm{E}(\xi_{ri,t}^*)(b_i'b_j)$ from Lemma 3.1, and thus we need to show:

$$Q_{10}^t = \tfrac{1}{n}\sum_{i=1}^n \sum_{j\neq i}[(\hat{\boldsymbol{\pi}}_{j,t}\hat{\xi}_{i,t}^*)(\hat{b}_i'\hat{b}_j) - (\boldsymbol{\pi}_{j,t}\xi_{ri,t}^*)(b_i'b_j)] \xrightarrow{p} 0.$$

(vi) Finally by Lemma 3.1, $\mathrm{E}(g_{\Phi i} g_{\Psi j}) = \sigma_{\varepsilon 0}^2 \sigma_{v0}^2 \sum_{t=1}^T \big[(b_i' b_j)(a_{ri,t+}' w_{\nu j,t}^*) + w_{\nu ji,+}(b_j' c_{ri,+t}^\circ)\big] + \sigma_{\varepsilon 0}^4 (w_{ji,+} a_{ij,++}) + \mu_{v0}^{(3)} \sum_{t=1}^T (b_i \odot c_{ri,tt}^*)' b_j \mathrm{E}(\xi_{\nu j,t}^*)$, and thus we need to show:

$$
\begin{aligned}
Q_{11}^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j \neq i} [(\hat{b}_i' \hat{b}_j)(\hat{a}_{i,t+}' \hat{w}_{j,t}^*) - (b_i' b_j)(a_{i,t+}' w_{j,t}^*)] \xrightarrow{p} 0, \\
Q_{12}^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j \neq i} [\hat{w}_{ji,+}(\hat{b}_j' \hat{c}_{i,+t}^\circ) - w_{ji,+}(b_j' c_{i,+t}^\circ)] \xrightarrow{p} 0, \\
Q_{13}^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j \neq i} [(\hat{w}_{ji,+} \hat{a}_{ij,++}) - (w_{ji,+} a_{ij,++})] \xrightarrow{p} 0, \\
Q_{14}^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j \neq i} [(\hat{b}_i \odot \hat{c}_{i,tt}^*)' \hat{b}_j (\hat{\xi}_{j,t}^*) - (b_i \odot c_{i,tt}^*)' b_j (\xi_{j,t}^*)] \xrightarrow{p} 0.
\end{aligned}
$$

All the terms in (iii)-(vi) are similar to the terms in (i) and (ii), and therefore their convergence in probability to zero is proved similarly to that of the terms in (i) and (ii).

Proof of (b): The proofs for the terms not involving y_0^* are trivial. We focus on the terms which involve y_0^*. Therefore, we need to prove:

$$
\begin{aligned}
R_1^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j \neq i} (b_i' b_j)[\xi_{i,t}^* \xi_{j,t}^* - \mathrm{E}(\xi_{i,t}^* \xi_{j,t}^*)] \xrightarrow{p} 0, \\
R_2^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j \neq i} \boldsymbol{\pi}_{j,t}[\xi_{i,t}^* - \mathrm{E}(\xi_{i,t}^*)] \xrightarrow{p} 0, \\
R_3^t &= \tfrac{1}{n}\textstyle\sum_{i=1}^n \sum_{j \neq i} [(b_i \odot c_{i,tt}^*)' b_j][\xi_{j,t}^* - \mathrm{E}(\xi_{j,t}^*)] \xrightarrow{p} 0.
\end{aligned}
$$

Recall: $\xi_{i,t}^* = w_{it}^{*\prime} y_0^*$ where $w_{it}^{*\prime}$ is the ith row of Ψ_t^*, and $y_0^* = V_m^* + \eta_m^* + \varepsilon$. We have $\xi_{j,t}^* - \mathrm{E}(\xi_{j,t}^*) = w_{it}^{*\prime}(V_m^* + \varepsilon)$. The convergence of R_2^t and R_3^t thus follow by Assumption F(ii), Lemma A.4(vi) and Lemma A.6(ii). To show $R_1^t \xrightarrow{p} 0$, note that $\Psi_t^* = \Psi_t K_m$, and $y_0^* = K_m^{-1} y_0$, so we can write, $\xi_{i,t}^* = a_{it}' y_0$, where a_{it}' is the ith row of Ψ_t. Then we have, $\sum_{i=1}^n \sum_{j \neq i} (b_i' b_j) \xi_{i,t}^* \xi_{j,t}^* = y_0' [\sum_{i=1}^n (a_{it} b_i') \sum_{j \neq i} (b_j a_{jt}')] y_0 = y_0' A_t y_0$, where $A_t = \Psi_t' \mathbb{B} \Psi_t - \Psi_t' \mathtt{diag}(\mathbb{B}) \Psi_t$, and $\mathbb{B} = (B_3' B_3)^{-1}$. Clearly, A_t is bounded in both row and column sums by Assuption E(iii) and Lemma A.1(i). Therefore, $R_1^t = \frac{1}{n}[y_0' A_t y_0 - \mathrm{E}(y_0' A_t y_0)] = o_p(1)$, by Assumption F($iii$). These complete the prove of convergence in the double summation part of Theorem 3.1.

■