Online Supplement to 'Deviance Information Criterion for Model Selection: Justification and Variation'*

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November 23, 2017

1 High Order Analytical Expansions

The purpose of this online supplement is to prove Lemma 3.2 in Li, et al (2017). Suppose Θ is a compact subset of R^P. For any $\theta \in \Theta$, $\{h_n(\theta) : n = 1, 2, \ldots\}$ is a sequence of eight-times continuously differentiable functions of θ, having an interior global minimum $\left\{\widehat{\theta}_n : n = 1, 2, \ldots\right\}$; $b(\theta)$ is a six-times continuously differentiable real function of θ. For any function $f(\theta)$, let $\widehat{f}$ be the value of function f evaluated at $\widehat{\theta}_n$, i.e., $\widehat{f} := f\left(\widehat{\theta}_n\right)$. When there is no confusion, we write $h_n(\theta)$ as $h(\theta)$ or h_n or even h and $b(\theta)$ as b. We use $B_\delta(\theta)$ to denote the open ball of radius δ centered at θ. So $B_{\sqrt{n}\delta}(0)$ is an open ball of radius $\sqrt{n}\delta$ centered at the origin. For convenience of exposition, we write $\frac{\partial^d}{\partial\theta_{j_1}\partial\theta_{j_2}\cdots\partial\theta_{j_d}} f(\theta)$ as $f_{j_1\cdots j_d}$. The Hessian of h_n at θ is denoted by $\nabla^2 h_n(\theta)$, and its (i,j)-component is written as h_{ij} while the components of its inverse is written as h^{ij}. Let μ^4_{ijkq}, μ^6_{ijkqrs}, $\mu^8_{ijkqrstw}$, $\mu^{10}_{ijkqrstwv\beta}$, $\mu^{12}_{ijkqrstwv\beta\tau\phi}$ be the fourth, sixth, eighth, tenth, and twelfth central moments of a multivariate Normal distribution whose covariance matrix is $\left(\nabla^2\widehat{h}\right)^{-1} := \left(\nabla^2 h_n(\theta)\right)^{-1}|_{\theta=\widehat{\theta}_n}$. Note that we require $h_n(\theta)$ be eight-times continuously differentiable and $b(\theta)$ be six-times continuously differentiable. These two conditions are stronger than what have typically been assumed in the literature on the Laplace approximation as we would like to develop higher order expansions.

Following Kass et al (1990), we call the pair $(\{h_n\}, b)$ satisfy the analytical assumptions for Laplace's method if the following assumptions are met. There exists positive numbers ε, M and η, and an integer n_0 such that $n \geq n_0$ implies (i) for all $\theta \in B_\varepsilon\left(\widehat{\theta}_n\right)$ and all

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$1 \leq j_1, \cdots, j_d \leq P$ with $0 \leq d \leq 8$, $\|h_n(\theta)\| < M$ and $\|h_{j_1 \cdots j_d}(\theta)\| < M$; (ii) $\nabla^2 \widehat{h}$ is positive definite and $\det\left(\nabla^2 \widehat{h}\right) > \eta$; (iii) $\int_\Theta b(\theta) \exp[-nh(\theta)]\, d\theta$ exists and is finite, and for all δ for which $0 < \delta < \varepsilon$ and $B_\delta\left(\widehat{\theta}_n\right) \subseteq \Theta$,

$$\left[\det\left(n\nabla^2 \widehat{h}\right)\right]^{\frac{1}{2}} \int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b(\theta) \exp\left[-nh_n(\theta) - n\widehat{h}\right] d\theta = O\left(n^{-3}\right).$$

If one sets $-nh_n$ to be the sequence of log-likelihood functions of a model (as a sequence of sample size n), we say the model is Laplace regular. Lemma 1.1 below and Lemma 2.1 in the next section extend Theorem 1 and Theorem 5 of Kass et al (1990) to a higher order. They will be used to prove Lemma 3.2 in Li, et al (2017).

Lemma 1.1 *If $(\{h_n\}, b)$ satisfy the analytical assumptions for Laplace's method, then*

$$\int_\Theta b(\theta) \exp[-nh(\theta)]\, d\theta = (2\pi)^{\frac{P}{2}} \left[\det\left(n\nabla^2 \widehat{h}\right)\right]^{-\frac{1}{2}} \exp\left[-n\widehat{h}\right] \left(\widehat{b} + \frac{1}{n} A_1 + \frac{1}{n^2} A_2 + O\left(n^{-3}\right)\right),$$

where

$$A_1 = -\frac{1}{24} \sum_{ijkq} \widehat{h}_{ijkq} \mu^4_{ijkq} \widehat{b} + \frac{1}{72} \sum_{ijkqrs} \widehat{h}_{ijk} \widehat{h}_{qrs} \mu^6_{ijkqrs} \widehat{b} - \frac{1}{6} \sum_{ijk\zeta} \widehat{h}_{ijk} \mu^4_{ijk\zeta} \widehat{b}_\zeta + \frac{1}{2} \sum_{\zeta\eta} \widehat{b}_{\zeta\eta} \widehat{h}^{\zeta\eta},$$

$$\begin{aligned}
A_2 \quad = \quad & -\frac{1}{720} \sum_{ijkqrs} \widehat{h}_{ijkqrs} \mu^6_{ijkqrs} \widehat{b} + \frac{1}{1152} \sum_{ijkqrstw} \widehat{h}_{ijkq} \widehat{h}_{rstw} \mu^8_{ijkqrstw} \widehat{b} \\
& + \frac{1}{720} \sum_{ijkqrstw} \widehat{h}_{ijk} \widehat{h}_{qrstw} \mu^8_{ijkqrstw} \widehat{b} - \frac{1}{1728} \sum_{ijkqrstwv\beta} \widehat{h}_{ijk} \widehat{h}_{qrs} \widehat{h}_{twv\beta} \mu^{10}_{ijkqrstwv\beta} \widehat{b} \\
& + \frac{1}{31104} \sum_{ijkqrstwv\beta\tau\phi} \widehat{h}_{ijk} \widehat{h}_{qrs} \widehat{h}_{twv} \widehat{h}_{\beta\tau\phi} \mu^{12}_{ijkqrstwv\beta\tau\phi} \widehat{b} - \frac{1}{120} \sum_{ijkqr\zeta} \widehat{h}_{ijkqr} \mu^6_{ijkqr\zeta} \widehat{b}_\zeta \\
& + \frac{1}{144} \sum_{ijkqrst\zeta} \widehat{h}_{ijk} \widehat{h}_{qrst} \mu^8_{ijkqrst\zeta} \widehat{b}_\zeta - \frac{1}{1296} \sum_{ijkqrstwv\zeta} \widehat{h}_{ijk} \widehat{h}_{qrs} \widehat{h}_{twv} \mu^{10}_{ijkqrstwv\zeta} \widehat{b}_\zeta \\
& - \frac{1}{48} \sum_{ijkq\zeta\eta} \widehat{h}_{ijkq} \mu^6_{ijkq\zeta\eta} \widehat{b}_{\zeta\eta} + \frac{1}{144} \sum_{ijkqrs\zeta\eta} \widehat{h}_{ijk} \widehat{h}_{qrs} \mu^8_{ijkqrs\zeta\eta} \widehat{b}_{\zeta\eta} \\
& - \frac{1}{36} \sum_{ijk\zeta\eta\xi} \widehat{h}_{ijk} \mu^6_{ijk\zeta\eta\xi} \widehat{b}_{\zeta\eta\xi} + \frac{1}{24} \sum_{\zeta\eta\xi\omega} \widehat{b}_{\zeta\eta\xi\omega} \mu^4_{\zeta\eta\xi\omega}.
\end{aligned}$$

Proof. Note that

$$\int_\Theta b(\theta) \exp[-nh(\theta)]\, d\theta = \int_{B_\delta\left(\widehat{\theta}_n\right)} b(\theta) \exp[-nh(\theta)]\, d\theta + \int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b(\theta) \exp[-nh(\theta)]\, d\theta,$$

and

$$\int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b(\theta) \exp[-nh(\theta)]\, d\theta$$

$$
\begin{aligned}
&= \left[\det\left(n\nabla^2\widehat{h}\right)\right]^{-\frac{1}{2}} \exp\left[-n\widehat{h}\right] \left[\det\left(n\nabla^2\widehat{h}\right)\right]^{\frac{1}{2}} \int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b(\theta) \exp\left[-n\left(h(\theta) - \widehat{h}\right)\right] d\theta \\
&= \left[\det\left(n\nabla^2\widehat{h}\right)\right]^{-\frac{1}{2}} \exp\left[-n\widehat{h}\right] O\left(n^{-3}\right)
\end{aligned}
$$

by assumption (iii) above.

Let $u = \sqrt{n}\left(\theta - \widehat{\theta}_n\right)$. Applying the Taylor expansion to $h(\theta)$ at $\widehat{\theta}_n$, we have

$$
\begin{aligned}
nh(\theta) &= n\widehat{h} + \frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j + \frac{1}{6}n^{-\frac{1}{2}}\sum_{ijk}\widehat{h}_{ijk}u_iu_ju_k + \frac{1}{24}n^{-1}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_q \\
&+ \frac{1}{120}n^{-\frac{3}{2}}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_r + \frac{1}{720}n^{-2}\sum_{ijkqrs}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_s \\
&+ \frac{1}{5040}n^{-\frac{5}{2}}\sum_{ijkqrst}\widehat{h}_{ijkqrst}u_iu_ju_ku_qu_ru_su_t + r_n(u),
\end{aligned}
$$

where

$$
r_n(u) = \frac{1}{40320}n^{-3}\sum_{ijkqrstw} h_{ijkqrstw}\left(\theta'\right) u_iu_ju_ku_qu_ru_su_tu_w,
$$

and θ' lies between θ and $\widehat{\theta}_n$.

Define

$$
\begin{aligned}
x &= \frac{1}{6}n^{-\frac{1}{2}}\sum_{ijk}\widehat{h}_{ijk}u_iu_ju_k + \frac{1}{24}n^{-1}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_q + \frac{1}{120}n^{-\frac{3}{2}}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_r \\
&+ \frac{1}{720}n^{-2}\sum_{ijkqrs}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_s + \frac{1}{5040}n^{-\frac{5}{2}}\sum_{ijkqrst}\widehat{h}_{ijkqrst}u_iu_ju_ku_qu_ru_su_t + r_n(u).
\end{aligned}
$$

Applying the Taylor expansion to $\exp(-x)$ at the origin, we have

$$
\begin{aligned}
\exp(-nh) &= \exp\left\{-n\widehat{h}\right\}\exp\left(-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right) \times \\
&\quad \left(1 + \Xi_1 + \frac{1}{2}\Xi_2 + \frac{1}{6}\Xi_3 + \frac{1}{24}\Xi_4 + \frac{1}{120}\Xi_5 + R_{1,n}\left(\theta, \widehat{\theta}_n\right)\right),
\end{aligned}
$$

where,

$$
\begin{aligned}
\Xi_1 &= -\frac{1}{6}n^{-\frac{1}{2}}\sum_{ijk}\widehat{h}_{ijk}u_iu_ju_k - \frac{1}{24}n^{-1}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_q - \frac{1}{120}n^{-\frac{3}{2}}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_r \\
&- \frac{1}{720}n^{-2}\sum_{ijkqrs}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_s - \frac{1}{5040}n^{-\frac{5}{2}}\sum_{ijkqrst}\widehat{h}_{ijkqrst}u_iu_ju_ku_qu_ru_su_t,
\end{aligned}
$$

$$
\Xi_2 = \frac{1}{36}n^{-1}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}u_iu_ju_ku_qu_ru_s + \frac{1}{24^2}n^{-2}\sum_{ijkqrstw}\widehat{h}_{ijkq}\widehat{h}_{rstw}u_iu_ju_ku_qu_ru_su_tu_w
$$

$$
\begin{aligned}
&+\frac{1}{72}n^{-\frac{3}{2}}\sum_{ijkqrst}\widehat{h}_{ijk}\widehat{h}_{qrst}u_iu_ju_ku_qu_ru_su_t+\frac{1}{360}n^{-2}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrstw}u_iu_ju_ku_qu_ru_su_tu_w\\
&+\frac{1}{1440}n^{-\frac{5}{2}}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrstwv}u_iu_ju_ku_qu_ru_su_tu_wu_v\\
&+\frac{1}{2160}n^{-\frac{5}{2}}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrstwv}u_iu_ju_ku_qu_ru_su_tu_wu_v,
\end{aligned}
$$

$$
\begin{aligned}
\Xi_3 &= -\frac{1}{216}n^{-\frac{3}{2}}\sum_{ijkqrstwv}h_{ijk}h_{qrs}h_{twv}u_iu_ju_ku_qu_ru_su_tu_wu_v\\
&\quad -\frac{1}{288}n^{-2}\sum_{ijkqrstwv\beta}h_{ijk}h_{qrs}h_{twv\beta}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta\\
&\quad -\frac{1}{1152}n^{-\frac{5}{2}}\sum_{ijkqrstwv\beta\tau}h_{ijk}h_{qrst}h_{wv\beta\tau}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau\\
&\quad -\frac{1}{1440}n^{-\frac{5}{2}}\sum_{ijkqrstwv\beta\tau}h_{ijk}h_{qrs}h_{twv\beta\tau}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau,
\end{aligned}
$$

$$
\begin{aligned}
\Xi_4 &= \frac{1}{1296}n^{-2}\sum_{ijkqrstwv\beta\tau\phi}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi\\
&\quad +\frac{1}{1296}n^{-\frac{5}{2}}\sum_{ijkqrstwv\beta\tau\phi\alpha}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi\alpha}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi u_\alpha,
\end{aligned}
$$

$$
\Xi_5=-\frac{1}{7776}n^{-\frac{5}{2}}\sum_{ijkqrstwv\beta\tau\phi\alpha\varkappa\varrho}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}\widehat{h}_{\alpha\varkappa\varrho}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi u_\alpha u_\varkappa u_\varrho.
$$

Taylor-expanding $b(\theta)$ at $\widehat{\theta}_n$, we have

$$
\begin{aligned}
b(\theta) &= \widehat{b}+n^{-\frac{1}{2}}\sum_i\widehat{b}_iu_i+\frac{1}{2}n^{-1}\sum_{ij}\widehat{b}_{ij}u_iu_j+\frac{1}{6}n^{-\frac{3}{2}}\sum_{ijk}\widehat{b}_{ijk}u_iu_ju_k\\
&\quad +\frac{1}{24}n^{-2}\sum_{ijkq}\widehat{b}_{ijkq}u_iu_ju_ku_q+\frac{1}{120}n^{-\frac{5}{2}}\sum_{ijkqr}\widehat{b}_{ijkqr}u_iu_ju_ku_qu_r+R_{2,n}\left(\theta,\widehat{\theta}_n\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
b(\theta)\exp\left[-nh(\theta)\right] &= \left\{\exp\left[-n\widehat{h}\right]\right\}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\left\{I_n\left(\theta,\widehat{\theta}_n\right)+R_n\left(\theta,\widehat{\theta}_n\right)\right\}\\
&= \left(\sqrt{2\pi}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{-\frac{1}{2}}\exp\left(-n\hat{h}\right)\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\\
&\quad \times\left\{I_n\left(\theta,\widehat{\theta}_n\right)+R_n\left(\theta,\widehat{\theta}_n\right)\right\},
\end{aligned}
$$

where $I_n\left(\theta,\widehat{\theta}_n\right)$ and $R_n\left(\theta,\widehat{\theta}_n\right)$ will be specified below. Thus,

$$
\begin{aligned}
& \int_{B_\delta\left(\widehat{\theta}_n\right)} b(\theta)\exp\left[-nh(\theta)\right]d\theta \\
= & \left(\sqrt{2\pi}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{-\frac{1}{2}}\exp\left(-n\widehat{h}\right)\times \\
& \int_{B_\delta\left(\widehat{\theta}_n\right)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\left\{I_n\left(\theta,\widehat{\theta}_n\right)+R_n\left(\theta,\widehat{\theta}_n\right)\right\}d\theta \\
= & \left(\sqrt{2\pi}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{-\frac{1}{2}}\exp\left(-n\widehat{h}\right)\left(\sqrt{n}\right)^{-\frac{P}{2}}\times \\
& \int_{B_{\sqrt{n}\delta}(0)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\left\{I_n\left(\theta,\widehat{\theta}_n\right)+R_n\left(\theta,\widehat{\theta}_n\right)\right\}du \\
= & \left(\sqrt{2\pi}\right)^{\frac{P}{2}}\left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}}\exp\left(-n\widehat{h}\right)\times \\
& \int_{B_{\sqrt{n}\delta}(0)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\left\{\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\right\}\left\{I_n\left(\theta,\widehat{\theta}_n\right)+R_n\left(\theta,\widehat{\theta}_n\right)\right\}du,
\end{aligned}
$$

where $R_n\left(\theta,\widehat{\theta}_n\right)$ is the sum of terms involving $R_{1,n}\left(\theta,\widehat{\theta}_n\right)$, $R_{2,n}\left(\theta,\widehat{\theta}_n\right)$ and terms of order equal to or smaller than $O(n^{-3})$. Furthermore, we can get

$$
R_{2,n}\left(\theta,\widehat{\theta}_n\right)=\frac{1}{720}n^{-3}\sum b_{ijkqrs}\left(\widetilde{\theta}\right)u_iu_ju_ku_qu_ru_s,
$$

where $\widetilde{\theta}$ lies between θ and $\widehat{\theta}_n$. So the leading term of $R_n\left(\theta,\widehat{\theta}_n\right)$ are $\widehat{b}R_{1n}\left(\theta,\widehat{\theta}_n\right)+R_{2n}\left(\theta,\widehat{\theta}_n\right)$ which include $r_n(u)$. The integral of $r_n(u)$ over $B_\varepsilon\left(\widehat{\theta}_n\right)$ can be expressed as

$$
\begin{aligned}
& \left|n^{-3}\int_{B_{\sqrt{n}\delta}(0)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\sum_{ijkqrstw}h_{ijkqrstw}\left(\theta'\right)u_iu_ju_ku_qu_ru_su_tu_wdu\right| \\
\leq\ & n^{-3}\int_{B_{\sqrt{n}\delta}(0)}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\sum_{ijkqrstw}\left|h_{ijkqrstw}\left(\theta'\right)\right|\left|u_iu_ju_ku_qu_ru_su_tu_w\right|du \\
\leq\ & n^{-3}M\int_{R^P}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\sum_{ijkqrstw}\left|u_iu_ju_ku_qu_ru_su_tu_w\right|du \\
=\ & O\left(n^{-3}\right),
\end{aligned}
$$

where $\int_{R^P}\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\sum_{ijkqrstw}\left|u_iu_ju_ku_qu_ru_su_tu_w\right|du$ is the eighth order moment of folded multivariate folded normal distribution with mean 0 and covariance $\left(\nabla^2\widehat{h}\right)^{-1}$ which is finite; see Kamat (1953) and Kan and Robotti (2017). Then we have

$$
\left(\sqrt{2\pi}\right)^{\frac{P}{2}}\left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}}\exp\left(-n\widehat{h}\right)\int_{B_{\sqrt{n}\delta}(0)}\left[I_n\left(\theta,\widehat{\theta}\right)+R_n\left(\theta,\widehat{\theta}\right)\right]f(u)\,du
$$

$$= \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right) \left[\int_{B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du + O\left(n^{-3}\right)\right].$$

For $I_n\left(\theta,\widehat{\theta}_n\right)$ we have

$$I_n\left(\theta,\widehat{\theta}_n\right) = I_n^0\left(\theta,\widehat{\theta}_n\right) + I_n^1\left(\theta,\widehat{\theta}_n\right) + I_n^2\left(\theta,\widehat{\theta}_n\right) + I_n^3\left(\theta,\widehat{\theta}_n\right) + I_n^4\left(\theta,\widehat{\theta}_n\right) + I_n^5\left(\theta,\widehat{\theta}_n\right),$$

where

$$\begin{aligned} I_n^0\left(\theta,\widehat{\theta}_n\right) &= \widehat{b}\left(1 + \Xi_1 + \frac{1}{2}\Xi_2 + \frac{1}{6}\Xi_3 + \frac{1}{24}\Xi_4 + \frac{1}{120}\Xi_5\right) \\ &= \widehat{b}\left\{1 + n^{-\frac{1}{2}}I_n^{01} + n^{-1}I_n^{02} + n^{-\frac{3}{2}}I_n^{03} + n^{-2}I_n^{04} + n^{-\frac{5}{2}}I_n^{05}\right\}, \end{aligned}$$

$$I_n^{01} = -\frac{1}{6}\sum_{ijk}\widehat{h}_{ijk}u_iu_ju_k, \quad I_n^{02} = -\frac{1}{24}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_q + \frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}u_iu_ju_ku_qu_ru_s,$$

$$\begin{aligned} I_n^{03} &= -\frac{1}{120}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_r + \frac{1}{144}\sum_{ijkqrst}\widehat{h}_{ijk}\widehat{h}_{qrst}u_iu_ju_ku_qu_ru_su_t \\ &\quad -\frac{1}{1296}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}u_iu_ju_ku_qu_ru_su_tu_wu_v, \end{aligned}$$

$$\begin{aligned} I_n^{04} &= -\frac{1}{720}\sum_{ijkqrs}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_s + \frac{1}{1152}\sum_{ijkqrstw}\widehat{h}_{ijkq}\widehat{h}_{rstw}u_iu_ju_ku_qu_ru_su_tu_w \\ &\quad +\frac{1}{720}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrstw}u_iu_ju_ku_qu_ru_su_tu_w \\ &\quad -\frac{1}{1728}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv\beta}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta \\ &\quad +\frac{1}{31104}\sum_{ijkqrstwv\beta\tau\phi}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi, \end{aligned}$$

$$\begin{aligned} I_n^{05} &= -\frac{1}{5040}\sum_{ijkqrst}\widehat{h}_{ijkqrst}u_iu_ju_ku_qu_ru_su_t + \frac{1}{2880}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrstwv}u_iu_ju_ku_qu_ru_su_tu_wu_v \\ &\quad +\frac{1}{5120}n^{-\frac{5}{2}}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrstwv}u_iu_ju_ku_qu_ru_su_tu_wu_v \\ &\quad -\frac{1}{6912}\sum_{ijkqrstwv\beta\tau}h_{ijk}h_{qrst}h_{wv\beta\tau}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau \\ &\quad -\frac{1}{8640}\sum_{ijkqrstwv\beta\tau}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv\beta\alpha}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau \\ &\quad +\frac{1}{31104}\sum_{ijkqrstwv\beta\tau\phi\alpha}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi\alpha}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi u_\alpha \end{aligned}$$

$$-\frac{1}{933120}\sum_{ijkqrstwv\beta\tau\phi\alpha\varkappa\varrho}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}\widehat{h}_{\alpha\varkappa\varrho}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi u_\alpha u_\varkappa u_\varrho,$$

$$\begin{aligned}
I_n^1\left(\theta,\widehat{\theta}_n\right) &= n^{-\frac{1}{2}}\sum\widehat{b}_\zeta u_\zeta\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\
&= n^{-\frac{1}{2}}\sum\widehat{b}_\zeta u_\zeta+n^{-1}I_n^{11}\left(\theta,\widehat{\theta}_n\right)+n^{-\frac{3}{2}}I_n^{12}\left(\theta,\widehat{\theta}_n\right)+n^{-2}I_n^{13}\left(\theta,\widehat{\theta}_n\right)\\
&\quad +n^{-\frac{5}{2}}I_n^{14}\left(\theta,\widehat{\theta}_n\right),
\end{aligned}$$

$$\begin{aligned}
I_n^{11}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{6}\sum_{ijk\zeta}\widehat{h}_{ijk}u_iu_ju_ku_\zeta\widehat{b}_\zeta,\\
I_n^{12}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{24}\sum_{ijkq\zeta}\widehat{h}_{ijkq}u_iu_ju_ku_qu_\zeta\widehat{b}_\zeta+\frac{1}{72}\sum_{ijkqrs\zeta}\widehat{h}_{ijk}\widehat{h}_{qrs}u_iu_ju_ku_qu_ru_su_\zeta\widehat{b}_\zeta,
\end{aligned}$$

$$\begin{aligned}
I_n^{13}\left(\theta,\widehat{\theta}\right) &= -\frac{1}{120}\sum_{ijkqr}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_ru_\zeta\widehat{b}_\zeta+\frac{1}{144}\sum_{ijkqrst}\widehat{h}_{ijk}\widehat{h}_{qrst}u_iu_ju_ku_qu_ru_su_tu_\zeta\widehat{b}_\zeta\\
&\quad -\frac{1}{1296}\sum_{ijkqrstwv}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\zeta\widehat{b}_\zeta,
\end{aligned}$$

$$\begin{aligned}
I_n^{14}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{720}\sum_{ijkqrs\zeta}\widehat{h}_{ijkqrs}u_iu_ju_ku_qu_ru_su_\zeta\widehat{b}_\zeta+\frac{1}{1152}\sum_{ijkqrstw\zeta}\widehat{h}_{ijkq}\widehat{h}_{rstw}u_iu_ju_ku_qu_ru_su_tu_wu_\zeta\widehat{b}_\zeta\\
&\quad +\frac{1}{720}\sum_{ijkqrstw\zeta}\widehat{h}_{ijk}\widehat{h}_{qrstw}u_iu_ju_ku_qu_ru_su_tu_wu_\zeta\widehat{b}_\zeta\\
&\quad -\frac{1}{1728}\sum_{ijkqrstwv\beta\zeta ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv\beta}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\zeta\widehat{b}_\zeta\\
&\quad +\frac{1}{31104}\sum_{ijkqrstwv\beta\tau\phi\zeta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\beta u_\tau u_\phi u_\zeta\widehat{b}_\zeta,
\end{aligned}$$

$$\begin{aligned}
I_n^2\left(\theta,\widehat{\theta}_n\right) &= n^{-1}\frac{1}{2}\sum\widehat{b}_{\zeta\eta}u_\zeta u_\eta\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\
&= \frac{1}{2}\left[n^{-1}\sum\widehat{b}_{\zeta\eta}u_\zeta u_\eta+n^{-\frac{3}{2}}I_n^{21}\left(\theta,\widehat{\theta}_n\right)+n^{-2}I_n^{22}\left(\theta,\widehat{\theta}_n\right)+n^{-\frac{5}{2}}I_n^{23}\left(\theta,\widehat{\theta}_n\right)\right],
\end{aligned}$$

$$\begin{aligned}
I_n^{21}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{6}\sum_{ijk\zeta\eta}\widehat{h}_{ijk}u_iu_ju_ku_\zeta u_\eta\widehat{b}_{\zeta\eta},\\
I_n^{22}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{24}\sum_{ijkq\zeta\eta}\widehat{h}_{ijkq}u_iu_ju_ku_qu_\zeta u_\eta\widehat{b}_{\zeta\eta}+\frac{1}{72}\sum_{ijkqrs\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrs}u_iu_ju_ku_qu_ru_su_\zeta u_\eta\widehat{b}_{\zeta\eta},
\end{aligned}$$

$$I_n^{23}\left(\theta,\widehat{\theta}_n\right) = -\frac{1}{120}\sum_{ijkqr\zeta\eta}\widehat{h}_{ijkqr}u_iu_ju_ku_qu_ru_\zeta u_\eta\widehat{b}_{\zeta\eta}+\frac{1}{144}\sum_{ijkqrst\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrst}u_iu_ju_ku_qu_ru_su_tu_\zeta u_\eta\widehat{b}_{\zeta\eta}$$

$$-\frac{1}{1296}\sum_{ijkqrstwv\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}u_iu_ju_ku_qu_ru_su_tu_wu_vu_\zeta u_\eta\widehat{b}_{\zeta\eta},$$

$$\begin{aligned}
I_n^3\left(\theta,\widehat{\theta}_n\right) &= n^{-\frac{3}{2}}\frac{1}{6}\sum\widehat{b}_{\zeta\eta\xi}u_\zeta u_\eta u_\xi\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\
&= \frac{1}{6}\left[n^{-\frac{3}{2}}\sum\widehat{b}_{\zeta\eta\xi}u_\zeta u_\eta u_\xi+n^{-2}I_n^{31}\left(\theta,\widehat{\theta}_n\right)+n^{-\frac{5}{2}}I_n^{32}\left(\theta,\widehat{\theta}_n\right)\right],
\end{aligned}$$

$$\begin{aligned}
I_n^{31}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{6}\sum_{ijk\zeta\eta}\widehat{h}_{ijk}u_iu_ju_ku_\zeta u_\eta u_\xi\widehat{b}_{\zeta\eta\xi},\\
I_n^{32}\left(\theta,\widehat{\theta}_n\right) &= -\frac{1}{24}\sum_{ijkq\zeta\eta}\widehat{h}_{ijkq}u_iu_ju_ku_qu_\zeta u_\eta u_\xi\widehat{b}_{\zeta\eta\xi}+\frac{1}{72}\sum_{ijkqrs\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrs}u_iu_ju_ku_qu_ru_su_\zeta u_\eta u_\xi\widehat{b}_{\zeta\eta\xi},
\end{aligned}$$

$$\begin{aligned}
&I_n^4\left(\theta,\widehat{\theta}_n\right)\\
&= n^{-2}\frac{1}{24}\sum\widehat{b}_{\zeta\eta\xi\omega}u_\zeta u_\eta u_\xi u_\omega\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\
&= \frac{1}{24}\left[n^{-2}\sum\widehat{b}_{\zeta\eta\xi\omega}u_\zeta u_\eta u_\xi u_\omega+n^{-\frac{5}{2}}I_n^{41}\left(\theta,\widehat{\theta}_n\right)\right],
\end{aligned}$$

$$I_n^{41}\left(\theta,\widehat{\theta}_n\right)=-\frac{1}{6}\sum_{ijk\zeta\eta}\widehat{h}_{ijk}u_iu_ju_ku_\zeta u_\eta u_\xi u_\omega\widehat{b}_{\zeta\eta\xi\omega},$$

$$\begin{aligned}
&I_n^5\left(\theta,\widehat{\theta}_n\right)\\
&= n^{-\frac{5}{2}}\frac{1}{120}\sum_{ijkqr}\widehat{b}_{ijkqr}u_iu_ju_ku_qu_r\left(1+\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{6}\Xi_3+\frac{1}{24}\Xi_4+\frac{1}{120}\Xi_5\right)\\
&= \frac{1}{120}\left[n^{-\frac{5}{2}}\sum_{ijkqr}\widehat{b}_{ijkqr}u_iu_ju_ku_qu_r\right].
\end{aligned}$$

Let

$$f(u)=\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\left\{\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\right\},$$

be the density function of the Normal distribution with mean 0 and covariance matrix $\left(\nabla^2\widehat{h}\right)^{-1}$. Then we have

$$\begin{aligned}
&\int_{R^p}I_n\left(\theta,\widehat{\theta}_n\right)f(u)\,du\\
&= \widehat{b}+\frac{1}{n}\left(-\frac{1}{24}\sum_{ijkq}\widehat{h}_{ijkq}\mu^4_{ijkq}+\frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs}-\frac{1}{6}\sum_{ijk\zeta}\widehat{h}_{ijk}\mu^4_{ijk\zeta}\widehat{b}_\zeta+\frac{1}{2}\sum_{\zeta\eta}\widehat{b}_{\zeta\eta}\widehat{h}^{\zeta\eta}\right)
\end{aligned}$$

$$
+\frac{1}{n^2}\begin{pmatrix}
-\frac{1}{720}\sum_{ijkqrs}\widehat{h}_{ijkqrs}\mu^6_{ijkqrs}+\frac{1}{1152}\sum_{ijkqrstw}\widehat{h}_{ijkq}\widehat{h}_{rstw}\mu^8_{ijkqrstw} \\
+\frac{1}{720}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrstw}\mu^8_{ijkqrstw}-\frac{1}{1728}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv\beta}\mu^{10}_{ijkqrstwv\beta} \\
+\frac{1}{31104}\sum_{ijkqrstwv\beta\tau\phi}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}\mu^{12}_{ijkqrstwv\beta\tau\phi} \\
-\frac{1}{120}\sum_{ijkqr\zeta}\widehat{h}_{ijkqr}\mu^6_{ijkqr\zeta}\widehat{b}_{\zeta}+\frac{1}{144}\sum_{ijkqrst\zeta}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^8_{ijkqrst\zeta}\widehat{b}_{\zeta} \\
-\frac{1}{1296}\sum_{ijkqrstwv\zeta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\zeta}\widehat{b}_{\zeta} \\
-\frac{1}{48}\sum_{ijkq\zeta\eta}\widehat{h}_{ijkq}\mu^6_{ijkq\zeta\eta}\widehat{b}_{\zeta\eta}+\frac{1}{144}\sum_{ijkqrs\zeta\eta}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrs\zeta\eta}\widehat{b}_{\zeta\eta} \\
-\frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{\zeta\eta\xi}+\frac{1}{24}\sum_{\zeta\eta\xi\omega}\widehat{b}_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}
\end{pmatrix}
$$
$$
+O\left(n^{-3}\right).
$$

Note that the odd order central moments of the multivariate normal distribution vanish and, by expanding the domain from $B_{\sqrt{n}\delta}(0)$ to R^P, the error can be expressed as

$$
\begin{aligned}
& \int_{R^p-B_{\sqrt{n}\delta}(0)}\sum_{ijkq}\widehat{h}_{ijkq}u_iu_ju_ku_qf(u)\,du \\
= & \sum_{ijkq}\widehat{h}_{ijkq}\int_{R^p-B_{\sqrt{n}\delta}(0)}u_iu_ju_ku_qf(u)\,du \\
= & \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^P-B_{\sqrt{n}\delta}(0)}u_iu_ju_ku_q\exp\left[-\frac{1}{2}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]du \\
= & \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^P-B_{\sqrt{n}\delta}(0)}u_iu_ju_ku_q\exp\left[-\frac{1}{4}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]\exp\left[-\frac{1}{4}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]du \\
\leq & \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^p-B_{\sqrt{n}\delta}(0)}u_iu_ju_ku_q\exp\left[-\frac{1}{4}\sum_{ij}\lambda_{\min}u_iu_j\right]\exp\left[-\frac{1}{4}\sum_{ij}\widehat{h}_{ij}u_iu_j\right]du \\
\leq & \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{4}\lambda_{\min}\delta^2n\right]\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^p}u_iu_ju_ku_q\exp\left[-\frac{1}{2}\sum_{ij}\left(\frac{1}{2}\widehat{h}_{ij}\right)u_iu_j\right]du \\
= & M'\exp\left[-\frac{1}{4}\lambda_{\min}\delta^2n\right],
\end{aligned}
$$

where $\lambda_{\min}>0$ is the smallest eigenvalue of $\nabla^2\widehat{h}$ and

$$
M'=\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{P}{2}}\left|\nabla^2\widehat{h}\right|^{\frac{1}{2}}\sum_{ijkq}\widehat{h}_{ijkq}\int_{R^p}u_iu_ju_ku_q\exp\left[-\sum_{ij}\left(\frac{1}{4}\widehat{h}_{ij}\right)u_iu_j\right]du<\infty,
$$

by the imposed assumptions. The first inequality follows from the fact that

$$
\lambda_{\min}=\min_{\|e\|=1}f(e)=e^TAe,
$$

where A is a positive definite matrix, and $\lambda_{\min}$ is the the smallest eigenvalue of A. Here, we only express one term of $I_n\left(\theta,\widehat{\theta}_n\right)$. The other terms can be analyzed similarly.

Hence, we have

$$\int_{B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du = \int_{R^P} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du - \int_{R^p - B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du,$$

and

$$\begin{aligned}
& \int_{\Theta} b(\theta)\exp\left[-nh(\theta)\right] d\theta \\
= \;& \int_{B_\delta\left(\widehat{\theta}_n\right)} b(\theta)\exp\left[-nh(\theta)\right] d\theta + \int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b(\theta)\exp\left[-nh(\theta)\right] d\theta \\
= \;& \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right)\left[\int_{B_{\sqrt{n}\delta}(0)} \left[I_n\left(\theta,\widehat{\theta}\right) + R_n\left(\theta,\widehat{\theta}\right)\right] f(u)\,du + O\left(n^{-3}\right)\right] \\
= \;& \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right)\left[\int_{B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du + O\left(n^{-3}\right)\right] \\
= \;& \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right)\left[\begin{array}{c} \int_{R^P} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du \\ -\int_{R^P - B_{\sqrt{n}\delta}(0)} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du \end{array} + O\left(n^{-3}\right)\right] \\
= \;& \left(\sqrt{2\pi}\right)^{\frac{P}{2}} \left|n\nabla^2\widehat{h}\right|^{-\frac{1}{2}} \exp\left(-n\widehat{h}\right)\left[\int_{R^P} I_n\left(\theta,\widehat{\theta}\right) f(u)\,du + O\left(n^{-3}\right)\right].
\end{aligned}$$

Hence, this lemma is proved. ■

2 High Order Stochastic Expansions

In this section we will develop high order stochastic Laplace expansions. Suppose $\mathbf{y} = (y_1, y_2, \cdots, y_n)'$ is a collection of random variables defined on a common probability space $\{\Omega, \mathcal{F}, \wp_\theta\}$, where Ω is a sample space, $\mathcal{F}$ is a sigma-algebra, and $\wp_\theta$ is a probability measure that depends on parameter $\theta \in \Theta$, a compact subset of R^P. Assume $\{y_i, i = 1, 2, \cdots\}$ take values in the same mathematical space $\Im$, which must be measurable with respect to some sigma-algebra, Σ. Let $h_n(\mathbf{y},\theta)$ be a sequence of functions, each of which is eight-times continuously differentiable with respect to θ and has an interior global minimum $\left\{\widehat{\theta}_n : n = 1, 2, \ldots\right\}$; $b(\theta)$ is a six-times continuously differentiable real function of θ. When there is no confusion, we write $h_n(\mathbf{y},\theta)$ as $h(\theta)$ or h_n or even h and $b(\theta)$ as b. For any function $f(\theta)$, let $\widehat{f}$ be the value of function f evaluated at $\widehat{\theta}_n$, i.e., $\widehat{f} := f\left(\widehat{\theta}_n\right)$.

We call the pair $(\{h_n\}, b)$ satisfy the analytical assumptions for the stochastic Laplace method on $\wp_\theta$ if the following assumptions are satisfied. There exists positive numbers ε, M and η such that (i) with probability approach one (w.p.a.1), for all $\theta \in B_\varepsilon\left(\widehat{\theta}_n\right)$ and all $1 \leq j_1, \cdots, j_d \leq P$ with $0 \leq d \leq 8$, $\|h_n(\theta)\| < M$ and $\|h_{j_1\cdots j_d}(\theta)\| < M$; (ii) w.p.a.1, $\nabla^2\widehat{h}$ is positive definite and $\det\left(\nabla^2\widehat{h}\right) > \eta$; (iii) $\int_\Theta b(\theta)\exp\left(-nh_n(\theta)\right) d\theta$ exists and is finite, and for all δ for which $0 < \delta < \varepsilon$ and $B_\delta\left(\widehat{\theta}_n\right) \subseteq \Theta$,

$$\left[\det\left(n\nabla^2\widehat{h}\right)\right]^{\frac{1}{2}} \int_{\Theta - B_\delta\left(\widehat{\theta}_n\right)} b\left(\theta\right) \exp\left[-n\left(h_n\left(\theta\right) - \widehat{h}\right)\right] d\theta = O_p\left(n^{-3}\right).$$

Note that our assumptions are different from those in Section 3 of Kass et al (1990) in two aspects. First, we require $h_n(\theta)$ be eight-times continuously differentiable and $b(\theta)$ be six-times continuously differentiable. Second, for conditions (ii) and (iii), instead of almost sure boundedness and almost sure convergence, we assume they hold w.p.a.1. We do so because we are interested in convergence in probability only. Following the result in Theorem 7 of Kass et al (1990), $(\{h_n\}, b)$ satisfy the analytical assumptions for stochastic Laplace's method on $\wp_\theta$ and Lemma 1.1 above, it is straightforward to show that

$$\int_\Theta b\left(\theta\right) \exp\left[-nh\left(\theta\right)\right] d\theta = \left(2\pi\right)^{\frac{P}{2}} \left[\det\left(n\nabla^2\widehat{h}\right)\right]^{-\frac{1}{2}} \exp\left[-n\widehat{h}\right] \left(\widehat{b} + \frac{1}{n}A_1 + \frac{1}{n^2}A_2 + O_p\left(n^{-3}\right)\right), \tag{1}$$

where the expressions for A_1 and A_2 are given in Lemma 1.1.

Lemma 2.1 *If both $(\{h_n\}, g \times b_D)$ and $(\{h_n\}, b_D)$ satisfy the analytical assumptions for the stochastic Laplace method on $\wp_\theta$, then*

$$\frac{\int g\left(\theta\right) b_D\left(\theta\right) \exp\left(-nh_n\left(\theta\right)\right) d\theta}{\int b_D\left(\theta\right) \exp\left(-nh_n\left(\theta\right)\right) d\theta} = \widehat{g} + \frac{1}{n}B_1 + \frac{1}{n^2}\left(B_2 - B_3\right) + O_p\left(\frac{1}{n^3}\right),$$

where

$$B_1 = \frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij} + \frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,j}\widehat{g}_i}{\widehat{b}_D} - \frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\widehat{g}_q,$$

$$\begin{aligned}
B_2 \quad = \quad & -\frac{1}{120}\sum_{ijkqrs}\widehat{h}_{ijkqr}\mu^6_{ijkqrs}\widehat{g}_s + \frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^8_{ijkqrstw}\widehat{g}_w \\
& -\frac{1}{1296}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{g}_\beta - \frac{1}{24}\frac{\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{b}_{D,s}\widehat{g}_r}{\widehat{b}_D} \\
& +\frac{1}{72}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{D,w}\widehat{g}_t}{\widehat{b}_D} - \frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{D,\eta\xi}\widehat{g}_\zeta}{\widehat{b}_D} \\
& +\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{D,\eta\xi\omega}\widehat{g}_\zeta}{\widehat{b}_D} - \frac{1}{48}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_{rs} \\
& +\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{g}_{tw} - \frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi} \\
& +\frac{1}{24}\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi\omega} - \frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D}
\end{aligned}$$

$$+\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi}\widehat{b}_{D,\omega}}{\widehat{b}_D}+\frac{1}{4}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D},$$

$$B_3=\left(\frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\frac{\widehat{b}_{D,ij}}{\widehat{b}_D}-\frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\frac{\widehat{b}_{D,q}}{\widehat{b}_D}+\frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs}-\frac{1}{24}\sum_{ijkq}\widehat{h}_{ijkq}\mu^4_{ijkq}\right)B_1,$$

where $\sigma_{ij}=h^{ij}$.

Proof. If $\left\{h^N(\theta),b_N\right\}$ and $\left\{h^D(\theta),b_D\right\}$ satisfy the analytical assumptions for the stochastic Laplace method on $\wp_\theta$, then by (1)

$$\frac{\int b_N(\theta)\exp\left[-nh^N(\theta)\right]d\theta}{\int b_D(\theta)\exp\left[-nh^D(\theta)\right]d\theta}=\frac{\left|\nabla^2\widehat{h}^N\right|^{-\frac{1}{2}}\exp\left[-nh\left(\widehat{\theta}^N_n\right)\right]}{\left|\nabla^2\widehat{h}^D\right|^{-\frac{1}{2}}\exp\left[-nh\left(\widehat{\theta}^D_n\right)\right]}\frac{b_k\left(\widehat{\theta}^N_n\right)+\frac{1}{n}c_N+\frac{1}{n^2}d_N+O_p\left(n^{-3}\right)}{b_k\left(\widehat{\theta}^D_n\right)+\frac{1}{n}c_D+\frac{1}{n^2}d_D+O_p\left(n^{-3}\right)}.$$

From Tierney and Kadane (1986) and Miyata (2004, 2010), we have

$$\frac{b_N\left(\widehat{\theta}^N_n\right)+\frac{1}{n}c_N+\frac{1}{n^2}d_N+O_p\left(n^{-3}\right)}{b_D\left(\widehat{\theta}^D_n\right)+\frac{1}{n}c_D+\frac{1}{n^2}d_D+O_p\left(n^{-3}\right)}=\frac{b_N\left(\widehat{\theta}^N_n\right)}{b_D\left(\widehat{\theta}^D_n\right)}\left[\frac{1+\frac{1}{n}\frac{c_N}{b_N\left(\widehat{\theta}^N_n\right)}+\frac{1}{n^2}\frac{d_N}{b_N\left(\widehat{\theta}^N_n\right)}+O_p\left(n^{-3}\right)}{1+\frac{1}{n}\frac{c_D}{b_D\left(\widehat{\theta}^D_n\right)}+\frac{1}{n^2}\frac{d_D}{b_D\left(\widehat{\theta}^D_n\right)}+O_p\left(n^{-3}\right)}\right]$$

$$=\frac{b_N\left(\widehat{\theta}^N_n\right)}{b_D\left(\widehat{\theta}^D_n\right)}\left\{\begin{array}{c}1+\frac{1}{n}\left(\frac{c_N}{b_N\left(\widehat{\theta}^N_n\right)}-\frac{c_D}{b_D\left(\widehat{\theta}^D_n\right)}\right)\\+\frac{1}{n^2}\left(\frac{d_N}{b_N\left(\widehat{\theta}^N_n\right)}-\frac{d_D}{b_D\left(\widehat{\theta}^D_n\right)}-\frac{c_D}{b_D\left(\widehat{\theta}^D_n\right)}\left(\frac{c_N}{b_N\left(\widehat{\theta}^N_n\right)}-\frac{c_D}{b_D\left(\widehat{\theta}^D_n\right)}\right)\right)+O_p\left(n^{-3}\right)\end{array}\right\},$$

where

$$\begin{aligned}\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{c_N}{\widehat{b}_N}-\frac{c_D}{\widehat{b}_D}\right)&=\frac{\widehat{b}_N\left(c_N\widehat{b}_D-c_D\widehat{b}_N\right)}{\widehat{b}^2_D\widehat{b}_N}=\frac{c_N\widehat{b}_D-c_D\widehat{b}_N}{\widehat{b}^2_D}\\&=\frac{1}{2}\frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{N,ij}\widehat{b}_D-\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,ij}\widehat{b}_N}{\widehat{b}^2_D}\\&\quad-\frac{1}{6}\frac{\sum_{ijkq}\widehat{h}^N_{ijk}\mu^4_{ijkq}\widehat{b}_{N,q}\widehat{b}_D-\sum_{ijkq}\widehat{h}^D_{ijk}\mu^4_{ijkq}\widehat{b}_{D,q}\widehat{b}_N}{\widehat{b}^2_D}\\&\quad+\frac{1}{72}\frac{\widehat{b}_N\sum_{ijkqrs}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\mu^6_{ijkqrs}-\sum_{ijkqrs}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\mu^6_{ijkqrs}\widehat{b}_N}{\widehat{b}_D}\\&\quad-\frac{1}{24}\frac{\widehat{b}_N\sum_{ijkq}\widehat{h}^N_{ijkq}\mu^4_{ijkq}-\sum_{ijkq}\widehat{h}^D_{ijkq}\mu^4_{ijkq}\widehat{b}_N}{\widehat{b}_D},\end{aligned}$$

where $\sigma_{ij}=h^{ij}$.

$$\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{d_N}{\widehat{b}_N}-\frac{d_D}{\widehat{b}_D}\right)$$

$$
\begin{aligned}
= \quad & -\frac{1}{720}\frac{\widehat{b}_N \sum_{ijkqrs}\widehat{h}^N_{ijkqrs}\mu^6_{ijkqrs} - \sum_{ijkqrs}\widehat{h}^D_{ijkqrs}\mu^6_{ijkqrs}\widehat{b}_N}{\widehat{b}_D} \\
& +\frac{1}{1152}\frac{\widehat{b}_N \sum_{ijkqrstw}\widehat{h}^N_{ijkq}\widehat{h}^N_{rstw}\mu^8_{ijkqrstw} - \sum_{ijkqrstw}\widehat{h}^D_{ijkq}\widehat{h}^D_{rstw}\mu^8_{ijkqrstw}\widehat{b}_N}{\widehat{b}_D} \\
& +\frac{1}{720}\frac{\widehat{b}_N \sum_{ijkqrstw}\widehat{h}^N_{ijk}\widehat{h}^N_{qrstw}\mu^8_{ijkqrstw} - \sum_{ijkqrstw}\widehat{h}^D_{ijk}\widehat{h}^D_{qrstw}\mu^8_{ijkqrstw}\widehat{b}_N}{\widehat{b}_D} \\
& -\frac{1}{1728}\frac{\widehat{b}_N \sum_{ijkqrstwv\beta}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\widehat{h}^N_{twv}\mu^{10}_{ijkqrstwv\beta} - \sum_{ijkqrstwv\beta}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\widehat{h}^D_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{b}_N}{\widehat{b}_D} \\
& +\frac{1}{31104}\frac{\left(\begin{array}{c}\widehat{b}_N \sum_{ijkqrstwv\beta\tau\phi}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}\mu^{12}_{ijkqrstwv\beta\tau\phi} \\ -\sum_{ijkqrstwv\beta\tau\phi}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\widehat{h}_{\beta\tau\phi}\mu^{12}_{ijkqrstwv\beta\tau\phi}\widehat{b}_N\end{array}\right)}{\widehat{b}_D} \\
& -\frac{1}{120}\frac{\sum_{ijkqrs}\widehat{h}^N_{ijkqr}\mu^6_{ijkqrs}\widehat{b}_{N,s}\widehat{b}_D - \sum_{ijkqrs}\widehat{h}^D_{ijkqr}\mu^6_{ijkqrs}\widehat{b}_{D,s}\widehat{b}_N}{\widehat{b}^2_D} \\
& +\frac{1}{144}\frac{\sum_{ijkqrstw}\widehat{h}^N_{ijk}\widehat{h}^N_{qrst}\mu^8_{ijkqrstw}\widehat{b}_{N,w}\widehat{b}_D - \sum_{ijkqrstw}\widehat{h}^N_{ijk}\widehat{h}^N_{qrst}\mu^8_{ijkqrstw}\widehat{b}_{D,w}\widehat{b}_N}{\widehat{b}^2_D} \\
& -\frac{1}{1296}\frac{\sum_{ijkqrstwv\beta}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\widehat{h}^N_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{b}_{N,\beta}\widehat{b}_D - \sum_{ijkqrstwv\beta}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\widehat{h}^D_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{b}_{D,\beta}\widehat{b}_N}{\widehat{b}^2_D} \\
& -\frac{1}{48}\frac{\sum_{ijkqrs}\widehat{h}^N_{ijkq}\mu^6_{ijkqrs}\widehat{b}_{N,rs}\widehat{b}_D - \sum_{ijkqrs}\widehat{h}^D_{ijkq}\mu^6_{ijkqrs}\widehat{b}_{D,rs}\widehat{b}_N}{\widehat{b}^2_D} \\
& +\frac{1}{144}\frac{\sum_{ijkqrstw}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{N,tw}\widehat{b}_D - \sum_{ijkqrstw}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{D,tw}\widehat{b}_N}{\widehat{b}^2_D} \\
& -\frac{1}{36}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}^N_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{N,\zeta\eta\xi}\widehat{b}_D - \sum_{ijk\zeta\eta\xi}\widehat{h}^D_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{D,\zeta\eta\xi}\widehat{b}_N}{\widehat{b}^2_D} \\
& +\frac{1}{24}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{N,\zeta\eta\xi\omega}\widehat{b}_D - \sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{D,\zeta\eta\xi\omega}\widehat{b}_N}{\widehat{b}^2_D}
\end{aligned}
$$

$$
\begin{aligned}
\frac{c_D}{\widehat{b}_D}\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{c_N}{\widehat{b}_N}-\frac{c_D}{\widehat{b}_D}\right) \quad = \quad & \frac{1}{2}c_D\frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{N,ij}\widehat{b}_D - \sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,ij}\widehat{b}_N}{\widehat{b}^3_D} \\
& -\frac{1}{6}c_D\frac{\sum_{ijkq}\widehat{h}^N_{ijk}\mu_{ijkq}\widehat{b}_{N,q}\widehat{b}_D - \sum_{ijkq}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\mu^6_{ijkqrs}\widehat{b}_{D,q}\widehat{b}_N}{\widehat{b}^3_D} \\
& +\frac{1}{72}c_D\frac{\widehat{b}_N \sum_{ijkqrs}\widehat{h}^N_{ijk}\widehat{h}^N_{qrs}\mu^6_{ijkqrs} - \sum_{ijkq}\widehat{h}^D_{ijk}\widehat{h}^D_{qrs}\mu^6_{ijkqrs}\widehat{b}_N}{\widehat{b}^2_D} \\
& -\frac{1}{24}c_D\frac{\widehat{b}_N \sum_{ijkq}\widehat{h}^N_{ijkq}\mu^4_{ijkq} - \sum_{ijkq}\widehat{h}^D_{ijkq}\mu^4_{ijkq}\widehat{b}_N}{\widehat{b}^2_D}.
\end{aligned}
$$

If we set $b_N(\theta) = g(\theta)\, b_D(\theta)$ and $h^N(\theta) = h^D(\theta) = h(\theta)$, we can show the following results for the derivatives of $b_N(\theta)$:

$$b_{N,i}(\theta) = g_i(\theta)\, b_D(\theta) + g(\theta)\, b_{D,i}(\theta), \tag{2}$$

$$b_{N,ij}(\theta) = g_{ij}(\theta)\, b_D(\theta) + g_i(\theta)\, b_{D,j}(\theta) + g_j(\theta)\, b_{D,i}(\theta) + g(\theta)\, b_{D,ij}(\theta), \tag{3}$$

$$\begin{aligned} b_{N,ijk}(\theta) &= g_{ijk}(\theta)\, b_D(\theta) + g_{ij}(\theta)\, b_{D,k}(\theta) + g_{ik}(\theta)\, b_{D,j}(\theta) + g_i(\theta)\, b_{D,jk}(\theta) + g_{jk}(\theta)\, b_{D,i}(\theta) \\ &\quad + g_j(\theta)\, b_{D,ik}(\theta) + g_k(\theta)\, b_{D,ij}(\theta) + g(\theta)\, b_{D,ijk}(\theta), \end{aligned} \tag{4}$$

$$\begin{aligned} b_{N,ijkq}(\theta) &= g_{ijkq}(\theta)\, b_D(\theta) + g_{ijk}(\theta)\, b_{D,q}(\theta) + g_{ijq}(\theta)\, b_{D,k}(\theta) + g_{ij}(\theta)\, b_{D,kq}(\theta) \\ &\quad + g_{ikq}(\theta)\, b_{D,j}(\theta) + g_{ik}(\theta)\, b_{D,jq}(\theta) + g_{iq}(\theta)\, b_{D,jk}(\theta) + g_i(\theta)\, b_{D,jkq}(\theta) \\ &\quad + g_{jkq}(\theta)\, b_{D,i}(\theta) + g_{jk}(\theta)\, b_{D,iq}(\theta) + g_{jq}(\theta)\, b_{D,ik}(\theta) + g_j(\theta)\, b_{D,ikq}(\theta) \\ &\quad + g_{kq}(\theta)\, b_{D,ij}(\theta) + g_k(\theta)\, b_{D,ijq}(\theta) + g_q(\theta)\, b_{D,ijk}(\theta) + g(\theta)\, b_{D,ijkq}(\theta), \end{aligned} \tag{5}$$

$$\begin{aligned} & b_{N,ij}(\theta)\, b_D(\theta) - b_{D,ij}(\theta)\, b_N(\theta) \\ = {} & [g_{ij}(\theta)\, b_D(\theta) + g_i(\theta)\, b_{D,j}(\theta) + g_j(\theta)\, b_{D,i}(\theta) + g(\theta)\, b_{D,ij}(\theta)]\, b_D(\theta) - b_{D,ij}(\theta)\, g(\theta)\, b_D(\theta) \\ = {} & g_{ij}(\theta)\, b_D(\theta)^2 + g_i(\theta)\, b_{D,j}(\theta)\, b_D(\theta) + g_j(\theta)\, b_{D,i}(\theta)\, b_D(\theta) \\ & + g(\theta)\, b_{D,ij}(\theta)\, b_D(\theta) - b_{D,ij}(\theta)\, g(\theta)\, b_D(\theta) \\ = {} & g_{ij}(\theta)\, b_D(\theta)^2 + g_i(\theta)\, b_{D,j}(\theta)\, b_D(\theta) + g_j(\theta)\, b_{D,i}(\theta)\, b_D(\theta), \end{aligned} \tag{6}$$

$$\begin{aligned} & b_{N,i}(\theta)\, b_D(\theta) - b_{D,i}(\theta)\, b_N(\theta) \\ = {} & (g_i(\theta)\, b_D(\theta) + g(\theta)\, b_{D,i}(\theta))\, b_D(\theta) - b_{D,i}(\theta)\, g(\theta)\, b_D(\theta) = g_i(\theta)\, b_D(\theta)^2, \end{aligned} \tag{7}$$

$$\begin{aligned} & b_{N,ijk}(\theta)\, b_D(\theta) - b_{D,ijk}(\theta)\, b_N(\theta) \\ = {} & \left[\begin{array}{c} g_{ijk}(\theta)\, b_D(\theta) + g_{ij}(\theta)\, b_{D,k}(\theta) + g_{ik}(\theta)\, b_{D,j}(\theta) + g_i(\theta)\, b_{D,jk}(\theta) \\ + g_{jk}(\theta)\, b_{D,i}(\theta) + g_j(\theta)\, b_{D,ik}(\theta) + g_k(\theta)\, b_{D,ij}(\theta) \end{array} \right] b_D(\theta), \end{aligned} \tag{8}$$

$$\begin{aligned} & b_{N,ijkq}(\theta)\, b_D(\theta) - b_{D,ijkq}(\theta)\, b_N(\theta) \\ = {} & \left[\begin{array}{c} g_{ijkq}(\theta)\, b_D(\theta) + g_{ijk}(\theta)\, b_{D,q}(\theta) + g_{ijq}(\theta)\, b_{D,k}(\theta) + g_{ij}(\theta)\, b_{D,kq}(\theta) \\ + g_{ikq}(\theta)\, b_{D,j}(\theta) + g_{ik}(\theta)\, b_{D,jq}(\theta) + g_{iq}(\theta)\, b_{D,jk}(\theta) + g_i(\theta)\, b_{D,jkq}(\theta) \end{array} \right] b_D(\theta) \\ & + \left[\begin{array}{c} g_{jkq}(\theta)\, b_{D,i}(\theta) + g_{jk}(\theta)\, b_{D,iq}(\theta) + g_{jq}(\theta)\, b_{D,ik}(\theta) + g_j(\theta)\, b_{D,ikq}(\theta) \\ + g_{kq}(\theta)\, b_{D,ij}(\theta) + g_k(\theta)\, b_{D,ijq}(\theta) + g_q(\theta)\, b_{D,ijk}(\theta) \end{array} \right] b_D(\theta). \end{aligned} \tag{9}$$

Thus, we have

$$\begin{aligned} \frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{c_N}{\widehat{b}_N} - \frac{c_D}{\widehat{b}_D}\right) &= \frac{1}{2}\frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{N,ij}\widehat{b}_D - \sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,ij}\widehat{b}_N}{\widehat{b}_D^2} \\ & -\frac{1}{6}\frac{\sum_{ijkq}\widehat{h}_{ijk}^N \mu_{ijkq}^4 \widehat{b}_{N,q}\widehat{b}_D - \sum_{ijkq}\widehat{h}_{ijk}^D \mu_{ijkq}^4 \widehat{b}_{D,q}\widehat{b}_N}{\widehat{b}_D^2} \end{aligned}$$

$$= \frac{1}{2}\frac{\sum_{ij}\widehat{\sigma}_{ij}\left(\widehat{b}_{N,ij}\widehat{b}_D - \widehat{b}_{D,ij}\widehat{b}_N\right)}{\widehat{b}_D^2} - \frac{1}{6}\frac{\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\left(\widehat{b}_{N,q}\widehat{b}_D - \widehat{b}_{D,q}\widehat{b}_N\right)}{\widehat{b}_D^2},$$

where

$$\begin{aligned}
\frac{\sum_{ij}\widehat{\sigma}_{ij}\left(\widehat{b}_{N,ij}\widehat{b}_D - \widehat{b}_{D,ij}\widehat{b}_N\right)}{\widehat{b}_D^2} &= \frac{\sum_{ij}\widehat{\sigma}_{ij}\left(\widehat{g}_{ij}\widehat{b}_D^2 + \widehat{g}_i\widehat{b}_{D,j}\widehat{b}_D + \widehat{g}_j\widehat{b}_{D,i}\widehat{b}_D\right)}{\widehat{b}_D^2} \\
&= \frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij}\widehat{b}_D^2 + 2\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_i\widehat{b}_{D,j}\widehat{b}_D}{\widehat{b}_D^2} \\
&= \sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij} + \frac{2\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,j}\widehat{g}_i}{\widehat{b}_D},
\end{aligned}$$

from (6) and

$$\frac{\widehat{b}_{N,q}\widehat{b}_D - \widehat{b}_{D,q}\widehat{b}_N}{\widehat{b}_D^2} = \frac{\left(\widehat{g}_q\widehat{b}_D + \widehat{g}\widehat{b}_{D,q}\right)\widehat{b}_D - \widehat{b}_{D,q}\widehat{g}\widehat{b}_D}{\widehat{b}_D^2} = \widehat{g}_q$$

by (7). Hence,

$$\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{c_N}{\widehat{b}_N} - \frac{c_D}{\widehat{b}_D}\right) = \frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij} + \frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,j}\widehat{g}_i}{\widehat{b}_D} - \frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\widehat{g}_q.$$

From (8) and (9), we can get

$$\begin{aligned}
&\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}^N_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{N,\zeta\eta\xi}\widehat{b}_D - \sum_{ijk\zeta\eta\xi}\widehat{h}^D_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{D,\zeta\eta\xi}\widehat{b}_N}{\widehat{b}_D^2} \\
=\ & \frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\left[\widehat{g}_{\zeta\eta\xi}\widehat{b}_D + \widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi} + \widehat{g}_{\zeta\xi}\widehat{b}_{D,\eta} + \widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi} + \widehat{g}_{\eta\xi}\widehat{b}_{D,\zeta} + \widehat{g}_{\eta}\widehat{b}_{D,\zeta\xi} + \widehat{g}_{\xi}\widehat{b}_{D,\zeta\eta}\right]}{\widehat{b}_D} \\
=\ & \sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi} + \frac{3\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D} + \frac{3\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi}}{\widehat{b}_D}, \quad (10)
\end{aligned}$$

$$\begin{aligned}
&\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{N,\zeta\eta\xi\omega}\widehat{b}_D - \sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{D,\zeta\eta\xi\omega}\widehat{b}_N}{\widehat{b}_D^2} \\
=\ & \sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi\omega} + \frac{4\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi}\widehat{b}_{D,\omega}}{\widehat{b}_D} + \frac{6\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D} \\
& + \frac{4\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_D} + \frac{4\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_D}. \quad (11)
\end{aligned}$$

We can also show that

$$\frac{\widehat{b}_N}{\widehat{b}_D}\left(\frac{d_N}{\widehat{b}_N} - \frac{d_D}{\widehat{b}_D}\right)$$

$$
\begin{aligned}
= \quad & -\frac{1}{120}\frac{\sum_{ijkqrs}\widehat{h}_{ijkqr}\mu^{6}_{ijkqrs}\left(\widehat{b}_{N,s}\widehat{b}_{D}-\widehat{b}_{D,s}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& +\frac{1}{144}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^{8}_{ijkqrstw}\left(\widehat{b}_{N,w}\widehat{b}_{D}-\widehat{b}_{D,w}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& -\frac{1}{1296}\frac{\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\left(\widehat{b}_{N,\beta}\widehat{b}_{D}-\widehat{b}_{D,\beta}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& -\frac{1}{48}\frac{\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^{6}_{ijkqrs}\left(\widehat{b}_{N,rs}\widehat{b}_{D}-\widehat{b}_{D,rs}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& +\frac{1}{144}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^{8}_{ijkqrstw}\left(\widehat{b}_{N,tw}\widehat{b}_{D}-\widehat{b}_{D,tw}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& -\frac{1}{36}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\left(\widehat{b}_{N,\zeta\eta\xi}\widehat{b}_{D}-\widehat{b}_{D,\zeta\eta\xi}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}} \\
& +\frac{1}{24}\frac{\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\left(\widehat{b}_{N,\zeta\eta\xi\omega}\widehat{b}_{D}-\widehat{b}_{D,\zeta\eta\xi\omega}\widehat{b}_{N}\right)}{\widehat{b}^{2}_{D}}.
\end{aligned}
$$

since $h^{N}(\theta)=h^{D}(\theta)=h(\theta)$. Hence, with (6), (7), (10), and (11), it can be shown that

$$
\begin{aligned}
& \frac{\widehat{b}_{N}}{\widehat{b}_{D}}\left(\frac{d_{N}}{\widehat{b}_{N}}-\frac{d_{D}}{\widehat{b}_{D}}\right) \\
= \quad & -\frac{1}{120}\sum_{ijkqrs}\widehat{h}_{ijkqr}\mu^{6}_{ijkqrs}\widehat{g}_{s}+\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^{8}_{ijkqrstw}\widehat{g}_{w} \\
& -\frac{1}{1296}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{g}_{\beta}-\frac{1}{24}\frac{\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^{6}_{ijkqrs}\widehat{b}_{D,s}\widehat{g}_{r}}{\widehat{b}_{D}} \\
& +\frac{1}{72}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^{8}_{ijkqrstw}\widehat{b}_{D,w}\widehat{g}_{t}}{\widehat{b}_{D}}-\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi}}{\widehat{b}_{D}} \\
& +\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\widehat{g}_{\zeta}\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_{D}}-\frac{1}{48}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^{6}_{ijkqrs}\widehat{g}_{rs} \\
& +\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^{8}_{ijkqrstw}\widehat{g}_{tw}-\frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi} \\
& +\frac{1}{24}\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi\omega}-\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^{6}_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_{D}} \\
& +\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi}\widehat{b}_{D,\omega}}{\widehat{b}_{D}}+\frac{1}{4}\frac{\sum_{\zeta\eta\xi\omega}\mu^{4}_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_{D}}.
\end{aligned}
$$

■

Using matrix notation for high order derivatives used in Magnus and Neudecker (1999) (with the exception that the first order derivative of a scalar function in our setting is a column vector), we can write Lemma 2.1 in matrix form. Before we do that, let us first introduce the following Generalized Isserlis theorem.

Theorem 2.1 (Generalized Isserlis Theorem) *If $A = \{\alpha_1, \ldots, \alpha_{2N}\}$ is a set of integers such that $1 \leq \alpha_i \leq P$, for each $i \in [1, 2N]$ and $X \in R^P$ is a zero mean multivariate normal random vector, then*

$$E(X_A) = \underset{A}{\Sigma\Pi} E(X_i X_j), \tag{12}$$

where $EX_A = E\left(\Pi_{i=1}^{2N} X_{\alpha_i}\right) = \mu_{\alpha_1,\ldots,\alpha_{2N}}$ and the notation $\Sigma\Pi$ means summing over all distinct ways of partitioning $X_{\alpha_1}, \ldots, X_{\alpha_{2N}}$ into pairs (X_i, X_j) and each summand is the product of the N pairs. This yields $(2N)!/\left(2^N N!\right) = (2N-1)!!$ terms in the sum where $(2N-1)!!$ is the double factorial defined by $(2N-1)!! = (2N-1) \times (2N-3) \times \cdots \times 1$.

The Isserlis theorem, first obtained by Isserlis (1918), expresses the higher order moments of a zero mean Gaussian vector $X \in R^P$ in terms of its covariance matrix. The generalized Isserlis theorem is due to Withers (1985) and Vignat (2012). For example, if $2N = 4$, then

$$E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

where $\alpha_i = i$ for $i \in [1, 4]$ and there are $(2 \times 2 - 1)!! = 3 \times 1 = 3$ terms in the sum. If $2N = 6$, there are $5 \times 3 \times 1 = 15$ terms in the sum.

Lemma 2.2 *Let $\nabla^j \widehat{h}$, $\nabla^j \widehat{g}$ and $\nabla^j \widehat{b}_D$ be the jth order derivatives of $h(\theta)$, $g(\theta)$ and $b_D(\theta)$ evaluated at $\widehat{\theta}_n$ respectively. If both $(\{h_n\}, g \times b_D)$ and $(\{h_n\}, b_D)$ satisfy the analytical assumptions for the stochastic Laplace method on $\wp_\theta$, then*

$$\frac{\int g(\theta) b_D(\theta) \exp(-nh_n(\theta)) d\theta}{\int b_D(\theta) \exp(-nh_n(\theta)) d\theta} = \widehat{g} + \frac{1}{n}B_1 + \frac{1}{n^2}(B_2 - B_3) + O_p\left(\frac{1}{n^3}\right),$$

where

$$B_1 = \frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right] + (\nabla\widehat{g})'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} - \frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g},$$

$$\begin{aligned}
B_2 \;=\; & -\frac{1}{8}(\nabla\widehat{g})'\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^5\widehat{h}\right)' vec\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right] \\
& + \frac{35}{48}(\nabla\widehat{g})'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]'\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right) \\
& - \frac{35}{48}\nabla\widehat{g}'\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\times \\
& \left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right] \\
& - \frac{5}{8}(\nabla\widehat{g})'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} tr\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{35}{24}\left(\nabla\widehat{g}\right)'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\\
&-\frac{5}{4}\left(\nabla\widehat{g}\right)'\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)tr\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right]\\
&+\frac{1}{2}\left(\nabla\widehat{g}\right)'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\left(\nabla^3\widehat{b}_D\right)'}{\widehat{b}_D}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\\
&-\frac{5}{16}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]\\
&+\frac{35}{48}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)tr\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right]\\
&-\frac{5}{12}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{g}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\\
&+\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\nabla^4\widehat{g}'\right]\\
&-\frac{5}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\\
&+\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}+\frac{3}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right],
\end{aligned}
$$

$$
B_3 = B_4\times B_1,
$$

$$
\begin{aligned}
B_4 \;=\;& \frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right]-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\widehat{h}^{(3)}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\\
&+\frac{5}{24}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\\
&-\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right],
\end{aligned}
$$

where $\sigma_{ij}=h^{ij}$ and $\mathbf{tr}$ denotes the trace of a matrix.

Proof. From (2.1), we first write each term of B_1 into matrix form by (12)

$$
\frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij}=\frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\widehat{g}_{ij}=\frac{1}{2}tr\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right],
$$

$$
\frac{\sum_{ij}\widehat{\sigma}_{ij}\widehat{b}_{D,j}\widehat{g}_i}{\widehat{b}_D}=\sum_{ij}\widehat{g}_i\widehat{\sigma}_{ij}\frac{\widehat{b}_{D,j}}{\widehat{b}_D}=\left(\nabla\widehat{g}\right)'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\nabla^3\widehat{h}\nabla\widehat{g},
$$

$$
-\frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\widehat{g}_q=-\frac{1}{2}\sum_{ijkq}\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{g}_q=-\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla\widehat{g}.
$$

Then we can similarly write each term of B_2 into matrix form by (12)

$$
-\frac{1}{120}\sum_{ijkqrs}\widehat{h}_{ijkqr}\mu^6_{ijkqrs}\widehat{g}_s \;=\; -\frac{15}{120}\sum_{ijkqrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{h}_{ijkqr}\widehat{\sigma}_{rs}\widehat{g}_s
$$

$$= \quad -\frac{1}{8}\left(\nabla\widehat{g}\right)'\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^5\widehat{h}\right)' vec\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right],$$

$$\begin{aligned}
&\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrst}\mu^8_{ijkqrstw}\widehat{g}_w \\
=\ &\frac{105}{144}\sum_{ijkqrstw}\widehat{g}_w\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{h}_{qrst}\widehat{\sigma}_{tw} = \frac{35}{48}\sum_{ijkqrstw}\widehat{g}_w\left(\widehat{\sigma}_{wt}\widehat{\sigma}_{rs}\widehat{h}_{trsq}\right)\widehat{\sigma}_{qk}\left(\widehat{h}_{kij}\widehat{\sigma}_{ij}\right) \\
=\ &\frac{35}{48}\left(\nabla\widehat{g}\right)'\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]'\nabla^4\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right),
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{1296}\sum_{ijkqrstwv\beta}\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{h}_{twv}\mu^{10}_{ijkqrstwv\beta}\widehat{g}_\beta \\
=\ &-\frac{945}{1296}\sum_{ijkqrstwv\beta}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{h}_{qrs}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\widehat{h}_{twv}\widehat{\sigma}_{v\beta}\widehat{g}_\beta \\
=\ &-\frac{35}{48}\sum_{twv\beta}\widehat{g}_\beta\widehat{\sigma}_{\beta v}\left(\widehat{\sigma}_{tw}\widehat{h}_{twv}\right)\sum_{ijkqrs}\left(\widehat{\sigma}_{ij}\widehat{h}_{ijk}\right)\widehat{\sigma}_{kq}\left(\widehat{h}_{qrs}\widehat{\sigma}_{rs}\right) \\
=\ &-\frac{35}{48}\nabla\widehat{g}'\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\times \\
&\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right],
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{24}\frac{\sum_{ijkqrs}\widehat{g}_r\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{b}_{D,s}\widehat{g}_r}{\widehat{b}_D} \\
=\ &-\frac{15}{24}\sum_{ijkqrs}\widehat{g}_r\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\frac{\widehat{b}_{D,s}}{\widehat{b}_D} = -\frac{5}{8}\sum_{rs}\widehat{g}_r\widehat{\sigma}_{rs}\frac{\widehat{b}_{D,s}}{\widehat{b}_D}\sum_{ijkq}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq} \\
=\ &-\frac{5}{8}\left(\nabla\widehat{g}\right)'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right],
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{72}\frac{\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{b}_{D,w}\widehat{g}_t}{\widehat{b}_D} \\
=\ &\frac{105}{72}\sum_{ijkqrstw}\widehat{g}_t\widehat{h}_{ijk}\widehat{h}_{qrs}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{\sigma}_{tw}\frac{\widehat{b}_{D,w}}{\widehat{b}_D} = \frac{35}{24}\sum_{tw}\left(\widehat{g}_t\widehat{\sigma}_{wt}\frac{\widehat{b}_{D,w}}{\widehat{b}_D}\right)\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{\sigma}_{rs}\widehat{h}_{qrs} \\
=\ &\frac{35}{24}\left(\nabla\widehat{g}\right)'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^3\widehat{h}' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right],
\end{aligned}$$

$$-\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{b}_{D,\eta\xi}\widehat{g}_\zeta}{\widehat{b}_D}$$

$$
\begin{aligned}
&= -\frac{15}{12}\sum_{ijk\zeta\eta\xi}\widehat{g}_\zeta\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{k\zeta}\widehat{\sigma}_{\eta\xi}\frac{\widehat{b}_{D,\eta\xi}}{\widehat{b}_D} = -\frac{5}{4}\sum_{ijk\zeta}\widehat{g}_\zeta\widehat{\sigma}_{\zeta k}\left(\widehat{h}_{ijk}\widehat{\sigma}_{ij}\right)\sum_{\eta\xi}\widehat{\sigma}_{\eta\xi}\frac{\widehat{b}_{D,\eta\xi}}{\widehat{b}_D} \\
&= -\frac{5}{4}(\nabla\widehat{g})'\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{b}_{D,\eta\xi\omega}\widehat{g}_\zeta}{\widehat{b}_D} &= \frac{3}{6}\sum_{\zeta\eta\xi\omega}\widehat{g}_\zeta\widehat{\sigma}_{\zeta\eta}\widehat{\sigma}_{\xi\omega}\frac{\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_D} = \frac{1}{2}\sum_{\zeta\eta\xi\omega}\widehat{g}_\zeta\widehat{\sigma}_{\zeta\eta}\frac{\widehat{b}_{D,\eta\xi\omega}}{\widehat{b}_D}\widehat{\sigma}_{\xi\omega} \\
&= \frac{1}{2}(\nabla\widehat{g})'\left(\nabla^2\widehat{h}\right)^{-1}\frac{\left(\nabla^3\widehat{b}_D\right)'}{\widehat{b}_D}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{48}\sum_{ijkqrs}\widehat{h}_{ijkq}\mu^6_{ijkqrs}\widehat{g}_{rs} &= -\frac{15}{48}\sum_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\widehat{h}_{ijkq}\sum_{rs}\widehat{\sigma}_{rs}\widehat{g}_{rs} \\
&= -\frac{5}{16}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{144}\sum_{ijkqrstw}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^8_{ijkqrstw}\widehat{g}_{tw} \\
&= \frac{105}{144}\sum_{ijkqrs}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{h}_{qrs}\widehat{\sigma}_{rs}\sum_{tw}\widehat{\sigma}_{tw}\widehat{g}_{tw} = \frac{35}{48}\sum_{ijkqrs}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{h}_{qrs}\widehat{\sigma}_{rs}\sum_{tw}\widehat{\sigma}_{tw}\widehat{g}_{tw} \\
&= \frac{35}{48}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\right],
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{36}\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta\xi} \\
&= -\frac{15}{36}\sum_{ijk\zeta\eta\xi}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{k\zeta}\widehat{g}_{\zeta\eta\xi}\widehat{\sigma}_{\eta\xi} = -\frac{5}{12}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{g}\right)' vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{24}\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi\omega} &= \frac{3}{24}\sum_{\zeta\eta\xi\omega}\widehat{\sigma}_{\zeta\eta}\widehat{\sigma}_{\xi\omega}\widehat{g}_{\zeta\eta\xi\omega} \\
&= \frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{g}\right)'\right],
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{12}\frac{\sum_{ijk\zeta\eta\xi}\widehat{h}_{ijk}\mu^6_{ijk\zeta\eta\xi}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi}}{\widehat{b}_D} \\
&= -\frac{15}{12}\sum_{ijk\zeta\eta\xi}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{k\zeta}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\eta\xi}\frac{\widehat{b}_{D,\xi}}{\widehat{b}_D} = -\frac{5}{4}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D},
\end{aligned}
$$

$$\begin{aligned}
\frac{1}{6}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta\xi}\widehat{b}_{D,\omega}}{\widehat{b}_D} &= \frac{3}{6}\sum_{\zeta\eta\xi\omega}\widehat{\sigma}_{\zeta\eta}\widehat{g}_{\zeta\eta\xi}\widehat{\sigma}_{\xi\omega}\frac{\widehat{b}_{D,\omega}}{\widehat{b}_D} \\
&= \frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4}\frac{\sum_{\zeta\eta\xi\omega}\mu^4_{\zeta\eta\xi\omega}\widehat{g}_{\zeta\eta}\widehat{b}_{D,\xi\omega}}{\widehat{b}_D} &= \frac{3}{4}\sum_{\zeta\eta}\widehat{g}_{\zeta\eta}\widehat{\sigma}_{\zeta\eta}\sum_{\xi\omega}\widehat{\sigma}_{\xi\omega}\frac{\widehat{b}_{D,\xi\omega}}{\widehat{b}_D} \\
&= \frac{3}{4}\mathbf{tr}\left[\nabla^2\widehat{g}\left(\nabla^2\widehat{h}\right)^{-1}\right]\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right].
\end{aligned}$$

And we have

$$\frac{1}{2}\sum_{ij}\widehat{\sigma}_{ij}\frac{\widehat{b}_{D,ij}}{\widehat{b}_D} = \frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right], \tag{13}$$

$$\begin{aligned}
-\frac{1}{6}\sum_{ijkq}\widehat{h}_{ijk}\mu^4_{ijkq}\frac{\widehat{b}_{D,q}}{\widehat{b}_D} &= -\frac{3}{6}\sum_{ijkq}\widehat{h}_{ijk}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq}\frac{\widehat{b}_{D,q}}{\widehat{b}_D} \qquad (14)\\
&= -\frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{72}\sum_{ijkqrs}\widehat{h}_{ijk}\widehat{h}_{qrs}\mu^6_{ijkqrs} \\
= &\frac{15}{72}\sum_{ijkqrs}\widehat{\sigma}_{ij}\widehat{h}_{ijk}\widehat{\sigma}_{kq}\widehat{h}_{qrs}\widehat{\sigma}_{rs} \\
= &\frac{5}{24}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right] \qquad (15)
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{24}\sum_{ijkq}\widehat{h}_{ijkq}\mu^4_{ijkq} &= -\frac{3}{24}\sum_{ijkq}\widehat{h}_{ijkq}\widehat{\sigma}_{ij}\widehat{\sigma}_{kq} \qquad (16)\\
&= -\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right].
\end{aligned}$$

From (13), (14), (15) and (16), we define

$$\begin{aligned}
B_4 = &\frac{1}{2}\mathbf{tr}\left[\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla^2\widehat{b}_D}{\widehat{b}_D}\right] - \frac{1}{2}vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\frac{\nabla\widehat{b}_D}{\widehat{b}_D} \\
&+\frac{5}{24}\left[vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)'\nabla^3\widehat{h}\left(\nabla^2\widehat{h}\right)^{-1}\left(\nabla^3\widehat{h}\right)'vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right] \\
&-\frac{1}{8}\mathbf{tr}\left[\left[\left(\nabla^2\widehat{h}\right)^{-1}\otimes vec\left(\left(\nabla^2\widehat{h}\right)^{-1}\right)\right]\left(\nabla^4\widehat{h}\right)'\right].
\end{aligned}$$

■

Proof of Lemma 3.2 in Li et al (2017) can be obtained by directly applying Lemma 2.2 above by setting $b_D(\theta) = p(\theta)$, $g(\theta) = l_t(\theta)$, $-nh^N(\theta) = -nh^D(\theta) = \ln p(\mathbf{y}|\theta)$.

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