Online Supplement to ‘Specification Tests based on MCMC Output’

Yong Li
Renmin University of China

Jun Yu
Singapore Management University

Tao Zeng
Zhejiang University

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The purpose of this online supplement is to prove Theorem 3.2 in Li, et al (2017), that is, to show under H_0 that $\widetilde{BIMT}$ has the same asymptotic distribution as BIMT and that $\widetilde{BMT}$ has the same asymptotic distribution as BMT. Based on Proposition 3.1, the relationship $\widetilde{BIMT}$ =BIMT+$o_p(n^{-1/2})$ is enough to guarantee that $\widetilde{BIMT}$ and BIMT have the same asymptotic distribution. Based on Theorem 3.1, $\tilde{J}_1 = J_1 + o_p(1)$ and $\tilde{J}_0 = J_0 + o_p(1)$ are enough to guarantee that $\widetilde{BMT}$ and BMT will have the same asymptotic distribution. Therefore, what we try to find are an order condition for M to ensure $\widetilde{BIMT}$ =BIMT+$o_p(n^{-1/2})$ and order conditions for M and M_L to ensure $\tilde{J}_1 = J_1 + o_p(1)$ and $\tilde{J}_0 = J_0 + o_p(1)$. Note that $\widetilde{BIMT}$ and $\tilde{J}_0$ are based on MCMC output obtained from the null model while $\tilde{J}_1$ is based on MCMC output obtained from both the null model and the expanded model because $J_1 = \mathbf{tr}\left\{C_E\left(\mathbf{y}, \left(\bar{\boldsymbol{\theta}}, \boldsymbol{\theta}_E = 0\right)\right) V_E\left(\bar{\boldsymbol{\theta}}_L\right)\right\}$.

We organize this supplement as follows. In Section 1, we give an order condition for M to ensure that $\widetilde{BIMT}$ =BIMT+$o_p(n^{-1/2})$. In Section 2, we give order conditions for M and M_L to ensure that $\tilde{J}_1 = J_1 + o_p(1)$. In Section 3, we give an order condition for M to ensure that $\tilde{J}_0 = J_0 + o_p(1)$. Section 4 proves Theorem 3.2. Throughout this supplement, the sample size n is assumed to go to infinity.

1 Order Condition for M to Ensure $\widetilde{BIMT}$ =BIMT+$o_p(n^{-1/2})$

Under H_0, $\hat{\mathbf{J}}_n(\bar{\boldsymbol{\theta}}) = O_p(1)$ and $nV(\bar{\boldsymbol{\theta}}) = O_p(1)$. If $\hat{\mathbf{J}}_n(\widetilde{\boldsymbol{\theta}}) - \hat{\mathbf{J}}_n(\bar{\boldsymbol{\theta}}) = o_p(n^{-1/2})$ and $n\left(\widetilde{V}(\widetilde{\boldsymbol{\theta}}) - V(\bar{\boldsymbol{\theta}})\right) = o_p(n^{-1/2})$, then we will have

$$
\begin{aligned}
&\widetilde{BIMT} = n\mathbf{tr}\left\{\hat{\mathbf{J}}_n\left(\widetilde{\boldsymbol{\theta}}\right)\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right)\right\} = \mathbf{tr}\left\{\hat{\mathbf{J}}_n\left(\widetilde{\boldsymbol{\theta}}\right)n\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right)\right\} \\
= \;& \mathbf{tr}\left\{\left[\hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right) + o_p(n^{-1/2})\right]\left[n\left(\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right) - V\left(\bar{\boldsymbol{\theta}}\right)\right) + nV\left(\bar{\boldsymbol{\theta}}\right)\right]\right\} \\
= \;& \mathbf{tr}\left\{\left[\hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right) + o_p(n^{-1/2})\right]\left[nV\left(\bar{\boldsymbol{\theta}}\right) + o_p(n^{-1/2})\right]\right\} \\
= \;& \mathbf{tr}\left\{n\hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right)V\left(\bar{\boldsymbol{\theta}}\right)\right\} + \mathbf{tr}\left\{\hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right)o_p(n^{-1/2})\right\} + \mathbf{tr}\left\{nV\left(\bar{\boldsymbol{\theta}}\right)o_p(n^{-1/2})\right\} + o_p(n^{-1})
\end{aligned}
$$

$$= \quad \text{BIMT}+o_p(n^{-1/2}) = \text{IOS}_A + o_p(n^{-1/2}) = q \times \text{IR} + o_p(n^{-1/2}).$$

Together with Proposition 3.1, this will ensure that $\widetilde{BIMT}$ has the same asymptotic distribution as BIMT. In Section 1.1 we give an order condition for M to ensure $\hat{\mathbf{J}}_n(\widetilde{\boldsymbol{\theta}}) - \hat{\mathbf{J}}_n(\bar{\boldsymbol{\theta}}) = o_p(n^{-1/2})$. In Section 1.2 we then give an order condition for M to ensure $n\left(\widetilde{V}(\widetilde{\boldsymbol{\theta}}) - V(\bar{\boldsymbol{\theta}})\right) = o_p(n^{-1/2})$.

1.1 Order condition for M to ensure $\hat{\mathbf{J}}_n\left(\widetilde{\boldsymbol{\theta}}\right) - \hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right) = o_p(n^{-1/2})$

Let us first assume $\boldsymbol{\theta}$ is a scalar. Let σ_{1n}^2 be the long run variance of Markov chain, $\left\{\boldsymbol{\theta}_n^{(m)}\right\}_{m=1}^M$, i.e., $\sigma_{1n}^2 = Var\left(\boldsymbol{\theta}_n^{(1)}|\mathbf{y}\right) + 2\sum_{k=1}^{\infty} \gamma_n\left(k|\mathbf{y}\right)$ where $\gamma_{1n}\left(k|\mathbf{y}\right)$ is the k^{th} order autocovariance. Note that $Var\left(\boldsymbol{\theta}_n^{(1)}|\mathbf{y}\right)$ is the posterior variance $V\left(\bar{\boldsymbol{\theta}}\right)$. We can rewrite σ_{1n}^2 as

$$\begin{aligned}
\sigma_{1n}^2 &= Var\left(\boldsymbol{\theta}_n^{(1)}|\mathbf{y}\right) + 2\sum\nolimits_{k=1}^{\infty} \gamma_{1n}\left(k|\mathbf{y}\right) = 2\sum\nolimits_{k=0}^{\infty} \gamma_{1n}\left(k|\mathbf{y}\right) - Var\left(\boldsymbol{\theta}_n^{(1)}|\mathbf{y}\right) \\
&= \left(2\sum\nolimits_{k=0}^{\infty} \frac{\gamma_{1n}\left(k|\mathbf{y}\right)}{Var\left(\boldsymbol{\theta}_n^{(1)}|\mathbf{y}\right)} - 1\right) Var\left(\boldsymbol{\theta}_n^{(1)}|\mathbf{y}\right) = \left(2\sum\nolimits_{k=0}^{\infty} \rho\left(k\right) - 1\right) Var\left(\boldsymbol{\theta}_n^{(1)}|\mathbf{y}\right).
\end{aligned}$$

According to Jones (2004), under Assumption 13, as $M \to \infty$, we have

$$\sqrt{M}\sigma_{1n}^{-1}\left(\widetilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right) \xrightarrow{d} N\left(0,1\right). \tag{1}$$

By the Taylor expansion and (1), we have

$$\begin{aligned}
\hat{\mathbf{J}}_n\left(\widetilde{\boldsymbol{\theta}}\right) &= \frac{1}{n}\sum_{t=1}^{n} s_t\left(\widetilde{\boldsymbol{\theta}}\right)^2 = \frac{1}{n}\sum_{t=1}^{n}\left[s_t\left(\bar{\boldsymbol{\theta}}\right) + h_t\left(\widetilde{\boldsymbol{\theta}}_4\right)\left(\widetilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right)\right]^2 \\
&= \frac{1}{n}\sum_{t=1}^{n} s_t\left(\bar{\boldsymbol{\theta}}\right)^2 + \frac{2}{n}\sum_{t=1}^{n} h_t\left(\widetilde{\boldsymbol{\theta}}_4\right)\left(\widetilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right) s_t\left(\bar{\boldsymbol{\theta}}\right) + \frac{1}{n}\sum_{t=1}^{n}\left[h_t\left(\widetilde{\boldsymbol{\theta}}_4\right)\left(\widetilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right)\right]^2 \\
&= \hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right) + O\left(\frac{1}{\sqrt{M}}\sigma_{1n}\right)\frac{2}{n}\sum_{t=1}^{n} h_t\left(\widetilde{\boldsymbol{\theta}}_4\right) s_t\left(\bar{\boldsymbol{\theta}}\right) + O\left(\frac{1}{M}\sigma_{1n}^2\right)\frac{1}{n}\sum_{t=1}^{n} h_t\left(\widetilde{\boldsymbol{\theta}}_4\right)^2 \\
&= \hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right) + O\left(\frac{1}{\sqrt{M}}\sigma_{1n}\right) O_p(1) + O\left(\frac{1}{M}\sigma_{1n}^2\right) O_p(1),
\end{aligned}$$

where $\widetilde{\boldsymbol{\theta}}_4$ lies between $\widetilde{\boldsymbol{\theta}}$ and $\bar{\boldsymbol{\theta}}$ and $\frac{2}{n}\sum_{t=1}^{n} h_t\left(\widetilde{\boldsymbol{\theta}}_4\right) s_t\left(\bar{\boldsymbol{\theta}}\right) = O_p\left(1\right)$ and $\frac{1}{n}\sum_{t=1}^{n} h_t\left(\widetilde{\boldsymbol{\theta}}_4\right)^2 = O_p\left(1\right)$ by Assumptions 10-12.

To show

$$\hat{\mathbf{J}}_n\left(\widetilde{\boldsymbol{\theta}}\right) = \hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right) + o_p(n^{-1/2}), \tag{2}$$

it is enough to have

$$\frac{1}{\sqrt{M}}\sigma_{1n} = o_p\left(n^{-1/2}\right), \tag{3}$$

which is equivalent to

$$M = O\left(n^{1+c_1^*}\sigma_{1n}^2\right), \text{ for any } c_1^* > 0. \tag{4}$$

The condition (3) is also used in Chen, Gao and Phillips (2017) to obtain the asymptotic normality of $\widetilde{\boldsymbol{\theta}}$ when $M \to \infty$ and $n \to \infty$.

In this paper, $\boldsymbol{\theta}$ is a q-dimensional vector. That is why we require

$$M = n^{1+c_1^*}\sigma_{1n}^{2*}, \tag{5}$$

with $\sigma_{1n}^{2*} = \max_{a\in\{1,\ldots,q\}}\sigma_{1n,a}^2$ and $\sigma_{1n,a}^2$ being the long run variance of $\left\{\boldsymbol{\theta}_a^{(m)}\right\}_{m=1}^M$ for $a = 1, \ldots, q$.

1.2 Order condition for M to ensure $n\left[\widetilde{V}(\widetilde{\boldsymbol{\theta}}) - V\left(\bar{\boldsymbol{\theta}}\right)\right] = o_p(n^{-1/2})$

Again, let us first assume $\boldsymbol{\theta}$ is a scalar. Let $\sigma_{2n}^2 = Var\left(\left(\boldsymbol{\theta}_n^{(1)} - \bar{\boldsymbol{\theta}}\right)^2 |\mathbf{y}\right) + 2\sum_{k=1}^{\infty}\gamma_{2n}'(k|\mathbf{y})$ be the long run variance of $\left\{\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}}\right)^2\right\}_{m=1}^M$ where $\gamma_{2n}'(k|\mathbf{y})$ is the k^{th} order autocovariance of $\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}}\right)^2$.

Note that

$$\begin{aligned}
\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right) &= \frac{1}{M}\sum_{m=1}^M\left(\boldsymbol{\theta}_n^{(m)} - \widetilde{\boldsymbol{\theta}}\right)^2 = \frac{1}{M}\sum_{m=1}^M\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}} + \bar{\boldsymbol{\theta}} - \widetilde{\boldsymbol{\theta}}\right)^2 \\
&= \frac{1}{M}\sum_{m=1}^M\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}}\right)^2 - \frac{2}{M}\sum_{m=1}^M\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}}\right)\left(\widetilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right) + \frac{1}{M}\sum_{m=1}^M\left(\bar{\boldsymbol{\theta}} - \widetilde{\boldsymbol{\theta}}\right)^2 \\
&= \frac{1}{M}\sum_{m=1}^M\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}}\right)^2 - \frac{1}{M}\sum_{m=1}^M\left(\bar{\boldsymbol{\theta}} - \widetilde{\boldsymbol{\theta}}\right)^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
\sqrt{M}\left(\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right) - V\left(\bar{\boldsymbol{\theta}}\right)\right) &= \frac{1}{\sqrt{M}}\sum_{m=1}^M\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}}\right)^2 - \sqrt{M}V\left(\bar{\boldsymbol{\theta}}\right) - \sqrt{M}\left(\widetilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right)^2 \\
&= \sqrt{M}\left(\frac{1}{M}\sum_{m=1}^M\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}}\right)^2 - V\left(\bar{\boldsymbol{\theta}}\right)\right) - \sqrt{M}\left(\widetilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right)^2.
\end{aligned}$$

Thus,

$$\sqrt{M}\sigma_{2n}^{-1}\left(\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right) - V\left(\bar{\boldsymbol{\theta}}\right)\right) = \sqrt{M}\sigma_{2n}^{-1}\left(\frac{1}{M}\sum_{m=1}^M\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}}\right)^2 - V\left(\bar{\boldsymbol{\theta}}\right)\right) - \sqrt{M}\sigma_{2n}^{-1}\left(\widetilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right)^2.$$

By (1), $\sqrt{M}\sigma_{2n}^{-1}\left(\widetilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right)^2 \xrightarrow{p} 0$ as $M \to \infty$. Note that

$$\sqrt{M}\sigma_{2n}^{-1}\left(\frac{1}{M}\sum_{m=1}^M\left(\boldsymbol{\theta}_n^{(m)} - \bar{\boldsymbol{\theta}}\right)^2 - V\left(\bar{\boldsymbol{\theta}}\right)\right) \xrightarrow{d} N(0, 1), \tag{6}$$

by the central limit theorem for Markov chains (Jones, 2004) under Assumption 13. Hence, we have

$$\sqrt{M}\sigma_{2n}^{-1}\left(\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right)-V\left(\bar{\boldsymbol{\theta}}\right)\right)\xrightarrow{d}N\left(0,1\right),$$

by the Slusky Theorem. It can be shown that

$$n\left(\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right)-V\left(\bar{\boldsymbol{\theta}}\right)\right)=\frac{n}{\sqrt{M}}\sigma_{2n}\left[\sqrt{M}\sigma_{2n}^{-1}\left(\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right)-V\left(\bar{\boldsymbol{\theta}}\right)\right)\right]=o_p(n^{-1/2}),\tag{7}$$

if

$$\frac{n}{\sqrt{M}}\sigma_{2n}=o(n^{-1/2}),\tag{8}$$

which is equivalent to

$$M=O\left(n^{3+c_2^*}\sigma_{2n}^2\right),\text{ for any } c_2^*>0.\tag{9}$$

Since $\boldsymbol{\theta}$ is q-dimensional, we require

$$M=n^{3+c_2^*}\sigma_{2n}^{2*},\tag{10}$$

with $\sigma_{2n}^{2*}=\max_{b\in\{1,\ldots,r\}}\sigma_{2n,b}^2$ and $\sigma_{2n,b}^2$ being the long run variance of $\left\{\boldsymbol{\vartheta}_b^{(m)}\right\}_{m=1}^M$ for $b=1,\ldots,r$, where $\boldsymbol{\vartheta}=vech\left[\left(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}}\right)'\right]$.

2 Order Conditions for M and M_L to Ensure $\tilde{J}_1-J_1=o_p(1)$

Following Li, et al (2015), $\frac{1}{n}C\left(\mathbf{y},\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)=O_p(1)$ and $nV(\bar{\boldsymbol{\theta}}_L)=O_p(1)$. If

$$\frac{1}{n}\left[C\left(\mathbf{y},\left(\widetilde{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)-C\left(\mathbf{y},\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)\right]=o_p(1),\tag{11}$$

and

$$n\left(\widetilde{V}(\widetilde{\boldsymbol{\theta}}_L)-V(\bar{\boldsymbol{\theta}}_L)\right)=o_p(1),\tag{12}$$

we will have

$$\begin{aligned}
\tilde{J}_1 &= \mathbf{tr}\left\{C_E\left(\mathbf{y},\left(\widetilde{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)\widetilde{V}_E\left(\widetilde{\boldsymbol{\theta}}_L\right)\right\}\\
&= \mathbf{tr}\left\{\frac{1}{n}C_E\left(\mathbf{y},\left(\widetilde{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)n\widetilde{V}_E\left(\widetilde{\boldsymbol{\theta}}_L\right)\right\}\\
&= \mathbf{tr}\left\{\left[\frac{1}{n}C_E\left(\mathbf{y},\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)+o_p\left(1\right)\right]\left[nV_E\left(\bar{\boldsymbol{\theta}}_L\right)+o_p\left(1\right)\right]\right\}\\
&= \mathbf{tr}\left\{C_E\left(\mathbf{y},\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)V_E\left(\bar{\boldsymbol{\theta}}_L\right)\right\}+o_p\left(1\right)\\
&= J_1+o_p\left(1\right).
\end{aligned}$$

Hence, for $\widetilde{BMT}$ =BMT+$o_p(1)$, we need to obtain an order condition for M in the original model to ensure (11) and an order condition for M_L in the expanded model to ensure (12).

2.1 Order condition for M to ensure $\frac{1}{n}\left[C\left(\mathbf{y},\left(\widetilde{\boldsymbol{\theta}},\ \boldsymbol{\theta}_E=0\right)\right)-C\left(\mathbf{y},\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)\right]=o_p(1)$

Let us first assume $\boldsymbol{\theta}$ is a scalar. By the Taylor expansion, we have

$$
\begin{aligned}
& \frac{1}{n}C\left(\mathbf{y},\left(\widetilde{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right) \\
= & \frac{1}{n}\sum_{t=1}^{n}s_t\left(\widetilde{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\sum_{t=1}^{n}s_t\left(\widetilde{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)' \\
= & \frac{1}{n}\sum_{t=1}^{n}\left[s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)+h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)'\right]\times \\
& \sum_{t=1}^{n}\left[s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)+h_t\left(\tilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\tilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)'\right]' \\
= & \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)+h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)'\right]\times \\
& \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)+h_t\left(\tilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\tilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)'\right]' \\
= & \frac{1}{\sqrt{n}}\sum_{t=1}^{n}s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)' \\
& +\frac{1}{\sqrt{n}}\sum_{t=1}^{n}s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)'\right)' \\
& +\frac{1}{\sqrt{n}}\sum_{t=1}^{n}h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)'\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)' \\
& +\frac{1}{\sqrt{n}}\sum_{t=1}^{n}h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)'\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)'\right)',
\end{aligned}
$$

where $\widetilde{\boldsymbol{\theta}}_5$ lies between $\widetilde{\boldsymbol{\theta}}$ and $\bar{\boldsymbol{\theta}}$. Under the null hypothesis, we have

$$
\begin{aligned}
& \frac{1}{\sqrt{n}}\sum_{t=1}^{n}s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)'\right)' \\
= & \frac{1}{\sqrt{n}}\sum_{t=1}^{n}s_t\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\left(\frac{1}{n}\sum_{t=1}^{n}h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\frac{\sqrt{n}}{\sqrt{M}}\sigma_{1n}\left(\sqrt{M}\sigma_{1n}^{-1}\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)\right)\right)' \\
= & O_p\left(1\right)O_p\left(1\right)O\left(\frac{\sqrt{n}}{\sqrt{M}}\sigma_{1n}\right)O_p(1)=O_p\left(\frac{\sqrt{n}}{\sqrt{M}}\sigma_{1n}\right),
\end{aligned}
$$

and

$$
\frac{1}{\sqrt{n}}\sum_{t=1}^{n}h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}h_t\left(\widetilde{\boldsymbol{\theta}}_5,\boldsymbol{\theta}_E=0\right)\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}},\mathbf{0}\right)\right)'
$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{t=1}^{n} h_t\left(\widetilde{\boldsymbol{\theta}}_5, \boldsymbol{\theta}_E=0\right)\sqrt{n}\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}}, \mathbf{0}\right)\left(\frac{1}{n}\sum_{t=1}^{n} h_t\left(\widetilde{\boldsymbol{\theta}}_5, \boldsymbol{\theta}_E=0\right)\sqrt{n}\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}}, \mathbf{0}\right)\right)' \\
&= \frac{1}{n}\sum_{t=1}^{n} h_t\left(\widetilde{\boldsymbol{\theta}}_5, \boldsymbol{\theta}_E=0\right)\frac{\sqrt{n}}{\sqrt{M}}\sigma_{1n}\left(\sqrt{M}\sigma_{1n}^{-1}\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}}, \mathbf{0}\right)\right)\times \\
&\quad \left(\frac{1}{n}\sum_{t=1}^{n} h_t\left(\widetilde{\boldsymbol{\theta}}_5, \boldsymbol{\theta}_E=0\right)\frac{\sqrt{n}}{\sqrt{M}}\sigma_{1n}\left(\sqrt{M}\sigma_{1n}^{-1}\left(\widetilde{\boldsymbol{\theta}}-\bar{\boldsymbol{\theta}}, \mathbf{0}\right)\right)\right)' \\
&= O_p(1)\, O\left(\frac{\sqrt{n}}{\sqrt{M}}\sigma_{1n}\right)O_p(1)O_p(1)O\left(\frac{\sqrt{n}}{\sqrt{M}}\sigma_{1n}\right)O_p(1) = O_p\left(\frac{n}{M}\sigma_{1n}^2\right)
\end{aligned}
$$

Hence, if $O\left(\frac{\sqrt{n}}{\sqrt{M}}\sigma_{1n}\right)=o(1)$, that is,

$$
M = O(n^{1+c_3^*}\sigma_{1n}^2), \text{ for any } c_3^*>0, \tag{13}
$$

then

$$
\frac{1}{n}C_E\left(\mathbf{y}, \left(\widetilde{\boldsymbol{\theta}}, \boldsymbol{\theta}_E=0\right)\right) = \frac{1}{n}C_E\left(\mathbf{y}, \left(\bar{\boldsymbol{\theta}}, \boldsymbol{\theta}_E=0\right)\right)+o_p(1).
$$

Again, since $\boldsymbol{\theta}$ is q-dimensional, we set

$$
M = n^{1+c_3^*}\sigma_{1n}^{2*}, \text{ for any } c_3^*>0. \tag{14}
$$

2.2 Order condition for M_L to ensure $n\left(\widetilde{V}(\tilde{\boldsymbol{\theta}}_L)-V(\bar{\boldsymbol{\theta}}_L)\right)=o_p(1)$

Since $\left\{\boldsymbol{\theta}_{Ln}^{(m)}\right\}_{m=1}^{M_L}$ is a geometrically ergodic Markov chain with stationary distribution as the posterior distribution of $\boldsymbol{\theta}_L$, the MCMC estimators of posterior mean $\bar{\boldsymbol{\theta}}_L$ and posterior variance $V\left(\bar{\boldsymbol{\theta}}_L\right)$ can be given by

$$
\widetilde{\boldsymbol{\theta}}_L = \frac{1}{M_L}\sum_{m=1}^{M_L}\boldsymbol{\theta}_{Ln}^{(m)},\ \widetilde{V}\left(\widetilde{\boldsymbol{\theta}}_L\right) = \frac{1}{M_L}\sum_{m=1}^{M_L}\left(\boldsymbol{\theta}_{Ln}^{(m)}-\widetilde{\boldsymbol{\theta}}_L\right)\left(\boldsymbol{\theta}_{Ln}^{(m)}-\widetilde{\boldsymbol{\theta}}_L\right)'.
$$

Let $\sigma_{Ln,b}^2$ be the long run variance of $\left\{\boldsymbol{\vartheta}_{Ln,b}^{(m)}\right\}_{m=1}^{M_L}$ for $b=1,2,\cdots,r_L$ where $\boldsymbol{\vartheta}_L = vech\left[\left(\boldsymbol{\theta}_L-\bar{\boldsymbol{\theta}}_L\right)\left(\boldsymbol{\theta}_L-\bar{\boldsymbol{\theta}}_L\right)'\right]$. If we choose

$$
M_L = n^{2+c_5^*}\sigma_{Ln}^{2*}, \text{ for any } c_5^*>0, \tag{15}
$$

with $\sigma_{Ln}^{2*}=\max_{b\in\{1,2,\ldots,q_L(q_L+1)/2\}}\sigma_{Ln,b}^2$, then using the same proof as in Section 1.2, we can show that

$$
n\left(\widetilde{V}(\widetilde{\boldsymbol{\theta}}_L)-V\left(\bar{\boldsymbol{\theta}}_L\right)\right)=o_p(n^{-1/2}),
$$

$$
n\left(\widetilde{V}(\widetilde{\boldsymbol{\theta}}_L)-V\left(\bar{\boldsymbol{\theta}}_L\right)\right)=o_p(1).
$$

3 Order Condition for M to Ensure $\tilde{J}_0 - J_0 = o_p(1)$

$\widetilde{BIMT}$ =BIMT+$o_p(n^{-1/4})$ is a sufficient condition to ensure $\tilde{J}_0 - J_0 = o_p(1)$. This is because if $\widetilde{BIMT}$ =BIMT+$o_p(n^{-1/4})$, then we have

$$\begin{aligned} \tilde{J}_0 &= \sqrt{n}\left(\widetilde{BIMT}/q - 1\right)^2 \\ &= \sqrt{n}\left(\text{BIMT}/q + o_p\left(n^{-1/4}\right)/q - 1\right)^2 \\ &= \sqrt{n}(\text{BIMT}/q-1)^2 + \sqrt{n}o_p\left(n^{-1/2}\right) - 2\sqrt{n}(\text{BIMT}/q-1)o_p\left(n^{-1/4}\right)/q \\ &= J_0 + o_p(1) + o_p\left(n^{-1/4}\right) = J_0 + o_p(1). \end{aligned}$$

In Section 1, we have shown that under the null hypothesis, $\hat{\mathbf{J}}_n(\bar{\boldsymbol{\theta}}) = O_p(1)$ and $nV(\bar{\boldsymbol{\theta}}) = O_p(1)$. If $\hat{\mathbf{J}}_n(\widetilde{\boldsymbol{\theta}}) - \hat{\mathbf{J}}_n(\bar{\boldsymbol{\theta}}) = o_p(n^{-1/4})$ and $n\left(\widetilde{V}(\widetilde{\boldsymbol{\theta}}) - V(\bar{\boldsymbol{\theta}})\right) = o_p(n^{-1/4})$, then we have

$$\begin{aligned} & \widetilde{BIMT} = n\mathbf{tr}\left\{\hat{\mathbf{J}}_n\left(\widetilde{\boldsymbol{\theta}}\right)\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right)\right\} = \mathbf{tr}\left\{\hat{\mathbf{J}}_n\left(\widetilde{\boldsymbol{\theta}}\right)n\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right)\right\} \\ = \;& \mathbf{tr}\left\{\left[\hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right) + o_p(n^{-1/4})\right]\left[n\left(\widetilde{V}\left(\widetilde{\boldsymbol{\theta}}\right) - V\left(\bar{\boldsymbol{\theta}}\right)\right) + nV\left(\bar{\boldsymbol{\theta}}\right)\right]\right\} \\ = \;& \mathbf{tr}\left\{\left[\hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right) + o_p(n^{-1/4})\right]\left[nV\left(\bar{\boldsymbol{\theta}}\right) + o_p(n^{-1/4})\right]\right\} \\ = \;& \mathbf{tr}\left\{n\hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right)V\left(\bar{\boldsymbol{\theta}}\right)\right\} + \mathbf{tr}\left\{\hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right)o_p(n^{-1/4})\right\} + \mathbf{tr}\left\{nV\left(\bar{\boldsymbol{\theta}}\right)o_p(n^{-1/4})\right\} + o_p(n^{-1/2}) \\ = \;& \text{BIMT}+o_p(n^{-1/4})O_p(1) + o_p(n^{-1/4})O_p(1) + o_p(n^{-1/2}) \\ = \;& \text{BIMT}+o_p(n^{-1/4}). \end{aligned}$$

According to (8) and (9), to ensure $n\left[\widetilde{V}(\widetilde{\boldsymbol{\theta}}) - V\left(\bar{\boldsymbol{\theta}}\right)\right] = o_p(n^{-1/4})$, we only need

$$\frac{n}{\sqrt{M}}\sigma^*_{2n} = o(n^{-1/4}), \tag{16}$$

which is equivalent to

$$M = O\left(n^{2.5+c_4^*}\sigma_{2n}^{2*}\right), \text{ for any } c_4^* > 0. \tag{17}$$

This order condition is weaker than that specified in (10).

Furthermore, according to (3) and (4), to ensure $\hat{\mathbf{J}}_n\left(\widetilde{\boldsymbol{\theta}}\right) = \hat{\mathbf{J}}_n\left(\bar{\boldsymbol{\theta}}\right) + o_p(n^{-1/4})$, we only need

$$\frac{1}{\sqrt{M}}\sigma^*_{1n} = o\left(n^{-1/4}\right), \tag{18}$$

which is equivalent to

$$M = O\left(n^{0.5+c_6^*}\sigma_{1n}^{2*}\right), \text{ for any } c_6^* > 0. \tag{19}$$

This order condition is weaker than that specified in (5).

4 Proof of Theorem 3.2

Combining the order conditions given by (5) and (10) provides the order condition for M_{BIMT} given by Equation (8) in Theorem 3.2 when BIMT is used.

Regarding BMT, according Section 2, if $M = n^{1+c_3^*}\sigma_{1n}^{2*}$ in the original model, then

$$\frac{1}{n}C_E\left(\mathbf{y},\left(\widetilde{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)=\frac{1}{n}C_E\left(\mathbf{y},\left(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}_E=0\right)\right)+o_p\left(1\right).$$

If $M_L = n^{2+c_5^*}\sigma_{Ln}^{2*}$ in the expanded model, then

$$n\left(\widetilde{V}(\widetilde{\boldsymbol{\theta}}_L)-V\left(\bar{\boldsymbol{\theta}}_L\right)\right)=o_p(1).$$

Under these two order conditions (one for M and one for M_L) we have $\tilde{J}_1 = J_1 + o_p(1)$.

According to Section 3, if $M = \max\left\{n^{0.5+c_6^*}\sigma_{1n}^{2*}, n^{2.5+c_4^*}\sigma_{2n}^{2*}\right\}$, we have $\widetilde{J}_0 = J_0 + o_p(1)$. From Section 2, if $M = n^{1+c_3^*}\sigma_{1n}^{2*}$ and $M_L = n^{2+c_5^*}\sigma_{Ln}^{2*}$, we have $\tilde{J}_1 = J_1 + o_p(1)$. Hence, if we set the number of MCMC draws to

$$M_{BMT}=\max\left\{n^{0.5+c_6^*}\sigma_{1n}^{2*},n^{1+c_3^*}\sigma_{1n}^{2*},n^{2.5+c_4^*}\sigma_{2n}^{2*}\right\}=\max\left\{n^{1+c_3^*}\sigma_{1n}^{2*},n^{2.5+c_4^*}\sigma_{2n}^{2*}\right\}$$

in the original model and to

$$M_L = n^{2+c_5^*}\sigma_{Ln}^{2*}$$

in the expanded model, then we have

$$\widetilde{BMT}=\tilde{J}_1+\tilde{J}_0=\text{BMT}+o_p\left(1\right).$$

This proves the asymptotic equivalence of $\widetilde{BMT}$ and BMT. Furthermore, under H_0, $\widetilde{BMT}$ converges to $\chi^2(q_E)$.

To derive the power property of $\widetilde{BMT}$, note that $\widetilde{BIMT}$ =BIMT+$o_p(n^{-1/4})$ also holds true for misspecified models when $M = \max\left\{n^{1+c_3^*}\sigma_{1n}^{2*}, n^{2.5+c_4^*}\sigma_{2n}^{2*}\right\}$. In addition, when the model is misspecified so that $q^* \neq q$, from Theorem 3.1, we can show that

$$\begin{aligned}
\tilde{J}_0 &= \sqrt{n}\left(\widetilde{BIMT}/q-1\right)^2\\
&= \sqrt{n}\left(\text{BIMT}/q+o_p\left(n^{-1/4}\right)/q-1\right)^2\\
&= \sqrt{n}(\text{BIMT}/q-1)^2+\sqrt{n}o_p\left(n^{-1/2}\right)-2\sqrt{n}(\text{BIMT}/q-1)o_p\left(n^{-1/4}\right)/q\\
&= J_0+o_p\left(1\right)+o_p\left(n^{-1/4}\right)=J_0+o_p\left(1\right)=O_p(\sqrt{n}).
\end{aligned}$$

Hence, the order of the power of $\widetilde{BMT}$ is no less than $O_p(\sqrt{n})$. This completes the proofs of Theorem 3.2.

5 Reference

1. Cheng, T., Gao, J., and Phillips, P.C.B. (2017). Bayesian estimation based on summary statistics: double asymptotics and practice. Working Paper.

2. Jones, G. L. (2004). On the Markov chain central limit theorem. *Probability Surveys*, **1**, 299-320, 5-1.

3. Li, Y., Liu, X. B., and Yu, J. (2015). A Bayesian chi-squared test for hypothesis testing. *Journal of Econometrics*, **189(1)**, 54-69.