Online Supplement to "Limit Theory for Continuous Time Systems with Mildly Explosive Regressors"*

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July 13, 2016

1 Introduction

This online Supplement contains two sections. First, we provide details of the proofs of the theorems in Chen, Phillips and Yu (2016) for the univariate case. Second, we include proofs of the theorems for the multivariate cases.

2 Univariate explosive regressor

2.1 Proof of Theorem 2.1

Proof. The arguments here and in much of what follows closely mirror those of MP (2009) in the mildly explosive case. We therefore provide only the main new details here. The limit theory of $\sum_{t=1}^{N} x_t^2$ and $\sum_{t=1}^{N} x_t u_{0t}$ is obtained using split sample arguments replacing summations in $\sum_{t=1}^{N}$ by $\left(\sum_{t=1}^{m_N} + \sum_{t=m_N+1}^{N}\right)$ where m_N is such that $\frac{m_N}{N^\alpha} + \frac{N}{m_N} \to \infty$ so that with $c > 0$ and $\alpha \in (0,1)$ we have

$$R_N^{-m_N} \sim \left(1 + \frac{c}{N^\alpha}\right)^{-m_N} \to 0,\ \frac{N^\alpha}{R_N^{N-m_N}} \to 0. \tag{2.1}$$

*Phillips acknowledges support from the NSF under Grant No. SES 12-58258. Yu acknowledges financial support from Singapore Ministry of Education Academic Research Fund Tier 2 under the grant number MOE2011-T2-2-096.

(i) Start by writing x_t in (2.9) as:

$$x_t = R_N^t x_0 + \sum_{j=1}^{t} R_N^{t-j} u_{xj} + \frac{1-R_N^t}{1-R_N}\mu, \tag{2.2}$$

so the standardized numerator can be decomposed as

$$\begin{aligned} & \left(R_N^N N^\alpha\right)^{-1} \sum_{t=1}^{N} x_t u_{0t} \\ = \; & \left(R_N^N N^\alpha\right)^{-1} \sum_{t=1}^{N} R_N^t u_{0t}\left(x_0 - \frac{\mu}{1-R_N}\right) + \left(R_N^N N^\alpha\right)^{-1} \sum_{t=1}^{N} u_{0t} \frac{\mu}{1-R_N} \\ & + \frac{R_N^{-N}}{\sqrt{N^\alpha}} \left(\sum_{t=1}^{m_N} + \sum_{t=m_N+1}^{N}\right) u_{0t} \left(\frac{1}{\sqrt{N^\alpha}} \sum_{j=1}^{t} R_N^{t-j} u_{xj}\right). \end{aligned} \tag{2.3}$$

For the first term on the right hand side of (2.3), since $\widetilde{x}_0 = x_{0N} N^{-\alpha/2} \Rightarrow X^*$ and $\widetilde{\mu} = N^{\alpha/2}\mu \Rightarrow \mu^*$, we have

$$D_N = N^{-\alpha/2} x_0 - \frac{N^{-\alpha/2}\mu}{1-R_N} = N^{-\alpha/2} x_0 - \frac{N^{\alpha/2}\mu}{-c + o(1)} \Rightarrow X^* + \frac{\mu^*}{c} = D. \tag{2.4}$$

Setting $D_N = N^{-\alpha/2}\left(x_0 - \frac{\mu}{1-R_N}\right)$, we then have,

$$\begin{aligned} & \left(R_N^N N^\alpha\right)^{-1} \sum_{t=1}^{N} R_N^t u_{0t}\left(x_0 - \frac{\mu}{1-R_N}\right) \\ = \; & N^{-\alpha/2} \sum_{t=1}^{N} R_N^{-(N-t)} u_{0t} \left\{N^{-\alpha/2}\left(x_0 - \frac{\mu}{1-R_N}\right)\right\} = D_N N^{-\alpha/2} \sum_{t=1}^{N} R_N^{-(N-t)} u_{0t} \\ = \; & D N^{-\alpha/2} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} u_{0t} + o_p(1), \text{ with } D := X^* + \frac{\mu^*}{c}. \end{aligned} \tag{2.5}$$

where we assume the probability space is expanded in such a way so that the weak convergence $\Rightarrow$ can be replaced by $\xrightarrow{p}$. Also note that

$$\begin{aligned} \mathbb{E}\left\{N^{-\alpha/2} \sum_{t=1}^{m_N} R_N^{-(N-t)} u_{0t}\right\}^2 &= \sigma_{00}^2 N^{-\alpha} \sum_{t=1}^{m_N} R_N^{-2(N-t)} = \sigma_{00}^2 N^{-\alpha} R_N^{-2(N+1)} \frac{1-R_N^{2m_N}}{1-R_N^2} \\ &= \sigma_{00}^2 \frac{R_N^{-2(N+1)} - R_N^{-2(N+1)+2m_N}}{-2c} = o(1), \end{aligned}$$

so that

$$N^{-\alpha/2} \sum_{t=1}^{m_N} R_N^{-(N-t)} u_{0t} = o_p(1), \tag{2.6}$$

and then

$$\left(R_N^N N^\alpha\right)^{-1} \sum_{t=1}^{m_N} R_N^t u_{0t} \left(x_0 - \frac{\mu}{1-R_N}\right) = o_p(1).$$

Hence, for the first term of (2.3) we have by virtue of the martingale central limit theorem (MCLT), as in Phillips and Magdalinos (2007),

$$\begin{aligned}
\left(R_N^N N^\alpha\right)^{-1} \sum_{t=1}^{N} R_N^t u_{0t} \left(x_0 - \frac{\mu}{1-R_N}\right) &= D_N \left(R_N^N N^{\alpha/2}\right)^{-1} \sum_{t=1}^{N} R_N^t u_{0t} \\
= \quad D \frac{1}{\sqrt{N^\alpha}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} u_{0t} + o_p(1) &= D\left(\frac{1}{\sqrt{N^\alpha}} \sum_{k=0}^{N-m_N-1} R_N^{-k} u_{0N-k}\right) \\
= \quad D\left(\frac{1}{\sqrt{N^\alpha}} \sum_{k=0}^{N-m_N-1} R_N^{-k} u'_{0k}\right) &\Rightarrow \frac{D\sigma_{00}}{(2c)^{1/2}} U_0,
\end{aligned}$$

where $u'_{0k} := u_{0N-k} \overset{i.i.d.}{\sim} \left(0, \sigma_{00}^2\right)$ and $U_0 = N(0,1)$ since $\frac{1}{N^\alpha} \sum_{k=0}^{N-m_N-1} R_N^{-2k} = \frac{1}{N^\alpha} \frac{1-R_N^{-2N}}{1-R_N^{-2}} \sim \frac{1}{N^\alpha} \frac{1}{R_N^2-1} \to \frac{1}{2c}$.

For the second term on the right hand side of (2.3), noting that $R_N^{-N}\sqrt{N} = \left(1+\frac{c}{N^\alpha}\right)^{-N}\sqrt{N} = O\left(e^{-c\frac{N}{N^\alpha}}\sqrt{N}\right) = o(1)$ for all $\alpha \in (0,1)$ we obtain

$$\left(R_N^N N^\alpha\right)^{-1} \sum_{t=1}^{N} u_{0t} \frac{\mu}{1-R_N} \sim R_N^{-N}\sqrt{N} \frac{\mu}{-c} \frac{1}{\sqrt{N}} \sum_{t=1}^{N} u_{0t} = O\left(e^{-c\frac{N}{N^\alpha}} \sqrt{\frac{N}{N^\alpha}}\right) O_p(1) = o_p(1). \tag{2.7}$$

The third term on the right hand side of (2.3) is

$$\begin{aligned}
&\frac{R_N^{-N}}{\sqrt{N^\alpha}} \left(\sum_{t=1}^{m_N} + \sum_{t=m_N+1}^{N}\right) u_{0t} \left(\frac{1}{\sqrt{N^\alpha}} \sum_{j=1}^{t} R_N^{t-j} u_{xj}\right) \\
= \quad & \frac{1}{\sqrt{N^\alpha}} \sum_{t=1}^{m_N} R_N^{-(N-t)} u_{0t} \left(\frac{1}{\sqrt{N^\alpha}} \sum_{j=1}^{t} R_N^{-j} u_{xj}\right) + \frac{1}{\sqrt{N^\alpha}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} u_{0t} \left(\frac{1}{\sqrt{N^\alpha}} \sum_{j=1}^{t} R_N^{-j} u_{xt}\right) \\
= \quad & \frac{1}{\sqrt{N^\alpha}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} u_{0t} \left(\frac{1}{\sqrt{N^\alpha}} \sum_{j=1}^{m_N} R_N^{-j} u_{xj}\right) + o_p(1),
\end{aligned}$$

where we use the fact that $N^{-\alpha/2} \sum_{t=1}^{m_N} R_N^{-(N-t)} u_{0t} = o_p(1)$ from (2.6). We now use a joint MCLT for the components

$$(U_{0N}, U_{xN}) \quad = \quad \left(\frac{1}{\sqrt{N^\alpha}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} u_{0t}, \frac{1}{\sqrt{N^\alpha}} \sum_{j=1}^{m_N} R_N^{-j} u_{xj}\right)$$

$$
\begin{aligned}
&= \left(\frac{1}{\sqrt{N^{\alpha}}} \sum_{t=1}^{N} R_N^{-(N-t)} u_{0t}, \frac{1}{\sqrt{N^{\alpha}}} \sum_{j=1}^{N} R_N^{-j} u_{xj} \right) + o_p(1) \\
&\Rightarrow \left(\frac{\sigma_{00}}{(2c)^{1/2}} U_0, \frac{\sigma_{xx}}{(2c)^{1/2}} U_x \right) \text{ with } \left(U_0', U_x'\right)' \sim N(0, I_2),
\end{aligned}
$$

just as in Phillips and Magdalinos (2007) and MP (2009), using the fact that the limit variates (U_0, U_x) are independent because

$$
\mathbb{E}\left\{ \left(\frac{1}{\sqrt{N^{\alpha}}} \sum_{t=1}^{N} R_N^{-(N-t)} u_{0t} \right) \left(\frac{1}{\sqrt{N^{\alpha}}} \sum_{j=1}^{N} R_N^{-j} u_{xj} \right) \right\} = \frac{N^{1-\alpha}}{R_N^N} \sigma_{0x} \to 0.
$$

Hence

$$
\begin{aligned}
& \frac{R_N^{-N}}{\sqrt{N^{\alpha}}} \left(\sum_{t=1}^{m_N} + \sum_{t=m_N+1}^{N} \right) u_{0t} \left(\frac{1}{\sqrt{N^{\alpha}}} \sum_{j=1}^{t} R_N^{t-j} u_{xj} \right) \\
&= \frac{1}{\sqrt{N^{\alpha}}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} u_{0t} \left(\frac{1}{\sqrt{N^{\alpha}}} \sum_{j=1}^{m_N} R_N^{-j} u_{xj} \right) + o_p(1) \\
&\Rightarrow \left(\frac{\sigma_{00}}{(2c)^{1/2}} U_0 \right) \left(\frac{\sigma_{xx}}{(2c)^{1/2}} U_x \right) = \frac{\sigma_{00}\sigma_{xx}}{2c} U_0 U_x.
\end{aligned}
$$

Combining the above results and using (2.4) we obtain

$$
\begin{aligned}
& \left(R_N^N N^{\alpha}\right)^{-1} \sum_{t=1}^{N} x_t u_{0t} \\
&= \left(R_N^N N^{\alpha}\right)^{-1} \sum_{t=1}^{N} R_N^t u_{0t} \left(x_0 - \frac{\mu}{1-R_N} \right) + \left(R_N^N N^{\alpha}\right)^{-1} \sum_{t=1}^{N} u_{0t} \frac{\mu}{1-R_N} \\
&\quad + \frac{R_N^{-N}}{\sqrt{N^{\alpha}}} \sum_{t=m_N+1}^{N} u_{0t} \left(\frac{1}{\sqrt{N^{\alpha}}} \sum_{j=1}^{t} R_N^{t-j} u_{xj} \right) + o_p(1) \\
&= \left(R_N^N N^{\alpha}\right)^{-1} \sum_{t=1}^{N} R_N^t u_{0t} \left(x_0 - \frac{\mu}{1-R_N} \right) + \frac{R_N^{-N}}{\sqrt{N^{\alpha}}} \sum_{t=m_N+1}^{N} u_{0t} \left(\frac{1}{\sqrt{N^{\alpha}}} \sum_{j=1}^{t} R_N^{t-j} u_{xj} \right) + o_p(1) \\
&\Rightarrow \frac{D\sigma_{00}}{(2c)^{1/2}} U_0 + \frac{\sigma_{00}\sigma_{xx}}{2c} U_0 U_x = \frac{\sigma_{00}}{(2c)^{1/2}} U_0 \left(D + \frac{\sigma_{xx}}{(2c)^{1/2}} U_x \right), \qquad (2.8)
\end{aligned}
$$

giving the limit of the numerator.

(ii) From the identity

$$
x_t^2 = R_N^2 x_{t-1}^2 + \mu^2 + u_{xt}^2 + 2R_N \mu x_{t-1} + 2R_N x_{t-1} u_{xt} + 2\mu u_{xt},
$$

we have

$$\left(R_N^2-1\right)\sum_{t=1}^{N}x_t^2=R_N^2x_N^2-R_N^2x_0^2-2R_N\mu\sum_{t=1}^{N}x_{t-1}-2R_N\sum_{t=1}^{N}x_{t-1}u_{xt}-N\mu^2-\sum_{t=1}^{N}u_{xt}^2-2\mu\sum_{t=1}^{N}u_{xt}. \tag{2.9}$$

We show in the following that each of the following standardized terms

$$\frac{R_N^2x_0^2}{R_N^{2N}N^\alpha},\frac{N\mu^2}{R_N^{2N}N^\alpha},\frac{2\mu\sum_{t=1}^N u_{xt}}{R_N^{2N}N^\alpha},\frac{\sum_{t=1}^N u_{xt}^2}{R_N^{2N}N^\alpha},\frac{R_N\mu\sum_{t=1}^N x_{t-1}}{R_N^{2N}N^\alpha},\ \frac{R_N\sum_{t=1}^N x_{t-1}u_{xt}}{R_N^{2N}N^\alpha}$$

are asymptotically negligible. In particular, since the standardized initial condition and drift satisfy $\widetilde{x}_0=x_{0N}N^{-\alpha/2}\Rightarrow X^*$ and $\widetilde{\mu}=N^{\alpha/2}\mu\Rightarrow\mu^*$, we find that

$$\frac{R_N^2x_0^2}{R_N^{2N}N^\alpha}=O_p\left(\left(\frac{x_0}{N^{\alpha/2}}\right)^2\frac{1}{R_N^{2N}}\right)=o_p\left(1\right),$$

$$\frac{N\mu^2}{R_N^{2N}N^\alpha}=O_p\left(\frac{N^{1-2\alpha}}{R_N^{2N}}\right)=o_p\left(1\right),$$

$$\frac{2\mu\sum_{t=1}^N u_{xt}}{R_N^{2N}N^\alpha}=\left(\frac{1}{R_N^{2N}}\right)\left(\frac{2\mu\sqrt{N}}{N^\alpha}\right)\left(\frac{1}{\sqrt{N}}\sum_{t=1}^N u_{xt}\right)=O_p\left(\frac{1}{R_N^{2N}}\right)\times O_p\left(N^{\frac{1}{2}-\frac{3}{2}\alpha}\right)\times O_p\left(1\right)=o_p\left(1\right),$$

$$\frac{\sum_{t=1}^N u_{xt}^2}{R_N^{2N}N^\alpha}=\frac{N}{R_N^{2N}N^\alpha}\frac{1}{N}\sum_{t=1}^N u_{xt}^2=O_p\left(\frac{N^{1-\alpha}}{R_N^{2N}}\right)\times O_p\left(1\right)=o_p\left(1\right),$$

$$\frac{R_N\sum_{t=1}^N x_{t-1}u_{xt}}{R_N^{2N}N^\alpha}=O_p\left(\frac{\sum_{t=1}^N x_{t-1}u_{xt}}{R_N^NN^\alpha}\right)\times O_p\left(\frac{1}{R_N^N}\right)=o_p\left(1\right),$$

since $\left(R_N^NN^\alpha\right)^{-1}\sum_{t=1}^N x_{t-1}u_{xt}=O_p\left(1\right)$ just as in the analysis of $\left(R_N^NN^\alpha\right)^{-1}\sum_{t=1}^N x_tu_{0t}$ in part (i); and finally

$$\frac{R_N\mu\sum_{t=1}^N x_t}{R_N^{2N}N^\alpha}=\frac{\mu}{N^{\alpha/2}R_N^N}\sum_{t=1}^N\frac{x_t}{N^{\alpha/2}R_N^t}\frac{R_N^t}{R_N^N}=O_p\left(\frac{1}{N^\alpha R_N^N}\right)\times O_p\left(N\right)=o_p\left(1\right).$$

Hence, from (2.9) and (2.2) we deduce that

$$\begin{aligned}\frac{\left(R_N^2-1\right)\sum_{t=1}^N x_t^2}{R_N^{2N}N^\alpha} &= \frac{R_N^2x_N^2}{R_N^{2N}N^\alpha}\left\{1+o_p\left(1\right)\right\}=\left(\frac{x_N}{R_N^NN^{\alpha/2}}\right)^2\left\{1+o_p\left(1\right)\right\}\\ &= \left\{\frac{x_0}{N^{\alpha/2}}+\frac{1}{N^{\alpha/2}}\sum_{j=1}^N R_N^{-j}u_{xj}+\frac{N^{\alpha/2}}{c}\mu\right\}^2\left\{1+o_p\left(1\right)\right\}\\ &\Rightarrow \left(\frac{\sigma_{xx}}{\left(2c\right)^{1/2}}U_x+D\right)^2.\end{aligned} \tag{2.10}$$

(iii) Combining the results (2.8) and (2.10), we have

$$\frac{\left(R_N^N N^\alpha\right)^{-1}\sum_{t=1}^N x_t u_{xt}}{\frac{\left(R_N^2-1\right)}{R_N^{2N}N^\alpha}\sum_{t=1}^N x_t^2} \sim \frac{\left(R_N^N N^\alpha\right)^{-1}\sum_{t=1}^N x_t u_{xt}}{\frac{2c}{R_N^{2N}N^{2\alpha}}\sum_{t=1}^N x_t^2} \Rightarrow \frac{\frac{\sigma_{00}}{(2c)^{1/2}}U_0\left(D+\frac{\sigma_{xx}}{(2c)^{1/2}}U_x\right)}{\left(\frac{\sigma_{xx}}{(2c)^{1/2}}U_x+D\right)^2}$$

$$= \frac{\frac{\sigma_{00}}{(2c)^{1/2}}U_0}{\frac{\sigma_{xx}}{(2c)^{1/2}}U_x+D} = \frac{\sigma_{00}U_0}{\sigma_{xx}U_x+(2c)^{1/2}D}.$$

Therefore,

$$R_N^N N^\alpha\left(\widehat{A}-A\right) = \frac{\left(R_N^N N^\alpha\right)^{-1}\sum_{t=1}^N x_t u_{xt}}{\left(R_N^{2N}N^{2\alpha}\right)^{-1}\sum_{t=1}^N x_t^2} \Rightarrow 2c\frac{\sigma_{00}U_0}{\sigma_{xx}U_x+(2c)^{1/2}D},$$

giving the stated result. ■

2.2 Proof of Corollary 2.2

Proof. The proof follows from Theorem 2.1 by noting the mappings

$$\sigma_{00}^2 \mapsto \sigma_{00}^2, \sigma_{xx}^2 \mapsto 1, R_N \mapsto a_h = e^{-\kappa h}, X^* \mapsto \frac{x_0}{\sigma_{xx}}, \mu \mapsto \frac{\mu\kappa}{\sigma_{xx}}h^{1/2}, \mu^* \mapsto \frac{\mu\kappa}{\sigma_{xx}},$$

$$D_N \mapsto D_h = \tilde{x}_{0h}h^{1/2} - \frac{h^{-1/2}\tilde{g}_h}{\kappa} \to D^* = \frac{x_0}{\sigma_{xx}} - \frac{\mu}{\sigma_{xx}},$$

with $h = 1/N^\alpha$. It follows that

$$a_h^{-N}h\sum_{t=1}^N \tilde{x}_{th}u_{0,th} \Rightarrow \frac{\sigma_{00}}{-2\kappa}U_0\left(U_x+(-2\kappa)^{1/2}D^*\right), \tag{2.11}$$

$$a_h^{-2N}h^2\sum_{t=1}^N \tilde{x}_{th}^2 \Rightarrow \left(\frac{1}{-2\kappa}\right)^2\left(U_x+(-2\kappa)^{1/2}D^*\right)^2, \tag{2.12}$$

and hence

$$\frac{a_h^N}{\sqrt{h}}\left(\widehat{\beta}-\beta\right) \Rightarrow (-2\kappa)\frac{\frac{\sigma_{00}}{\sigma_{xx}}U_0}{U_x+(-2\kappa)^{1/2}D^*} = (-2\kappa)\frac{\sigma_{00}U_0}{\sigma_{xx}U_x+(-2\kappa)^{1/2}(x_0-\mu)}.$$

■

3 Multivariate explosive regressor

3.1 Proof of Thorem 3.1

Proof. First, we rewrite x_t by backward recursion as,

$$x_t = (I-R_N)^{-1}\mu + R_N^t\left(x_0-(I-R_N)^{-1}\mu\right) + \sum_{j=1}^t R_N^{t-j}u_{xj}.$$

(i) With this expression, we have

$$
\begin{aligned}
& vec\left(\frac{1}{N^{\alpha}}\sum_{t=1}^{N}u_{0t}x_t' R_N^{-N}\right) \qquad (3.1)\\
= \;& vec\left(\frac{1}{N^{\alpha}}\sum_{t=1}^{N}u_{0t}\left\{(I-R_N)^{-1}\mu + R_N^t\left(x_0-(I-R_N)^{-1}\mu\right)+\sum_{j=1}^{t}R_N^{t-j}u_{xj}\right\}' R_N^{-N}\right)\\
= \;& \frac{1}{N^{\alpha}}vec\left(\sum_{t=1}^{N}u_{0t}\left\{x_0-(I-R_N)^{-1}\mu\right\}' R_N^{t-N}\right)+\frac{1}{N^{\alpha}}vec\left(\sum_{t=1}^{N}u_{0t}\mu'(I-R_N)^{-1}R_N^{-N}\right)\\
& +\frac{1}{N^{\alpha}}vec\left(\left(\sum_{t=1}^{m_N}+\sum_{t=m_N+1}^{N}\right)u_{0t}\left(\sum_{j=1}^{t}R_N^{t-j}u_{xj}\right)' R_N^{-N}\right).
\end{aligned}
$$

For the first item on the right side of (3.1), letting $D_N = N^{-\alpha/2}\left(x_0 + N^{\alpha}C^{-1}\mu\right)$, we obtain

$$
\begin{aligned}
& \frac{1}{N^{\alpha}}vec\left(\sum_{t=1}^{N}u_{0t}\left\{x_0-(I-R_N)^{-1}\mu\right\}' R_N^{t-N}\right)\\
= \;& \frac{1}{N^{\alpha/2}}vec\left(\sum_{t=1}^{N}u_{0t}D_N' R_N^{t-N}\right) = \frac{1}{N^{\alpha/2}}\sum_{t=1}^{N}\left(R_N^{t-N}\otimes u_{0t}\right)vec\left(D_N'\right)\\
= \;& \frac{1}{N^{\alpha/2}}\sum_{t=m_N+1}^{N}\left(R_N^{t-N}\otimes u_{0t}\right)D \; + o_p(1) \text{ with } D := X^* + C^{-1}\mu^*,
\end{aligned}
$$

since $D_N = N^{-\alpha/2}x_0 + N^{\alpha/2}C^{-1}\mu \Rightarrow X^* + C^{-1}\mu^* = D$ and by replacing the weak convergence with convergence in probability in an expanded space for the final step. In addition, we have

$$
\begin{aligned}
& \mathbb{E}\left\|\frac{1}{N^{\alpha/2}}\sum_{t=1}^{m_N}\left(R_N^{t-N}\otimes u_{0t}\right)\right\|^2\\
= \;& N^{-\alpha}\sum_{t=1}^{m_N}\left\|R_N^{2(t-N)}\right\|\mathbb{E}\left\|u_{0t}\right\|^2 = \frac{N^{-\alpha}\left\|R_N\right\|^{-2(N-1)}\left(1-\left\|R_N\right\|^{2(m_N-2)}\right)}{1-\left\|R_N\right\|^2}\mathbb{E}\left\|u_{0t}\right\|^2\\
= \;& \frac{\mathbb{E}\left\|u_{0t}\right\|^2}{-\max_{1\le i\le K}c_i}o(1) = o(1) \text{ assuming } \mathbb{E}\left\|u_{0t}\right\|^2 < \infty.
\end{aligned}
$$

The result implies

$$
\frac{1}{N^{\alpha/2}}\sum_{t=1}^{m_N}\left(R_N^{t-N}\otimes u_{0t}\right) = o_p(1) \text{ and } \frac{1}{N^{\alpha/2}}\sum_{t=1}^{m_N}\left(R_N^{t-N}\otimes u_{0t}\right)D = o_p(1).
$$

Hence, for the first item of (3.1), we have

$$
\frac{1}{N^{\alpha}}vec\left(\sum_{t=1}^{N}u_{0t}\left\{x_0-(I-R_N)^{-1}\mu\right\}' R_N^{t-N}\right)
$$

$$= \frac{1}{N^{\alpha/2}} \sum_{t=m_N+1}^{N} \left(R_N^{t-N} \otimes u_{0t} \right) D + o_p(1) = \frac{1}{N^{\alpha/2}} \sum_{j=0}^{N-m_N-1} \left(R_N^{-j} D \otimes u_{0N-j} \right)$$

$$= \frac{1}{N^{\alpha/2}} \sum_{j=0}^{N-m_N-1} \left(R_N^{-j} D \otimes \widetilde{u}_{0j} \right),$$

where $\widetilde{u}_{0j} = u_{0N-j} \overset{d}{=} N(0, \Omega_{00})$.

For the second item of (3.1), we have

$$\frac{1}{N^{\alpha}} vec \left(\sum_{t=1}^{N} u_{0t} \mu' (I - R_N)^{-1} R_N^{-N} \right)$$

$$= \frac{-1}{N^{\alpha}} vec \left(\sum_{t=1}^{N} u_{0t} \mu' C^{-1} R_N^{-N} N^{\alpha} \right) = vec \left(- \sum_{t=1}^{N} \frac{u_{0t}}{\sqrt{N}} \mu' C^{-1} R_N^{-N} \sqrt{N} \right) = o_p(1),$$

since $R_N^{-N} \sqrt{N} = O\left(e^{-CN^{1-\alpha}} \sqrt{N} \right) = o_p(1)$.

For the third item of (3.1), we have

$$\frac{1}{N^{\alpha}} vec \left(\left(\sum_{t=1}^{m_N} + \sum_{t=m_N+1}^{N} \right) u_{0t} \left(\sum_{j=1}^{t} R_N^{t-j} u_{xj} \right)' R_N^{-N} \right)$$

$$= \frac{1}{N^{\alpha}} vec \left(\left(\sum_{t=1}^{m_N} + \sum_{t=m_N+1}^{N} \right) u_{0t} \left(\sum_{j=1}^{N} R_N^{t-j} u_{xj} \right)' R_N^{-N} \right) + o_p(1)$$

$$= \left(\frac{1}{N^{\alpha/2}} \sum_{t=1}^{m_N} R_N^{-(N-t)} \otimes u_{0t} \right) vec \left(\frac{1}{N^{\alpha/2}} \sum_{j=1}^{N} R_N^{-j} u_{xj} \right)' +$$

$$\left(\frac{1}{N^{\alpha/2}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} \otimes u_{0t} \right) vec \left(\frac{1}{N^{\alpha/2}} \sum_{j=1}^{N} R_N^{-j} u_{xj} \right)'$$

$$= \left(\frac{1}{N^{\alpha/2}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} \otimes u_{0t} \right) \left(\frac{1}{N^{\alpha/2}} \sum_{j=1}^{m_N} R_N^{-j} u_{xt} \right) + o_p(1)$$

$$= \frac{1}{N^{\alpha/2}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} U_{xN} \otimes u_{0t} + o_p(1), \text{ with } U_{xN} = \frac{1}{N^{\alpha/2}} \sum_{j=1}^{m_N} R_N^{-j} u_{xt},$$

since we have shown $\sum_{t=1}^{m_N} R_N^{-(N-t)} \otimes u_{0t} = o_p(1)$. Note that, from MP

$$U_{xN} \Rightarrow \left(\int_0^{\infty} e^{-pC} \Omega_{xx} e^{-pC} dp \right)^{1/2} U_x =: \widetilde{U}_x,$$

with $U_x = N(0, I_K)$. Hence the third item has the following form:

$$\begin{aligned}
&\frac{1}{N^{\alpha}} vec \left(\sum_{t=1}^{m_N} + \sum_{t=m_N+1}^{N} \right) u_{0t} \left(\sum_{j=1}^{t} R_N^{t-j} u_{xj} \right)' R_N^{-N} \\
= &\frac{1}{N^{\alpha/2}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} U_{xN} \otimes u_{0t} + o_p(1) .
\end{aligned}$$

Combining the above results and using (3.1), we have the limit result for the numerator as,

$$\begin{aligned}
&vec \left(\frac{1}{N^{\alpha}} \sum_{t=1}^{N} u_{0t} x_t' R_N^{-N} \right) \\
= &\frac{1}{N^{\alpha/2}} \sum_{t=m_N+1}^{N} R_N^{-(N-t)} (U_{xN} + D) \otimes u_{0t} + o_p(1) \\
\Rightarrow &\left(\int_0^{\infty} e^{-pC} \left(D + \widetilde{U}_x \right) \left(D + \widetilde{U}_x \right)' e^{-pC} dp \otimes \Omega_{00} \right)^{1/2} W_0 \\
\overset{d}{=} &MN \left(0, \int_0^{\infty} e^{-pC} \left(D + \widetilde{U}_x \right) \left(D + \widetilde{U}_x \right)' e^{-pC} dp \otimes \Omega_{00} \right), \qquad (3.2)
\end{aligned}$$

where $W_0 = N(0, I_{mK})$. Due to the sample splitting at $t = m_N$, as $N \to \infty$ the limit variate W_0 is independent of the limit variate U_x.

(ii) From the identity

$$x_t x_t' = \mu\mu' + R_N x_{t-1}\mu' + u_{xt}\mu' + \mu x_{t-1}' R_N + R_N x_{t-1} x_{t-1}' R_N' + u_{xt} x_{t-1}' R_N + \mu u_{xt}' + R_N x_{t-1} u_{xt}' + u_{xt} u_{xt}',$$

we have

$$\begin{aligned}
&(R_N \otimes R_N - I_{K\times K}) \sum_{t=1}^{N} vec(x_t x_t') \qquad (3.3) \\
= &(R_N \otimes R_N) vec(x_N x_N') - (R_N \otimes R_N) vec(x_0 x_0') - N vec(\mu\mu') - \sum_{t=1}^{N} vec(R_N x_{t-1}\mu') - \sum_{t=1}^{N} vec\left(u_{xt}\mu'\right) \\
&- \sum_{t=1}^{N} vec(\mu x_{t-1}' R_N) - \sum_{t=1}^{N} vec(u_{xt} x_{t-1}' R_N) - \sum_{t=1}^{N} vec\left(\mu u_{xt}'\right) - \sum_{t=1}^{N} vec\left(R_N x_{t-1} u_{xt}'\right) - \sum_{t=1}^{N} vec(u_{xt} u_{xt}') .
\end{aligned}$$

We show in the following that each of the following terms standardized by $N^{-\alpha}\left(R_N^{-N} \otimes R_N^{-N}\right)$

$$(R_N \otimes R_N) vec(x_0 x_0'), N vec(\mu\mu'), \sum_{t=1}^{N} vec(R_N x_{t-1}\mu'), \sum_{t=1}^{N} vec(\mu u_{xt}'), \sum_{t=1}^{N} vec(u_{xt} x_{t-1}' R_N), \sum_{t=1}^{N} vec(u_{xt} u_{xt}')$$

are asymptotically negligible. In particular, we have

$$\begin{aligned}
& N^{-\alpha}\left(R_N^{-N}\otimes R_N^{-N}\right)\left(R_N\otimes R_N\right)vec\left(x_0x_0'\right) \\
= & \left(R_N^{-(N-1)}\otimes R_N^{-(N-1)}\right)vec\left(N^{-\alpha/2}x_0x_0'N^{-\alpha/2}\right)=o_p\left(1\right),
\end{aligned}$$

$$N^{-\alpha}\left(R_N^{-N}\otimes R_N^{-N}\right)Nvec\left(\mu\mu'\right)=O_p\left(N^{1-2\alpha}R_N^{-N}\otimes R_N^{-N}\right)=o_p\left(1\right),$$

$$\begin{aligned}
& N^{-\alpha}\left(R_N^{-N}\otimes R_N^{-N}\right)\sum_{t=1}^{N}vec\left(\mu u_{xt}'\right) \\
= & N^{\frac{1}{2}-\alpha}\left(R_N^{-N}\otimes R_N^{-N}\right)\left(I_K\otimes\mu\right)\frac{1}{\sqrt{N}}\sum_{t=1}^{N}u_{xt} \\
= & O_p\left(N^{\frac{1-3\alpha}{2}}\right)O_p\left(R_N^{-N}\otimes R_N^{-N}\right)O_p\left(1\right)=o_p\left(1\right),
\end{aligned}$$

$$\begin{aligned}
& N^{-\alpha}\left(R_N^{-N}\otimes R_N^{-N}\right)\sum_{t=1}^{N}vec\left(u_{xt}x_{t-1}'R_N\right) \\
= & N^{-\alpha}\left(R_N^{-N}\otimes R_N^{-N}\right)\left(R_N\otimes I_K\right)\sum_{t=1}^{N}vec\left(u_{xt}x_{t-1}'\right) \\
= & R_N^{-(N-1)}\otimes R_N^{-N}N^{-\alpha}\sum_{t=1}^{N}vec\left(u_{xt}x_{t-1}'\right)=O_p\left(R_N^{-(N-1)}\right)\otimes O_p\left(1\right)=o_p\left(1\right),
\end{aligned}$$

since $R_N^{-N}N^{-\alpha}\sum_{t=1}^{N}vec\left(u_{xt}x_{t-1}'\right)=O_p\left(1\right)$ following the same argument that $R_N^{-N}N^{-\alpha}\sum_{t=1}^{N}vec\left(u_{0t}x_t'\right)=O_p\left(1\right)$. Finally,

$$\begin{aligned}
& N^{-\alpha}\left(R_N^{-N}\otimes R_N^{-N}\right)\sum_{t=1}^{N}vec\left(R_Nx_{t-1}\mu'\right) \\
= & N^{-\alpha}\left(R_N^{-N}\otimes R_N^{-N}\right)\left(\mu\otimes R_N\right)\sum_{t=1}^{N}x_{t-1}=N^{-\alpha}\left(R_N^{-N}\mu\otimes R_N^{-(N-1)}\right)\sum_{t=1}^{N}x_{t-1} \\
= & R_N^{-N}\mu N^{-\alpha/2}\otimes\sum_{t=1}^{N}R_N^{-(N-1-t)}N^{-\alpha/2}R_N^{-t}x_{t-1}=R_N^{-N}\mu N^{-\alpha/2}\otimes O_p\left(N^{\alpha}\right).
\end{aligned}$$

Therefore, for the denominator, we have from (3.3) that

$$\begin{aligned}
& N^{-\alpha}\left(R_N\otimes R_N-I_{K\times K}\right)\left(R_N^{-N}\otimes R_N^{-N}\right)\sum_{t=1}^{N}vec\left(x_tx_t'\right) \\
= & N^{-\alpha}\left(R_N^{-(N-1)}\otimes R_N^{-(N-1)}\right)vec\left(x_Nx_N'\right)+o_p\left(1\right)
\end{aligned}$$

$$
\begin{aligned}
&= vec\left\{\left(N^{-\alpha/2}R_N^{-(N-1)}x_N\right)\left(N^{-\alpha/2}R_N^{-(N-1)}x_N\right)'\right\}+o_p(1)\\
&\Rightarrow vec\left(\left(D+\widetilde{U}_x\right)\left(D+\widetilde{U}_x\right)'\right),
\end{aligned}
$$

since

$$
\begin{aligned}
& N^{-\alpha/2}R_N^{-N}x_N\\
&= N^{-\alpha/2}R_N^{-N}\left((I-R_N)^{-1}\mu+R_N^N\left(x_0-(I-R_N)^{-1}\mu\right)+\sum_{j=1}^N R_N^{N-j}u_{xj}.\right)\\
&= R_N^{-N}C^{-1}N^{\alpha/2}\mu+N^{-\alpha/2}x_0+N^{\alpha/2}C^{-1}\mu+N^{-\alpha/2}\sum_{j=1}^N R_N^{-j}u_{xj}\\
&\Rightarrow D+\left(\int_0^\infty e^{-pC}\Omega_{xx}e^{-pC}dp\right)^{1/2}U_x=D+\widetilde{U}_x.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
& N^{-2\alpha}\left(R_N^{-N}\otimes R_N^{-N}\right)\sum_{t=1}^N vec\left(x_tx_t'\right)\\
&= (C\otimes I_K+I_K\otimes C)^{-1}vec\left(\left(D+\widetilde{U}_x\right)\left(D+\widetilde{U}_x\right)'\right)+o_p(1)\\
&= vec\left(\int_0^\infty e^{-pC}\left(D+\widetilde{U}_x\right)\left(D+\widetilde{U}_x\right)'e^{-pC}dp\right)+o_p(1). \qquad (3.4)
\end{aligned}
$$

(iii) Combining results from (3.2) and (3.4), we obtain

$$
\begin{aligned}
& vec\left\{N^\alpha\left(\widehat{A}-A\right)R_N^N\right\}\\
&= vec\left\{N^\alpha\left(\sum_{t=1}^N u_{0t}x_t'\right)\left(\sum_{t=1}^N x_tx_t'\right)^{-1}R_N^N\right\}\\
&= \left[\left\{N^{-2\alpha}\sum_{t=1}^N R_N^{-N}x_tx_t'R_N^{-N}\right\}^{-1}\otimes I_m\right]vec\left\{N^{-\alpha}\sum_{t=1}^N u_{0t}x_t'R_N^{-N}\right\}\\
&\Rightarrow \left[\left(\int_0^\infty e^{-pC}\left(D+\widetilde{U}_x\right)\left(D+\widetilde{U}_x\right)'e^{-pC}\right)^{-1}\otimes I_m\right]\left(\int_0^\infty e^{-pC}\left(D+\widetilde{U}_x\right)\left(D+\widetilde{U}_x\right)'e^{-pC}dp\otimes\Omega_{00}\right)^{1/2}W_0\\
&= \left[\left(\int_0^\infty e^{-pC}\left(D+\widetilde{U}_x\right)\left(D+\widetilde{U}_x\right)'e^{-pC}\right)^{-1/2}\otimes\Omega_{00}^{1/2}\right]W_0,
\end{aligned}
$$

giving the stated result. ■

3.2 Proof of Theorem 3.2

Proof. Given the following limit result obtained in the proof of Theorem 3.1

$$N^{-\alpha/2}R_N^{-N}x_N \Rightarrow D+\left(\int_0^\infty e^{-pC}\Omega_{xx}e^{-pC}dp\right)^{1/2}U_x = D+\widetilde{U}_x,$$

we have

$$\begin{aligned} H_{\perp N}H_{\perp N}' &= I_K - \frac{x_N x_N'}{x_N' x_N} \Rightarrow I_K - \frac{\left(D+\widetilde{U}_x\right)\left(D+\widetilde{U}_x\right)'}{\left(D+\widetilde{U}_x\right)'\left(D+\widetilde{U}_x\right)} \\ &= I_K - X_c X_c' := H_\perp H_\perp', \end{aligned}$$

where $X_c := \left(D+\widetilde{U}_x\right)/\left\{\left(D+\widetilde{U}_x\right)'\left(D+\widetilde{U}_x\right)\right\}^{1/2}$ and $D+\widetilde{U}_x$ is the same as the limit given in Theorem 3.1 but with $C = cI_K$. The subvector z_{2t} can be written as

$$z_{2t} = -H_{\perp N}'\mu\sum_{j=1}^{N-t}\rho_N^{-j} - H_{\perp N}'\sum_{j=1}^{N-t}\rho_N^{-j}u_{xt+j}, \tag{3.5}$$

by the reverse autoregression

$$z_{2t} = -\rho_N^{-1}H_{\perp N}'\mu + \rho_N^{-1}z_{2t+1} - \rho_N^{-1}H_{\perp N}'u_{xt+1}.$$

Using the following expression for the scaled error in the LS estimator of A

$$N^{(1+\alpha)/2}\left(\widehat{A}_N - A\right) = \left(N^{-(1+\alpha)/2}\sum_{t=1}^N u_{0t}z_t'\right)\left(\sum_{t=1}^N z_t z_t'\right)^{-1}H_N',$$

we can write the expression in component form as

$$\begin{aligned} &\left(\frac{1}{N^{(1+\alpha)/2}}\sum_{t=1}^N u_{0t}z_t'\right)\begin{pmatrix} O_p\left(N^{1-\alpha}\rho_N^{-2N}\right) & O_p\left(\rho_N^{-N}\right) \\ O_p\left(\rho_N^{-N}\right) & \left(N^{-(1+\alpha)}\sum_{t=1}^N z_{2t}z_{2t}'\right)^{-1} + o_p(1)\end{pmatrix}H_N' \\ =\ &\left(\frac{1}{N^{(1+\alpha)/2}}\sum_{t=1}^N u_{0t}z_{2t}'\right)\left(\frac{1}{N^{1+\alpha}}\sum_{t=1}^N z_{2t}z_{2t}'\right)^{-1}H_{\perp N}' + o_p(1). \end{aligned}$$

(i) Letting $Z_1 = [z_{11}, z_{12}, \dots z_{1N}]'$ and $Z_2 = [z_{11}, z_{12}, \dots z_{1N}]'$, we have

$$\left(\frac{1}{N^{1+\alpha}}\sum_{t=1}^N z_t z_t'\right)^{-1} = \begin{pmatrix} \left(\frac{Z_1'Z_1}{N^{1+\alpha}}\right)^{-1} + \Pi_{1N}\left(\frac{Z_2'Q_1Z_2}{N^{1+\alpha}}\right)^{-1}\Pi_{1N}' & -\Pi_{1N}\left(\frac{Z_2'Q_1Z_2}{N^{1+\alpha}}\right)^{-1} \\ -\left(\frac{Z_2'Q_1Z_2}{N^{1+\alpha}}\right)^{-1}\Pi_{1N}' & \left(\frac{Z_2'Q_1Z_2}{N^{1+\alpha}}\right)^{-1}\end{pmatrix},$$

with $Q_1 = I_N - Z_1 (Z_1' Z_1)^{-1} Z_1'$ and $\Pi_{1N} = (Z_1' Z_1)^{-1} Z_1' Z_2$. We show in the following that

$$\left(\frac{1}{N^{1+\alpha}} \sum_{t=1}^{N} z_t z_t' \right)^{-1} = \begin{pmatrix} O_p\left(N^{1-\alpha}\rho_N^{-2N}\right) & O_p\left(\rho_N^{-N}\right) \\ O_p\left(\rho_N^{-N}\right) & \left(\frac{1}{N^{1+\alpha}}\sum_{t=1}^{N} z_{2t} z_{2t}'\right)^{-1} + o_p(1) \end{pmatrix}. \tag{3.6}$$

First, $Z_1' Z_1 = O_p\left(\rho_N^{2N} N^{2\alpha}\right)$ since

$$\left\| \frac{\rho_N^{-2N}}{N^{2\alpha}} Z_1' Z_1 \right\| = \frac{\rho_N^{-2N}}{N^{2\alpha}} \left\| \sum_{t=1}^{N} z_{1t} z_{1t}' \right\| = \frac{\rho_N^{-2N}}{N^{2\alpha}} \|H_{cN}\| \|H_{cN}'\| \left\| \sum_{t=1}^{N} x_t x_t' \right\| = O_p(1), \tag{3.7}$$

and $\sum_{t=1}^{N} x_t x_t' = O_p\left(\rho_N^{2N} N^{2\alpha}\right)$.

Second, we show $Z_1' Z_2 = O_p\left(\rho_N^{N} N^{2\alpha}\right)$. Using (3.5) and

$$z_{1t} = H_{cN}' \left\{ \frac{N^\alpha}{-c}\mu + \rho_N^t \left(x_0 + \frac{N^\alpha}{c}\mu \right) + \sum_{j=1}^{t} \rho_N^{t-j} u_{xj} \right\},$$

we have the following representation for $Z_1' Z_2$:

$$\begin{aligned} \sum_{t=1}^{N} z_{1t} z_{2t}' &= \sum_{t=1}^{N} H_{cN}' \left\{ \begin{array}{c} \left(\frac{N^\alpha}{-c}\mu + \rho_N^t \left(x_0 + \frac{N^\alpha}{c}\mu\right) + \sum_{j=1}^{t} \rho_N^{t-j} u_{xj} \right) \\ \left(-\mu' \frac{1-\rho_N^{t-N}}{\rho_N - 1} - \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u_{xj}' \right) \end{array} \right\} H_{\perp N} \\ &= \sum_{t=1}^{N} H_{cN}' \left\{ \begin{array}{c} \frac{N^\alpha}{c} \frac{1-\rho_N^{t-N}}{\rho_N - 1} \mu\mu' - \frac{\rho_N^t - \rho_N^{2t-N}}{\rho_N - 1} \left(x_0 + \frac{N^\alpha}{c}\mu\right)\mu' \\ - \frac{1-\rho_N^{t-N}}{\rho_N - 1} \sum_{j=1}^{t} \rho_N^{t-j} u_{xj} \mu' + \frac{N^\alpha}{c}\mu \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u_{xj}' \\ -\rho_N^t \left(x_0 + \frac{N^\alpha}{c}\mu\right) \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u_{xj}' - \sum_{j=1}^{t} \rho_N^{t-j} u_{xj} \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u_{xj}' \end{array} \right\} H_{\perp N}. \end{aligned}$$

Hence,

$$\begin{aligned} &\left\| \frac{\rho_N^{-N}}{N^{2\alpha}} \sum_{t=1}^{N} z_{1t} z_{2t}' \right\| \leqslant \frac{\rho_N^{-N}}{N^{2\alpha}} \left\| H_{cN}' \right\| \left\| H_{\perp N} \right\| \\ &\left\{ \begin{array}{c} \left(\frac{N^\alpha}{c}\right)^2 \left\| \sum_{t=1}^{N} \left(1 - \rho_N^{t-N}\right) \mu\mu' \right\| + \left\| \sum_{t=1}^{N} \frac{\rho_N^t - \rho_N^{2t-N}}{\rho_N - 1} \left(x_0 + \frac{N^\alpha}{c}\mu\right)\mu' \right\| \\ + \left\| \sum_{t=1}^{N} \frac{1-\rho_N^{t-N}}{\rho_N - 1} \sum_{j=1}^{t} \rho_N^{t-j} u_{xj} \mu' \right\| + \left\| \sum_{t=1}^{N} \frac{N^\alpha}{c}\mu \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u_{xj}' \right\| \\ + \left\| \sum_{t=1}^{N} \rho_N^t \left(x_0 + \frac{N^\alpha}{c}\mu\right) \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u_{xj}' \right\| + \left\| \sum_{t=1}^{N} \sum_{j=1}^{t} \rho_N^{t-j} u_{xj} \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u_{xj}' \right\| \end{array} \right\} \\ = \; & O_p(1), \end{aligned}$$

since

$$\frac{\rho_N^{-N}}{N^{2\alpha}} \left(\frac{N^\alpha}{c}\right)^2 \left\| \sum_{t=1}^{N} \left(1 - \rho_N^{t-N}\right) \mu\mu' \right\|$$

$$= \frac{\rho_N^{-N}}{c^2}\left(N - \frac{\rho_N^{-(N-1)} - \rho_N}{1-\rho_N}\right)\|\mu\mu'\|$$

$$= \frac{\rho_N^{-N}}{c^2}\left(N - \frac{\rho_N^{-(N-1)} - \rho_N}{1-\rho_N}\right)N^{-\alpha}\|\mu^*\mu^{*\prime}\| + o_p(1) = o_p(1),$$

$$\frac{\rho_N^{-N}}{N^{2\alpha}}\left\|\sum_{t=1}^{N}\frac{\rho_N^t - \rho_N^{2t-N}}{\rho_N - 1}\left(x_0 + \frac{N^\alpha}{c}\mu\right)\mu'\right\|$$

$$= \frac{\rho_N^{-N}}{N^{2\alpha}(\rho_N - 1)}\left(\frac{\rho_N\left(1-\rho_N^N\right)}{1-\rho_N} - \frac{\rho_N^{-N}\rho_N^2\left(1-\rho_N^{2N}\right)}{1-\rho_N^2}\right)\left\|\left(x_0 + \frac{N^\alpha}{c}\mu\right)\mu'\right\|$$

$$= -\frac{2\rho_N + \rho_N^2}{2c^2}\left\|x^*\mu^{*\prime} + \frac{1}{c}\mu^*\mu^{*\prime}\right\| = O_p(1),$$

and

$$\frac{\rho_N^{-N}}{N^{2\alpha}}\left\|\sum_{t=1}^{N}\frac{1-\rho_N^{t-N}}{\rho_N - 1}\sum_{j=1}^{t}\rho_N^{t-j}u_{xj}\mu'\right\| = o_p(1),$$

as

$$\frac{\rho_N^{-N}}{N^{2\alpha}}\mathbb{E}\left\|\sum_{t=1}^{N}\frac{1-\rho_N^{t-N}}{\rho_N - 1}\sum_{j=1}^{t}\rho_N^{t-j}u_{xj}\mu'\right\| = 0.$$

Similarly,

$$\left\|\sum_{t=1}^{N}\frac{N^\alpha}{c}\mu\sum_{j=t+1}^{N}\rho_N^{-(j-t)}u'_{xj}\right\| = o_p(1) \text{ and } \left\|\sum_{t=1}^{N}\rho_N^t\left(x_0 + \frac{N^\alpha}{c}\mu\right)\sum_{j=t+1}^{N}\rho_N^{-(j-t)}u'_{xj}\right\| = o_p(1).$$

Further, we have from MP (2009) that

$$\frac{\rho_N^{-N}}{N^{2\alpha}}\mathbb{E}\left\|\sum_{t=1}^{N}\sum_{j=1}^{t}\rho_N^{t-j}u_{xj}\sum_{j=t+1}^{N}\rho_N^{-(j-t)}u'_{xj}\right\| \leqslant B, \text{ where } B \text{ is some constant.}$$

In summary, combining the above results, we have

$$Z_1'Z_2 = O_p\left(\rho_N^N N^{2\alpha}\right), Z_1'Z_1 = O_p\left(\rho_N^{2N}N^{2\alpha}\right), \Pi_{1N} = \left(Z_1'Z_1\right)^{-1}\left(Z_1'Z_2\right) = O_p\left(\rho_N^{-N}\right),$$

$$\frac{Z_2'Q_1Z_2}{N^{1+\alpha}} = \frac{Z_2'Z_2}{N^{1+\alpha}} + O_p\left(N^{\alpha-1}\right).$$

Next, we derive the limit distribution for $\frac{Z_2'Z_2}{N^{1+\alpha}}$. Considering that $z_{2N} = 0$ by construction, we have

$$Z_2'Z_2 = \sum_{t=1}^{N-1}\left(-\rho_N^{-1}H_{\perp N}'\mu + \rho_N^{-1}z_{2t+1} - \rho_N^{-1}H_{\perp N}'u_{xt+1}\right)\left(-\rho_N^{-1}H_{\perp N}'\mu + \rho_N^{-1}z_{2t+1} - \rho_N^{-1}H_{\perp N}'u_{xt+1}\right)',$$

which leads to

$$
\begin{aligned}
& \frac{\rho_N^2-1}{N}\sum_{t=1}^{N-1} z_{2t}z_{2t}' \\
= & \frac{z_{2N}z_{2N}'}{N}-\frac{z_{0N}z_{0N}'}{N}+\frac{1}{N}H_{\perp N}\sum_{t=1}^{N-1}\mu\mu' H_{\perp N}'-\frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1}\mu z_{2t+1}' \\
& +\frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1}\mu u_{xt+1}' H_{\perp N}-\frac{1}{N}\sum_{t=1}^{N-1} z_{2t+1}\mu' H_{\perp N}-\frac{1}{N}\sum_{t=1}^{N-1} z_{2t+1}u_{xt+1}' H_{\perp N} \\
& +\frac{1}{N}H_{\perp N}\sum_{t=1}^{N-1} u_{xt+1}\mu' H_{\perp N}'-\frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1} u_{xt+1}z_{2t+1}'+\frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1} u_{xt+1}u_{xt+1}' H_{\perp N} \\
= & \frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1} u_{xt+1}u_{xt+1}' H_{\perp N}-\frac{1}{N}\sum_{t=1}^{N-1} z_{2t+1}\mu' H_{\perp N}-\frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1}\mu z_{2t+1}'+o_p(1) \\
= & \frac{1}{N}H_{\perp N}'\Omega_{xx}H_{\perp N}+\frac{2}{c}H_{\perp N}'\mu^*\mu^{*\prime} H_{\perp N}+o_p(1),
\end{aligned}
$$

since the following hold:

(1)

$$
\frac{z_{2N}z_{2N}'}{N}=O_p\left(N^{-1}\right)=o_p(1),
$$

(2)

$$
\frac{z_{0N}z_{0N}'}{N}=O_p\left(N^{\alpha-1}\right)=o_p(1),
$$

(3)

$$
\begin{aligned}
& \frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1} u_{xt+1}z_{2t+1}' \\
= & \frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1} u_{xt+1}\left\{-\mu\sum_{j=1}^{N-1-t}\rho_N^{-j}-\sum_{j=1}^{N-1-t}\rho_N^{-j}u_{xt+1+j}\right\}' H_{\perp N} \\
= & o_p(1),
\end{aligned}
$$

(4)

$$
\begin{aligned}
\frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1}\mu z_{2t+1}' &= \frac{1}{N}H_{\perp N}'\mu\sum_{t=1}^{N-1}\left\{-\mu\sum_{j=1}^{N-t-1}\rho_N^{-j}-\sum_{j=1}^{N-t-1}\rho_N^{-j}u_{xt+1+j}\right\}' H_{\perp N} \\
&= -\frac{1}{N}H_{\perp N}'\sum_{t=1}^{N-1}\sum_{j=1}^{N-1-t}\rho_N^{-j}\mu\mu' H_{\perp N}+o_p(1)
\end{aligned}
$$

$$= -\frac{1}{N^{1+\alpha}} H'_{\perp N} \sum_{t=1}^{N-1} \frac{\rho_N^{-1}\left(1-\rho_N^{-(N-t-1)}\right)}{1-\rho_N^{-1}} \mu^* \mu^{*\prime} H_{\perp N} + o_p(1)$$
$$= -\frac{1}{c} H'_{\perp N} \mu^* \mu^{*\prime} H_{\perp N} + o_p(1),$$

(5)

$$\frac{1}{N} H'_{\perp N} \sum_{t=1}^{N-1} u_{xt+1} u'_{xt+1} H_{\perp N} \Longrightarrow \frac{1}{N} H'_{\perp N} \Omega_{xx} H_{\perp N},$$

and

(6)

$$\frac{1}{N} H_{\perp N} \sum_{t=1}^{N-1} \mu \mu' H'_{\perp N} = o_p(1).$$

Hence, by the same argument as in Lemma 4.3 of MP, we have

$$\begin{aligned} &\frac{\rho_N^2 - 1}{N} \sum_{t=1}^{N-1} z_{2t} z'_{2t} \\ = \; & H'_{\perp N} \Omega_{xx} H_{\perp N} + \frac{2}{c} H'_{\perp N} \mu^* \mu^{*\prime} H_{\perp N} + o_p(1) \\ \Rightarrow \; & H'_{\perp} \Omega_{xx} H_{\perp} + \frac{2}{c} H'_{\perp} \mu^* \mu^{*\prime} H_{\perp}, \end{aligned}$$

where $H_\perp$ is a $K \times (K-1)$ matrix (an orthogonal complement of the vector X_c) satisfying

$$H_\perp H'_\perp = I_K - \frac{\left(D + \widetilde{U}_x\right)\left(D + \widetilde{U}_x\right)'}{\left(D + \widetilde{U}_x\right)'\left(D + \widetilde{U}_x\right)}.$$

Therefore,

$$\frac{1}{N^{1+\alpha}} \sum_{t=1}^{N} z_{2t} z'_{2t} \Rightarrow H'_\perp \left(\frac{1}{2c} \Omega_{xx} + \frac{\mu^*}{c} \frac{\mu^{*\prime}}{c}\right) H_\perp.$$

(ii) Normalizing by N^{-1}, the component $N^{-1} \sum_{t=1}^{N} u_{0t} z'_{2t}$ is asymptotically negligible, since

$$\frac{1}{N} \sum_{t=1}^{N} u_{0t} z'_{2t} = \frac{1}{N} \sum_{t=1}^{N} u_{0t} \left(-H'_{\perp N} \mu \sum_{j=1}^{N-t} \rho_N^{-j} - H'_{\perp N} \sum_{j=1}^{N-t} \rho_N^{-j} u_{xt+j}\right)' = o_p(1).$$

Hence, when normalized by $\frac{1}{N^{(1+\alpha)/2}}$, we have

$$\frac{1}{N^{(1+\alpha)/2}} vec\left(\sum_{t=1}^{N} u_{0t} z'_{2t}\right) \tag{3.8}$$

$$= -\frac{1}{N^{(1+\alpha)/2}} vec\left(\sum_{t=1}^{N} u_{0t}\mu' \sum_{j=1}^{N-t} \rho_N^{-j} H_{\perp N}\right) - \frac{1}{N^{(1+\alpha)/2}} vec\left(\sum_{t=1}^{N} u_{0t} \sum_{j=1}^{N-t} \rho_N^{-j} u'_{xt+j} H_{\perp N}\right).$$

For the first item on the right side of (3.8), we have

$$\begin{aligned}
& -\frac{1}{N^{(1+\alpha)/2}} vec\left(\sum_{t=1}^{N} u_{0t}\mu' \sum_{j=1}^{N-t} \rho_N^{-j} H_{\perp N}\right) \\
= & -\frac{1}{N^{(1+\alpha)/2}} vec\left(\sum_{t=1}^{N} u_{0t}\mu' \sum_{j=t+1}^{N} \rho_N^{-(j-t)} H_{\perp N}\right) \\
= & -\frac{1}{N^{(1+\alpha)/2}} vec\left(\sum_{t=1}^{N} u_{0t} \left(1 - \rho_N^{-(N-t)}\right) \mu' H_{\perp N} \frac{N^{\alpha}}{c}\right) \\
= & -\frac{1}{c}\frac{1}{\sqrt{N}} vec\left(\sum_{t=1}^{N} u_{0t}\mu^{*\prime} H_{\perp N}\right) + \frac{1}{c}\frac{1}{\sqrt{N}} vec\left(\sum_{t=1}^{N} \rho_N^{-(N-t)} u_{0t}\mu^{*} H_{\perp N}\right) \\
= & -H'_{\perp N}\frac{\mu^*}{c} \otimes \left(\frac{1}{\sqrt{N}} vec\left(\sum_{t=1}^{N} u_{0t}\right)\right) \\
\Rightarrow & \; H'_{\perp}\frac{\mu^*}{c} \otimes N\left(0, \Omega_{00}\right) \overset{d}{=} N\left(0, H'_{\perp}\frac{\mu^*}{c}\frac{\mu^{*\prime}}{c} H_{\perp} \otimes \Omega_{00}\right).
\end{aligned} \tag{3.9}$$

The second item on the right handside of (3.8) is asymptotically negligible, since

$$-\frac{1}{N^{(1+\alpha)/2}} \sum_{t=1}^{N} u_{0t} \sum_{j=1}^{N-t} \rho_N^{-j} u'_{xt+j} H_{\perp N} = -\frac{1}{N^{(1+\alpha)/2}} \sum_{t=m_N+1}^{N} u_{0t} \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u'_{xj} H_{\perp N} + o_p\left(1\right), \tag{3.10}$$

and

$$-\frac{1}{N^{(1+\alpha)/2}} vec\left(\sum_{t=m_N+1}^{N} u_{0t} \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u'_{xj} H_{\perp N}\right) \Rightarrow N\left(0, \frac{1}{2c} H'_{\perp N}\Omega_{xx}H'_{\perp N} \otimes \Omega_{00}\right). \tag{3.11}$$

(1) For equation (3.10), we have

$$\begin{aligned}
& \mathbb{E}\left\|\frac{1}{N^{(1+\alpha)/2}} \sum_{t=1}^{m_N} u_{0t} \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u'_{xt+j}\right\|^2 \\
= & \frac{\mathbb{E}\left\|u_{0t}\right\|^2}{N^{1+\alpha}} \mathbb{E}\left\|\sum_{t=1}^{m_N} \sum_{j=t+1}^{N} \rho_N^{-(j-t)} u'_{xt+j}\right\| \\
= & \frac{\mathbb{E}\left\|u_{01}\right\|^2 \mathbb{E}\left\|u_{x1}\right\|^2}{N^{1+\alpha}} \sum_{t=1}^{m_N} \sum_{j=t+1}^{N} \rho_N^{-2(j-t)} = o\left(1\right).
\end{aligned}$$

(2) The result (3.11) follows from Lemma 4.4 of MP.

Combining (3.9) and (3.11), the limit distribution of $\frac{1}{N^{(1+\alpha)/2}}\sum_{t=1}^{N} u_{0t}z_{2t}'$ is

$$\begin{aligned}
&\frac{1}{N^{(1+\alpha)/2}} vec\left(\sum_{t=1}^{N} u_{0t}z_{2t}'\right) \\
\Rightarrow\;& \left(H_{\perp}'\frac{\mu^*}{c}\frac{\mu^{*\prime}}{c}H_{\perp}\otimes\Omega_{00}\right)^{1/2}\times N\left(0, I_{m\times(K-1)}\right)+\left(\frac{1}{2c}H_{\perp}'\Omega_{xx}H_{\perp}\otimes\Omega_{00}\right)^{1/2}\times N\left(0, I_{m\times(K-1)}\right) \\
=\;& \left(H_{\perp}'\left(\frac{\mu^*}{c}\frac{\mu^{*\prime}}{c}+\frac{1}{2c}\Omega_{xx}\right)H_{\perp}\otimes\Omega_{00}\right)^{1/2}\times N\left(0, I_{m\times(K-1)}\right),
\end{aligned}$$

since we have the following independent structure asymptotically

$$\lim_{N\to\infty}\mathbb{E}\left\{\left(\frac{1}{\sqrt{N}}\sum_{t=1}^{N} u_{0t}\mu^{*\prime}\right)\left(\frac{1}{N^{(1+\alpha)/2}}\sum_{t=m_N+1}^{N} u_{0t}\sum_{j=t+1}^{N}\rho_N^{-(j-t)}u_{xj}'\right)'\right\}=0.$$

(iii) Using the results from (i) and (ii), we obtain

$$\begin{aligned}
&N^{(1+\alpha)/2} vec\left(\widehat{A}-A\right) \\
=\;& vec\left\{N^{(1+\alpha)/2}\left(\sum_{t=1}^{N} u_{0t}z_{2t}'\right)\left(\sum_{t=1}^{N} z_{2t}z_{2t}'\right)^{-1}H_{\perp N}'\right\}+o_p(1) \\
=\;& \left[\left\{H_{\perp N}\left(\frac{1}{N^{1+\alpha}}\sum_{t=1}^{N} z_{2t}z_{2t}'\right)^{-1}\right\}\otimes I_m\right] vec\left\{\frac{1}{N^{(1+\alpha)/2}}\sum_{t=1}^{N} u_{0t}z_{2t}'\right\}+o_p(1) \\
\Rightarrow\;& \left[H_{\perp}\left\{H_{\perp}'\left(\frac{\mu^*}{c}\frac{\mu^{*\prime}}{c}+\frac{1}{2c}\Omega_{xx}\right)H_{\perp}\right\}^{-1}\otimes I_m\right]\left\{H_{\perp}'\left(\frac{\mu^*}{c}\frac{\mu^{*\prime}}{c}+\frac{1}{2c}\Omega_{xx}\right)H_{\perp}\otimes\Omega_{00}\right\}^{1/2}\times N\left(0, I_{m\times(K-1)}\right) \\
\stackrel{d}{=}\;& MN\left(0, H_{\perp}\left\{H_{\perp}'\left(\frac{\mu^*}{c}\frac{\mu^{*\prime}}{c}+\frac{1}{2c}\Omega_{xx}\right)H_{\perp}\right\}^{-1}H_{\perp}'\otimes\Omega_{00}\right),
\end{aligned}$$

giving the stated result. ■