Online Supplement to 'Deviance Information Criterion for Model Selection: Justification and Variation'*

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1 High Order Analytical Expansions

The purpose of this online supplement is to prove Lemma 3.2 in Li, et al (2017). Suppose Θ is a compact subset of R^P. For any $\theta \in \Theta$, $\{h_n(\theta):\ n = 1, 2, \ldots\}$ is a sequence of eight-times continuously differentiable functions of θ, having an interior global minimum $\left\{\widehat{\theta}_n : n = 1, 2, \ldots\right\}$; $b(\theta)$ is a six-times continuously differentiable real function of θ. For any function $f(\theta)$, let $\widehat{f}$ be the value of function f evaluated at $\widehat{\theta}_n$, i.e., $\widehat{f} := f\left(\widehat{\theta}_n\right)$. When there is no confusion, we write $h_n(\theta)$ as $h(\theta)$ or h_n or even h and $b(\theta)$ as b. We use $B_\delta(\theta)$ to denote the open ball of radius δ centered at θ. So $B_{\sqrt{n}\delta}(0)$ is an open ball of radius $\sqrt{n}\delta$ centered at the origin. For convenience of exposition, we write $\frac{\partial^d}{\partial\theta_{j_1}\partial\theta_{j_2}\cdots\partial\theta_{j_d}} f(\theta)$ as $f_{j_1\cdots j_d}$. The Hessian of h_n at θ is denoted by $\nabla^2 h_n(\theta)$, and its (i,j)-component is written as h_{ij} while the components of its inverse is written as h^{ij}. Let μ^4_{ijkq}, μ^6_{ijkqrs}, $\mu^8_{ijkqrstw}$, $\mu^{10}_{ijkqrstwv\beta}$, $\mu^{12}_{ijkqrstwv\beta\tau\phi}$ be the fourth, sixth, eighth, tenth, and twelfth central moments of a multivariate Normal distribution whose covariance matrix is $\left(\nabla^2\widehat{h}\right)^{-1} := \left(\nabla^2 h_n(\theta)\right)^{-1}|_{\theta=\widehat{\theta}_n}$. Note that we require $h_n(\theta)$ be eight-times continuously differentiable and $b(\theta)$ be six-times continuously differentiable. These two conditions are stronger than what have typically been assumed in the literature on the Laplace approximation as we would like to develop higher order expansions.

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Following Kass et al (1990), we call the pair $(\{h_n\}, b)$ satisfy the analytical assumptions for Laplace's method if the following assumptions are met. There exists positive numbers ε, M and η, and an integer n_0 such that $n \geq n_0$ implies (i) for all $\theta \in B_\varepsilon\left(\widehat{\theta}_n\right)$ and all $1 \leq j_1, \cdots, j_d \leq P$ with $0 \leq d \leq 8$, $\|h_n(\theta)\| < M$ and $\|h_{j_1 \cdots j_d}(\theta)\| < M$; (ii) $\nabla^2 \widehat{h}$ is positive definite and $\det\left(\nabla^2 \widehat{h}\right) > \eta$; (iii) $\int_\Theta b(\theta) \exp\left[-nh(\theta)\right] d\theta$ exists and is finite, and for all δ for which $0 < \delta < \varepsilon$ and $B_\delta\left(\widehat{\theta}_n\right) \subseteq \Theta$,

$$\left[\det\left(n\nabla^2 \widehat{h}\right)\right]^{\frac{1}{2}} \int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b(\theta) \exp\left[-nh_n(\theta) - n\widehat{h}\right] d\theta = O\left(n^{-3}\right).$$

If one sets $-nh_n$ to be the sequence of log-likelihood functions of a model (as a sequence of sample size n), we say the model is Laplace regular. Lemma 1.1 below and Lemma 2.1 in the next section extend Theorem 1 and Theorem 5 of Kass et al (1990) to a higher order. They will be used to prove Lemma 3.2 in Li, et al (2017).

Lemma 1.1 *If $(\{h_n\}, b)$ satisfy the analytical assumptions for Laplace's method, then*

$$\int_\Theta b(\theta) \exp\left[-nh(\theta)\right] d\theta = (2\pi)^{\frac{P}{2}} \left[\det\left(n\nabla^2 \widehat{h}\right)\right]^{-\frac{1}{2}} \exp\left[-n\widehat{h}\right] \left(\widehat{b} + \frac{1}{n} A_1 + \frac{1}{n^2} A_2 + O\left(n^{-3}\right)\right),$$

where

$$A_1 = -\frac{1}{24} \sum_{ijkq} \widehat{h}_{ijkq} \mu^4_{ijkq} \widehat{b} + \frac{1}{72} \sum_{ijkqrs} \widehat{h}_{ijk} \widehat{h}_{qrs} \mu^6_{ijkqrs} \widehat{b} - \frac{1}{6} \sum_{ijk\zeta} \widehat{h}_{ijk} \mu^4_{ijk\zeta} \widehat{b}_\zeta + \frac{1}{2} \sum_{\zeta\eta} \widehat{b}_{\zeta\eta} \widehat{h}^{\zeta\eta},$$

$$\begin{aligned}
A_2 \;=\; & -\frac{1}{720} \sum_{ijkqrs} \widehat{h}_{ijkqrs} \mu^6_{ijkqrs} \widehat{b} + \frac{1}{1152} \sum_{ijkqrstw} \widehat{h}_{ijkq} \widehat{h}_{rstw} \mu^8_{ijkqrstw} \widehat{b} \\
& + \frac{1}{720} \sum_{ijkqrstw} \widehat{h}_{ijk} \widehat{h}_{qrstw} \mu^8_{ijkqrstw} \widehat{b} - \frac{1}{1728} \sum_{ijkqrstwv\beta} \widehat{h}_{ijk} \widehat{h}_{qrs} \widehat{h}_{twv\beta} \mu^{10}_{ijkqrstwv\beta} \widehat{b} \\
& + \frac{1}{31104} \sum_{ijkqrstwv\beta\tau\phi} \widehat{h}_{ijk} \widehat{h}_{qrs} \widehat{h}_{twv} \widehat{h}_{\beta\tau\phi} \mu^{12}_{ijkqrstwv\beta\tau\phi} \widehat{b} - \frac{1}{120} \sum_{ijkqr\zeta} \widehat{h}_{ijkqr} \mu^6_{ijkqr\zeta} \widehat{b}_\zeta \\
& + \frac{1}{144} \sum_{ijkqrst\zeta} \widehat{h}_{ijk} \widehat{h}_{qrst} \mu^8_{ijkqrst\zeta} \widehat{b}_\zeta - \frac{1}{1296} \sum_{ijkqrstwv\zeta} \widehat{h}_{ijk} \widehat{h}_{qrs} \widehat{h}_{twv} \mu^{10}_{ijkqrstwv\zeta} \widehat{b}_\zeta \\
& - \frac{1}{48} \sum_{ijkq\zeta\eta} \widehat{h}_{ijkq} \mu^6_{ijkq\zeta\eta} \widehat{b}_{\zeta\eta} + \frac{1}{144} \sum_{ijkqrs\zeta\eta} \widehat{h}_{ijk} \widehat{h}_{qrs} \mu^8_{ijkqrs\zeta\eta} \widehat{b}_{\zeta\eta} \\
& - \frac{1}{36} \sum_{ijk\zeta\eta\xi} \widehat{h}_{ijk} \mu^6_{ijk\zeta\eta\xi} \widehat{b}_{\zeta\eta\xi} + \frac{1}{24} \sum_{\zeta\eta\xi\omega} \widehat{b}_{\zeta\eta\xi\omega} \mu^4_{\zeta\eta\xi\omega}.
\end{aligned}$$

Proof. Note that

$$\int_\Theta b(\theta) \exp\left[-nh(\theta)\right] d\theta = \int_{B_\delta\left(\widehat{\theta}_n\right)} b(\theta) \exp\left[-nh(\theta)\right] d\theta + \int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b(\theta) \exp\left[-nh(\theta)\right] d\theta,$$

and

$$\begin{aligned}
&\int_{\Theta-B_\delta\left(\widehat{\theta}_n\right)} b(\theta)\exp\left[-nh(\theta)\right]d\theta \\
&= \left[\det\left(n\nabla^2\widehat{h}\right)\right]^{-\frac{1}{2}}\exp\left[-n\widehat{h}\right]\left[\det\left(n\nabla^2\widehat{h}\right)\right]^{\frac{1}{2}}\int_{\Theta-B_\delta\left(\widehat{\theta}_n\right)} b(\theta)\exp\left[-n\left(h(\theta)-\widehat{h}\right)\right]d\theta \\
&= \left[\det\left(n\nabla^2\widehat{h}\right)\right]^{-\frac{1}{2}}\exp\left[-n\widehat{h}\right]O\left(n^{-3}\right)
\end{aligned}$$

by assumption (iii) above.

Let $u=\sqrt{n}\left(\theta-\widehat{\theta}_n\right)$. Applying the Taylor expansion to $h(\theta)$ at $\widehat{\theta}_n$, we have

$$\begin{aligned}
nh(\theta) &= n\widehat{h}+\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j+\frac{1}{6}n^{-\frac{1}{2}}\sum_{ijk}\widehat{h}_{ijk}u_iu_ju_k+\frac{1}{24}n^{-1}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_q \\
&\quad+\frac{1}{120}n^{-\frac{3}{2}}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_r+\frac{1}{720}n^{-2}\sum_{ijkqrs}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_s \\
&\quad+\frac{1}{5040}n^{-\frac{5}{2}}\sum_{ijkqrst}\widehat{h}_{ijkqrst}u_iu_ju_ku_qu_ru_su_t+r_n(u),
\end{aligned}$$

where

$$r_n(u)=\frac{1}{40320}n^{-3}\sum_{ijkqrstw}h_{ijkqrstw}\left(\theta'\right)u_iu_ju_ku_qu_ru_su_tu_w,$$

and θ' lies between θ and $\widehat{\theta}_n$.

Define

$$\begin{aligned}
x &= \frac{1}{6}n^{-\frac{1}{2}}\sum_{ijk}\widehat{h}_{ijk}u_iu_ju_k+\frac{1}{24}n^{-1}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_q+\frac{1}{120}n^{-\frac{3}{2}}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_r \\
&\quad+\frac{1}{720}n^{-2}\sum_{ijkqrs}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_s+\frac{1}{5040}n^{-\frac{5}{2}}\sum_{ijkqrst}\widehat{h}_{ijkqrst}u_iu_ju_ku_qu_ru_su_t+r_n(u).
\end{aligned}$$

Applying the Taylor expansion to $\exp(-x)$ at the origin, we have

$$\begin{aligned}
\exp(-nh) &= \exp\left\{-n\widehat{h}\right\}\exp\left(-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right)\times \\
&\qquad\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5+R_{1,n}\left(\theta,\widehat{\theta}_n\right)\right),
\end{aligned}$$

where,

$$\begin{aligned}
\Xi_1 &= -\frac{1}{6}n^{-\frac{1}{2}}\sum_{ijk}\widehat{h}_{ijk}u_iu_ju_k-\frac{1}{24}n^{-1}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_q-\frac{1}{120}n^{-\frac{3}{2}}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_r \\
&\quad-\frac{1}{720}n^{-2}\sum_{ijkqrs}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_s-\frac{1}{5040}n^{-\frac{5}{2}}\sum_{ijkqrst}\widehat{h}_{ijkqrst}u_iu_ju_ku_qu_ru_su_t,
\end{aligned}$$

$$
\begin{aligned}
\Xi_2 \quad = \quad & \frac{1}{36} n^{-1} \sum_{ijkqrs} \widehat{h}_{ijk} \widehat{h}_{qrs} u_i u_j u_k u_q u_r u_s + \frac{1}{24^2} n^{-2} \sum_{ijkqrstw} \widehat{h}_{ijkq} \widehat{h}_{rstw} u_i u_j u_k u_q u_r u_s u_t u_w \\
& + \frac{1}{72} n^{-\frac{3}{2}} \sum_{ijkqrst} \widehat{h}_{ijk} \widehat{h}_{qrst} u_i u_j u_k u_q u_r u_s u_t + \frac{1}{360} n^{-2} \sum_{ijkqrstw} \widehat{h}_{ijk} \widehat{h}_{qrstw} u_i u_j u_k u_q u_r u_s u_t u_w \\
& + \frac{1}{1440} n^{-\frac{5}{2}} \sum_{ijkqrstwv} \widehat{h}_{ijk} \widehat{h}_{qrstwv} u_i u_j u_k u_q u_r u_s u_t u_w u_v \\
& + \frac{1}{2160} n^{-\frac{5}{2}} \sum_{ijkqrstwv} \widehat{h}_{ijk} \widehat{h}_{qrstwv} u_i u_j u_k u_q u_r u_s u_t u_w u_v ,
\end{aligned}
$$

$$
\begin{aligned}
\Xi_3 \quad = \quad & -\frac{1}{216} n^{-\frac{3}{2}} \sum_{ijkqrstwv} h_{ijk} h_{qrs} h_{twv} u_i u_j u_k u_q u_r u_s u_t u_w u_v \\
& - \frac{1}{288} n^{-2} \sum_{ijkqrstwv\beta} h_{ijk} h_{qrs} h_{twv\beta} u_i u_j u_k u_q u_r u_s u_t u_w u_v u_\beta \\
& - \frac{1}{1152} n^{-\frac{5}{2}} \sum_{ijkqrstwv\beta\tau} h_{ijk} h_{qrst} h_{wv\beta\tau} u_i u_j u_k u_q u_r u_s u_t u_w u_v u_\beta u_\tau \\
& - \frac{1}{1440} n^{-\frac{5}{2}} \sum_{ijkqrstwv\beta\tau} h_{ijk} h_{qrs} h_{twv\beta\tau} u_i u_j u_k u_q u_r u_s u_t u_w u_v u_\beta u_\tau ,
\end{aligned}
$$

$$
\begin{aligned}
\Xi_4 \quad = \quad & \frac{1}{1296} n^{-2} \sum_{ijkqrstwv\beta\tau\phi} \widehat{h}_{ijk} \widehat{h}_{qrs} \widehat{h}_{twv} \widehat{h}_{\beta\tau\phi} u_i u_j u_k u_q u_r u_s u_t u_w u_v u_\beta u_\tau u_\phi \\
& + \frac{1}{1296} n^{-\frac{5}{2}} \sum_{ijkqrstwv\beta\tau\phi\alpha} \widehat{h}_{ijk} \widehat{h}_{qrs} \widehat{h}_{twv} \widehat{h}_{\beta\tau\phi\alpha} u_i u_j u_k u_q u_r u_s u_t u_w u_v u_\beta u_\tau u_\phi u_\alpha ,
\end{aligned}
$$

$$
\Xi_5 = -\frac{1}{7776} n^{-\frac{5}{2}} \sum_{ijkqrstwv\beta\tau\phi\alpha\varkappa\varrho} \widehat{h}_{ijk} \widehat{h}_{qrs} \widehat{h}_{twv} \widehat{h}_{\beta\tau\phi} \widehat{h}_{\alpha\varkappa\varrho} u_i u_j u_k u_q u_r u_s u_t u_w u_v u_\beta u_\tau u_\phi u_\alpha u_\varkappa u_\varrho .
$$

Taylor-expanding $b(\theta)$ at $\widehat{\theta}_n$, we have

$$
\begin{aligned}
b(\theta) \quad = \quad & \widehat{b} + n^{-\frac{1}{2}} \sum_i \widehat{b}_i u_i + \frac{1}{2} n^{-1} \sum_{ij} \widehat{b}_{ij} u_i u_j + \frac{1}{6} n^{-\frac{3}{2}} \sum_{ijk} \widehat{b}_{ijk} u_i u_j u_k \\
& + \frac{1}{24} n^{-2} \sum_{ijkq} \widehat{b}_{ijkq} u_i u_j u_k u_q + \frac{1}{120} n^{-\frac{5}{2}} \sum_{ijkqr} \widehat{b}_{ijkqr} u_i u_j u_k u_q u_r + R_{2,n}\left(\theta, \widehat{\theta}_n\right) .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
b(\theta) \exp\left[-nh(\theta)\right] \quad = \quad & \left\{ \exp\left[-n\widehat{h}\right] \right\} \exp\left[-\frac{1}{2} \sum_{ij} \widehat{h}_{ij} u_i u_j \right] \left\{ I_n\left(\theta, \widehat{\theta}_n\right) + R_n\left(\theta, \widehat{\theta}_n\right) \right\} \\
= \quad & \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|\nabla^2 \widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\hat{h}\right) \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}} \left|\nabla^2 \widehat{h}\right|^{\frac{1}{2}} \exp\left[-\frac{1}{2} \sum_{ij} \widehat{h}_{ij} u_i u_j \right] \\
& \times \left\{ I_n\left(\theta, \widehat{\theta}_n\right) + R_n\left(\theta, \widehat{\theta}_n\right) \right\} ,
\end{aligned}
$$

where $I_n\left(\theta,\widehat{\theta}_n\right)$ and $R_n\left(\theta,\widehat{\theta}_n\right)$ will be specified below. Thus,

$$\begin{aligned}
&\int_{B_\delta\left(\widehat{\theta}_n\right)} b\left(\theta\right)\exp\left[-nh\left(\theta\right)\right]d\theta \\
=\ & \left(\sqrt{2\pi}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{-\frac{1}{2}}\exp\left(-n\widehat{h}\right)\times \\
&\int_{B_\delta\left(\widehat{\theta}_n\right)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\left\{I_n\left(\theta,\widehat{\theta}_n\right)+R_n\left(\theta,\widehat{\theta}_n\right)\right\}d\theta \\
=\ & \left(\sqrt{2\pi}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{-\frac{1}{2}}\exp\left(-n\widehat{h}\right)\left(\sqrt{n}\right)^{-\frac{P}{2}}\times \\
&\int_{B_{\sqrt{n}\delta}(0)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\left\{I_n\left(\theta,\widehat{\theta}_n\right)+R_n\left(\theta,\widehat{\theta}_n\right)\right\}du \\
=\ & \left(\sqrt{2\pi}\right)^{\frac{P}{2}}\left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}}\exp\left(-n\widehat{h}\right)\times \\
&\int_{B_{\sqrt{n}\delta}(0)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\left\{\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\right\}\left\{I_n\left(\theta,\widehat{\theta}_n\right)+R_n\left(\theta,\widehat{\theta}_n\right)\right\}du,
\end{aligned}$$

where $R_n\left(\theta,\widehat{\theta}_n\right)$ is the sum of terms involving $R_{1,n}\left(\theta,\widehat{\theta}_n\right)$, $R_{2,n}\left(\theta,\widehat{\theta}_n\right)$ and terms of order equal to or smaller than $O(n^{-3})$. Furthermore, we can get

$$R_{2,n}\left(\theta,\widehat{\theta}_n\right)=\frac{1}{720}n^{-3}\sum b_{ijkqrs}\left(\widetilde{\theta}\right)u_iu_ju_ku_qu_ru_s,$$

where $\widetilde{\theta}$ lies between θ and $\widehat{\theta}_n$. So the leading term of $R_n\left(\theta,\widehat{\theta}_n\right)$ are $\widehat{b}R_{1n}\left(\theta,\widehat{\theta}_n\right)+R_{2n}\left(\theta,\widehat{\theta}_n\right)$ which include $r_n\left(u\right)$. The integral of $r_n\left(u\right)$ over $B_\varepsilon\left(\widehat{\theta}_n\right)$ can be expressed as

$$\begin{aligned}
&\left|n^{-3}\int_{B_{\sqrt{n}\delta}(0)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\sum_{ijkqrstw}h_{ijkqrstw}\left(\theta'\right)u_iu_ju_ku_qu_ru_su_tu_wdu\right| \\
\leq\ & n^{-3}\int_{B_{\sqrt{n}\delta}(0)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\sum_{ijkqrstw}\left|h_{ijkqrstw}\left(\theta'\right)\right|\left|u_iu_ju_ku_qu_ru_su_tu_w\right|du \\
\leq\ & n^{-3}M\int_{R^P}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\sum_{ijkqrstw}\left|u_iu_ju_ku_qu_ru_su_tu_w\right|du \\
=\ & O\left(n^{-3}\right),
\end{aligned}$$

where $\int_{R^P}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\sum_{ijkqrstw}\left|u_iu_ju_ku_qu_ru_su_tu_w\right|du$ is the eighth order moment of folded multivariate folded normal distribution with mean 0 and covariance $\left(\nabla^2\widehat{h}\right)^{-1}$ which is finite; see Kamat (1953) and Kan and Robotti (2017). Then we have

$$\left(\sqrt{2\pi}\right)^{\frac{P}{2}}\left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}}\exp\left(-n\widehat{h}\right)\int_{B_{\sqrt{n}\delta}(0)}\left[I_n\left(\theta,\widehat{\theta}\right)+R_n\left(\theta,\widehat{\theta}\right)\right]f\left(u\right)du$$

$$= \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right) \left[\int_{B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\, du + O\left(n^{-3}\right)\right].$$

For $I_n\left(\theta,\widehat{\theta}_n\right)$ we have

$$I_n\left(\theta,\widehat{\theta}_n\right) = I_n^0\left(\theta,\widehat{\theta}_n\right) + I_n^1\left(\theta,\widehat{\theta}_n\right) + I_n^2\left(\theta,\widehat{\theta}_n\right) + I_n^3\left(\theta,\widehat{\theta}_n\right) + I_n^4\left(\theta,\widehat{\theta}_n\right) + I_n^5\left(\theta,\widehat{\theta}_n\right),$$

where

$$\begin{aligned} I_n^0\left(\theta,\widehat{\theta}_n\right) &= \widehat{b}\left(1 + \Xi_1 + \frac{1}{2}\Xi_2 + \frac{1}{6}\Xi_3 + \frac{1}{24}\Xi_4 + \frac{1}{120}\Xi_5\right) \\ &= \widehat{b}\left\{1 + n^{-\frac{1}{2}} I_n^{01} + n^{-1} I_n^{02} + n^{-\frac{3}{2}} I_n^{03} + n^{-2} I_n^{04} + n^{-\frac{5}{2}} I_n^{05}\right\}, \end{aligned}$$

$$I_n^{01} = -\frac{1}{6}\sum_{ijk}\widehat{h}_{ijk}u_iu_ju_k, \quad I_n^{02} = -\frac{1}{24}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_q + \frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}u_iu_ju_ku_qu_ru_s,$$

$$\begin{aligned} I_n^{03} &= -\frac{1}{120}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_r + \frac{1}{144}\sum_{ijkqrst}\widehat{h}_{ijk}\widehat{h}_{qrst}u_iu_ju_ku_qu_ru_su_t \\ &\quad -\frac{1}{1296}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}u_iu_ju_ku_qu_ru_su_tu_wu_v, \end{aligned}$$

$$\begin{aligned} I_n^{04} &= -\frac{1}{720}\sum_{ijkqrs}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_s + \frac{1}{1152}\sum_{ijkqrstw}\widehat{h}_{ijkq}\widehat{h}_{rstw}u_iu_ju_ku_qu_ru_su_tu_w \\ &\quad +\frac{1}{720}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrstw}u_iu_ju_ku_qu_ru_su_tu_w \\ &\quad -\frac{1}{1728}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv\beta}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta \\ &\quad +\frac{1}{31104}\sum_{ijkqrstwv\beta\tau\phi}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi, \end{aligned}$$

$$\begin{aligned} I_n^{05} &= -\frac{1}{5040}\sum_{ijkqrst}\widehat{h}_{ijkqrst}u_iu_ju_ku_qu_ru_su_t + \frac{1}{2880}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrstwv}u_iu_ju_ku_qu_ru_su_tu_wu_v \\ &\quad +\frac{1}{5120}n^{-\frac{5}{2}}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrstwv}u_iu_ju_ku_qu_ru_su_tu_wu_v \\ &\quad -\frac{1}{6912}\sum_{ijkqrstwv\beta\tau}h_{ijk}h_{qrst}h_{wv\beta\tau}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau \\ &\quad -\frac{1}{8640}\sum_{ijkqrstwv\beta\tau}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv\beta\alpha}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau \\ &\quad +\frac{1}{31104}\sum_{ijkqrstwv\beta\tau\phi\alpha}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi\alpha}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi u_\alpha \end{aligned}$$

$$-\frac{1}{933120}\sum_{ijkqrstwv\beta\tau\phi\alpha\varkappa\varrho}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}\widehat{h}_{\alpha\varkappa\varrho}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi u_\alpha u_\varkappa u_\varrho,$$

$$\begin{aligned}
I_n^1\left(\theta,\widehat{\theta}_n\right) &= n^{-\frac{1}{2}}\sum\widehat{b}_\zeta u_\zeta\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\
&= n^{-\frac{1}{2}}\sum\widehat{b}_\zeta u_\zeta+n^{-1}I_n^{11}\left(\theta,\widehat{\theta}_n\right)+n^{-\frac{3}{2}}I_n^{12}\left(\theta,\widehat{\theta}_n\right)+n^{-2}I_n^{13}\left(\theta,\widehat{\theta}_n\right)\\
&\quad +n^{-\frac{5}{2}}I_n^{14}\left(\theta,\widehat{\theta}_n\right),
\end{aligned}$$

$$\begin{aligned}
I_n^{11}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{6}\sum_{ijk\zeta}\widehat{h}_{ijk}u_iu_ju_ku_\zeta\widehat{b}_\zeta,\\
I_n^{12}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{24}\sum_{ijkq\zeta}\widehat{h}_{ijkq}u_iu_ju_ku_qu_\zeta\widehat{b}_\zeta+\frac{1}{72}\sum_{ijkqrs\zeta}\widehat{h}_{ijk}\widehat{h}_{qrs}u_iu_ju_ku_qu_ru_su_\zeta\widehat{b}_\zeta,
\end{aligned}$$

$$\begin{aligned}
I_n^{13}\left(\theta,\widehat{\theta}\right) &= -\frac{1}{120}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_ru_\zeta\widehat{b}_\zeta+\frac{1}{144}\sum_{ijkqrst}\widehat{h}_{ijk}\widehat{h}_{qrst}u_iu_ju_ku_qu_ru_su_tu_\zeta\widehat{b}_\zeta\\
&\quad -\frac{1}{1296}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\zeta\widehat{b}_\zeta,
\end{aligned}$$

$$\begin{aligned}
I_n^{14}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{720}\sum_{ijkqrs\zeta}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_su_\zeta\widehat{b}_\zeta+\frac{1}{1152}\sum_{ijkqrstw\zeta}\widehat{h}_{ijkq}\widehat{h}_{rstw}u_iu_ju_ku_qu_ru_su_tu_wu_\zeta\widehat{b}_\zeta\\
&\quad +\frac{1}{720}\sum_{ijkqrstw\zeta}\widehat{h}_{ijk}\widehat{h}_{qrstw}u_iu_ju_ku_qu_ru_su_tu_wu_\zeta\widehat{b}_\zeta\\
&\quad -\frac{1}{1728}\sum_{ijkqrstwv\beta\zeta ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv\beta}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\zeta\widehat{b}_\zeta\\
&\quad +\frac{1}{31104}\sum_{ijkqrstwv\beta\tau\phi\zeta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi u_\zeta\widehat{b}_\zeta,
\end{aligned}$$

$$\begin{aligned}
I_n^2\left(\theta,\widehat{\theta}_n\right) &= n^{-1}\frac{1}{2}\sum\widehat{b}_{\zeta\eta}u_\zeta u_\eta\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\
&= \frac{1}{2}\left[n^{-1}\sum\widehat{b}_{\zeta\eta}u_\zeta u_\eta+n^{-\frac{3}{2}}I_n^{21}\left(\theta,\widehat{\theta}_n\right)+n^{-2}I_n^{22}\left(\theta,\widehat{\theta}_n\right)+n^{-\frac{5}{2}}I_n^{23}\left(\theta,\widehat{\theta}_n\right)\right],
\end{aligned}$$

$$\begin{aligned}
I_n^{21}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{6}\sum_{ijk\zeta\eta}\widehat{h}_{ijk}u_iu_ju_ku_\zeta u_\eta\widehat{b}_{\zeta\eta},\\
I_n^{22}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{24}\sum_{ijkq\zeta\eta}\widehat{h}_{ijkq}u_iu_ju_ku_qu_\zeta u_\eta\widehat{b}_{\zeta\eta}+\frac{1}{72}\sum_{ijkqrs\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrs}u_iu_ju_ku_qu_ru_su_\zeta u_\eta\widehat{b}_{\zeta\eta},
\end{aligned}$$

$$I_n^{23}\left(\theta,\widehat{\theta}_n\right) = -\frac{1}{120}\sum_{ijkqr\zeta\eta}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_ru_\zeta u_\eta\widehat{b}_{\zeta\eta}+\frac{1}{144}\sum_{ijkqrst\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrst}u_iu_ju_ku_qu_ru_su_tu_\zeta u_\eta\widehat{b}_{\zeta\eta}$$

$$-\frac{1}{1296}\sum_{ijkqrstwv\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\zeta u_\eta\widehat{b}_{\zeta\eta},$$

$$\begin{aligned}I_n^3\left(\theta,\widehat{\theta}_n\right) &= n^{-\frac{3}{2}}\frac{1}{6}\sum\widehat{b}_{\zeta\eta\xi}u_\zeta u_\eta u_\xi\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\ &= \frac{1}{6}\left[n^{-\frac{3}{2}}\sum\widehat{b}_{\zeta\eta\xi}u_\zeta u_\eta u_\xi+n^{-2}I_n^{31}\left(\theta,\widehat{\theta}_n\right)+n^{-\frac{5}{2}}I_n^{32}\left(\theta,\widehat{\theta}_n\right)\right],\end{aligned}$$

$$\begin{aligned}I_n^{31}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{6}\sum_{ijk\zeta\eta}\widehat{h}_{ijk}u_iu_ju_ku_\zeta u_\eta u_\xi\widehat{b}_{\zeta\eta\xi},\\ I_n^{32}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{24}\sum_{ijkq\zeta\eta}\widehat{h}_{ijkq}u_iu_ju_ku_qu_\zeta u_\eta u_\xi\widehat{b}_{\zeta\eta\xi}+\frac{1}{72}\sum_{ijkqrs\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrs}u_iu_ju_ku_qu_ru_su_\zeta u_\eta u_\xi\widehat{b}_{\zeta\eta\xi},\end{aligned}$$

$$\begin{aligned}&I_n^4\left(\theta,\widehat{\theta}_n\right)\\ &= n^{-2}\frac{1}{24}\sum\widehat{b}_{\zeta\eta\xi\omega}u_\zeta u_\eta u_\xi u_\omega\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\ &= \frac{1}{24}\left[n^{-2}\sum\widehat{b}_{\zeta\eta\xi\omega}u_\zeta u_\eta u_\xi u_\omega+n^{-\frac{5}{2}}I_n^{41}\left(\theta,\widehat{\theta}_n\right)\right],\end{aligned}$$

$$I_n^{41}\left(\theta,\widehat{\theta}_n\right)=-\frac{1}{6}\sum_{ijk\zeta\eta}\widehat{h}_{ijk}u_iu_ju_ku_\zeta u_\eta u_\xi u_\omega\widehat{b}_{\zeta\eta\xi\omega},$$

$$\begin{aligned}&I_n^5\left(\theta,\widehat{\theta}_n\right)\\ &= n^{-\frac{5}{2}}\frac{1}{120}\sum_{ijkqr}\widehat{b}_{ijkqr}u_iu_ju_ku_qu_r\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\ &= \frac{1}{120}\left[n^{-\frac{5}{2}}\sum_{ijkqr}\widehat{b}_{ijkqr}u_iu_ju_ku_qu_r\right].\end{aligned}$$

Let

$$f(u)=\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\left\{\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\right\},$$

be the density function of the Normal distribution with mean 0 and covariance matrix $\left(\nabla^2\widehat{h}\right)^{-1}$. Then we have

$$\begin{aligned}&\int_{R^p}I_n\left(\theta,\widehat{\theta}_n\right)f(u)\,du\\ &= \widehat{b}+\frac{1}{n}\left(-\frac{1}{24}\sum_{ijkq}\widehat{h}_{ijkq}\mu^4_{ijkq}+\frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs}-\frac{1}{6}\sum_{ijk\zeta}\widehat{h}_{ijk}\mu^4_{ijk\zeta}\widehat{b}_\zeta+\frac{1}{2}\sum_{\zeta\eta}\widehat{b}_{\zeta\eta}\widehat{h}^{\zeta\eta}\right)\end{aligned}$$

$$
+\frac{1}{n^2}\left(\begin{array}{c}
-\frac{1}{720}\sum_{ijkqrs}\widehat{h}_{ijkqrs}\mu^6_{ijkqrs}+\frac{1}{1152}\sum_{ijkqrstw}\widehat{h}_{ijkq}\widehat{h}_{rstw}\mu^8_{ijkqrstw}\\
+\frac{1}{720}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrstw}\mu^8_{ijkqrstw}-\frac{1}{1728}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv\beta}\mu^{10}_{ijkqrstwv\beta}\\
+\frac{1}{31104}\sum_{ijkqrstwv\beta\tau\phi}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}\mu^{12}_{ijkqrstwv\beta\tau\phi}\\
-\frac{1}{120}\sum_{ijkqr\zeta}\widehat{h}_{ijkqr}\mu^6_{ijkqr\zeta}\widehat{b}_{\zeta}+\frac{1}{144}\sum_{ijkqrst\zeta}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^8_{ijkqrst\zeta}\widehat{b}_{\zeta}\\
-\frac{1}{1296}\sum_{ijkqrstwv\zeta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\zeta}\widehat{b}_{\zeta}\\
-\frac{1}{48}\sum_{ijkq\zeta\eta}\widehat{h}_{ijkq}\mu^6_{ijkq\zeta\eta}\widehat{b}_{\zeta\eta}+\frac{1}{144}\sum_{ijkqrs\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrs\zeta\eta}\widehat{b}_{\zeta\eta}\\
-\frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{\zeta\eta\xi}+\frac{1}{24}\sum_{\zeta\eta\xi\omega}\widehat{b}_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}
\end{array}\right)
+O\left(n^{-3}\right).
$$

Note that the odd order central moments of the multivariate normal distribution vanish and, by expanding the domain from $B_{\sqrt{n}\delta}(0)$ to R^P, the error can be expressed as

$$
\begin{aligned}
&\int_{R^p-B_{\sqrt{n}\delta}(0)}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_qf(u)\,du\\
=\ &\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^p-B_{\sqrt{n}\delta}(0)}u_iu_ju_ku_qf(u)\,du\\
=\ &\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^P-B_{\sqrt{n}\delta}(0)}u_iu_ju_ku_q\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]du\\
=\ &\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^P-B_{\sqrt{n}\delta}(0)}u_iu_ju_ku_q\exp\left[-\frac{1}{4}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\exp\left[-\frac{1}{4}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]du\\
\leq\ &\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^p-B_{\sqrt{n}\delta}(0)}u_iu_ju_ku_q\exp\left[-\frac{1}{4}\sum_{ij}\lambda_{\min}u_iu_j\right]\exp\left[-\frac{1}{4}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]du\\
\leq\ &\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{4}\lambda_{\min}\delta^2n\right]\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^p}u_iu_ju_ku_q\exp\left[-\frac{1}{2}\sum_{ij}\left(\frac{1}{2}\widehat{h}_{ij}\right)u_iu_j\right]du\\
=\ &M'\exp\left[-\frac{1}{4}\lambda_{\min}\delta^2n\right],
\end{aligned}
$$

where $\lambda_{\min}>0$ is the smallest eigenvalue of $\nabla^2\widehat{h}$ and

$$
M'=\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^p}u_iu_ju_ku_q\exp\left[-\sum_{ij}\left(\frac{1}{4}\widehat{h}_{ij}\right)u_iu_j\right]du<\infty,
$$

by the imposed assumptions. The first inequality follows from the fact that

$$
\lambda_{\min}=\min_{\|e\|=1}f(e)=e^TAe,
$$

where A is a positive definite matrix, and $\lambda_{\min}$ is the the smallest eigenvalue of A. Here, we only express one term of $I_n\left(\theta,\widehat{\theta}_n\right)$. The other terms can be analyzed similarly.

Hence, we have

$$\int_{B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du = \int_{R^P} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du - \int_{R^p - B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du,$$

and

$$\begin{aligned}
& \int_{\Theta} b(\theta)\exp\left[-nh(\theta)\right] d\theta \\
= \; & \int_{B_\delta\left(\widehat{\theta}_n\right)} b(\theta)\exp\left[-nh(\theta)\right] d\theta + \int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b(\theta)\exp\left[-nh(\theta)\right] d\theta \\
= \; & \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right)\left[\int_{B_{\sqrt{n}\delta}(0)} \left[I_n\left(\theta,\widehat{\theta}\right) + R_n\left(\theta,\widehat{\theta}\right)\right] f(u)\,du + O\left(n^{-3}\right)\right] \\
= \; & \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right)\left[\int_{B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du + O\left(n^{-3}\right)\right] \\
= \; & \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right)\left[\begin{array}{c} \int_{R^P} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du \\ -\int_{R^P - B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du \end{array} + O\left(n^{-3}\right)\right] \\
= \; & \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right)\left[\int_{R^P} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du + O\left(n^{-3}\right)\right].
\end{aligned}$$

Hence, this lemma is proved. ■

2 High Order Stochastic Expansions

In this section we will develop high order stochastic Laplace expansions. Suppose $\mathbf{y} = (y_1, y_2, \cdots, y_n)'$ is a collection of random variables defined on a common probability space $\{\Omega, \mathcal{F}, \wp_\theta\}$, where Ω is a sample space, $\mathcal{F}$ is a sigma-algebra, and $\wp_\theta$ is a probability measure that depends on parameter $\theta \in \Theta$, a compact subset of R^P. Assume $\{y_i, i = 1, 2, \cdots\}$ take values in the same mathematical space $\Im$, which must be measurable with respect to some sigma-algebra, Σ. Let $h_n(\mathbf{y},\theta)$ be a sequence of functions, each of which is eight-times continuously differentiable with respect to θ and has an interior global minimum $\left\{\widehat{\theta}_n : n = 1, 2, \ldots\right\}$; $b(\theta)$ is a six-times continuously differentiable real function of θ. When there is no confusion, we write $h_n(\mathbf{y},\theta)$ as $h(\theta)$ or h_n or even h and $b(\theta)$ as b. For any function $f(\theta)$, let $\widehat{f}$ be the value of function f evaluated at $\widehat{\theta}_n$, i.e., $\widehat{f} := f\left(\widehat{\theta}_n\right)$.

We call the pair $(\{h_n\}, b)$ satisfy the analytical assumptions for the stochastic Laplace method on $\wp_\theta$ if the following assumptions are satisfied. There exists positive numbers ε, M and η such that (i) with probability approach one (w.p.a.1), for all $\theta \in B_\varepsilon\left(\widehat{\theta}_n\right)$ and all $1 \le j_1, \cdots, j_d \le P$ with $0 \le d \le 8$, $\|h_n(\theta)\| < M$ and $\|h_{j_1 \cdots j_d}(\theta)\| < M$; (ii) w.p.a.1, $\nabla^2\widehat{h}$ is positive definite and $\det\left(\nabla^2\widehat{h}\right) > \eta$; (iii) $\int_\Theta b(\theta)\exp\left(-nh_n(\theta)\right) d\theta$ exists and is finite, and for all δ for which $0 < \delta < \varepsilon$ and $B_\delta\left(\widehat{\theta}_n\right) \subseteq \Theta$,

$$\left[\det\left(n\nabla^2\widehat{h}\right)\right]^{\frac{1}{2}} \int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b(\theta)\exp\left[-n\left(h_n(\theta) - \widehat{h}\right)\right] d\theta = O_p\left(n^{-3}\right).$$

Note that our assumptions are different from those in Section 3 of Kass et al (1990) in two aspects. First, we require $h_n(\theta)$ be eight-times continuously differentiable and $b(\theta)$ be six-times continuously differentiable. Second, for conditions (ii) and (iii), instead of almost sure boundedness and almost sure convergence, we assume they hold w.p.a.1. We do so because we are interested in convergence in probability only. Following the result in Theorem 7 of Kass et al (1990), $(\{h_n\}, b)$ satisfy the analytical assumptions for stochastic Laplace's method on $\wp_\theta$ and Lemma 1.1 above, it is straightforward to show that

$$\int_\Theta b(\theta)\exp\left[-nh(\theta)\right] d\theta = (2\pi)^{\frac{P}{2}}\left[\det\left(n\nabla^2\widehat{h}\right)\right]^{-\frac{1}{2}}\exp\left[-n\widehat{h}\right]\left(\widehat{b} + \frac{1}{n}A_1 + \frac{1}{n^2}A_2 + O_p\left(n^{-3}\right)\right), \tag{1}$$

where the expressions for A_1 and A_2 are given in Lemma 1.1.

Lemma 2.1 *If both $(\{h_n\}, g \times b_D)$ and $(\{h_n\}, b_D)$ satisfy the analytical assumptions for the stochastic Laplace method on $\wp_\theta$, then*

$$\frac{\int g(\theta)\, b_D(\theta)\exp\left(-nh_n(\theta)\right) d\theta}{\int b_D(\theta)\exp\left(-nh_n(\theta)\right) d\theta} = \widehat{g} + \frac{1}{n}B_1 + \frac{1}{n^2}\left(B_2 - B_3\right) + O_p\left(\frac{1}{n^3}\right),$$

where

$$B_1 = \frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij} + \frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,j}\widehat{g}_i}{\widehat{b}_D} - \frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\widehat{g}_q,$$

$$\begin{aligned}
B_2 = & -\frac{1}{120}\sum_{ijkqrs}\widehat{h}_{ijkqr}\mu^6_{ijkqrs}\widehat{g}_s + \frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^8_{ijkqrstw}\widehat{g}_w \\
& -\frac{1}{1296}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{g}_\beta - \frac{1}{24}\frac{\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{b}_{D,s}\widehat{g}_r}{\widehat{b}_D} \\
& +\frac{1}{72}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{D,w}\widehat{g}_t}{\widehat{b}_D} - \frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{D,\eta\xi}\widehat{g}_\zeta}{\widehat{b}_D} \\
& +\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{D,\eta\xi\omega}\widehat{g}_\zeta}{\widehat{b}_D} - \frac{1}{48}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_{rs} \\
& +\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{g}_{tw} - \frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi} \\
& +\frac{1}{24}\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi\omega} - \frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D}
\end{aligned}$$

$$+\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi}\widehat{b}_{D,\omega}}{\widehat{b}_D}+\frac{1}{4}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D},$$

$$B_3=\left(\frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\frac{\widehat{b}_{D,ij}}{\widehat{b}_D}-\frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\frac{\widehat{b}_{D,q}}{\widehat{b}_D}+\frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs}-\frac{1}{24}\sum_{ijkq}\widehat{h}_{ijkq}\mu^4_{ijkq}\right)B_1,$$

where $\sigma_{ij}=h^{ij}$.

Proof. If $\left\{h^N(\theta),b_N\right\}$ and $\left\{h^D(\theta),b_D\right\}$ satisfy the analytical assumptions for the stochastic Laplace method on $\wp_\theta$, then by (1)

$$\frac{\int b_N(\theta)\exp\left[-nh^N(\theta)\right]d\theta}{\int b_D(\theta)\exp\left[-nh^D(\theta)\right]d\theta}=\frac{\left|\nabla^2\widehat{h}^N\right|^{-\frac{1}{2}}\exp\left[-nh\left(\widehat{\theta}^N_n\right)\right]}{\left|\nabla^2\widehat{h}^D\right|^{-\frac{1}{2}}\exp\left[-nh\left(\widehat{\theta}^D_n\right)\right]}\frac{b_k\left(\widehat{\theta}^N_n\right)+\frac{1}{n}c_N+\frac{1}{n^2}d_N+O_p\left(n^{-3}\right)}{b_k\left(\widehat{\theta}^D_n\right)+\frac{1}{n}c_D+\frac{1}{n^2}d_D+O_p\left(n^{-3}\right)}.$$

From Tierney and Kadane (1986) and Miyata (2004, 2010), we have

$$\frac{b_N\left(\widehat{\theta}^N_n\right)+\frac{1}{n}c_N+\frac{1}{n^2}d_N+O_p\left(n^{-3}\right)}{b_D\left(\widehat{\theta}^D_n\right)+\frac{1}{n}c_D+\frac{1}{n^2}d_D+O_p\left(n^{-3}\right)}=\frac{b_N\left(\widehat{\theta}^N_n\right)}{b_D\left(\widehat{\theta}^D_n\right)}\left[\frac{1+\frac{1}{n}\frac{c_N}{b_N\left(\widehat{\theta}^N_n\right)}+\frac{1}{n^2}\frac{d_N}{b_N\left(\widehat{\theta}^N_n\right)}+O_p\left(n^{-3}\right)}{1+\frac{1}{n}\frac{c_D}{b_D\left(\widehat{\theta}^D_n\right)}+\frac{1}{n^2}\frac{d_D}{b_D\left(\widehat{\theta}^D_n\right)}+O_p\left(n^{-3}\right)}\right]$$

$$=\frac{b_N\left(\widehat{\theta}^N_n\right)}{b_D\left(\widehat{\theta}^D_n\right)}\left\{\begin{array}{c}1+\frac{1}{n}\left(\frac{c_N}{b_N\left(\widehat{\theta}^N_n\right)}-\frac{c_D}{b_D\left(\widehat{\theta}^D_n\right)}\right)\\+\frac{1}{n^2}\left(\frac{d_N}{b_N\left(\widehat{\theta}^N_n\right)}-\frac{d_D}{b_D\left(\widehat{\theta}^D_n\right)}-\frac{c_D}{b_D\left(\widehat{\theta}^D_n\right)}\left(\frac{c_N}{b_N\left(\widehat{\theta}^N_n\right)}-\frac{c_D}{b_D\left(\widehat{\theta}^D_n\right)}\right)\right)+O_p\left(n^{-3}\right)\end{array}\right\},$$

where

$$\begin{aligned}\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{c_N}{\widehat{b}_N}-\frac{c_D}{\widehat{b}_D}\right)&=\frac{\widehat{b}_N\left(c_N\widehat{b}_D-c_D\widehat{b}_N\right)}{\widehat{b}^2_D\widehat{b}_N}=\frac{c_N\widehat{b}_D-c_D\widehat{b}_N}{\widehat{b}^2_D}\\&=\frac{1}{2}\frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{N,ij}\widehat{b}_D-\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,ij}\widehat{b}_N}{\widehat{b}^2_D}\\&\quad-\frac{1}{6}\frac{\sum_{ijkq}\widehat{h}^N_{ijk}\mu^4_{ijkq}\widehat{b}_{N,q}\widehat{b}_D-\sum_{ijkq}\widehat{h}^D_{ijk}\mu^4_{ijkq}\widehat{b}_{D,q}\widehat{b}_N}{\widehat{b}^2_D}\\&\quad+\frac{1}{72}\frac{\widehat{b}_N\sum_{ijkqrs}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\mu^6_{ijkqrs}-\sum_{ijkqrs}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\mu^6_{ijkqrs}\widehat{b}_N}{\widehat{b}_D}\\&\quad-\frac{1}{24}\frac{\widehat{b}_N\sum_{ijkq}\widehat{h}^N_{ijkq}\mu^4_{ijkq}-\sum_{ijkq}\widehat{h}^D_{ijkq}\mu^4_{ijkq}\widehat{b}_N}{\widehat{b}_D},\end{aligned}$$

where $\sigma_{ij}=h^{ij}$.

$$\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{d_N}{\widehat{b}_N}-\frac{d_D}{\widehat{b}_D}\right)$$

$$
\begin{aligned}
= \quad & -\frac{1}{720}\frac{\widehat{b}_N \sum_{ijkqrs}\widehat{h}^N_{ijkqrs}\mu^6_{ijkqrs} - \sum_{ijkqrs}\widehat{h}^D_{ijkqrs}\mu^6_{ijkqrs}\widehat{b}_N}{\widehat{b}_D} \\
& +\frac{1}{1152}\frac{\widehat{b}_N \sum_{ijkqrstw}\widehat{h}^N_{ijkq}\widehat{h}^N_{rstw}\mu^8_{ijkqrstw} - \sum_{ijkqrstw}\widehat{h}^D_{ijkq}\widehat{h}^D_{rstw}\mu^8_{ijkqrstw}\widehat{b}_N}{\widehat{b}_D} \\
& +\frac{1}{720}\frac{\widehat{b}_N \sum_{ijkqrstw}\widehat{h}^N_{ijk}\widehat{h}^N_{qrstw}\mu^8_{ijkqrstw} - \sum_{ijkqrstw}\widehat{h}^D_{ijk}\widehat{h}^D_{qrstw}\mu^8_{ijkqrstw}\widehat{b}_N}{\widehat{b}_D} \\
& -\frac{1}{1728}\frac{\widehat{b}_N \sum_{ijkqrstwv\beta}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\widehat{h}^N_{twv}\mu^{10}_{ijkqrstwv\beta} - \sum_{ijkqrstwv\beta}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\widehat{h}^D_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{b}_N}{\widehat{b}_D} \\
& +\frac{1}{31104}\frac{\left(\begin{array}{c}\widehat{b}_N \sum_{ijkqrstwv\beta\tau\phi}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}\mu^{12}_{ijkqrstwv\beta\tau\phi} \\ -\sum_{ijkqrstwv\beta\tau\phi}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}\mu^{12}_{ijkqrstwv\beta\tau\phi}\widehat{b}_N\end{array}\right)}{\widehat{b}_D} \\
& -\frac{1}{120}\frac{\sum_{ijkqrs}\widehat{h}^N_{ijkqr}\mu^6_{ijkqrs}\widehat{b}_{N,s}\widehat{b}_D - \sum_{ijkqrs}\widehat{h}^D_{ijkqr}\mu^6_{ijkqrs}\widehat{b}_{D,s}\widehat{b}_N}{\widehat{b}^2_D} \\
& +\frac{1}{144}\frac{\sum_{ijkqrstw}\widehat{h}^N_{ijk}\widehat{h}^N_{qrst}\mu^8_{ijkqrstw}\widehat{b}_{N,w}\widehat{b}_D - \sum_{ijkqrstw}\widehat{h}^N_{ijk}\widehat{h}^N_{qrst}\mu^8_{ijkqrstw}\widehat{b}_{D,w}\widehat{b}_N}{\widehat{b}^2_D} \\
& -\frac{1}{1296}\frac{\sum_{ijkqrstwv\beta}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\widehat{h}^N_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{b}_{N,\beta}\widehat{b}_D - \sum_{ijkqrstwv\beta}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\widehat{h}^D_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{b}_{D,\beta}\widehat{b}_N}{\widehat{b}^2_D} \\
& -\frac{1}{48}\frac{\sum_{ijkqrs}\widehat{h}^N_{ijkq}\mu^6_{ijkqrs}\widehat{b}_{N,rs}\widehat{b}_D - \sum_{ijkqrs}\widehat{h}^D_{ijkq}\mu^6_{ijkqrs}\widehat{b}_{D,rs}\widehat{b}_N}{\widehat{b}^2_D} \\
& +\frac{1}{144}\frac{\sum_{ijkqrstw}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{N,tw}\widehat{b}_D - \sum_{ijkqrstw}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{D,tw}\widehat{b}_N}{\widehat{b}^2_D} \\
& -\frac{1}{36}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}^N_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{N,\zeta\eta\xi}\widehat{b}_D - \sum_{ijk\zeta\eta\xi}\widehat{h}^D_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{D,\zeta\eta\xi}\widehat{b}_N}{\widehat{b}^2_D} \\
& +\frac{1}{24}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{N,\zeta\eta\xi\omega}\widehat{b}_D - \sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{D,\zeta\eta\xi\omega}\widehat{b}_N}{\widehat{b}^2_D}
\end{aligned}
$$

$$
\begin{aligned}
\frac{c_D}{\widehat{b}_D}\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{c_N}{\widehat{b}_N}-\frac{c_D}{\widehat{b}_D}\right) \quad = \quad & \frac{1}{2}c_D\frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{N,ij}\widehat{b}_D - \sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,ij}\widehat{b}_N}{\widehat{b}^3_D} \\
& -\frac{1}{6}c_D\frac{\sum_{ijkq}\widehat{h}^N_{ijk}\mu_{ijkq}\widehat{b}_{N,q}\widehat{b}_D - \sum_{ijkq}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\mu^6_{ijkqrs}\widehat{b}_{D,q}\widehat{b}_N}{\widehat{b}^3_D} \\
& +\frac{1}{72}c_D\frac{\widehat{b}_N\sum_{ijkqrs}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\mu^6_{ijkqrs} - \sum_{ijkq}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\mu^6_{ijkqrs}\widehat{b}_N}{\widehat{b}^2_D} \\
& -\frac{1}{24}c_D\frac{\widehat{b}_N\sum_{ijkq}\widehat{h}^N_{ijkq}\mu^4_{ijkq} - \sum_{ijkq}\widehat{h}^D_{ijkq}\mu^4_{ijkq}\widehat{b}_N}{\widehat{b}^2_D}.
\end{aligned}
$$

If we set $b_N(\theta) = g(\theta)\, b_D(\theta)$ and $h^N(\theta) = h^D(\theta) = h(\theta)$, we can show the following results for the derivatives of $b_N(\theta)$:

$$b_{N,i}(\theta) = g_i(\theta)\, b_D(\theta) + g(\theta)\, b_{D,i}(\theta), \tag{2}$$

$$b_{N,ij}(\theta) = g_{ij}(\theta)\, b_D(\theta) + g_i(\theta)\, b_{D,j}(\theta) + g_j(\theta)\, b_{D,i}(\theta) + g(\theta)\, b_{D,ij}(\theta), \tag{3}$$

$$\begin{aligned} b_{N,ijk}(\theta) &= g_{ijk}(\theta)\, b_D(\theta) + g_{ij}(\theta)\, b_{D,k}(\theta) + g_{ik}(\theta)\, b_{D,j}(\theta) + g_i(\theta)\, b_{D,jk}(\theta) + g_{jk}(\theta)\, b_{D,i}(\theta) \\ &\quad + g_j(\theta)\, b_{D,ik}(\theta) + g_k(\theta)\, b_{D,ij}(\theta) + g(\theta)\, b_{D,ijk}(\theta), \end{aligned} \tag{4}$$

$$\begin{aligned} b_{N,ijkq}(\theta) &= g_{ijkq}(\theta)\, b_D(\theta) + g_{ijk}(\theta)\, b_{D,q}(\theta) + g_{ijq}(\theta)\, b_{D,k}(\theta) + g_{ij}(\theta)\, b_{D,kq}(\theta) \\ &\quad + g_{ikq}(\theta)\, b_{D,j}(\theta) + g_{ik}(\theta)\, b_{D,jq}(\theta) + g_{iq}(\theta)\, b_{D,jk}(\theta) + g_i(\theta)\, b_{D,jkq}(\theta) \\ &\quad + g_{jkq}(\theta)\, b_{D,i}(\theta) + g_{jk}(\theta)\, b_{D,iq}(\theta) + g_{jq}(\theta)\, b_{D,ik}(\theta) + g_j(\theta)\, b_{D,ikq}(\theta) \\ &\quad + g_{kq}(\theta)\, b_{D,ij}(\theta) + g_k(\theta)\, b_{D,ijq}(\theta) + g_q(\theta)\, b_{D,ijk}(\theta) + g(\theta)\, b_{D,ijkq}(\theta), \end{aligned} \tag{5}$$

$$\begin{aligned} & b_{N,ij}(\theta)\, b_D(\theta) - b_{D,ij}(\theta)\, b_N(\theta) \\ = {} & [g_{ij}(\theta)\, b_D(\theta) + g_i(\theta)\, b_{D,j}(\theta) + g_j(\theta)\, b_{D,i}(\theta) + g(\theta)\, b_{D,ij}(\theta)]\, b_D(\theta) - b_{D,ij}(\theta)\, g(\theta)\, b_D(\theta) \\ = {} & g_{ij}(\theta)\, b_D(\theta)^2 + g_i(\theta)\, b_{D,j}(\theta)\, b_D(\theta) + g_j(\theta)\, b_{D,i}(\theta)\, b_D(\theta) \\ & + g(\theta)\, b_{D,ij}(\theta)\, b_D(\theta) - b_{D,ij}(\theta)\, g(\theta)\, b_D(\theta) \\ = {} & g_{ij}(\theta)\, b_D(\theta)^2 + g_i(\theta)\, b_{D,j}(\theta)\, b_D(\theta) + g_j(\theta)\, b_{D,i}(\theta)\, b_D(\theta), \end{aligned} \tag{6}$$

$$\begin{aligned} & b_{N,i}(\theta)\, b_D(\theta) - b_{D,i}(\theta)\, b_N(\theta) \\ = {} & (g_i(\theta)\, b_D(\theta) + g(\theta)\, b_{D,i}(\theta))\, b_D(\theta) - b_{D,i}(\theta)\, g(\theta)\, b_D(\theta) = g_i(\theta)\, b_D(\theta)^2, \end{aligned} \tag{7}$$

$$\begin{aligned} & b_{N,ijk}(\theta)\, b_D(\theta) - b_{D,ijk}(\theta)\, b_N(\theta) \\ = {} & \begin{bmatrix} g_{ijk}(\theta)\, b_D(\theta) + g_{ij}(\theta)\, b_{D,k}(\theta) + g_{ik}(\theta)\, b_{D,j}(\theta) + g_i(\theta)\, b_{D,jk}(\theta) \\ + g_{jk}(\theta)\, b_{D,i}(\theta) + g_j(\theta)\, b_{D,ik}(\theta) + g_k(\theta)\, b_{D,ij}(\theta) \end{bmatrix} b_D(\theta), \end{aligned} \tag{8}$$

$$\begin{aligned} & b_{N,ijkq}(\theta)\, b_D(\theta) - b_{D,ijkq}(\theta)\, b_N(\theta) \\ = {} & \begin{bmatrix} g_{ijkq}(\theta)\, b_D(\theta) + g_{ijk}(\theta)\, b_{D,q}(\theta) + g_{ijq}(\theta)\, b_{D,k}(\theta) + g_{ij}(\theta)\, b_{D,kq}(\theta) \\ + g_{ikq}(\theta)\, b_{D,j}(\theta) + g_{ik}(\theta)\, b_{D,jq}(\theta) + g_{iq}(\theta)\, b_{D,jk}(\theta) + g_i(\theta)\, b_{D,jkq}(\theta) \end{bmatrix} b_D(\theta) \\ & + \begin{bmatrix} g_{jkq}(\theta)\, b_{D,i}(\theta) + g_{jk}(\theta)\, b_{D,iq}(\theta) + g_{jq}(\theta)\, b_{D,ik}(\theta) + g_j(\theta)\, b_{D,ikq}(\theta) \\ + g_{kq}(\theta)\, b_{D,ij}(\theta) + g_k(\theta)\, b_{D,ijq}(\theta) + g_q(\theta)\, b_{D,ijk}(\theta) \end{bmatrix} b_D(\theta). \end{aligned} \tag{9}$$

Thus, we have

$$\begin{aligned} \frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{c_N}{\widehat{b}_N} - \frac{c_D}{\widehat{b}_D}\right) &= \frac{1}{2}\frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{N,ij}\widehat{b}_D - \sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,ij}\widehat{b}_N}{\widehat{b}_D^2} \\ &\quad - \frac{1}{6}\frac{\sum_{ijkq}\widehat{h}_{ijk}^N\mu_{ijkq}^4\widehat{b}_{N,q}\widehat{b}_D - \sum_{ijkq}\widehat{h}_{ijk}^D\mu_{ijkq}^4\widehat{b}_{D,q}\widehat{b}_N}{\widehat{b}_D^2} \end{aligned}$$

$$= \frac{1}{2}\frac{\sum_{ij}\widehat{\sigma}_{ij}\left(\widehat{b}_{N,ij}\widehat{b}_D-\widehat{b}_{D,ij}\widehat{b}_N\right)}{\widehat{b}_D^2}-\frac{1}{6}\frac{\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\left(\widehat{b}_{N,q}\widehat{b}_D-\widehat{b}_{D,q}\widehat{b}_N\right)}{\widehat{b}_D^2},$$

where

$$\begin{aligned}\frac{\sum_{ij}\widehat{\sigma}_{ij}\left(\widehat{b}_{N,ij}\widehat{b}_D-\widehat{b}_{D,ij}\widehat{b}_N\right)}{\widehat{b}_D^2} &= \frac{\sum_{ij}\widehat{\sigma}_{ij}\left(\widehat{g}_{ij}\widehat{b}_D^2+\widehat{g}_i\widehat{b}_{D,j}\widehat{b}_D+\widehat{g}_j\widehat{b}_{D,i}\widehat{b}_D\right)}{\widehat{b}_D^2}\\ &= \frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij}\widehat{b}_D^2+2\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_i\widehat{b}_{D,j}\widehat{b}_D}{\widehat{b}_D^2}\\ &= \sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij}+\frac{2\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,j}\widehat{g}_i}{\widehat{b}_D},\end{aligned}$$

from (6) and

$$\frac{\widehat{b}_{N,q}\widehat{b}_D-\widehat{b}_{D,q}\widehat{b}_N}{\widehat{b}_D^2}=\frac{\left(\widehat{g}_q\widehat{b}_D+\widehat{g}\widehat{b}_{D,q}\right)\widehat{b}_D-\widehat{b}_{D,q}\widehat{g}\widehat{b}_D}{\widehat{b}_D^2}=\widehat{g}_q$$

by (7). Hence,

$$\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{c_N}{\widehat{b}_N}-\frac{c_D}{\widehat{b}_D}\right)=\frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij}+\frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,j}\widehat{g}_i}{\widehat{b}_D}-\frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\widehat{g}_q.$$

From (8) and (9), we can get

$$\begin{aligned}&\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}^N_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{N,\zeta\eta\xi}\widehat{b}_D-\sum_{ijk\zeta\eta\xi}\widehat{h}^D_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{D,\zeta\eta\xi}\widehat{b}_N}{\widehat{b}_D^2}\\ =&\ \frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\left[\widehat{g}_{\zeta\eta\xi}\widehat{b}_D+\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}+\widehat{g}_{\zeta\xi}\widehat{b}_{D,\eta}+\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi}+\widehat{g}_{\eta\xi}\widehat{b}_{D,\zeta}+\widehat{g}_{\eta}\widehat{b}_{D,\zeta\xi}+\widehat{g}_{\xi}\widehat{b}_{D,\zeta\eta}\right]}{\widehat{b}_D}\\ =&\ \sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi}+\frac{3\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D}+\frac{3\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi}}{\widehat{b}_D},\end{aligned} \quad (10)$$

$$\begin{aligned}&\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{N,\zeta\eta\xi\omega}\widehat{b}_D-\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{D,\zeta\eta\xi\omega}\widehat{b}_N}{\widehat{b}_D^2}\\ =&\ \sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi\omega}+\frac{4\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi}\widehat{b}_{D,\omega}}{\widehat{b}_D}+\frac{6\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}\\ &+\frac{4\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_D}+\frac{4\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_D}.\end{aligned} \quad (11)$$

We can also show that

$$\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{d_N}{\widehat{b}_N}-\frac{d_D}{\widehat{b}_D}\right)$$

$$
\begin{aligned}
= \quad & -\frac{1}{120}\frac{\sum_{ijkqrs}\widehat{h}_{ijkqr}\mu^{6}_{ijkqrs}\left(\widehat{b}_{N,s}\widehat{b}_{D}-\widehat{b}_{D,s}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& +\frac{1}{144}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^{8}_{ijkqrstw}\left(\widehat{b}_{N,w}\widehat{b}_{D}-\widehat{b}_{D,w}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& -\frac{1}{1296}\frac{\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\left(\widehat{b}_{N,\beta}\widehat{b}_{D}-\widehat{b}_{D,\beta}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& -\frac{1}{48}\frac{\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^{6}_{ijkqrs}\left(\widehat{b}_{N,rs}\widehat{b}_{D}-\widehat{b}_{D,rs}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& +\frac{1}{144}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^{8}_{ijkqrstw}\left(\widehat{b}_{N,tw}\widehat{b}_{D}-\widehat{b}_{D,tw}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& -\frac{1}{36}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\left(\widehat{b}_{N,\zeta\eta\xi}\widehat{b}_{D}-\widehat{b}_{D,\zeta\eta\xi}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& +\frac{1}{24}\frac{\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\left(\widehat{b}_{N,\zeta\eta\xi\omega}\widehat{b}_{D}-\widehat{b}_{D,\zeta\eta\xi\omega}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}}.
\end{aligned}
$$

since $h^{N}(\theta)=h^{D}(\theta)=h(\theta)$. Hence, with (6), (7), (10), and (11), it can be shown that

$$
\begin{aligned}
& \frac{\widehat{b}_{N}}{\widehat{b}_{D}}\left(\frac{d_{N}}{\widehat{b}_{N}}-\frac{d_{D}}{\widehat{b}_{D}}\right) \\
= \quad & -\frac{1}{120}\sum_{ijkqrs}\widehat{h}_{ijkqr}\mu^{6}_{ijkqrs}\widehat{g}_{s}+\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^{8}_{ijkqrstw}\widehat{g}_{w} \\
& -\frac{1}{1296}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{g}_{\beta}-\frac{1}{24}\frac{\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^{6}_{ijkqrs}\widehat{b}_{D,s}\widehat{g}_{r}}{\widehat{b}_{D}} \\
& +\frac{1}{72}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^{8}_{ijkqrstw}\widehat{b}_{D,w}\widehat{g}_{t}}{\widehat{b}_{D}}-\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi}}{\widehat{b}_{D}} \\
& +\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_{D}}-\frac{1}{48}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^{6}_{ijkqrs}\widehat{g}_{rs} \\
& +\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^{8}_{ijkqrstw}\widehat{g}_{tw}-\frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi} \\
& +\frac{1}{24}\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi\omega}-\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_{D}} \\
& +\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi}\widehat{b}_{D,\omega}}{\widehat{b}_{D}}+\frac{1}{4}\frac{\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_{D}}.
\end{aligned}
$$

■

Using matrix notation for high order derivatives used in Magnus and Neudecker (1999) (with the exception that the first order derivative of a scalar function in our setting is a column vector), we can write Lemma 2.1 in matrix form. Before we do that, let us first introduce the following Generalized Isserlis theorem.

Theorem 2.1 (Generalized Isserlis Theorem) *If $A = \{\alpha_1, \ldots, \alpha_{2N}\}$ is a set of integers such that $1 \le \alpha_i \le P$, for each $i \in [1, 2N]$ and $X \in R^P$ is a zero mean multivariate normal random vector, then*

$$E(X_A) = \underset{A}{\Sigma\Pi} E(X_i X_j), \tag{12}$$

where $EX_A = E\left(\Pi_{i=1}^{2N} X_{\alpha_i}\right) = \mu_{\alpha_1,\ldots,\alpha_{2N}}$ and the notation $\Sigma\Pi$ means summing over all distinct ways of partitioning $X_{\alpha_1}, \ldots, X_{\alpha_{2N}}$ into pairs (X_i, X_j) and each summand is the product of the N pairs. This yields $(2N)!/\left(2^N N!\right) = (2N-1)!!$ terms in the sum where $(2N-1)!!$ is the double factorial defined by $(2N-1)!! = (2N-1) \times (2N-3) \times \cdots \times 1$.

The Isserlis theorem, first obtained by Isserlis (1918), expresses the higher order moments of a zero mean Gaussian vector $X \in R^P$ in terms of its covariance matrix. The generalized Isserlis theorem is due to Withers (1985) and Vignat (2012). For example, if $2N = 4$, then

$$E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

where $\alpha_i = i$ for $i \in [1, 4]$ and there are $(2 \times 2 - 1)!! = 3 \times 1 = 3$ terms in the sum. If $2N = 6$, there are $5 \times 3 \times 1 = 15$ terms in the sum.

Lemma 2.2 *Let $\nabla^j \widehat{h}$, $\nabla^j \widehat{g}$ and $\nabla^j \widehat{b}_D$ be the jth order derivatives of $h(\theta)$, $g(\theta)$ and $b_D(\theta)$ evaluated at $\widehat{\theta}_n$ respectively. If both $(\{h_n\}, g \times b_D)$ and $(\{h_n\}, b_D)$ satisfy the analytical assumptions for the stochastic Laplace method on $\wp_\theta$, then*

$$\frac{\int g(\theta) b_D(\theta) \exp(-nh_n(\theta)) d\theta}{\int b_D(\theta) \exp(-nh_n(\theta)) d\theta} = \widehat{g} + \frac{1}{n}B_1 + \frac{1}{n^2}(B_2 - B_3) + O_p\left(\frac{1}{n^3}\right),$$

where

$$B_1 = \frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] + (\nabla\widehat{g})'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} - \frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g},$$

$$B_2 = B_{21} + B_{22},$$

$$\begin{aligned}
B_{21} \;=\; & -\frac{1}{8}(\nabla\widehat{g})'\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^5\widehat{h}\right)'vec\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right] \\
& +\frac{1}{4}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\otimes\left(vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
& +\frac{1}{6}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
& +\frac{1}{16}\mathbf{tr}\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right)'\nabla^4\widehat{h}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\right)\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{3}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{4}vec\left(\nabla^3\widehat{h}'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{16}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{24}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\\
&\times vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla g\\
&-\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]\nabla\widehat{b}_D'\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{2}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\otimes\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{b}_D\right)'\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\frac{\nabla\widehat{b}_D}{\widehat{b}_D}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{12}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\frac{\nabla\widehat{b}_D}{\widehat{b}_D}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\widehat{b}_Dvec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\widehat{b}_Dvec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{2}vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\left(\nabla^2\widehat{h}\right)^{-1}\widehat{b}_D\right)'\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{4}\mathbf{tr}\left[\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\left(\nabla^3\widehat{b}_D\right)'}{\widehat{b}_D}\nabla\widehat{g},
\end{aligned}
$$

$$
\begin{aligned}
B_{22}=&-\frac{1}{16}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]\\
&-\frac{1}{4}\mathbf{tr}\left[\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]
\end{aligned}
$$

$$+\frac{1}{16}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]$$
$$+\frac{1}{24}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]$$
$$+\frac{1}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)$$
$$+\frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)$$
$$+\frac{1}{4}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)$$
$$-\frac{1}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{g}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)$$
$$-\frac{1}{6}vec\left(\nabla^3\widehat{g}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)$$
$$+\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\nabla^4\widehat{g}'\right]$$
$$-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}$$
$$-\frac{1}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}$$
$$-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}$$
$$+\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}$$
$$+\frac{1}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right]$$
$$+\frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right],$$

$$B_3 = B_4 \times B_1,$$

$$\begin{aligned}B_4 \quad = \quad & \frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right]-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\widehat{h}^{(3)}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\\ &+\frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\\ &+\frac{1}{12}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\\ &-\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right],\end{aligned}$$

where $\sigma_{ij} = h^{ij}$ *and* $\mathbf{tr}$ *denotes the trace of a matrix.*

Proof. From (2.1), we first write each term of B_1 into matrix form by (12)

$$\frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij}=\frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij}=\frac{1}{2}tr\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right],$$

$$\frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,j}\widehat{g}_i}{\widehat{b}_D}=\sum_{ij}\widehat{g}_i\widehat{\sigma}_{ij}\frac{\widehat{b}_{D,j}}{\widehat{b}_D}=(\nabla\widehat{g})'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D},$$

$$-\frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\widehat{g}_q=-\frac{1}{2}\sum_{ijkq}\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{g}_q=-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.$$

Then we can similarly write each term of B_2 into matrix form by (12)

$$\begin{aligned}
-\frac{1}{120}\sum_{ijkqrs}\widehat{h}_{ijkqr}\mu^6_{ijkqrs}\widehat{g}_s &= -\frac{15}{120}\sum_{ijkqrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{h}_{ijkqr}\widehat{\sigma}_{rs}\widehat{g}_s\\
&= -\frac{1}{8}(\nabla\widehat{g})'\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^5\widehat{h}\right)'vec\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right].
\end{aligned}$$

By (45),

$$\begin{aligned}
&\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^8_{ijkqrstw}\widehat{g}_w\\
=\ &\frac{1}{4}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\otimes\left(vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{6}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{16}\mathbf{tr}\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right)'\nabla^4\widehat{h}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{4}vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\right)\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

From (46), we have

$$\begin{aligned}
&-\frac{1}{1296}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{g}_\beta\\
=\ &-\frac{3}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{4}vec\left(\nabla^3\widehat{h}'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{16}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&-\frac{1}{24}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\\
&\times vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla g.
\end{aligned}$$

$$-\frac{1}{24}\sum_{ijkqrs}\frac{\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_r\widehat{b}_{D,s}}{\widehat{b}_D}$$

$$= -\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]\frac{\nabla\widehat{b}_D}{\widehat{b}_D}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}$$

$$-\frac{1}{2}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\otimes\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right)'\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.$$

Similiar to the proof of (40), we can get

$$\frac{1}{72}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{D,w}\widehat{g}_t}{\widehat{b}_D}$$

$$= \frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\frac{\nabla\widehat{b}_D}{\widehat{b}_D}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}$$

$$+\frac{1}{12}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\frac{\nabla\widehat{b}_D}{\widehat{b}_D}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}$$

$$+\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}$$

$$+\frac{1}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}$$

$$+\frac{1}{2}vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right)'\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.$$

By (18), we can show that

$$-\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{D,\eta\xi}\widehat{g}_\zeta}{\widehat{b}_D}$$

$$= -\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}$$

$$-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}$$

$$-\frac{1}{4}\mathbf{tr}\left[\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g},$$

$$\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{D,\eta\xi\omega}\widehat{g}_\zeta}{\widehat{b}_D} = \frac{3}{6}\sum_{\zeta\eta\xi\omega}\widehat{g}_\zeta\widehat{\sigma}_{\zeta\eta}\widehat{\sigma}_{\xi\omega}\frac{\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_D} = \frac{1}{2}\sum_{\zeta\eta\xi\omega}\widehat{g}_\zeta\widehat{\sigma}_{\zeta\eta}\frac{\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_D}\widehat{\sigma}_{\xi\omega}$$

$$= \frac{1}{2}\left(\nabla\widehat{g}\right)'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\left(\nabla^3\widehat{b}_D\right)'}{\widehat{b}_D}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right].$$

From (29),

$$-\frac{1}{48}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_{rs}$$

$$
\begin{aligned}
= & -\frac{1}{16}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] \\
& -\frac{1}{4}\mathbf{tr}\left[\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right].
\end{aligned}
$$

And by the formula (40),

$$
\begin{aligned}
& \frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{g}_{tw} \\
= & \frac{1}{16}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] \\
& +\frac{1}{24}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] \\
& +\frac{1}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right) \\
& +\frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right) \\
& +\frac{1}{4}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right).
\end{aligned}
$$

From (34),

$$
\begin{aligned}
& -\frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi} \\
= & -\frac{1}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{g}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right) \\
& -\frac{1}{6}vec\left(\nabla^3\widehat{g}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right).
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{24}\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi\omega} &= \frac{3}{24}\sum_{\zeta\eta\xi\omega}\widehat{\sigma}_{\zeta\eta}\widehat{\sigma}_{\xi\omega}\widehat{g}_{\zeta\eta\xi\omega} \\
&= \frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{g}\right)'\right].
\end{aligned}
$$

By the formula (18), we can get

$$
\begin{aligned}
& -\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D} \\
= & -\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} \\
& -\frac{1}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} \\
& -\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}.
\end{aligned}
$$

$$\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi}\widehat{b}_{D,\omega}}{\widehat{b}_D} = \frac{3}{6}\sum_{\zeta\eta\xi\omega}\widehat{\sigma}_{\zeta\eta}\widehat{g}_{\zeta\eta\xi}\widehat{\sigma}_{\xi\omega}\frac{\widehat{b}_{D,\omega}}{\widehat{b}_D}$$
$$= \frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}.$$

From (17),

$$\frac{1}{4}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}$$
$$= \frac{1}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right]+\frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right].$$

And we have

$$\frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\frac{\widehat{b}_{D,ij}}{\widehat{b}_D} = \frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right], \tag{13}$$

$$-\frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\frac{\widehat{b}_{D,q}}{\widehat{b}_D} = -\frac{3}{6}\sum_{ijkq}\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\frac{\widehat{b}_{D,q}}{\widehat{b}_D} \tag{14}$$
$$= -\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}.$$

From (35), we have

$$\frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs} \tag{15}$$
$$= \frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)$$
$$+\frac{1}{12}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right).$$

Note that

$$-\frac{1}{24}\sum_{ijkq}\widehat{h}_{ijkq}\mu^4_{ijkq} = -\frac{3}{24}\sum_{ijkq}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq} \tag{16}$$
$$= -\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right].$$

From (13), (14), (15) and (16), we define

$$B_4 = \frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right]-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\widehat{h}^{(3)}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}$$
$$+\frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)$$
$$+\frac{1}{12}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)$$

$$-\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right].$$

■

Proof of Lemma 3.2 in Li et al (2017) can be obtained by directly applying Lemma 2.2 above by setting $b_D(\theta) = p(\theta)$, $g(\theta) = l_t(\theta)$, $-nh^N(\theta) = -nh^D(\theta) = \ln p(\mathbf{y}|\theta)$.

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3 Appendix

In this appendix, we will rewrite each term of into matrix form.

3.1 For the term $\frac{1}{4}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}$

We can get

$$\begin{aligned}&\frac{1}{4}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}\\ =\ &\frac{1}{4}\left(\frac{\sum_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\eta}\widehat{\sigma}_{\xi\omega}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}+\frac{\sum_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\xi}\widehat{\sigma}_{\eta\omega}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}+\frac{\sum_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\xi}\widehat{\sigma}_{\eta\omega}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}\right)\end{aligned}$$

where

$$\frac{\sum_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\eta}\widehat{\sigma}_{\xi\omega}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}=\sum_{\zeta\eta}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\eta}\sum_{\xi\omega}\widehat{\sigma}_{\xi\omega}\frac{\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}=\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right],$$

$$\begin{aligned}&\frac{\sum_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\xi}\widehat{\sigma}_{\eta\omega}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}\\ =\ &\sum_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\xi}\widehat{\sigma}_{\eta\omega}\frac{\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}=\sum_{\zeta\eta\xi\omega}\widehat{\sigma}_{\xi\zeta}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\eta\omega}\frac{\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}=\sum_{\xi\omega}\frac{\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}\sum_{\zeta\eta}\widehat{\sigma}_{\xi\zeta}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\eta\omega}\\ =\ &\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right],\end{aligned}$$

$$\frac{\sum_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\xi}\widehat{\sigma}_{\eta\omega}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}=\frac{\sum_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\xi}\widehat{\sigma}_{\eta\omega}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D}=\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right],$$

then we have

$$\begin{aligned}&\frac{1}{4}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D} \qquad (17)\\ =\ &\frac{1}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right]+\frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right]\\ =\ &\frac{1}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right]+\frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]\end{aligned}$$

3.2 For the term $-\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D}$

Note that

$$-\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D}=-\frac{1}{12}\frac{\sum_{ijkqrs}\widehat{h}_{ijk}\mu^6_{ijkqrs}\widehat{g}_{qr}\widehat{b}_{D,s}}{\widehat{b}_D}$$

where

$$\begin{aligned}\mu^6_{ijkqrs}\ =\ &\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\sigma_{rs}+\widehat{\sigma}_{iq}\widehat{\sigma}_{kj}\widehat{\sigma}_{rs}+\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rs}\\ &+\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}+\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}+\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\end{aligned}$$

$$+\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr} + \widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr} + \widehat{\sigma}_{is}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}$$
$$+\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks} + \widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}$$
$$+\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr} + \widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}$$
$$+\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq} + \widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}.$$

then we can decompose into three groups. The first group has 6 elements without $\widehat{\sigma}_{qr}$ but with $\widehat{\sigma}_{ij}$, $\widehat{\sigma}_{ik}$ or $\widehat{\sigma}_{jk}$, that is

$$\sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{kj}\widehat{\sigma}_{rs}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rs}\widehat{g}_{qr}\widehat{b}_{D,s}$$
$$+ \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{g}_{qr}\widehat{b}_{D,s}.$$

Note that the 6 elements have the same quantity since

$$\begin{aligned}
\sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{kj}\widehat{\sigma}_{rs}\widehat{g}_{qr}\widehat{b}_{D,s} &= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rs}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{g}_{qr}\widehat{b}_{D,s} = \sum_{ijkqrs} \widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{g}_{qr}\widehat{\sigma}_{rs}\widehat{b}_{D,s} \\
&= vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{b}_D
\end{aligned}$$

and

$$\begin{aligned}
\sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{g}_{qr}\widehat{b}_{D,s} &= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{qr}\widehat{b}_{D,s} = \sum_{ijkqrs} \widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{qr}\widehat{b}_{D,s}
\end{aligned}$$

where

$$\begin{aligned}
\sum_{ijkqrs} \widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{qr}\widehat{b}_{D,s} &= \sum_{ijkqrs} \widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kr}\widehat{g}_{rq}\widehat{\sigma}_{qs}\widehat{b}_{D,s} = \sum_{ijkqrs} \widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{g}_{qr}\widehat{\sigma}_{rs}\widehat{b}_{D,s} \\
&= vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{b}_D.
\end{aligned}$$

The second group has 6 elements without $\widehat{\sigma}_{qr}$, $\widehat{\sigma}_{ij}$, $\widehat{\sigma}_{ik}$ and $\widehat{\sigma}_{jk}$, that is

$$\sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{g}_{qr}\widehat{b}_{D,s}$$
$$+ \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{g}_{qr}\widehat{b}_{D,s}.$$

These 6 elements have the same quantity since

$$\sum_{ijkqrs} \widehat{h}_{ijk}\left[\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks} + \widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\right]\widehat{g}_{qr}\widehat{b}_{D,s}$$

$$
\begin{aligned}
&= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= 2\sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{g}_{qr}\widehat{b}_{D,s} = 2\sum_{ijkqrs} \widehat{\sigma}_{iq}\widehat{g}_{qr}\widehat{\sigma}_{jr}\widehat{h}_{ijk}\widehat{\sigma}_{ks}\widehat{b}_{D,s} \\
&= 2vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{b}_D,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{ijkqrs} \widehat{h}_{ijk}\left[\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr} + \widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\right]\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= \sum_{ijkqrs} \widehat{h}_{ikj}\widehat{\sigma}_{iq}\widehat{\sigma}_{kr}\widehat{\sigma}_{js}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{kji}\widehat{\sigma}_{kr}\widehat{\sigma}_{jq}\widehat{\sigma}_{is}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= 2vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{b},
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{ijkqrs} \widehat{h}_{ijk}\left[\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq} + \widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\right]\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= \sum_{ijkqrs} \widehat{h}_{kij}\widehat{\sigma}_{kq}\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{jki}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{is}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= 2vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{b}.
\end{aligned}
$$

The third group has 3 elements without $\widehat{\sigma}_{qr}$, that is

$$
\sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr}\widehat{g}_{qr}\widehat{b}_{D,s} + \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{is}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{g}_{qr}\widehat{b}_{D,s}.
$$

We have

$$
\begin{aligned}
&\sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{g}_{qr}\widehat{b}_{D,s} \\
&= \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr}\widehat{g}_{qr}\widehat{b}_{D,s} = \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{is}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{g}_{qr}\widehat{b}_{D,s} = \sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{jk}\widehat{\sigma}_{is}\widehat{b}_{D,s}\widehat{\sigma}_{qr}\widehat{g}_{qr} \\
&= \sum_{qr}\widehat{\sigma}_{qr}\widehat{g}_{qr}\sum_{ijks}\widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{is}\widehat{b}_{D,s} = \mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{b}_D.
\end{aligned}
$$

Then we have

$$\begin{aligned}
& -\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D} \\
= & -\frac{1}{2}vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right. \\
& -\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} \\
& -\frac{1}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}.
\end{aligned}$$

And for the same reason, we have

$$\begin{aligned}
& -\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\widehat{b}_{D,\eta\xi}\widehat{g}_{\zeta}}{\widehat{b}_D} \\
= & -\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
& -\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
& -\frac{1}{4}\mathbf{tr}\left[\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned} \tag{18}$$

Note that

$$\begin{aligned}
& vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} \\
= & \ \mathbf{tr}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right] \\
= & \ \mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right] \\
= & \ \mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]
\end{aligned}$$

and we have

$$\begin{aligned}
& \sum_{ijkqrs}\widehat{\sigma}_{iq}\widehat{g}_{qr}\widehat{\sigma}_{jr}\widehat{h}_{ijk}\widehat{\sigma}_{ks}\frac{\widehat{b}_{D,s}}{\widehat{b}_D} \\
= & \sum_{ijkqrs}\widehat{\sigma}_{iq}\widehat{h}_{ijk}\widehat{\sigma}_{ks}\frac{\widehat{b}_{D,s}}{\widehat{b}_D}\widehat{\sigma}_{jr}\widehat{g}_{qr} = \sum_{qr}\left[\sum_{ijks}\widehat{\sigma}_{iq}\widehat{h}_{ijk}\widehat{\sigma}_{ks}\frac{\widehat{b}_{D,s}}{\widehat{b}_D}\widehat{\sigma}_{jr}\right]\widehat{g}_{qr} \\
= & \sum_{qr}\left[\sum_{ijks}\widehat{\sigma}_{qi}\widehat{\sigma}_{ks}\frac{\widehat{b}_{D,s}}{\widehat{b}_D}\widehat{h}_{ikj}\widehat{\sigma}_{jr}\right]\widehat{g}_{qr} = \sum_{qr}\left[\sum_{j}\left[\sum_{ik}\left[\widehat{\sigma}_{qi}\left[\sum_{s}\widehat{\sigma}_{ks}\frac{\widehat{b}_{D,s}}{\widehat{b}_D}\right]\widehat{h}_{ikj}\right]\widehat{\sigma}_{jr}\right]\right]\widehat{g}_{qr}
\end{aligned}$$

$$= \mathbf{tr}\left[\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right)'\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right],$$

it can also be derived as follow

$$\begin{aligned}
& vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} \\
= \; & vec\left(\nabla^2\widehat{g}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} \\
= \; & vec\left(\nabla^2\widehat{g}\right)' vec\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right] \\
= \; & vec\left(\nabla^2\widehat{g}\right)'\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right)'\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right] vec\left(\nabla^3\widehat{h}\right) \\
= \; & \mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right)\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\right] \\
= \; & \mathbf{tr}\left[\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right)'\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right].
\end{aligned}$$

Then we have

$$\begin{aligned}
& -\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D} \\
= \; & -\frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] \\
& -\frac{1}{2}\mathbf{tr}\left[\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right)'\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] \\
& -\frac{1}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right] vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}.
\end{aligned}$$

3.3 For the term $-\frac{1}{48}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_{rs}$

Note that

$$\begin{aligned}
\mu^6_{ijkqrs} = \; & \widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\sigma_{rs} + \widehat{\sigma}_{iq}\widehat{\sigma}_{kj}\widehat{\sigma}_{rs} + \widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rs} \\
& + \widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs} + \widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs} + \widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs} \\
& + \widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr} + \widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr} + \widehat{\sigma}_{is}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr} \\
& + \widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks} + \widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks} \\
& + \widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr} + \widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr} \\
& + \widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq} + \widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}.
\end{aligned}$$

We can decompose $\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_{rs}$ into two groups. The first group has 12 elements without $\widehat{\sigma}_{rs}$

$$\begin{aligned}
&\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs}+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}\widehat{g}_{rs}+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{g}_{rs}\\
&+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{g}_{rs}+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr}\widehat{g}_{rs}+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{is}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{g}_{rs}\\
&+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{g}_{rs}+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{g}_{rs}+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{g}_{rs}\\
&+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{g}_{rs}+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{g}_{rs}+\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{g}_{rs}.
\end{aligned}$$

The 12 elements in this group have the same quantility, that is

$$\begin{aligned}
\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs} &= \sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{g}_{rs}\widehat{\sigma}_{sq} \\
&= \mathbf{tr}\left[\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right].
\end{aligned} \tag{19}$$

The detail proof is as follows

$$\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{g}_{rs}=\sum_{ijkqsr}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{g}_{sr}=\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs}, \tag{20}$$

$$\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}\widehat{g}_{rs}=\sum_{ikjqrs}\widehat{h}_{ikjq}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}\widehat{g}_{rs}=\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs}, \tag{21}$$

$$\begin{aligned}
\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr}\widehat{g}_{rs} &= \sum_{ijkqsr}\widehat{h}_{ijkq}\widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr}\widehat{g}_{sr} \\
&= \sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}\widehat{g}_{rs}=\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs}
\end{aligned} \tag{22}$$

where the last equatility is because of (21),

$$\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{g}_{rs}=\sum_{iqjkrs}\widehat{h}_{iqjk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{g}_{rs}=\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs} \tag{23}$$

$$\begin{aligned}
\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{g}_{rs} &= \sum_{ijkqsr}\widehat{h}_{ijkq}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{g}_{sr} \\
&= \sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{g}_{rs}=\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs}
\end{aligned} \tag{24}$$

$$\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{g}_{rs}=\sum_{jkiqrs}\widehat{h}_{jkiq}\widehat{\sigma}_{jk}\widehat{\sigma}_{ir}\widehat{\sigma}_{qs}\widehat{g}_{rs}=\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs} \tag{25}$$

$$\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{g}_{rs}=\sum_{jqikrs}\widehat{h}_{jqik}\widehat{\sigma}_{jq}\widehat{\sigma}_{ir}\widehat{\sigma}_{ks}\widehat{g}_{rs}=\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs} \tag{26}$$

$$\sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{g}_{rs} = \sum_{kqijrs} \widehat{h}_{kqij}\widehat{\sigma}_{kq}\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{g}_{rs} = \sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs} \quad (27)$$

$$\sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{is}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{g}_{rs} = \sum_{jkqirs} \widehat{h}_{jkqi}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{is}\widehat{g}_{rs} = \sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs}$$

$$\sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{g}_{rs} = \sum_{jqkirs} \widehat{h}_{jqki}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{\sigma}_{is}\widehat{g}_{rs} = \sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs}$$

$$\sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{g}_{rs} = \sum_{kqjirs} \widehat{h}_{kqji}\widehat{\sigma}_{kq}\widehat{\sigma}_{jr}\widehat{\sigma}_{is}\widehat{g}_{rs} = \sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{rs}$$

$$\begin{aligned} & \sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{g}_{rs}\widehat{\sigma}_{qs} \\ = & \sum_{ijkq}\widehat{h}_{ijkq}\left[\widehat{\sigma}_{ij}\sum_{rs}\widehat{\sigma}_{kr}\widehat{g}_{rs}\widehat{\sigma}_{qs}\right] \\ = & \ \mathbf{tr}\left[\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right] \end{aligned}$$

We can illustrate the result by a simple example. Let

$$\hat{h}^{(4)} = \begin{bmatrix} h_{1111} & h_{1112} \\ h_{2111} & h_{2112} \\ h_{1211} & h_{1212} \\ h_{2211} & h_{2212} \\ h_{1121} & h_{1122} \\ h_{2121} & h_{2122} \\ h_{1221} & h_{1222} \\ h_{2221} & h_{2222} \end{bmatrix}$$

and

$$\widehat{e} = \left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}$$

and

$$\widehat{e} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

$$\left(\nabla^2\widehat{h}\right)^{-1} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right) = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix}$$

then

$$\widehat{e} \otimes vec\left(\left(\nabla^2 \widehat{h}\right)^{-1}\right) = \begin{bmatrix} \sigma_{11}e_{11} & \sigma_{11}e_{12} \\ \sigma_{21}e_{11} & \sigma_{21}e_{12} \\ \sigma_{12}e_{11} & \sigma_{12}e_{12} \\ \sigma_{22}e_{11} & \sigma_{22}e_{12} \\ \sigma_{11}e_{11} & \sigma_{11}e_{12} \\ \sigma_{21}e_{11} & \sigma_{21}e_{12} \\ \sigma_{12}e_{11} & \sigma_{12}e_{12} \\ \sigma_{22}e_{11} & \sigma_{22}e_{12} \end{bmatrix}.$$

The second group has 3 elements with $\widehat{\sigma}_{rs}$

$$\sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{g}_{rs} + \sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{iq}\widehat{\sigma}_{kj}\widehat{\sigma}_{rs}\widehat{g}_{rs} + \sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{iq}\widehat{\sigma}_{kj}\widehat{\sigma}_{rs}\widehat{g}_{rs}.$$

These 3 elements has the same quantity. We can get

$$\begin{aligned} & \sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{g}_{rs} \\ = \; & \mathbf{tr}\left[\left[\left(\nabla^2 \widehat{h}\right)^{-1} \otimes vec\left(\left(\nabla^2 \widehat{h}\right)^{-1}\right)\right]\left(\nabla^4 \widehat{h}\right)'\right] \mathbf{tr}\left[\left(\nabla^2 \widehat{h}\right)^{-1} \nabla^2 \widehat{g}\right]. \end{aligned} \tag{28}$$

From (19) and (28), we have

$$\begin{aligned} & -\frac{1}{48}\sum_{ijkqrs} \widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_{rs} \\ = \; & -\frac{3}{48}\sum_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{h}_{ijkq}\sum_{rs}\widehat{\sigma}_{rs}\widehat{g}_{rs} - \frac{12}{48}\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{g}_{rs}\widehat{\sigma}_{qs} \\ = \; & -\frac{1}{16}\mathbf{tr}\left[\left[\left(\nabla^2 \widehat{h}\right)^{-1} \otimes vec\left(\left(\nabla^2 \widehat{h}\right)^{-1}\right)\right]\left(\nabla^4 \widehat{h}\right)'\right] \mathbf{tr}\left[\left(\nabla^2 \widehat{h}\right)^{-1} \nabla^2 \widehat{g}\right] \\ & -\frac{1}{4}\mathbf{tr}\left[\left[\left(\left(\nabla^2 \widehat{h}\right)^{-1} \nabla^2 \widehat{g}\left(\nabla^2 \widehat{h}\right)^{-1}\right) \otimes vec\left(\left(\nabla^2 \widehat{h}\right)^{-1}\right)\right]\left(\nabla^4 \widehat{h}\right)'\right]. \end{aligned} \tag{29}$$

$$\begin{aligned} & \sum_{ijkqrs} \widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{g}_{rs}\widehat{\sigma}_{qs} \\ = \; & \sum_{ijkqrs}\widehat{\sigma}_{rk}\widehat{\sigma}_{ij}\widehat{h}_{ijkq}\widehat{\sigma}_{qs}\widehat{g}_{rs} = \sum_{rs}\left[\sum_{ijkq}\widehat{\sigma}_{rk}\widehat{\sigma}_{ij}\widehat{h}_{ijkq}\widehat{\sigma}_{qs}\right]\widehat{g}_{rs} \\ = \; & \mathbf{tr}\left[\left[\left(\nabla^2 \widehat{h}\right)^{-1} \otimes vec\left(\left(\nabla^2 \widehat{h}\right)^{-1}\right)'\right]\nabla^4 \widehat{h}\left(\nabla^2 \widehat{h}\right)^{-1}\nabla^2 \widehat{g}\right] \end{aligned}$$

then we have

$$-\frac{1}{48}\sum_{ijkqrs} \widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_{rs} \tag{30}$$

$$
\begin{aligned}
&= -\frac{3}{48}\sum_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{h}_{ijkq}\sum_{rs}\widehat{\sigma}_{rs}\widehat{g}_{rs} - \frac{12}{48}\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{g}_{rs}\widehat{\sigma}_{qs} \\
&= -\frac{1}{16}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] \\
&\quad -\frac{1}{4}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] \\
&= -\frac{1}{16}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\right]\nabla^4\widehat{h}\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] \\
&\quad -\frac{1}{4}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right].
\end{aligned}
$$

3.4 For the term $-\frac{1}{24}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_r\widehat{b}_{D,s}$

Similiar to the proof of (29), we have

$$
\begin{aligned}
&-\frac{1}{24}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_r\widehat{b}_{D,s} \qquad (31) \\
&= -\frac{3}{24}\sum_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{h}_{ijkq}\sum_{rs}\widehat{\sigma}_{rs}\widehat{g}_r\widehat{b}_{D,s} - \frac{12}{24}\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_r\widehat{b}_{D,s} \\
&= -\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\nabla\widehat{b}_D'\right] \\
&\quad -\frac{1}{2}\mathbf{tr}\left[\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\nabla\widehat{b}_D'\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right].
\end{aligned}
$$

And we can also written (31) as

$$
\begin{aligned}
&-\frac{1}{24}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_r\widehat{b}_{D,s} \\
&= -\frac{3}{24}\sum_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{h}_{ijkq}\sum_{rs}\widehat{\sigma}_{rs}\widehat{g}_r\widehat{b}_{D,s} - \frac{12}{24}\sum_{ijkqrs}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{sq}\widehat{g}_r\widehat{b}_{D,s} \\
&= -\frac{3}{24}\sum_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{h}_{ijkq}\sum_{rs}\widehat{\sigma}_{rs}\widehat{g}_r\widehat{b}_{D,s} - \frac{12}{24}\sum_{ijkqrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{sq}\widehat{b}_{D,s}\widehat{h}_{ijqk}\widehat{\sigma}_{kr}\widehat{g}_r \\
&= -\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]\nabla\widehat{b}_D'\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
&\quad -\frac{1}{2}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\otimes\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{b}_D\right)'\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}
$$

3.5 For the term $-\frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi}$

Note that $\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi} = \sum_{ijkqrs}\widehat{h}_{ijk}\mu^6_{ijkqrs}\widehat{g}_{qrs}$. We can decompose $\sum_{ijkqrs}\widehat{h}_{ijk}\mu^6_{ijkqrs}\widehat{g}_{qrs}$ into two groups. The first group consists of 9 elements which has one term from $(\widehat{\sigma}_{ij},\widehat{\sigma}_{ik},\widehat{\sigma}_{jk})$

and one term from $(\widehat{\sigma}_{qr},\widehat{\sigma}_{rs},\widehat{\sigma}_{qs})$, that is

$$\begin{aligned}
&\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{g}_{qrs}+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{kj}\widehat{\sigma}_{rs}\widehat{g}_{qrs}+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rs}\widehat{g}_{qrs}\\
&+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{qrs}+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}\widehat{g}_{qrs}+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{g}_{qrs}\\
&+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{g}_{qrs}+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr}\widehat{g}_{qrs}+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{is}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{g}_{qrs}.
\end{aligned}$$

These 9 elements have the same quantility and we have

$$\begin{aligned}
\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{g}_{qrs} &= \sum_{ijkqrs}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{g}_{qrs}\widehat{\sigma}_{rs} \qquad (32)\\
&= vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{g}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right).
\end{aligned}$$

The second group consists of 6 elements which doesn't include one term from $(\widehat{\sigma}_{ij},\widehat{\sigma}_{ik},\widehat{\sigma}_{jk})$, that is

$$\begin{aligned}
&\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{qs}\widehat{g}_{qrs}+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{\sigma}_{qs}\widehat{g}_{qrs}\\
&+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{qrs}+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{g}_{qrs}\\
&+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{\sigma}_{qs}\widehat{g}_{qrs}+\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{qs}\widehat{g}_{qrs}.
\end{aligned}$$

These 6 elements have the same quantility and we have

$$\begin{aligned}
&\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{qs}\widehat{g}_{qrs} \qquad (33)\\
=\;& vec\left(\nabla^3\widehat{g}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right] vec\left(\nabla^3\widehat{h}\right).
\end{aligned}$$

Then from (32) and (33), we can get

$$\begin{aligned}
&-\frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi} \qquad (34)\\
=\;& -\frac{9}{36}\sum_{ijk\zeta\eta\xi}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{k\zeta}\widehat{g}_{\zeta\eta\xi}\widehat{\sigma}_{\eta\xi}-\frac{6}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\widehat{\sigma}_{i\zeta}\widehat{\sigma}_{j\eta}\widehat{\sigma}_{k\xi}\widehat{g}_{\zeta\eta\xi}\\
=\;& -\frac{1}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{g}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\\
&-\frac{1}{6}vec\left(\nabla^3\widehat{g}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right] vec\left(\nabla^3\widehat{h}\right),
\end{aligned}$$

3.6 For the term $\frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs}$

Similiar to the proof of (34)

$$\begin{aligned}
& \frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs} \\
= \; & \frac{9}{72}\sum_{ijkqrs}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{h}_{qrs}\widehat{\sigma}_{rs}+\frac{6}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{h}_{qrs}
\end{aligned}$$

where

$$\sum_{ijkqrs}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{h}_{qrs}\widehat{\sigma}_{rs}=vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right),$$

$$\begin{aligned}
\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{h}_{qrs} &= \sum_{ijk}\widehat{h}_{ijk}\sum_{qrs}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{h}_{qrs} \\
&= vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right),
\end{aligned}$$

then we can get

$$\begin{aligned}
& \frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs} \\
= \; & \frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right) \\
& +\frac{1}{12}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right).
\end{aligned} \tag{35}$$

3.7 For the term $\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{g}_{tw}$

We can decompose $\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{g}_{tw}$ into seven groups. One group is the terms that have $\widehat{\sigma}_{tw}$. That is, we have

$$\begin{aligned}
& \sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs}\widehat{\sigma}_{tw}\widehat{g}_{tw} \\
= \; & \sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs}\sum_{tw}\widehat{\sigma}_{tw}\widehat{g}_{tw} \\
= \; & 9vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] \\
& +6vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]
\end{aligned} \tag{36}$$

by (29). Note that there are 15 terms in this group.

The other six groups have the same quantity. One of them which has $\widehat{\sigma}_{sw}$ in each element is as follows:

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{kt}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ik}\widehat{\sigma}_{jq}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ik}\widehat{\sigma}_{jt}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{iq}\widehat{\sigma}_{jt}\widehat{\sigma}_{kr}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ir}\widehat{\sigma}_{jt}\widehat{\sigma}_{kq}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{sw}\widehat{g}_{tw}.
\end{aligned}$$

Note that this group can be further decomposed into three sub-groups. The first sub-group consists of 6 elements which include $\widehat{\sigma}_{rt}$ or $\widehat{\sigma}_{qt}$, that is

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ik}\widehat{\sigma}_{jq}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{g}_{tw}.
\end{aligned}$$

These 6 elements has the same quantity and we have

$$\begin{aligned}
& \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{g}_{tw} \\
= & \sum_{ijkqrstw} \widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{ir}\widehat{h}_{rqs}\widehat{\sigma}_{qt}\widehat{g}_{tw}\widehat{\sigma}_{sw} \\
= & \; vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right).
\end{aligned} \tag{37}$$

The second sub-group consists of 3 elements which include one term from ($\widehat{\sigma}_{kt}$, $\widehat{\sigma}_{rt}$, $\widehat{\sigma}_{qt}$) and one term from ($\widehat{\sigma}_{ij}$, $\widehat{\sigma}_{ik}$, $\widehat{\sigma}_{jk}$), that is

$$\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{kt}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ik}\widehat{\sigma}_{jt}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{g}_{tw}.$$

These 3 elements has the same quantity and we have

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{g}_{tw} \qquad (38)\\
= &\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{it}\widehat{g}_{tw}\widehat{\sigma}_{sw} = \sum_{ijkqrstw} \widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{it}\widehat{g}_{tw}\widehat{\sigma}_{sw}\widehat{h}_{sqr}\widehat{\sigma}_{qr}\\
= &\, vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right).
\end{aligned}$$

The third sub-group consists of 6 elements which include one term from $(\widehat{\sigma}_{ij},\, \widehat{\sigma}_{ik},\, \widehat{\sigma}_{jk})$ but doesn't include any term from $(\widehat{\sigma}_{kt},\, \widehat{\sigma}_{rt},\, \widehat{\sigma}_{qt})$

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{iq}\widehat{\sigma}_{jt}\widehat{\sigma}_{kr}\widehat{\sigma}_{sw}\widehat{g}_{tw}\\
&+\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ir}\widehat{\sigma}_{jt}\widehat{\sigma}_{kq}\widehat{\sigma}_{sw}\widehat{g}_{tw}\\
&+\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{\sigma}_{sw}\widehat{g}_{tw} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{sw}\widehat{g}_{tw}.
\end{aligned}$$

These 6 elements has the same quantity and we have

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{sw}\widehat{g}_{tw} \qquad (39)\\
= &\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{it}\widehat{g}_{tw}\widehat{\sigma}_{sw} = \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{\sigma}_{it}\widehat{g}_{tw}\widehat{\sigma}_{ws}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{h}_{srq}\\
= &\, vec\left(\nabla^3\widehat{h}\right)'\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right).
\end{aligned}$$

Then from (36), (37), (38) and (39), we have

$$\begin{aligned}
&\frac{1}{144}\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{g}_{tw} \qquad (40)\\
= &\frac{9}{144}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]\\
&+\frac{6}{144}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]\\
&+\frac{36}{144}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)\\
&+\frac{18}{144}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\\
&+\frac{36}{144}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right).
\end{aligned}$$

We can rewrite (37) as

$$\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{g}_{tw} \tag{41}$$

$$= \sum_{ijkqrstw} \widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{ir}\widehat{h}_{rqs}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{g}_{tw} = \sum_{ijkqrstw} \widehat{\sigma}_{tq}\widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{ir}\widehat{h}_{qrs}\widehat{\sigma}_{sw}\widehat{g}_{tw}$$

$$= \sum_{tw}\left[\sum_{ijkqrs} \widehat{\sigma}_{tq}\widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{ir}\widehat{h}_{qrs}\widehat{\sigma}_{sw}\right]\widehat{g}_{tw} = \sum_{tw}\left[\sum_{ijkqrs} \widehat{\sigma}_{tq}\widehat{h}_{ijk}\widehat{\sigma}_{jk}\widehat{\sigma}_{ir}\widehat{h}_{qrs}\widehat{\sigma}_{sw}\right]\widehat{g}_{tw}$$

$$= \mathbf{tr}\left[\left[\left(vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right],$$

and

$$\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{g}_{tw} \tag{42}$$

$$= \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{it}\widehat{g}_{tw}\widehat{\sigma}_{sw} = \sum_{ijkqrstw} \widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{it}\widehat{g}_{tw}\widehat{\sigma}_{sw}\widehat{h}_{sqr}\widehat{\sigma}_{qr}$$

$$= \sum_{ijkqrstw} \widehat{\sigma}_{ti}\widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{qr}\widehat{h}_{qrs}\widehat{\sigma}_{sw}\widehat{g}_{tw}$$

$$= \mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^2\widehat{g}\right]$$

and

$$\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{sw}\widehat{g}_{tw} \tag{43}$$

$$= \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{it}\widehat{g}_{tw}\widehat{\sigma}_{sw} = \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{\sigma}_{it}\widehat{g}_{tw}\widehat{\sigma}_{ws}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{h}_{srq}$$

$$= \sum_{ijkqrstw} \widehat{\sigma}_{ti}\widehat{h}_{ikj}\widehat{\sigma}_{kq}\widehat{\sigma}_{jr}\widehat{h}_{qrs}\widehat{\sigma}_{sw}\widehat{g}_{tw}$$

$$= \mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]$$

Then we have

$$\frac{1}{144}\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{g}_{tw} \tag{44}$$

$$= \frac{9}{144}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]$$

$$+\frac{6}{144}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]$$

$$+\frac{36}{144}\mathbf{tr}\left[\left[\left(vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]$$

$$+\frac{18}{144}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^2\widehat{g}\right]$$

$$+\frac{36}{144}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right].$$

3.8 For the term $\frac{1}{72}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{D,w}\widehat{g}_t}{\widehat{b}_D}$

$$\begin{aligned}
&\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{b}_{D,w}\widehat{g}_t\\
=&\sum_{ijkqrstw}\widehat{\sigma}_{sw}\widehat{b}_{D,w}\widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{ir}\widehat{h}_{srq}\widehat{\sigma}_{qt}\widehat{g}_t\\
=&\ vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\widehat{b}_D vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{b}_{D,w}\widehat{g}_t\\
=&\sum_{ijkqrstw}\widehat{\sigma}_{qr}\widehat{h}_{qrs}\widehat{b}_{D,w}\widehat{\sigma}_{sw}\widehat{\sigma}_{jk}\widehat{h}_{jki}\widehat{\sigma}_{it}\widehat{g}_t\\
=&\ vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\widehat{b}_D vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{it}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{sw}\widehat{b}_{D,w}\widehat{g}_t\\
=&\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{it}\widehat{b}_{D,w}\widehat{g}_t\widehat{\sigma}_{sw}=\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{\sigma}_{it}\widehat{b}_{D,w}\widehat{g}_t\widehat{\sigma}_{sw}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{h}_{srq}\\
=&\sum_{ijkqrstw}\widehat{\sigma}_{jr}\widehat{\sigma}_{sw}\widehat{b}_{D,w}\widehat{h}_{srq}\widehat{\sigma}_{qk}\widehat{h}_{jki}\widehat{\sigma}_{it}\widehat{g}_t\\
=&\ vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\left(\nabla^2\widehat{h}\right)^{-1}\widehat{b}_D\right)'\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{72}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{D,w}\widehat{g}_t}{\widehat{b}_D}\\
=&\ \frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\nabla\widehat{g}'\right]\\
&+\frac{1}{12}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\nabla\widehat{g}'\right]
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\nabla\widehat{g}'\left(\nabla^2\widehat{h}\right)^{-1}\right)\right. \\
& +\frac{1}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\nabla\widehat{g}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right) \\
& +\frac{1}{2}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\nabla\widehat{g}'\left(\nabla^2\widehat{h}\right)^{-1}\right)\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right) \\
= \; & \frac{1}{8}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\frac{\nabla\widehat{b}_D}{\widehat{b}_D}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
& +\frac{1}{12}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right)\frac{\nabla\widehat{b}_D}{\widehat{b}_D}'\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
& +\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
& +\frac{1}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
& +\frac{1}{2}vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\right)'\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}
$$

3.9 For the term $\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^8_{ijkqrstw}\widehat{g}_w$

In this case, we can decompose $\mu^8_{ijkqrstw}$ into seven groups, each has 15 terms. The first four groups has $\widehat{\sigma}_{qw}$, $\widehat{\sigma}_{rw}$, $\widehat{\sigma}_{sw}$, $\widehat{\sigma}_{tw}$ in each term repectively which have the similiar structures. Out of these four groups, the group with $\widehat{\sigma}_{tw}$ is as follows

$$
\begin{aligned}
& \sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{g}_w+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{g}_w \\
& +\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{g}_w+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ik}\widehat{\sigma}_{jq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{g}_w \\
& +\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{g}_w+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{g}_w \\
& +\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{g}_w+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{g}_w \\
& +\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{g}_w+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{g}_w \\
& +\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{g}_w+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{\sigma}_{tw}\widehat{g}_w \\
& +\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{is}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{g}_w+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{\sigma}_{tw}\widehat{g}_w \\
& +\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{tw}\widehat{g}_w.
\end{aligned}
$$

The above 15 terms can be further decomposed into two different sub-groups by whether two of (i, j, k) are in the same combination of subscript (whether we have $\widehat{\sigma}_{ij}$ or $\widehat{\sigma}_{jk}$ or $\widehat{\sigma}_{ik}$). One sub-group is

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{g}_{w} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{g}_{w} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ik}\widehat{\sigma}_{jq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ik}\widehat{\sigma}_{jr}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{g}_{w} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ik}\widehat{\sigma}_{js}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{g}_{w} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{is}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{g}_{w}
\end{aligned}$$

which consists of 9 terms. Note that each term in this sub-group has the same value and we have

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
= &\sum_{ijkqrstw} \widehat{\sigma}_{rs}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{h}_{rsqt}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
= &\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \otimes \left(vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\right] \nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

Antoher sub-group is

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{g}_{w} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{g}_{w} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{ir}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{is}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{\sigma}_{tw}\widehat{g}_{w} + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{is}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{tw}\widehat{g}_{w}
\end{aligned}$$

which consists of 6 terms. Note that each term in this subgroup has the same value and we have

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
= &\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{h}_{qrst}\widehat{\sigma}_{tw}\widehat{g}_{w} \\
= &\ vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1} \otimes \left(\nabla^2\widehat{h}\right)^{-1} \otimes \left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

The second three groups has $\widehat{\sigma}_{iw}$, $\widehat{\sigma}_{jw}$, $\widehat{\sigma}_{kw}$ in each term repectively which have the similiar structures. Out of these three groups, the group with $\widehat{\sigma}_{iw}$ is as follows

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{st}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{\sigma}_{rt}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jk}\widehat{\sigma}_{qt}\widehat{\sigma}_{rs}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{\sigma}_{st}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{\sigma}_{rt}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jq}\widehat{\sigma}_{kt}\widehat{\sigma}_{rs}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{st}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{qt}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jr}\widehat{\sigma}_{kt}\widehat{\sigma}_{qs}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{\sigma}_{rt}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{\sigma}_{qt}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{js}\widehat{\sigma}_{kt}\widehat{\sigma}_{qr}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jt}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jt}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jt}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{\sigma}_{iw}\widehat{g}_w.
\end{aligned}$$

This group can be further decomposed into two sub-groups. The first sub-group is as follows

$$\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{st}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{\sigma}_{rt}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jk}\widehat{\sigma}_{qt}\widehat{\sigma}_{rs}\widehat{\sigma}_{iw}\widehat{g}_w.$$

Note that each element in this group include $\widehat{\sigma}_{jk}$. There are 3 terms in the first sub-group which have the same value and we have

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jk}\widehat{\sigma}_{qr}\widehat{\sigma}_{st}\widehat{\sigma}_{iw}\widehat{g}_w \\
= &\sum_{ijkqrstw} \widehat{g}_w\widehat{\sigma}_{iw}\widehat{\sigma}_{jk}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{qr}\widehat{\sigma}_{st} = \sum_{ijkw} \widehat{g}_w\widehat{\sigma}_{iw}\widehat{\sigma}_{jk}\widehat{h}_{ijk} \sum_{qrst} \widehat{h}_{qrst}\widehat{\sigma}_{qr}\widehat{\sigma}_{st} \\
= &\, vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\,\mathbf{tr}\left[\left(\left(\nabla^2\widehat{h}\right)^{-1} \otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right)' \nabla^4\widehat{h}\right].
\end{aligned}$$

The second sub-group consists of remained terms

$$\begin{aligned}
&\sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{\sigma}_{st}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{\sigma}_{rt}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jq}\widehat{\sigma}_{kt}\widehat{\sigma}_{rs}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jr}\widehat{\sigma}_{kq}\widehat{\sigma}_{st}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{qt}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jr}\widehat{\sigma}_{kt}\widehat{\sigma}_{qs}\widehat{\sigma}_{iw}\widehat{g}_w \\
&+ \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{\sigma}_{rt}\widehat{\sigma}_{iw}\widehat{g}_w + \sum_{ijkqrstw} \widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{\sigma}_{qt}\widehat{\sigma}_{iw}\widehat{g}_w
\end{aligned}$$

$$+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{js}\widehat{\sigma}_{kt}\widehat{\sigma}_{qr}\widehat{\sigma}_{iw}\widehat{g}_w+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jt}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{\sigma}_{iw}\widehat{g}_w$$
$$+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jt}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{\sigma}_{iw}\widehat{g}_w+\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jt}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{\sigma}_{iw}\widehat{g}_w.$$

There are 12 terms in the second sub-group which have the same value and we have

$$\begin{aligned}
&\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\widehat{\sigma}_{jq}\widehat{\sigma}_{kr}\widehat{\sigma}_{st}\widehat{\sigma}_{iw}\widehat{g}_w\\
=&\sum_{ijkqrstw}\widehat{\sigma}_{jq}\widehat{\sigma}_{st}\widehat{h}_{qstr}\widehat{\sigma}_{rk}\widehat{h}_{jki}\widehat{\sigma}_{iw}\widehat{g}_w\\
=&\ vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\right)\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^8_{ijkqrstw}\widehat{g}_w \qquad (45)\\
=&\ \frac{36}{144}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\otimes\left(vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{24}{144}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{9}{144}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\,\mathbf{tr}\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right)'\nabla^4\widehat{h}\right]\\
&+\frac{36}{144}vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\right)\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
=&\ \frac{1}{4}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\otimes\left(vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{6}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{16}\mathbf{tr}\left[\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right)'\nabla^4\widehat{h}\right]vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\\
&+\frac{1}{4}vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\right)\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

3.10 For the term $-\frac{1}{1296}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{g}_\beta$

Note that $\mu^{10}_{ijkqrstwv\beta}$ can be decomposed into 9 groups which have the same quantility. Each group has 105 elements which have $\widehat{\sigma}_{i\beta},\widehat{\sigma}_{j\beta},\widehat{\sigma}_{k\beta},\widehat{\sigma}_{q\beta},\widehat{\sigma}_{r\beta},\widehat{\sigma}_{s\beta},\widehat{\sigma}_{t\beta},\widehat{\sigma}_{w\beta}$ and $\widehat{\sigma}_{v\beta}$. We take the group with $\widehat{\sigma}_{v\beta}$ as an example. This group is further decomposed into 7 groups with 15 elements in each group. There are 6 groups out of 7 like the follows:

$$\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_\beta+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_\beta$$

$$
\begin{aligned}
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kt}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{jq}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{kr}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{kt}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jt}\widehat{\sigma}_{kr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{jt}\widehat{\sigma}_{kq}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{kj}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{kq}\widehat{\sigma}_{jr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{kr}\widehat{\sigma}_{jq}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}.
\end{aligned}
$$

The elements in these group don't include $\widehat{\sigma}_{tw}$. We can further decomposed the above group into two sub-groups. The first sub-group is as follows. The elements of this sub-group include $\widehat{\sigma}_{ij}$, $\widehat{\sigma}_{ik}$ or $\widehat{\sigma}_{jk}$

$$
\begin{aligned}
&\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kt}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{jq}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{kr}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{kt}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{kj}\widehat{\sigma}_{qr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}.
\end{aligned}
$$

There are 9 terms in the first sub-group which have the same quantity and we have

$$
\begin{aligned}
&\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
=&\sum_{ijkqrstwv\beta}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{h}_{qrs}\widehat{\sigma}_{rt}\widehat{\sigma}_{sw}\widehat{h}_{twv}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
=&\ vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}
$$

The second sub-group is as follows

$$
\begin{aligned}
&\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jt}\widehat{\sigma}_{kr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}\\
&+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}+\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{jt}\widehat{\sigma}_{kq}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}
\end{aligned}
$$

$$+ \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{kq}\widehat{\sigma}_{jr}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{kr}\widehat{\sigma}_{jq}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}$$

There are 6 terms in the second sub-group which have the same quantity and we have

$$\begin{aligned}
& \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
= & \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{h}_{qrs}\widehat{\sigma}_{kt}\widehat{\sigma}_{sw}\widehat{h}_{twv}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
= & \, vec\left(\nabla^3\widehat{h}'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

Note that out of the 7 groups, we have 1 group remained as follows

$$\begin{aligned}
& \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{jq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{is}\widehat{\sigma}_{kj}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{is}\widehat{\sigma}_{kq}\widehat{\sigma}_{jr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{is}\widehat{\sigma}_{kr}\widehat{\sigma}_{jq}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}.
\end{aligned}$$

We can further decomposed the above group into two sub-groups. The first sub-group is as follows

$$\begin{aligned}
& \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{jq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{kr}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ik}\widehat{\sigma}_{ks}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jk}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ir}\widehat{\sigma}_{jk}\widehat{\sigma}_{qs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
& + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{is}\widehat{\sigma}_{kj}\widehat{\sigma}_{qr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}.
\end{aligned}$$

The elements of this sub-group include $\widehat{\sigma}_{ij}$, $\widehat{\sigma}_{ik}$ or $\widehat{\sigma}_{jk}$. There are 9 terms in the first sub-group which have the same quantity and we have

$$\begin{aligned}
&\sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
= &\sum_{ijkqrs} \widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{h}_{qrs}\widehat{\sigma}_{rs} \sum_{twv\beta} \widehat{\sigma}_{tw}\widehat{h}_{twv}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
= &\, vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right) \\
&\times vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

The second sub-group is as follows

$$\begin{aligned}
&\sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{js}\widehat{\sigma}_{kr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
&+ \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{ir}\widehat{\sigma}_{jq}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{it}\widehat{\sigma}_{js}\widehat{\sigma}_{kq}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
&+ \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{is}\widehat{\sigma}_{kq}\widehat{\sigma}_{jr}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} + \sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{is}\widehat{\sigma}_{kr}\widehat{\sigma}_{jq}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta}.
\end{aligned}$$

There are 6 terms in the second sub-group which have the same value and we have

$$\begin{aligned}
&\sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{\sigma}_{tw}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
= &\sum_{ijkqrs} \widehat{h}_{ijk}\widehat{\sigma}_{iq}\widehat{\sigma}_{jr}\widehat{\sigma}_{ks}\widehat{h}_{qrs} \sum_{twv\beta} \widehat{\sigma}_{tw}\widehat{h}_{twv}\widehat{\sigma}_{v\beta}\widehat{g}_{\beta} \\
= &\, vec\left(\nabla^3\widehat{h}\right)' \left[\left(\nabla^2\widehat{h}\right)^{-1} \otimes \left(\nabla^2\widehat{h}\right)^{-1} \otimes \left(\nabla^2\widehat{h}\right)^{-1}\right] vec\left(\nabla^3\widehat{h}\right) \\
&\times vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&-\frac{1}{1296}\sum_{ijkqrstwv\beta} \widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{g}_{\beta} \qquad (46) \\
= &-\frac{9\times 6\times 9}{1296} vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}' \\
&\times\left[\left(\nabla^2\widehat{h}\right)^{-1} \otimes \left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
&-\frac{9\times 6\times 6}{1296} vec\left(\nabla^3\widehat{h}'\left[\left(\nabla^2\widehat{h}\right)^{-1} \otimes \left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\right)' \\
&\times\left[\left(\nabla^2\widehat{h}\right)^{-1} \otimes \left(\nabla^2\widehat{h}\right)^{-1}\right]\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g} \\
&-\frac{9\times 1\times 9}{1296} vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)' \nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}\right. \\
&-\frac{9\times 1\times 6}{1296}vec\left(\nabla^3\widehat{h}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\otimes\left(\nabla^2\widehat{h}\right)^{-1}\right]vec\left(\nabla^3\widehat{h}\right) \\
&\times vec\left(\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla g\right.
\end{aligned}
$$

Note that

$$
\frac{9\times 6\times 9}{1296}=\frac{3}{8},\frac{9\times 6\times 6}{1296}=\frac{1}{4},\frac{9\times 1\times 9}{1296}=\frac{1}{16},\frac{9\times 1\times 6}{1296}=\frac{1}{24}.
$$