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Lottery Rather than Waiting-Line Auction*

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Abstract

This paper studies the allocative efficiency of two non-price allocation mechanisms – the lottery (random allocation) and the waiting-line auction (queue system) – for the cases where individuals possess identical time costs (the homogeneous case), and where time costs are correlated with valuations (the heterogeneous case). The relative efficiency of the two mechanisms is shown to depend critically on a scarcity factor (measured by the ratio of the number of objects available for allocation over the number of participants) and on the shape of the distribution of valuations. We found that the lottery is more efficient than the waiting-line auction for a wide range of situations, and that while heterogeneity of time costs may improve the allocative efficiency of the waiting-line auction, the ranking on relative efficiency is not altered.

KEY WORDS: Lottery; Non-price allocation, Rent-seeking; Waiting-line auction

JEL CLASSIFICATION: C15; D44, D61

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1 Introduction

Governments often play a key role in the allocation of goods and services when prices are set below market clearing levels. Two commonly-used mechanisms are the lottery (random allocation) and the waiting-line auction (first-come first-served queue system). Lotteries have been used widely to allocate hunting permits, fishing berths, oil drill leases, or even admission to universities, while waiting line auctions have been used to allocate publicly-provided goods and services such as medical care services or subsidized public housing.

In selecting an allocation mechanism, one must consider its *equity* and *efficiency*. The *equity* of a mechanism is measured by the welfare impact of the allocation. The case for the lottery to allocate economic goods and burdens (e.g. military draft) is frequently made on the grounds of horizontal equity, i.e. individuals who possess the same relevant characteristics should be treated equally (see Eckhoff (1989), Elster (1991), Goodwin (1992) and Boyce (1994)). The *efficiency* of a mechanism is measured by the degree of rent dissipation, due to *resource misallocation* and the incurrence of *rent seeking costs*. In a waiting-line auction, individuals who queued up earlier may be the ones with lower opportunity cost of time rather than the ones with higher valuations, while in the case of the lottery, individuals who value the objects most may not receive an allocation.[1] There are no rent-seeking costs in a lottery, but waiting in line creates both disutility and potential loss of income.

In this paper, we study the allocative efficiency of the lottery versus the waiting-line auction, when individuals possess identical time costs (the homogeneous case), and when time costs are correlated with valuations (the heterogeneous case). For the homogeneous case, our results generalize the analysis in Taylor, et. al. (2003).[2] There are two key findings in this paper. Firstly, relative efficiency is critically dependent on the shape of the distribution of time valuations and a scarcity factor (measured by the ratio of the number of objects available for allocation over the number of participants). Secondly, the lottery is almost always more efficient than the waiting-line auction unless there are very few objects to be allocated and there are only a few participants possessing high values (i.e. the distribution of time valuations is L-shaped). For the heterogeneous case, we study a model where time costs and valuations are correlated. We show that if the correlation is positive (negative), the relative efficiency of the waiting-line auction improves (declines). However, the the ranking of allocative efficiency vis-a-vis the lottery is not altered.[3]

The rest of the paper is organized as follows. Section 2 presents the basic model.

[1] Studies on the economics of rationing and the queue system have been conducted by Tobin (1952), Nicholas, Smolensky and Tideman (1971), Barzel (1974) and Suen (1989).

[2] Using numerical analysis, Taylor, et. al. considered the local impact of mean-preserving dispersions in valuations on relative efficiency.

[3] These results are related to the analysis in Sah (1987), which concluded that the sufficiently poor would always prefer non-convertible rations (i.e. a lottery) to queuing.

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Section 3 analyzes the relative relative efficiency of the two mechanisms for the homogeneous case under different distributional assumptions for time valuations. Section 4 extends the analysis to the heterogeneous case. Section 5 concludes the paper.

2 The Model

There are m identical and indivisible objects to be distributed free of charge to $n(> m)$ individuals, at most one object per person, using either a lottery or a waiting-line auction. The opportunity costs of time of the n individuals (measured by their wage rates) are denoted by $w_1, w_2, \cdots, w_n$, and their monetary valuations (measured in dollars) are denoted by $v_1, v_2, \cdots, v_n$. The ratio $y_i = v_i/w_i$ describes an individual's valuation of an object measured in time units. We refer to v_i as *monetary valuation* and y_i as *time valuation*. In our analysis, it is often more convenient to work with time valuations.

Individuals are risk neutral; they know their own monetary valuations and time costs, but not those of the other $(n-1)$ individuals. However, each individual believes that the monetary valuations and time costs of the other rival claimants are independent realizations of a pair of continuous random variables $\{V, W\}$, which has a joint distribution function $F(v, w)$ with support $[\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$, for some finite non-negative $\underline{v}$ and positive $\underline{w}$. The marginal distributions of V and W are denoted by $F_V(v)$ and $F_W(w)$, respectively. Similarly, the marginal distribution of Y is denoted by $F_Y(y)$.

The allocative efficiency of an allocation mechanism is measured by the *expected social surplus*, defined as the sum of the expected payoffs for all n individuals.

2.1 Lottery

At a pre-specified time, m individuals are randomly chosen and allocated an object. The probability that the ith individual obtains an object is

$$H^R = \frac{m}{n}.$$

If the ith individual has a monetary valuation of v_i, his monetary payoff is

$$\pi^R(v_i) = v_i H^R = \frac{m v_i}{n}.$$

Given the symmetric treatment of all individuals, the expected social surplus is

$$S^R = nE[\pi^R(V)] = mE(V). \tag{1}$$

Hence, the expected social surplus generated by a lottery depends only on the number of objects available for allocation and the mean value of the distribution of monetary valuations.

2.2 Waiting-line auction

In a waiting-line auction, objects are allocated at a pre-specified time and location, on a first-come first-served basis. Following Holt and Sherman (1982), we consider an individual's decision whether (or not) to join the queue, conditional on an expected waiting time. Each individual occupies only one position in the queue. Individuals who arrive after the mth person will be notified so that no unsuccessful persons will spend time in the queue.[4] We assume that the time taken to reach the queue is negligible compared with the waiting time.

The equilibrium waiting time. For each individual i, there is an optimal waiting time $\tau(y_i)$, which is a strictly increasing function of time valuation y_i. Under the assumption that $\tau(y)$ is differentiable, $\tau(y)$ can be written as

$$\tau(y) = \frac{1}{H_Y^Q(y)} \int_{\underline{y}}^{y} x h_Y^Q(x)dx = y - \frac{1}{H_Y^Q(y)} \int_{\underline{y}}^{y} H_Y^Q(x)dx \qquad (2)$$

where $h_Y^Q(y)$ and $H_Y^Q(y)$ are, respectively, the density function and the distribution function of the mth largest order-statistic among the $(n-1)$ independent draws from the distribution of time valuations.[5] Denoting the marginal distribution of Y by $F_Y(y)$, we can verify that

$$H_Y^Q(y) = \sum_{k=n-m}^{n-1} \binom{n-1}{k} [F_Y(y)]^k [1-F_Y(y)]^{n-k-1}.$$

It is straightforward to verify that the optimal waiting time $\tau(v_i)$ is a decreasing function in m and an increasing function in n. As shown in Holt and Sherman(1982), if the arrival time at the queue is chosen according to $\tau(y)$, expected monetary payoff will be globally maximized. The probability that individual i will receive an object is simply $H_Y^Q(y_i)$.

The equilibrium expected payoff. The expected payoff, in time units, is

$$\pi^Q(y_i) = (y_i - \tau(y_i))H^Q(y_i) = \int_{\underline{y}}^{y_i} H_Y^Q(x)dx$$

for individual i. Multiplying $\pi^Q(y_i)$ by the time cost w_i yields the expected monetary payoff. The expected social surplus is

$$S^Q = nE\left(W \int_{\underline{y}}^{Y} H_Y^Q(x)dx\right). \qquad (3)$$

Note that S^Q depends on the joint distribution of time valuation Y and time cost W.

[4]Holt and Sherman (1982) shows that if unsuccessful individuals also have to wait, individuals will optimally reduce their waiting time. In equilibrium, expected waiting time as well as payoff remains unchanged.

[5]The second part of the equation follows from integration by parts.

3 Efficiency Comparison: Homogeneous Case

To compare allocative efficiency, we first derive closed-form expressions for S^R and S^Q, and then compute the ratio S^R/S^Q. When individuals have identical time costs (i.e. $w_i = w_c, i = 1, \cdots, n$), the expected social surplus of a waiting-line auction can be expressed in terms of $V = Yw_c$,

$$S^Q = nE\left(\int_{\underline{v}}^{V} H_V^Q(x)dx\right),$$

where,

$$H_V^Q(v) = \sum_{k=n-k}^{n-1} \binom{n-1}{k} [F_V(v)]^k [1 - F_V(v)]^{n-k-1}$$

By switching the order of integrations and then the order of integration and summation,

$$\begin{aligned} S^Q &= n\int_{\underline{v}}^{\overline{v}} \left(\int_{\underline{v}}^{v} H_V^Q(x)dx\right) f_V(v)dv \\ &= n\int_{\underline{v}}^{\overline{v}} \left(\int_{x}^{\overline{v}} f_V(v)dv\right) H_V^Q(x)dx \\ &= n\int_{\underline{v}}^{\overline{v}} [1 - F_V(v)]H_V^Q(v)dv \\ &= n\sum_{k=n-m}^{n-1} \binom{n-1}{k} \int_{\underline{v}}^{\overline{v}} F_V(v)^k[1 - F_V(v)]^{n-k}dv \end{aligned} \tag{4}$$

Thus, S^Q depends on the number of objects to be allocated m, the number of individuals n, and the distribution of monetary valuations $F_V(v)$.

When time costs are homogeneous, resource misallocation does not occur in a waiting-line auction; inefficiency results from the rent-seeking costs of waiting in line. In a lottery, rent dissipation is due solely to resource misallocation, as noted earlier. Hence, the lottery is more (less) efficient than the waiting-line auction if resource misallocation in the lottery is smaller (larger) than the rent-seeking costs incurred in a waiting-line auction.

To analyze the relative efficiency of the two allocation mechanisms, four classes of distributions for V are considered: power function, Weibull, logistic and beta.[6] We summarize the technical results in the lemmas (with proofs provided in the Appendix) and focus our discussion on the corresponding Propositions.

[6] These four classes cover the broad range of distributional forms: L-shaped (the majority have very low valuations and a few have very high valuations), U-shaped (the majority have either very high or very low valuations), J-shaped (the majority have high valuations and a few have low valuations), flat and unimodal, (the majority have valuations in the middle, with a few have very low or very high valuations).

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3.1 Monetary valuation with the power function distribution

The cumulative distribution function (cdf) takes the following form[7]:

$$F(v;\theta,\beta) = \left(\frac{v}{\theta}\right)^{\beta},\ 0\ < v < \theta; \theta > 0, \beta > 0, \tag{5}$$

where β determines the *shape* of the distribution and θ controls the range or *scale* of the V values. The probability density function (pdf) is decreasing (L-shaped) when $\beta < 1$, constant (uniform) when $\beta = 1$, and increasing (J-shaped) when $\beta > 1$. Figure 1 illustrates these cases. The mean and variance of the distribution are $E(V) = \frac{\theta\beta}{\beta+1}$ and $Var(V) = \frac{\theta^2\beta}{(\beta+2)(\beta+1)^2}$, respectively.

Figure 1

Lemma 1: *If monetary valuations are drawn from the power function distribution, the expected social surplus functions are* $S^R = m\theta\beta/(\beta+1)$ *and* $S^Q = S^R h(\beta,n,m)$*, where*

$$h(\beta,n,m) = \frac{n}{m} - \frac{n!\ (\beta n + m + 1)\Gamma(n - m + \frac{1}{\beta})}{\beta m(n-m-1)!\ \Gamma(n+1+\frac{1}{\beta})}, \tag{6}$$

with $\Gamma(\cdot)$ *being the gamma function. Furthermore,* $h(\beta,n,m)$ *satisfies: (i) it is strictly increasing in* m*, decreasing in* n*, and decreasing in* β*; (ii)* $h(1,n,m) = \frac{m+1}{n+1}$*; (iii)* $h(\frac{1}{2},n,m) = \frac{3mn+3n+2-2m^2}{(n+1)(n+2)}$*; (iv)* $h(\beta,n,1) = \frac{n(1+\beta)}{(1+n\beta)(1+n\beta-\beta)}$*; and (v)* $\lim_{\beta\to\infty} h(\beta,n,m) = 0$.

Proposition 1: *Suppose monetary valuations are drawn from the power function distribution. For any given* θ*,* m *and* n*, the lottery is more efficient than the waiting-line auction if* $\beta \geq 1$*. The degree of relative efficiency, as measured by* $h(\beta,n,m)$*, increases as* β *increases. If* $\beta < 1$*, the lottery remains more efficient than the waiting-line auction, provided the* m/n *ratio is sufficiently small (For instance,* $h(\frac{1}{2},n,m) \leq 1$ *if* $m/n \leq \frac{1}{2}$*).*

The function $h(\beta,n,m)$ measures the relative efficiency of the waiting-line auction versus the lottery. It depends on the *scarcity factor* (the m/n ratio) and a *distributional shape factor* (the parameter β in the power function distribution). When $\beta < 1$ and the m/n ratio is sufficiently small, the waiting-line auction may be more efficient than the lottery, if the rent-seeking costs of waiting in line is smaller than the resource misallocation in the lottery. As the m/n ratio rise (falls), the relative efficiency of the waiting-line auction improves (worsens), as the reduction (increase) in optimal waiting time is larger than the reduction (increase) in resource misallocation in a lottery. Similarly, as β rises and the

[7] Taylor, et al. (2003) considered the power function distribution with $m = 1$, and the beta distribution with $\beta = 1$. As noted in Section 3.4, the latter is really a special case of the power function distribution.

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distribution of V shifts to the right, there is a greater likelihood of facing competitors with higher valuations. This causes the optimal waiting times (and hence, rent-seeking costs) to increase, leading to a decline in relative efficiency of the waiting-line auction.

3.2 Monetary valuation with the Weibull distribution

The Weibull distribution models a range of valuation distributions: (i) extremely positively-skewed, (ii) unimodal and positively-skewed, and (iii) nearly symmetric. The cdf of the Weibull distribution takes the form

$$F(v;\theta,\beta) = 1 - \exp\left(-\left(\frac{v}{\theta}\right)^{\beta}\right),\ v > 0; \theta > 0, \beta > 0, \tag{7}$$

where β is the shape parameter and θ is the scale parameter.[8] The mean and variance of a Weibull random variable are, respectively, $E(V) = \theta\Gamma(1+\frac{1}{\beta})$ and $Var(V) = \theta^2[\Gamma(1+\frac{2}{\beta}) - [\Gamma(1+\frac{1}{\beta})]^2]$. Figure 2 illustrates the density function of the Weibull distribution.

Figure 2

Lemma 2: *If monetary valuations are drawn from the Weibull distribution, the expected social surplus functions are* $S^R = m\theta\Gamma(1+1/\beta)$ *and* $S^Q = S^R h(\beta, n, m)$, *where*

$$h(\beta,n,m) = \frac{n}{m}\sum_{n-m}^{n-1}\sum_{j=0}^{k}\binom{n-1}{k}\binom{k}{j}(-1)^j\left(\frac{1}{n-k+j}\right)^{1/\beta}. \tag{8}$$

Furthermore, $h(\beta, n, m)$ *is decreasing in* β *and* $h(1, n, m) = 1$.

Proposition 2: *Suppose monetary valuations are drawn from the Weibull distribution. The lottery is more efficient than the waiting-line auction if* $\beta > 1$, *while the waiting-line auction is more efficient than the lottery if* $\beta < 1$. *The two mechanisms are equally efficient when* $\beta = 1$, *which is the case where monetary valuations follow an exponential distribution.*

The above results indicate that if monetary valuations are drawn from a Weibull distribution, the relative efficiency of the waiting-line auction versus the lottery depends only on the shape parameter β. However, the degree of relative efficiency still depends on the m/n ratio. In Figure 3, we plot the relative efficiency of the lottery versus the waiting-line auction (i.e., $S^R/S^Q = 1/h(\beta, n, m)$) in Figure 3. The numerical results indicate that when $m/n \approx 1$, there is only a small difference in allocative efficiency.

[8] When $\beta < 1$, the pdf is decreasing in v. When $\beta > 1$, the pdf is unimodal with a longer tail to the right. When $\beta = 1$, the Weibull distribution is an exponential distribution. When $\beta = 3.768$, the pdf of the Weibull distribution is very similar to that of a normal distribution (See Hernandez and Johnson, 1980).

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Figure 3

3.3 Monetary valuation with the logistic distribution

The logistic distribution function represents the case where the distribution of valuations is symmetric and unimodal.[9] The cdf of the logistic distribution takes the form

$$F(v;\mu,\theta) = 1 - \frac{1}{1+\exp[(v-\mu)/\theta]}, \quad -\infty < v < \infty; -\infty < \mu < \infty, \theta > 0. \tag{9}$$

The logistic distribution is symmetric around the mean $E(V) = \mu$, with variance $Var(V) = \frac{1}{3}\pi^2\theta^2$.[10] A few plots of the logistic pdf are illustrated in Figure 4.[11]

Figure 4

Lemma 3: *If monetary valuations are drawn from the logistic distribution, the expected social surplus functions are*

$$S^R = m\mu \text{ and } S^Q = n\theta[\Psi(n) - \Psi(n-m)]$$

where $\Psi(\cdot)$ *is the digamma function defined as* $\Psi(z) = d\log\Gamma(z)/dz$. Taking $\mu/\theta = 10$,

$$\frac{S^Q}{S^R} = \frac{n}{10m}[\Psi(n) - \Psi(n-m)],$$

which is an increasing function of m. We can show that $\max_m(S^Q/S^R) < 1$ for $n < 10000$.

Proposition 3: *If monetary valuations are drawn from the logistic distribution with a negligible probability of negative valuations* ($\mu/\theta \geq 10$)*, the lottery is almost always more efficient than the waiting-line auction.*

This result is particularly striking as it indicates that when time costs are homogeneous and monetary valuations can be modeled as a symmetric distribution, the optimal allocation mechanism is almost always a lottery, regardless of the number of objects available for allocation or the number of individuals vying for the objects.

[9] The logistic distribution is chosen rather than the normal distribution for our analysis, as the normal distribution does not allow us to derive closed-form expressions for the expected surplus functions. With suitable choice of parameters, the logistic distribution may approximate a normal distribution.

[10] μ is a location parameter and θ is the scale parameter. The larger the value of θ, the flatter is the pdf.

[11] Note that the logistic distribution may assume negative values, which has no economic meaning, since an individual with a negative monetary valuation will not choose to participate in either the lottery or the waiting-line auction. However, we can make the probability of negative values negligible by having a large mean-to-scale ratio, i.e., $\mu/\theta \geq 10$. The probability of negative values is $F(0,\mu,\theta) = 1 - 1/[1+\exp(-\mu/\theta)]$. When $\mu/\theta \geq 10$, we have $F(0,\mu\theta) \leq 4.5 \times 10^{-5}$.

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3.4 Monetary valuation with the beta distribution

The pdf of the beta distribution has the following form

$$F(v;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1},\ 0 \le v \le 1, \alpha > 0, \beta > 0. \tag{10}$$

In terms of the potential shapes of the density function, the beta distribution is the richest family of distributions. It is U-shaped if $\alpha < 1$ and $\beta < 1$, uniform if $\alpha = 1$ and $\beta = 1$, L-shaped if $\alpha < 1$ and $\beta > 1$, J-shaped if $\alpha > 1$ and $\beta < 1$, and unimodal, otherwise. The mean and variance of the beta distribution are $E(V) = \frac{\alpha}{\alpha+\beta}$ and $Var(V) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, respectively. We provide plots of the beta pdf in Figure 5.

Figure 5

Since the cdf of the beta distribution does not have a closed-form expression, we are unable to derive general closed-form expressions of S^R and S^Q. Two special cases, when $\alpha = 1$ and when $\beta = 1$, allow us to obtain closed-form expression of the ratio S^R/S^Q. Using the general expression given in (4), we also compute the values of S^R/S^Q for a range of parameter configurations of β, n and m. These results are presented in Figure 6.

When $\beta = 1$, the beta distribution is a special case of the power function distribution. Lemma 1 and Proposition 1 apply. Hence, the lottery is more efficient than the waiting-line auction if $\alpha \ge 1$. The lottery remains more efficient if $\alpha < 1$, provided that m/n is sufficiently small. For the case when $\alpha = 1$, we obtain the following result.

Lemma 4: *If monetary valuations are drawn from the beta distribution and* $\alpha = 1$, *the expected social surplus functions are* $S^R = m/(1+\beta)$ *and* $S^Q = S^R h(\beta, n, m)$, *where*

$$h(\beta, n, m) = \frac{n!\Gamma(m+1+1/\beta)}{m!\Gamma(n+1+1/\beta)}. \tag{11}$$

Furthermore, $h(\beta, n, m)$ *is increasing in* β, *and* $\lim_{\beta\to\infty} h(\beta, n, m) = 1$.

When $\alpha = 1$, the lottery is more efficient than the waiting-line auction regardless of the value of β or m/n. This result can be generalized to the case when $\alpha > 1$. The intuition here is that as α increases above 1, the weight of the pdf shifts to the right, so that there is a greater likelihood that rival participants possess higher valuations. Competition intensifies in the waiting-line auction, and the optimal waiting times become longer. Hence, the $h(\beta, n, m)$ function decreases in value as α increases (see Figure 6). Combining Lemmas 1 and 4, we have the follwing result.

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Proposition 4: *If monetary valuations are drawn from the beta distribution, the lottery is more efficient than the waiting-line auction, except possibly in the case where $\alpha < 1$, $\beta \geq 1$, i.e., when the beta distribution is L-shaped, and the m/n ratio is sufficiently large.*

Figure 6

4 Efficiency Comparison: Heterogeneous Case

In this section, we consider the case where time costs are heterogeneous. The following special cases are straightforward to analyze: (i) monetary valuation V and time cost W are independent and (ii) time valutaion Y and time cost W are independent. Using the general expression derived in (3), some simple conditioning arguments show that when W is independent of V, all the results of Lemmas 1 to 4 go through. Again, using (3), a direct manipulation shows that the S^R/S^Q function remains the same if W is not constant but still independent of Y. Therefore, heterogeneity in time costs does not affect the earlier results on the relative efficiency of the two mechanisms if time costs are independent of monetary valuations or time valuations.

For the general case where Y and W are correlated, we first note that rent dissipation in the waiting-line auction includes potential resource misallocation as well. Our analysis suggests that the relative efficiency of the waiting-line auction improves when the correlation between Y and W is positive, and deteriorates when the correlation is negative.

4.1 Positively correlated time valuation and time cost

Consider the case where Y and W are uniformly distributed on the area $A = \{(y, w) : [0 \leq y \leq 1], [0 \leq w \leq \beta y^{\beta-1}]\}$.[12] The joint pdf of Y and W is

$$f(y, w) = 1, \qquad 0 \leq y \leq 1,\ 0 \leq w \leq \beta y^{\beta-1};\ \beta \geq 1. \tag{12}$$

It is easy to verify that $f_Y(y) = \beta y^{\beta-1}, 0 \leq y \leq 1$, i.e., the marginal distribution of Y is the power function distribution with the scale parameter equal to 1, and $f_W(w) = 1 - (w/\beta)^{\frac{1}{\beta-1}}, 0 \leq w \leq \beta$. The correlation coefficient between Y and W is

$$\rho(Y, W) = \frac{\beta - 1}{2\beta} \sqrt{\frac{(\beta+2)(9\beta-6)}{7\beta^2 - 2\beta + 4}}$$

For $\beta = 1, 2, 4, \infty$, we have $\rho(Y, W) = 0, 0.32, 0.48$, and 0.57, respectively. This indicates that Y and W are uncorrelated when $\beta = 1$, and that the correlation increases as β increases (with an upper limit of 0.57).

[12] From a modeling perspective, we could also consider a joint distribution of V and W. The current specification is chosen for its tractability.

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Lemma 5: *If time valuation Y and time cost W are drawn jointly from the distribution specified in (12), the expected social surplus functions are $S^R = \frac{1}{4}m\beta$, and $S^Q = S^R h(\beta, n, m)$, where,*

$$h(\beta, n, m) = \frac{2\beta}{2\beta - 1}\left(\frac{n}{m} - \frac{1}{\beta} + \frac{m+1}{2\beta(n+1)} - \frac{n!\Gamma(n - m + \frac{1}{\beta})}{m(n-m-1)!\Gamma(n + \frac{1}{\beta})}\right) \quad (13)$$

Furthermore, $h(\beta, n, m)$ is decreasing in β, $h(1, n, m) = \frac{m+1}{n+1}$ and $\lim_{\beta\to\infty} h(\beta, n, m) = 0$.

A comparison of the h functions, stated in Lemma 1 and Lemma 5, allows us to determine the effect of positive correlation of W and Y on allocative efficiency. When $\beta = 1$, so that $\rho(Y, W) = 0$, both h functions have the same value. Hence, Proposition 1 applies even though time costs are heterogeneous. As β increases above 1, causing the marginal distribution of time valuations (given by $f_Y(y)$) to become more negatively skewed, the waiting-line auction becomes less efficient, regardless of the degree of correlation between Y and W. This result follows directly from the fact that $h(\beta, n, m) \leq 1$ when $\beta = 1$, and the fact that $h(\beta, n, m)$ is decreasing in β (Lemma 5).[13]

4.2 Negatively correlated time valuation and time cost

To analyze the impact of negative correlation on relative efficiency, consider the following specification of time cost: $W^* = \beta - W$.[14] Then,

$$\rho(Y, W*) = -\rho(Y, W) = -\frac{\beta - 1}{2\beta}\sqrt{\frac{(\beta+2)(9\beta-6)}{7\beta^2 - 2\beta + 4}},$$

i.e., the time valuation Y and time cost W^* are negatively correlated.

Lemma 6: *If time valuation Y and time cost W follow the joint distribution specified in (12), the expected social surplus functions when time cost $W^* = \beta - W$ are $S^R = \frac{1}{4}m\beta$, and $S^Q = S^R h(\beta, n, m)$, where*

$$h(\beta, n, m) = \frac{4\beta}{3\beta - 1}h_1(\beta, n, m) - \frac{\beta+1}{3\beta - 1}h_2(\beta, n, m) \quad (14)$$

with h_1 being the h function defined in Lemma 1 and h_2 the h function defined in Lemma 5. Furthermore, $h(\beta, n, m)$ is decreasing in β, $h(1, n, m) = \frac{m+1}{n+1}$ and $\lim_{\beta\to\infty} h(\beta, n, m) = 0$.

[13] When β tends to ∞, both h functions are trivially identical, as both h functions tend to zero.

[14] This specification is chosen for its tractability and does not affect the generality of the results presented in this subsection.

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Together, Lemmas 5 and 6 allow us to analyze the relative efficiency of the two mechanisms under alternative scenarios of positive and negative correlations of time costs and time valuations. We illustrate the analysis graphically in Figure 7.

Figure 7

In Figure 7, we plotted the three h functions (defined in Lemmas 1, 5 and 6, respectively), against m, for a given value of β and for $n = 50$. The plots in Figure 7 indicate that when Y and W are positively correlated, the relative efficiency of the waiting-line auction is higher than in the case when Y and W are uncorrelated. The converse is true when Y and W are negatively correlated.[15]

We note that even though the allocative efficiency of the waiting-line auction improves when time costs and time valuations are correlated, the improvement is not sufficient to alter the ranking in relative efficiency. We summarize our findings in Proposition 5.

Proposition 5: *If time valuations Y and time costs W follow the joint distribution specified in (12), the allocative efficiency of the waiting-line auction improves (deteriorates) if Y and W are more positively (negatively) correlated, compared with the case when they are uncorrelated.*

An argument frequently raised against the use of the waiting-line auction as an allocation mechanism is that those individuals with lower valuations are also likely to have lower time costs. The assertion is that these individuals are likely to join the queue earlier, so that objects are not necessarily allocated to those who might possess higher valuations. Proposition 5 indicates that, on the contrary, the relative efficiency of the waiting-line auction in fact improves when time cost is positively correlated with valuation.[16]

5 Conclusion

By comparing the expected social surplus functions of the two allocation mechanisms, we are able to delineate the circumstances under which a lottery is more efficient than the waiting-line auction, and vice versa. Our analysis suggests that when time costs are

[15] Since monetary valuation is a product of time valuation and time cost, $V = YW$, a positive correlation between Y and W implies a positive correlation between V and W. Similarly, it is straightforward to show that V and W^* are negatively correlated.

[16] In the case of negative correlation, the decline in the relative efficiency of the waiting line auction is particularly easy to understand in the following example. Suppose all individuals possess the same monetary valuations. Then time costs and time valuations are negatively correlated. In this case, since everyone values the good identically, the costs of waiting in line have no offsetting benefit in terms of allocative efficiency.

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homogeneous, random allocation is the optimal mechanism in a wide range of circumstances (Propositions 1 to 4). We show that the relative efficiency of the waiting-line auction improves when the correlation between time costs and time valuations is positive, but deteriorates when the correlation is negative (Proposition 5).

Our results indicate that besides its *equity* appeal, the lottery is also the *more efficient* non-price allocation mechanism in a wide variety of situations. Although heterogeneity in time costs may improve the relative efficiency of the waiting-line auction when time valuations and time costs are positively correlated, our study shows that the improvement is not likely to be significant enough to reverse the efficiency ranking in most situations.

Appendix: Proofs of the Lemmas

Proof of Lemma 1. For the power function distribution, we have $F_V(v) = (v/\theta)^\beta$, and

$$\begin{aligned}
S^R &= mE(V) = \frac{m\theta\beta}{1+\beta} \\
S^Q &= n\sum_{k=n-m}^{n-1}\binom{n-1}{k}\int_0^\theta [(v/\theta)^\beta)]^k[1-(v/\theta)^\beta]^{n-k}dv \\
&= n\sum_{k=n-m}^{n-1}\binom{n-1}{k}\frac{\theta}{\beta}\int_0^1 u^{k+\frac{1}{\beta}-1}(1-u)^{n-k}du, \quad \text{(by letting } u=(v/\theta)^\beta\text{)} \\
&= n\sum_{k=n-m}^{n-1}\binom{n-1}{k}\frac{\theta}{\beta}\frac{\Gamma(k+\frac{1}{\beta})\Gamma(n-k+1)}{\Gamma(n+1+\frac{1}{\beta})} \\
&= \frac{\theta}{\beta}\frac{\Gamma(n+1)}{\Gamma(n+1+\frac{1}{\beta})}\sum_{k=n-m}^{n-1}\frac{\Gamma(k+\frac{1}{\beta})}{k!}(n-k) \\
&= \frac{\theta}{\beta}\frac{\Gamma(n+1)}{\Gamma(n+1+\frac{1}{\beta})}\left(\frac{\beta^2\Gamma(n+1+\frac{1}{\beta})}{(1+\beta)\Gamma(n)} - \frac{\beta(\beta n+m+1)\Gamma(n-m+\frac{1}{\beta})}{(1+\beta)\Gamma(n-m)}\right) \\
&= \frac{m\theta\beta}{1+\beta}\left(\frac{n}{m} - \frac{n!(\beta n+m+1)\Gamma(n-m+\frac{1}{\beta})}{\beta m(n-m-1)!\Gamma(n+1+\frac{1}{\beta})}\right).
\end{aligned}$$

It is straightforward to verify the properties of the h function stated in the Lemma.

Proof of Lemma 2. Forthe Weibull distribution, we have $F_V(v) = 1-\exp[-(v/\theta)^\beta]$, and

$$\begin{aligned}
S^R &= mE(V) = m\theta\Gamma(1+\frac{1}{\beta}) \\
S^Q &= n\sum_{k=n-m}^{n-1}\binom{n-1}{k}\int_0^\infty [1-\exp(-(v/\theta)^\beta)]^k[\exp(-v/\theta)^\beta]^{n-k}dv.
\end{aligned}$$

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Making a change of variable $u = (v/\theta)^\beta$, and then applying the binominal expansion to $[1-\exp(-u)]^k$, the integral in the summation for S^Q becomes

$$\begin{aligned}
& \int_0^\infty [1-\exp(-(v/\theta)^\beta)]^k [\exp(-v/\theta)^\beta]^{n-k} dv \\
= & \frac{\theta}{\beta}\int_0^\infty u^{1/\beta-1}[1-\exp(-u)]^k[\exp(-u)]^{n-k}du \\
= & \frac{\theta}{\beta}\int_0^\infty u^{1/\beta-1}\left\{\sum_{j=0}^k \binom{k}{j}(-1)^j\exp(-ju)\right\}[\exp(-(n-k)u)]du \\
= & \frac{\theta}{\beta}\int_0^\infty u^{1/\beta-1}\left\{\sum_{j=0}^k \binom{k}{j}(-1)^j\exp(-(n-k+j)u)\right\}du \\
= & \frac{\theta}{\beta}\sum_{j=0}^k \binom{k}{j}(-1)^j\int_0^\infty u^{1/\beta-1}\exp[-(n-k+j)u]du \\
= & \frac{\theta}{\beta}\sum_{j=0}^k \binom{k}{j}(-1)^j\Gamma(1/\beta)\left(\frac{1}{n-k+j}\right)^{1/\beta}.
\end{aligned}$$

Substituting this back into the expression for S^Q, we have

$$\begin{aligned}
S^Q &= n\frac{\theta}{\beta}\sum_{k=n-m}^{n-1}\binom{n-1}{k}\sum_{j=0}^k\binom{k}{j}(-1)^j\Gamma(1/\beta)\left(\frac{1}{n-k+j}\right)^{1/\beta} \\
&= n\theta\Gamma(1+1/\beta)\sum_{k=n-m}^{n-1}\sum_{j=0}^k\binom{n-1}{k}\binom{k}{j}(-1)^j\left(\frac{1}{n-k+j}\right)^{1/\beta} \\
&= m\theta\Gamma(1+1/\beta)h(\beta,n,m).
\end{aligned}$$

Since $1/(n-k+j) \le 1$ with equality occurring only when $k = n-1$, and $j = 0$, the terms in the summation of $h(\beta,n,m)$ are thus either constant or increasing in β. Hence h is an increasing function of β. Finally,

$$\begin{aligned}
h(1,n,m) &= \frac{n}{m}\sum_{k=n-m}^{n-1}\binom{n-1}{k}\sum_{j=0}^k\binom{k}{j}(-1)^j\left(\frac{1}{n-k+j}\right) \\
&= \frac{n}{m}\sum_{k=n-m}^{n-1}\binom{n-1}{k}\frac{k!}{(n-k)(n-k+1)\cdots(n-1)n} \\
&= \frac{n}{m}\sum_{k=n-m}^{n-1}\frac{1}{n} = 1.
\end{aligned}$$

Note that the first summation is handled by a combinatory formula

$$\sum_{j=0}^k\binom{k}{j}\frac{(-1)^j}{a+j} = \frac{k!}{a(a+1)\cdots(a+k)},\ \text{for } a \ne 0,-1,-2,\cdots,-k.$$

Proof of Lemma 3. For the logistic distribution, we have

$$\begin{aligned} S^R &= mE(V) = m\mu \\ & \int_{-\infty}^{\infty} F_V(v)^k[1-F_V(v)]^{n-k}dv \\ &= \int_{-\infty}^{\infty}\left[1-\frac{1}{1+\exp((v-\mu)/\theta)}\right]^k\left[\frac{1}{1+\exp((v-\mu)/\theta)}\right]^{n-k}dv. \end{aligned}$$

Letting $w=\{1+\exp[(v-\mu)/\theta]\}^{-1}$, the above integral becomes

$$\begin{aligned} & \int_0^1 (1-w)^k w^{n-k}\frac{\theta}{w(1-w)}dw \\ &= \theta\int_0^1 (1-w)^{k-1}w^{n-k-1}dw \\ &= \theta\frac{\Gamma(k)\Gamma(n-k)}{\Gamma(n)} \end{aligned}$$

This gives

$$\begin{aligned} S^Q &= n\sum_{k=n-m}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix}\frac{\theta\Gamma(k)\Gamma(n-k)}{\Gamma(n)} \\ &= n\theta\sum_{k=n-m}^{n-1}\frac{(n-1)!}{k!(n-k-1)!}\frac{(k-1)!(n-k-1)!}{(n-)!} \\ &= n\theta\sum_{k=n-m}^{n-1}\frac{1}{k} = n\theta[\Psi(n)-\Psi(n-m)]. \end{aligned}$$

Proof of Lemma 4. When $\alpha=1$, the beta distribution is $F_V(v)=1-(1-v)^{\beta}$. Hence,

$$\begin{aligned} S^R &= mE(V) = \frac{m}{1+\beta} \\ S^Q &= n\sum_{k=n-m}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix}\int_0^1 [1-(1-v)^{\beta}]^k(1-v)^{\beta(n-k)}dv \\ &= n\sum_{k=n-m}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix}\frac{1}{\beta}\frac{\Gamma(n-k+\frac{1}{\beta})}{\Gamma(n+1+\frac{1}{\beta})} \\ &= \frac{n!}{\beta\Gamma(n+1+\frac{1}{\beta})}\sum_{k=n-m}^{n-1}\frac{\Gamma(n-k+\frac{1}{\beta})}{\Gamma(n-k)} \\ &= \frac{n!}{\beta\Gamma(n+1+\frac{1}{\beta})}\frac{\beta\Gamma(m+1+\frac{1}{\beta})}{(1+\beta)\Gamma(m)}, \\ &= \frac{m}{1+\beta}\frac{n!\Gamma(m+1+\frac{1}{\beta})}{m!\Gamma(n+1+\frac{1}{\beta})} \end{aligned}$$

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The rest of the proof is straightforward.

Proof of Lemma 5.

$$\begin{aligned}
S^Q &= nE\left(W\int_0^Y H_Y^Q(x)dx\right) \\
&= n\int_0^1\int_0^{\beta y^{\beta-1}} w\left(\int_0^y H_Y^Q(x)dx\right) f(y,w)dwdy \\
&= n\int_0^1\left(\int_0^y H_Y^Q(x)dx\right)\frac{1}{2}\beta^2 y^{2(\beta-1)}dy \\
&= \frac{n\beta^2}{2}\int_0^1\left(\int_x^1 y^{2(\beta-1)}dy\right)H_Y^Q(x)dx \\
&= \frac{n\beta^2}{2(2\beta-1)}\int_0^1(1-x^{2\beta-1})H_Y^Q(x)dx \\
&= \frac{n\beta^2}{2(2\beta-1)}\sum_{k=n-m}^{n-1}\binom{n-1}{k}\int_0^1(1-x^{2\beta-1})(x^\beta)^k(1-x^\beta)^{n-k-1}dx, \qquad (\text{letting } u = x^\beta) \\
&= \frac{n\beta}{2(2\beta-1)}\sum_{k=n-m}^{n-1}\binom{n-1}{k}\left(\int_0^1 u^{k+\frac{1}{\beta}-1(1-u)^{n-k-1}} - \int_0^1 u^{k+1}(1-u)^{n-k-1}\right)du \\
&= \frac{n\beta}{2(2\beta-1)}\sum_{k=n-m}^{n-1}\binom{n-1}{k}\left(\frac{\Gamma(k+\frac{1}{\beta})\Gamma(n-k)}{\Gamma(n+\frac{1}{\beta})} - \frac{\Gamma(k+2)\Gamma(n-k)}{\Gamma(n+2)}\right) \\
&= \frac{\beta}{2(2\beta-1)}\sum_{k=n-m}^{n-1}\left(\frac{\Gamma(n+1)\Gamma(k+\frac{1}{\beta})}{\Gamma(n+\frac{1}{\beta})\Gamma(k+1)} - \frac{k+1}{n+1}\right) \\
&= \frac{\beta}{2(2\beta-1)}\left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{\beta})}\left(\frac{\beta\Gamma(n+\frac{1}{\beta})}{\Gamma(n)} - \frac{\beta\Gamma(n-m+\frac{1}{\beta})}{\Gamma(n-m)}\right) - \frac{m(n+1)-\frac{1}{2}m(m+1)}{n+1}\right) \\
&= \left(\frac{m\beta}{4}\right)\frac{2\beta}{2\beta-1}\left(\frac{n}{m} - \frac{1}{\beta} + \frac{m+1}{2\beta(n+1)} - \frac{n!\Gamma(n-m+\frac{1}{\beta})}{\beta(n-m-1)!\Gamma(n+\frac{1}{\beta})}\right)
\end{aligned}$$

It is straightforward to verify the properties of the h function stated in the Lemma.

Proof of Lemma 6.

$$\begin{aligned}
S^Q &= nE\left(W^*\int_0^Y H_Y^Q(x)dx\right) = nE\left((\beta - W)\int_0^Y H_Y^Q(x)dx\right) \\
&= n\beta E\left(\int_0^Y H_Y^Q(x)dx\right) - nE\left(W\int_0^Y H_Y^Q(x)dx\right)
\end{aligned}$$

The first part can be obtained from Lemma 1 and the second part can be obtained from Lemma 5.

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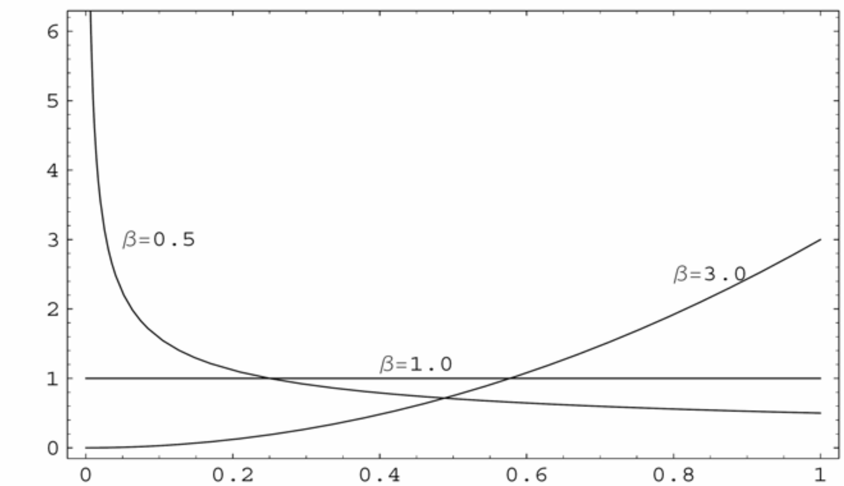

Figure 1: Plots of pdf of power function distribution: $\theta = 1.0$

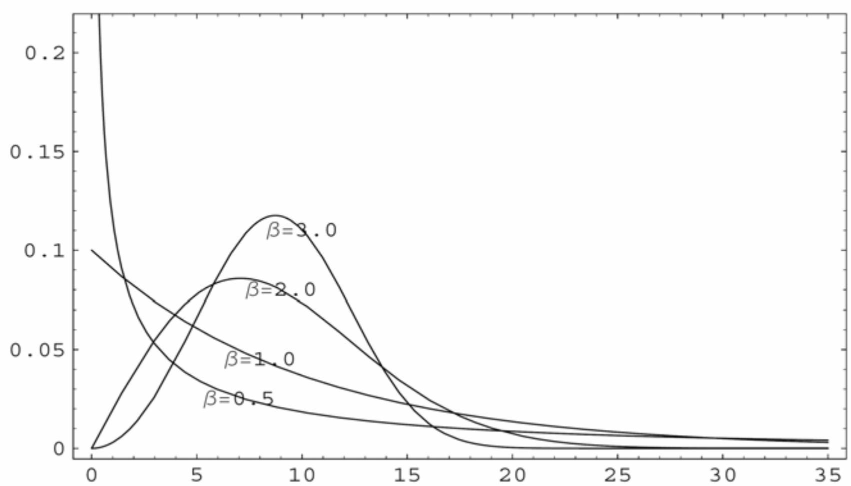

Figure 2: Plots of pdf of Weibull distribution: $\theta = 10.0$

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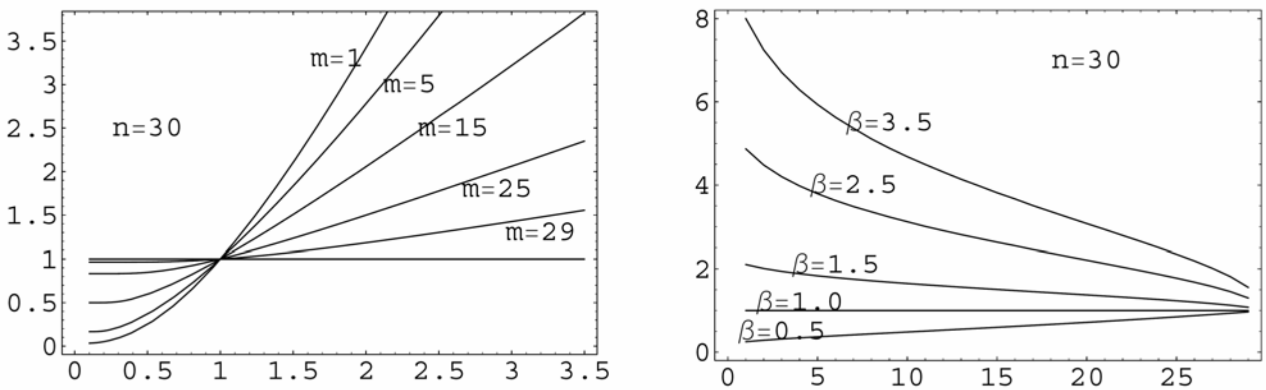

Figure 3: Plots of S^R/S^Q vs β (left), and vs m for Weibull distribution

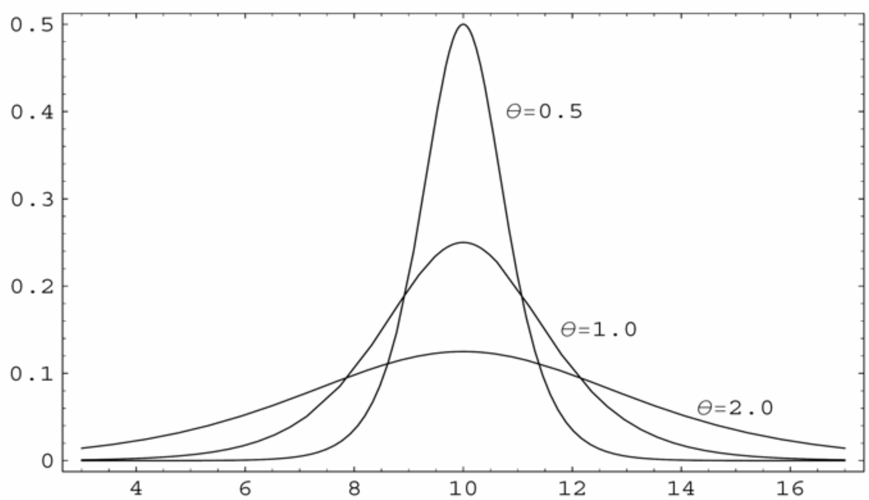

Figure 4: Plots of pdf of logistic distribution: $\mu = 10.0$

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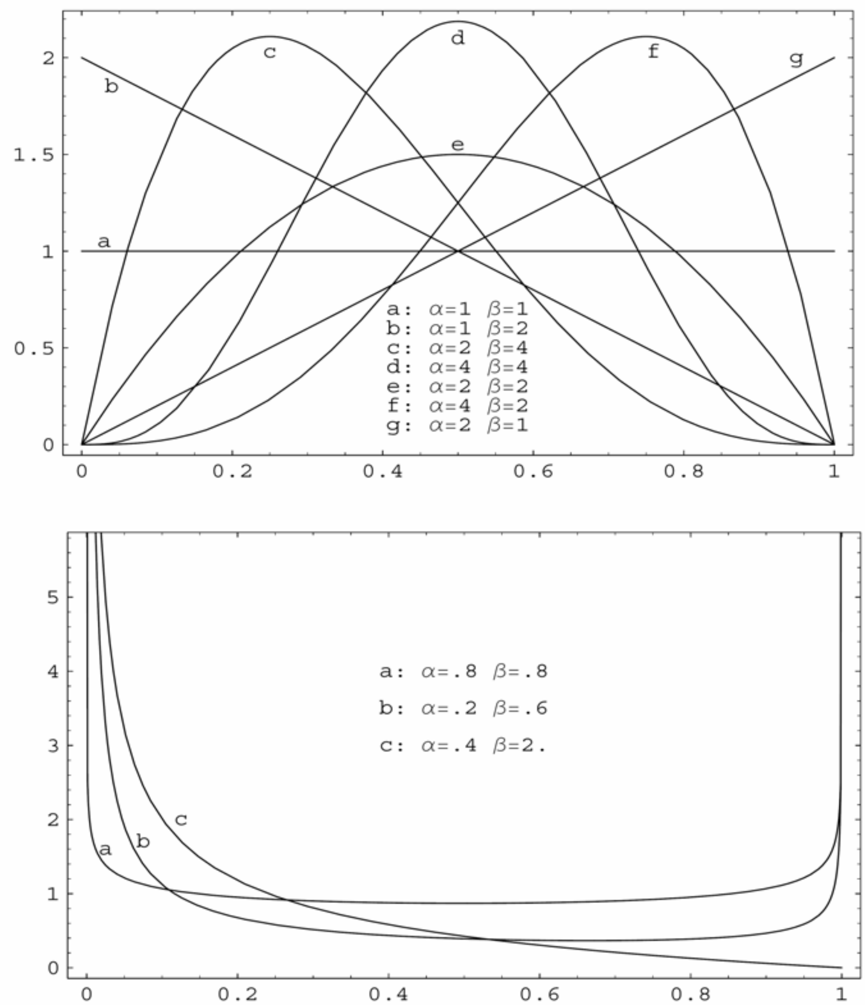

Figure 5: Plots of pdf of beta distribution

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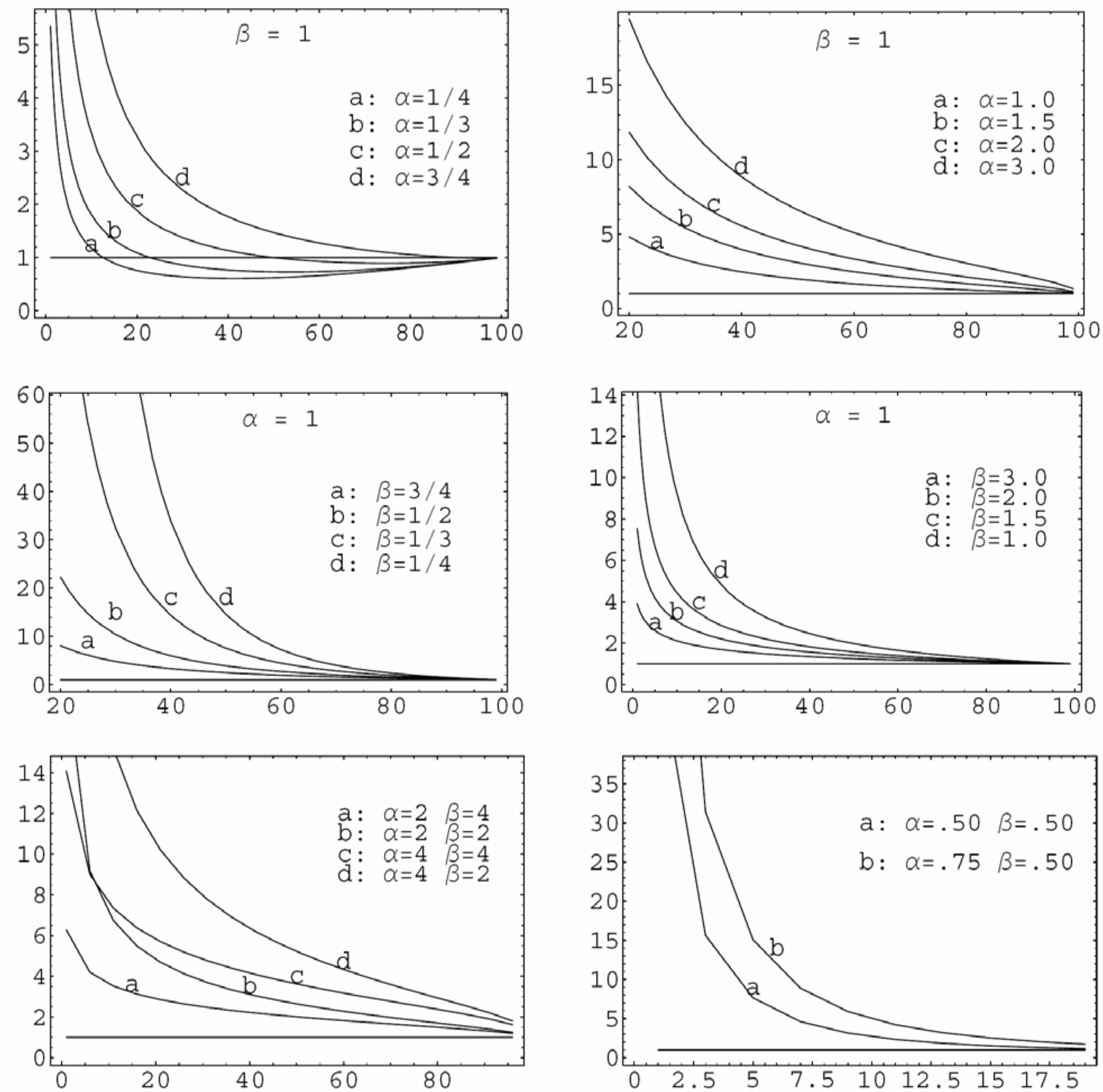

Figure 6: Plots of S^R/S^Q vs m for beta distribution, $n = 100$ for the first five plots and $n = 20$ for the last one. The last two plots are based on a few selected points.

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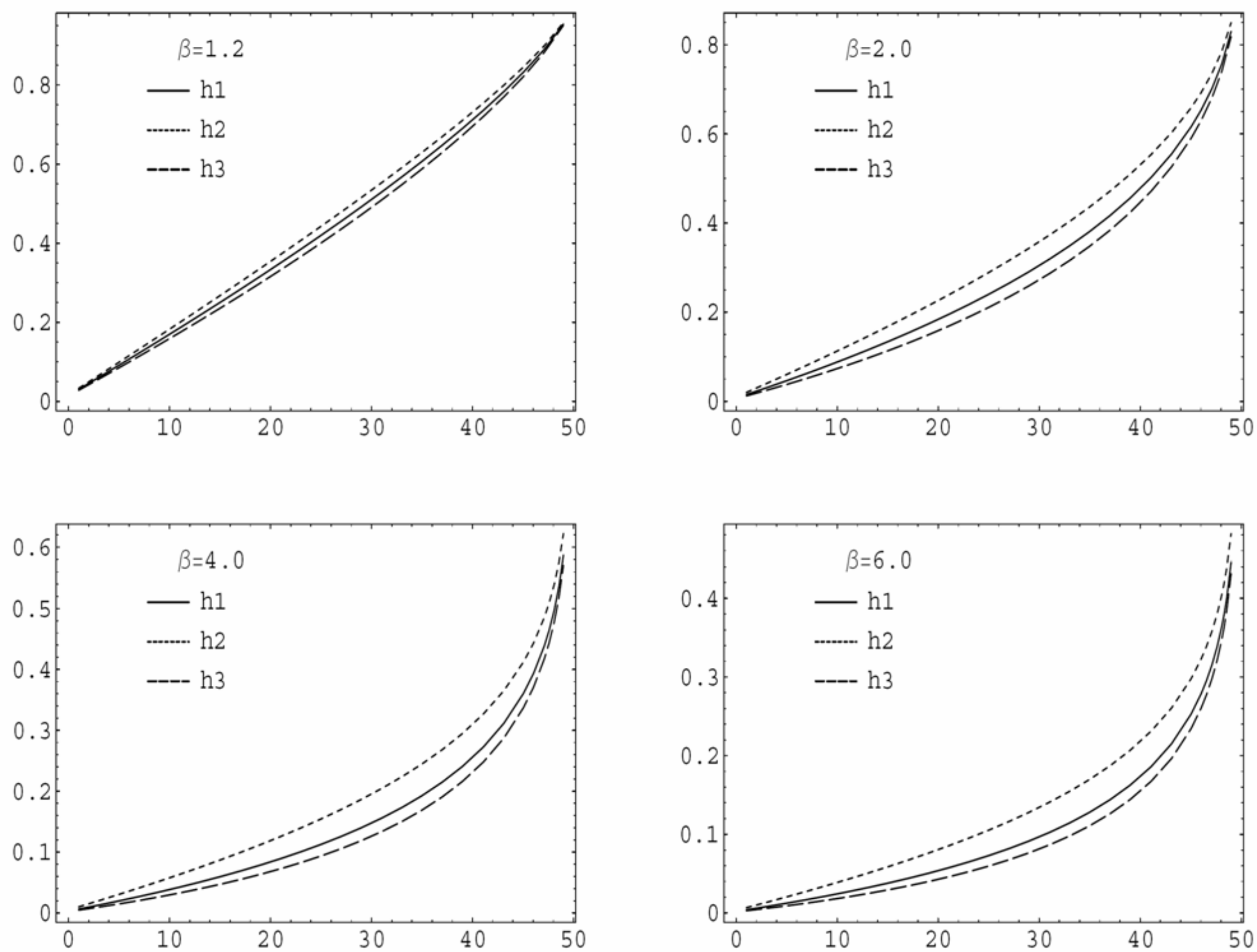

Figure 7: Plots of the three h functions defined in Lemmas 1, 5 and 6: $n = 50$