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Limit theory for an explosive autoregressive process ☆

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HIGHLIGHTS

- Establish large sample properties for AR(1) with an intercept and an explosive root.
- Show that the LS estimate of intercept and its t-statistic are asymptotically normal.
- Show that no invariance principle applies to autoregressive coefficient estimate.
- Show that tests have better power for the zero intercept in the explosive case.

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ABSTRACT

Large sample properties are studied for a first-order autoregression (AR(1)) with a root greater than unity. It is shown that, contrary to the AR coefficient, the least-squares (LS) estimator of the intercept and its t-statistic are asymptotically normal without requiring the Gaussian error distribution, and hence an invariance principle applies. The coefficient based test and the t test have better power for testing the hypothesis of zero intercept in the explosive process than in the stationary process.

1. Introduction

Consider a first-order autoregression defined by

$$x_t = d + \alpha x_{t-1} + u_t, \quad x_0 \sim O_p(1), \tag{1.1}$$

where u_t is a sequence of independent and identically distributed (i.i.d.) random errors with $E(u_t) = 0, E\left(u_t^2\right) = \sigma^2 \in (0, \infty)$ (i.e., $u_t \overset{iid}{\sim} (0, \sigma^2)$). The available sample is $\{x_t\}_{t=1}^T$. Let $\sum$ denote $\sum_{t=1}^T$. If d is known a priori and assumed zero without loss of generality, based on the available sample, the least-squares (LS) estimator of α is,

$$\widehat{\alpha} = \frac{\sum x_t x_{t-1}}{\sum x_{t-1}^2}. \tag{1.2}$$

If the value of d is unknown a priori, the LS estimators of α and d are, respectively,

$$\widehat{\alpha} = \frac{\sum \left(x_t - \overline{X}\right)\left(x_{t-1} - \overline{X}_-\right)}{\sum \left(x_{t-1} - \overline{X}_-\right)^2} \quad \text{and} \quad \widehat{d} = \overline{X} - \widehat{\alpha}\overline{X}_-, \tag{1.3}$$

where $\overline{X} = \sum x_t / T, \overline{X}_- = \sum x_{t-1}/T$.

The limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics have been developed in the literature in several special cases of Model (1.1), including the stationary case ($|\alpha| < 1$), the unit root case ($\alpha = 1$), and the explosive case ($|\alpha| > 1$). Hamilton (1994) provides the textbook treatment of the unit root case in the page range 490–494 and the stationary case in page 216.

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If $|\alpha| > 1, x_0 = 0, u_t \overset{iid}{\sim} N(0, \sigma^2)$, and $d = 0$ and is known a priori, White (1958) showed that

$$\frac{\alpha^T}{\alpha^{2-1}} \left(\widehat{\alpha} - \alpha\right) \Rightarrow Cauchy.$$

When $u_t \overset{iid}{\sim}(0, \sigma^2)$ but is not necessarily normally distributed, Anderson (1959) showed that

$$\frac{\alpha^T}{a^2 - 1} \left(\widehat{\alpha} - \alpha\right) \Rightarrow y/z,$$

where y and z are the limits of y_T and z_T defined by

$$y_T = \sum_{t=1}^{T} \alpha^{-(T-t)} u_t \quad \text{and} \quad z_T = \alpha \sum_{t=1}^{T-1} \alpha^{-t} u_t + \alpha x_0. \tag{1.4}$$

Obviously the limiting distributions of y_T and z_T, and hence of $\widehat{\alpha}$, depend on the distribution of u's, so no central limit theorem (CLT) or invariance principle is applicable. The role played by the initial condition in the limiting distribution could be found in z. In this case the rate of the convergence depends on both T and α.

In this paper, we extend the literature by establishing the limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics for the explosive AR(1) process with an unknown intercept. We show that the asymptotic normality and, hence, an invariance principle hold true for $\widehat{d}$ and its t-statistic without assuming the Gaussian error distribution. The motivation for our study comes from a recent literature on econometric analysis of bubbles; see for example, Phillips et al. (2011, 2014, 2015a, forthcoming, 2015b, forthcoming). All proofs are in the Appendix.

2. The model

We now focus our attention on Model (1.1) with $|\alpha| > 1$. An equivalent representation of x_t is

$$x_t = \frac{1 - \alpha^t}{1 - \alpha} d + \alpha^t x_0 + \sum_{j=0}^{t-1} \alpha^j u_{t-j}. \tag{2.1}$$

Obviously, $\left(1 - \alpha^t\right) d / (1 - \alpha)$ and $\alpha^t x_0$ have the same order of $O_p\left(\alpha^t\right)$ if $d \neq 0$. It becomes clear later that $\sum_{j=0}^{t-1} \alpha^j u_{t-j}$ has the order of $O_p\left(\alpha^t\right)$ too. This is the reason why both the intercept and the initial condition play an important role in the asymptotic theory for the explosive process. The model can also be expressed as

$$x_t = \frac{1 - \alpha^t}{1 - \alpha} d + x_t^0, \tag{2.2}$$

where x_t^0 is an explosive AR(1) process with no intercept.

Denote

$$w_T = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} u_t. \tag{2.3}$$

Following the Lindeberg–Feller CLT, the limiting distribution of w_T is $N\left(0, \sigma^2\right)$. Following Anderson (1959), we define y_T and z_T as in Eq. (1.4). In the following lemma whose proof is in the Appendix, we give the limits of w_T, y_T, and z_T, and show that they are independent from each other.

Lemma 2.1. *Define w_T, y_T, and z_T as in Eqs. (1.4) and (2.3). Then we have* (a) $y_T \Rightarrow y$, $z_T \Rightarrow z$, *and y and z are independent;* (b) $w_T \Rightarrow w \overset{d}{=} N\left(0, \sigma^2\right)$ *and w is independent of (y, z).*

To obtain the limiting distribution of the LS estimator of α in the explosive AR(1) model without intercept, Anderson (1959) proved that

$$\left(\alpha^{-(T-2)} \sum x_{t-1}^0 u_t,\ \left(\alpha^2 - 1\right) \alpha^{-2(T-1)} \sum \left(x_{t-1}^0\right)^2\right) \Rightarrow \left(yz, z^2\right). \tag{2.4}$$

Using this result together with the independence of w, y, z, we obtain the following results.

Theorem 2.2. *For Model (1.1) with $|\alpha| > 1$, we have, as $T \to \infty$,*

(a) $\alpha^{-(T-1)} x_T \Rightarrow z + \alpha d / (\alpha - 1)$;
(b) $\alpha^{-(T-2)} \sum x_{t-1} u_t \Rightarrow y \left[z + \alpha d / (\alpha - 1)\right]$;
(c) $(\alpha - 1) \alpha^{-(T-1)} \sum x_{t-1} \Rightarrow z + \alpha d / (\alpha - 1)$;
(d) $\left(\alpha^2 - 1\right) \alpha^{-2(T-1)} \sum x_{t-1}^2 \Rightarrow \left[z + \alpha d / (\alpha - 1)\right]^2$.

Since $z_T = \alpha \sum_{t=1}^{T-1} \alpha^{-t} u_t + \alpha x_0$, not surprisingly, the initial condition αx_0 appears in the limit, z. According to Theorem 2.2, the intercept term d appears in all the asymptotic distributions. In particular, the intercept and the initial condition affect the asymptotic distributions in the same manner. This observation is consistent with the one in Eq. (2.1) where the three terms on the right hand side have the same order of magnitude.

The centered LS estimators of d and α and their t-statistics are given by

$$\begin{pmatrix} \widehat{d} - d \\ \widehat{\alpha} - \alpha \end{pmatrix} = \begin{pmatrix} T & \sum x_{t-1} \\ \sum x_{t-1} & \sum x_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum u_t \\ \sum x_{t-1} u_t \end{pmatrix},$$

and

$$t_d = \frac{\left(\widehat{d} - d\right) \left[T \sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[\sum x_{t-1}^2 \times \hat{\sigma}^2\right]^{1/2}},$$

$$t_\alpha = \frac{\left(\widehat{\alpha} - \alpha\right) \left[T \sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[T \times \hat{\sigma}^2\right]^{1/2}},$$

where $\hat{\sigma}^2 = T^{-1} \sum \left(x_t - \hat{d} - \hat{\alpha} x_{t-1}\right)^2$.

Since $\sum x_{t-1} u_t$ and $\sum x_{t-1}$ have the same rate of convergence, α^{-T}, we have

$$\begin{aligned}
\begin{pmatrix} \sqrt{T}\left(\widehat{d} - d\right) \\ \alpha^T \left(\widehat{\alpha} - \alpha\right) \end{pmatrix} &= \begin{pmatrix} 1 & T^{-1/2} \alpha^{-T} \sum x_{t-1} \\ T^{-1/2} \alpha^{-T} \sum x_{t-1} & \alpha^{-2T} \sum x_{t-1}^2 \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} T^{-1/2} \sum u_t \\ \alpha^{-T} \sum x_{t-1} u_t \end{pmatrix} \\
&= \begin{pmatrix} 1 & o_p(1) \\ o_p(1) & \alpha^{-2T} \sum x_{t-1}^2 \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} T^{-1/2} \sum u_t \\ \alpha^{-T} \sum x_{t-1} u_t \end{pmatrix}.
\end{aligned}$$

Consequently, we have the following theorem which extends Anderson's results to the explosive AR(1) model with intercept.

Theorem 2.3. *For Model (1.1) with $|\alpha| > 1$, if $Pr\{z + \alpha d / (\alpha - 1) = 0\} = 0$, the following limits apply as $T \to \infty$:*

(a)

$$\sqrt{T}\left(\widehat{d} - d\right) = T^{-1/2} \sum u_t + o_p(1) \Rightarrow w \overset{d}{=} N\left(0, \sigma^2\right), \tag{2.5}$$

(b)

$$\frac{\alpha^T}{\alpha^2-1}(\widehat{\alpha}-\alpha) = \frac{\alpha^{-(T-2)}\sum x_{t-1}u_t}{\left(\alpha^2-1\right)\alpha^{-2(T-1)}\sum x_{t-1}^2} + o_p(1) \Rightarrow \frac{y}{z+\alpha d/(\alpha-1)}, \tag{2.6}$$

(c)

$$\hat{\sigma}^2 = T^{-1}\sum\left(x_t - \hat{d} - \hat{\alpha}x_{t-1}\right)^2 \xrightarrow{p} \sigma^2, \tag{2.7}$$

(d)

$$t_d = \frac{\sqrt{T}\left(\widehat{d}-d\right)}{\left\{\hat{\sigma}^2\right\}^{1/2}} + o_p(1) \Rightarrow \frac{w}{\sigma} \overset{d}{=} N(0,1), \tag{2.8}$$

(e)

$$t_\alpha \Rightarrow \frac{y}{z+\alpha d/(\alpha-1)} \times \left|z+\frac{\alpha d}{\alpha-1}\right| \times \left\{\frac{\alpha^2-1}{\alpha^2\sigma^2}\right\}^{1/2}. \tag{2.9}$$

Remark 2.4. An invariance principle exists for $\widehat{d}$ and its t-statistic as Eqs. (2.5) and (2.8) hold true even when u_t is not normally distributed.

Remark 2.5. In Eq. (2.6), if $d = 0$, the limiting distribution becomes y/z which is the same as that derived by Anderson (1959) for the model without intercept and the intercept is not estimated. It implies that when $d = 0$ the limiting distribution is the same regardless of whether or not d is estimated. This is not surprising as $x_t = x_t^0$ when $d = 0$. Hence, $\alpha^{-(T-2)}\sum x_{t-1}u_t = \alpha^{-(T-2)}\sum x_{t-1}^0 u_t$, $\left(\alpha^2-1\right)\alpha^{-2(T-1)}\sum x_{t-1}^2 = \left(\alpha^2-1\right)\alpha^{-2(T-1)}\sum\left(x_{t-1}^0\right)^2$, suggesting that the middle term in Eq. (2.6) is the same as the ratio of the two terms in Eq. (2.4). This result is in sharp contrast to the unit root model.

Remark 2.6. With the same intuition as before, the distributions of both z and y depend on the distribution of u_t. Hence, no invariance principle applies to $\widehat{\alpha}$ and its t-statistic.

Remark 2.7. The independence of w, y and z suggests $\sqrt{T}\left(\widehat{d}-d\right)$ and $\alpha^T(\widehat{\alpha}-\alpha)/\left(\alpha^2-1\right)$ are asymptotically independent. Similarly, t_d and t_α are asymptotically independent.

Remark 2.8. As apparent in Theorem 2.3 (a) and (d), neither the initial condition (x_0) nor the intercept (d) can be found in the limiting distributions of $\sqrt{T}\left(\widehat{d}-d\right)$ and t_d. In sharp contrast, both the initial condition and the intercept appear in the limiting distributions of $\frac{\alpha^T}{\alpha^2-1}(\widehat{\alpha}-\alpha)$ and t_α. In fact, they play the same role in the limiting distributions. It is worth noting that what matters in the limiting distributions is not x_0 or d, but x_0/σ and d/σ. This point can be seen more clearly by studying a special case where $u_t \overset{iid}{\sim} N(0,\sigma^2)$. In this case, we get

$$y = N(0, \alpha^2\sigma^2/\left(\alpha^2-1\right)) \quad \text{and}$$
$$z = N(0, \alpha^2\sigma^2/\left(\alpha^2-1\right)) + \alpha x_0,$$

which are independently distributed. Let

$$\xi := \left(\frac{\alpha^2-1}{\alpha^2\sigma^2}\right)^{1/2} y, \quad \text{and}$$
$$\eta := \left(\frac{\alpha^2-1}{\alpha^2\sigma^2}\right)^{1/2} z - \left(\frac{\alpha^2-1}{\alpha^2\sigma^2}\right)^{1/2}\alpha x_0,$$

be two independent $N(0,1)$ random variables. Then, Theorem 2.3 (b) becomes

$$\frac{\alpha^T}{\alpha^2-1}(\widehat{\alpha}-\alpha) \Rightarrow \frac{\xi}{\eta+\sqrt{\left(\alpha^2-1\right)/\alpha^2}\left[\alpha x_0/\sigma+\alpha d/\sigma(\alpha-1)\right]}.$$

It can be seen that both x_0/σ and d/σ, but not x_0 and d, determine the limiting distribution of $\widehat{\alpha}$. When $x_0 = d = 0$, we obtain the standard Cauchy limiting distribution. The dependence on the ratio of x_0/σ and d/σ was also found in the unit root and local-to-unity literature. See, for example, Phillips (1987) and Perron (1991).

Remark 2.9. While in general the limiting distribution of t_α depends on both the initial value and the intercept as shown in Eq. (2.9), the result is remarkably different when $u_t \overset{iid}{\sim} N(0,\sigma^2)$. In this case, we have

$$t_\alpha \Rightarrow \frac{\xi}{z+\alpha d/(\alpha-1)}\left|z+\alpha d/(\alpha-1)\right|.$$

Let $P_+ = \Pr\{z+\alpha d/(\alpha-1) > 0\}$ and $P_- = \Pr\{z+\alpha d/(\alpha-1) < 0\}$. Then, from the independence of ξ and z, we obtain the moment generating function for the limit of t_α,

$$P_+ \cdot E(\exp\{t\xi\}) + P_- \cdot E(\exp\{-t\xi\}) = P_+ \cdot \exp\left\{t^2/2\right\} + P_- \cdot \exp\left\{t^2/2\right\} = \exp\left\{t^2/2\right\}.$$

Therefore, $t_\alpha \Rightarrow N(0,1)$ which does not depend on the initial condition nor the intercept.

3. Comparison with other models

The limit theory for the explosive model and that for the unit root model are distinctively different. First, for a unit root model, we have

$$x_t = x_0 + \sum_{j=0}^{t-1} u_{t-j} = O_p\left(\sqrt{t}\right), \quad \text{when } d = 0,$$

$$x_t = dt + x_0 + \sum_{j=0}^{t-1} u_{t-j} = O_p(t), \quad \text{when } d \neq 0.$$

Obviously, the presence of a nonzero intercept changes the asymptotic property of x_t, and consequently, leads to a change in the limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics. The discontinuity of the limiting distributions of $\widehat{d}$ and t_d at $d = 0$ makes it hard to analyze the local power behavior when they are used to test $d = 0$. To analyze the local power, we often use the limit theory for the unit root model with an intercept dependent on T:

$$x_t = d_T + \alpha x_{t-1} + u_t \quad \text{with } d_T = d/\sqrt{T},\ \alpha = 1.$$

In contrast, for the explosive process, the limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics become continuous at the point $d = 0$ as we have shown in Theorem 2.3. Hence, the local power can be obtained directly.

Second, for the explosive process, $\widehat{d}$ and $\widehat{\alpha}$ are asymptotically independent, and t_d and t_α are also asymptotically independent, regardless of the value of d. In contrast, when a unit root process is considered, the asymptotic distributions of $\widehat{d}$ and $\widehat{\alpha}$ (as well as t_d and t_α) are always correlated, and the strength of the correlation varies as the value of d changes.

For the explosive process, the comparison of the limit theory between Anderson (1959) and Theorem 2.3 reveals that, when the intercept is zero, the limiting distribution of $\widehat{\alpha}$ is the same regardless of whether or not the intercept is estimated. On the other hand, for unit root process, the estimation of the intercept changes the limiting distribution of $\widehat{\alpha}$.

The differences between the explosive process and the stationary model are more subtle and have important implications. First, for the stationary AR(1) process, the limiting distribution of $\sqrt{T}\left(\widehat{d}-d\right)$ is a linear combination of the limiting distribution of $T^{-1/2}\sum u_t$ and that of $T^{-1/2}\sum x_{t-1}u_t$. As a result, the asymptotic variance of $\sqrt{T}\left(\widehat{d}-d\right)$ is $\sigma^2+d^2(1+\alpha)/(1-\alpha)$ which depends on d. For the explosive AR(1) process, the limiting distribution of $\sqrt{T}\left(\widehat{d}-d\right)$ is dominated by $T^{-1/2}\sum u_t$, the asymptotic distribution of which, as shown in Eq. (2.5), is $N\left(0,\sigma^2\right)$ whose variance is independent of d. This distinction sheds insights on the differences in the finite sample power behavior of the test of the null hypothesis $H_0 : d = 0$ in the context of the explosive process and the stationary AR(1) process. Under the null, $\sqrt{T}\widehat{d} \Rightarrow w \overset{d}{=} N\left(0,\sigma^2\right)$, for both the explosive and the stationary models. Under the alternative hypothesis $H_1 : d \neq 0$, the finite sample distribution of $\sqrt{T}\widehat{d}$ can be approximated by

$$\sqrt{T}\widehat{d} \overset{d}{\approx} \sqrt{T}d + w \overset{d}{=} N\left(\sqrt{T}d,\ \sigma^2\right), \quad \text{if } |\alpha| > 1,$$

and by

$$\sqrt{T}\widehat{d} \overset{d}{\approx} \sqrt{T}d + N\left(0,\ \sigma^2 + \frac{d^2(1+\alpha)}{1-\alpha}\right) \overset{d}{=} N\left(\sqrt{T}d,\ \sigma^2 + \frac{d^2(1+\alpha)}{1-\alpha}\right), \quad \text{if } |\alpha| < 1.$$

Note that the shift of the mean is the same in both cases. However, when $|\alpha| < 1$, the variance of the finite sample distribution increases with $|d|$ whereas when $|\alpha| > 1$, the variance of the finite sample distribution remains unchanged. Therefore, we expect the test to have a better power for the explosive model than for the stationary model.

A similar observation applies to the t test. Under the null hypothesis $H_0 : d = 0$, for both the explosive process and the stationary process, we have

$$\tilde{t}_d = \frac{\widehat{d}\left[T\sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[\sum x_{t-1}^2 \times \hat{\sigma}^2\right]^{1/2}} \Rightarrow N(0,1).$$

Under the alternative hypothesis that $H_1 : d \neq 0$, Theorem 2.3 (d) gives us an approximation of the finite sample distribution of $\tilde{t}_d$ for the explosive case:

$$\begin{aligned}\tilde{t}_d &= t_d + \frac{\sqrt{T}d\left[\sum x_{t-1}^2 - T^{-1}\left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[\sum x_{t-1}^2 \times \hat{\sigma}^2\right]^{1/2}} \\ &= \frac{\sqrt{T}\left(\widehat{d}-d\right)}{\left\{\hat{\sigma}^2\right\}^{1/2}} + \frac{\sqrt{T}d}{\left\{\hat{\sigma}^2\right\}^{1/2}} + o_p(1) \\ &\overset{d}{\approx} \frac{w}{\sigma} + \frac{\sqrt{T}d}{\sigma} \overset{d}{=} N\left(\frac{\sqrt{T}d}{\sigma}, 1\right).\end{aligned}$$

For the stationary case, the approximation of the finite sample distribution of $\tilde{t}_d$ is given by

$$\begin{aligned}\tilde{t}_d &= t_d + \frac{\sqrt{T}d\left[\sum x_{t-1}^2 - T^{-1}\left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[\sum x_{t-1}^2 \times \hat{\sigma}^2\right]^{1/2}} \\ &\overset{d}{\approx} N\left(\sqrt{T}d\sqrt{\frac{1}{\sigma^2 + d^2(1+\alpha)/(1-\alpha)}}, 1\right).\end{aligned}$$

Note that in both cases, the variance of the approximate finite sample distribution is the same but the means are different. Since $\sigma^2 + d^2(1+\alpha)/(1-\alpha) > \sigma^2$ when $|\alpha| < 1$, we have

$$\sqrt{T}d\sqrt{\frac{1}{\sigma^2 + d^2(1+\alpha)/(1-\alpha)}} < \frac{\sqrt{T}d}{\sigma},$$

the shift of the mean of $\tilde{t}_d$ from H_0 to H_1 under the explosive model is greater than that under the stationary model. Therefore, the t test is expected to have better power for the explosive process than for the stationary process.

Second, for the explosive process, the results in Theorem 2.3 suggest that, regardless of the value of d, $\widehat{d}$ and $\widehat{\alpha}$ are asymptotically independent, and t_d and t_α are also asymptotically independent. On the contrary, for the stationary process, the asymptotic independence between $\widehat{d}$ and $\widehat{\alpha}$ and that between t_d and t_α can only be guaranteed by the condition of $d = 0$.

Third, for the explosive process with intercept, the value of d affects the limiting distributions of $\sum x_{t-1}u_t$ and $\sum x_{t-1}^2$, and, hence, the limiting distribution of $\widehat{\alpha}$, as shown in Theorem 2.3. The value of d has no impact on the limiting distribution of $\widehat{d}$ because it is decided by the unique dominating term, $T^{-1/2}\sum u_t$. On the contrary, for the stationary process with an intercept, the magnitude of d does not change the limiting distribution of $\widehat{\alpha}$, but only affect the limiting distribution of $\widehat{d}$.

4. Conclusions

In this paper the asymptotic theory is developed for the explosive AR(1) process with intercept. The results extend the literature in several directions. First, it is proved that an invariance principle applies to the intercept and its t-statistic while it continues to fail to apply to the AR coefficient. Second, the asymptotic independence between LS estimators of the intercept and the AR coefficient and the asymptotic independence between their t-statistics are established. Third, the comparison conducted in the paper reveals that the coefficient based test and the t test have better power for testing $H_0 : d = 0$ under the explosive process than under the stationary process. However, our theory does not cover the model with a time trend. How to include a time trend into a model with explosive behavior and how to establish the asymptotic theory for the new model are beyond the scope of the present paper.

Appendix

Proof of Lemma 2.1. (a) has been proved by Anderson (1959).

(b): The fact of $w_T \Rightarrow w \overset{d}{=} N\left(0,\sigma^2\right)$ simply follows the Lindeberg–Feller CLT. To prove the independence between w and z, let

$$z_T^* = \alpha x_0 + \alpha\sum_{s=1}^{\left\lfloor\sqrt{T}\right\rfloor}\alpha^{-s}u_s, \qquad \tilde{z}_T = \alpha\sum_{s=\left\lfloor\sqrt{T}\right\rfloor+1}^{T-1}\alpha^{-s}u_s,$$

and

$$w_T^* = \frac{1}{\sqrt{T}}\sum_{s=\left\lfloor\sqrt{T}\right\rfloor+1}^{T-1}u_s, \qquad \tilde{w}_T = \frac{1}{\sqrt{T}}\sum_{s=1}^{\left\lfloor\sqrt{T}\right\rfloor}u_s,$$

where $\left\lfloor\sqrt{T}\right\rfloor$ is the largest integer not greater than $\sqrt{T}$. Then z_T^* and w_T^* are independently distributed because they involve disjoint sets of u's. As T goes to infinity, we have

$$E\left(z_T - z_T^*\right)^2 = E\left(\tilde{z}_T\right)^2 = \left(\alpha^2\sum_{s=\left\lfloor\sqrt{T}\right\rfloor+1}^{T-1}\alpha^{-2s}\right)\sigma^2 \to 0,$$

and

$$E\left(w_T - w_T^*\right)^2 = E\left(\tilde{w}_T\right)^2 = \frac{\left\lfloor\sqrt{T}\right\rfloor}{T}\sigma^2 \to 0.$$

Then, $z_T - z_T^*$ and $w_T - w_T^*$ converge with probability 1 to 0, therefore, the asymptotic independence between z_T and w_T follows. The independence between w and y can be proved in a similar way. ■

Proof of Theorem 2.2. (a): Starting from Eq. (2.2), we have

$$\begin{aligned}
\alpha^{-(T-1)}x_T &= \alpha^{-(T-1)}\left(\frac{1-\alpha^T}{1-\alpha}d + x_T^0\right) \\
&= \alpha^{-(T-1)}\left(\frac{1-\alpha^T}{1-\alpha}d + \alpha x_{T-1}^0 + u_T\right) \\
&= \frac{\alpha d}{\alpha-1} + \alpha^{-(T-2)}x_{T-1}^0 + o_p(1) \\
&= \frac{\alpha d}{\alpha-1} + z_T + o_p(1) \Rightarrow z + \frac{\alpha d}{\alpha-1},
\end{aligned}$$

where the fourth equality comes from the definition of z_T in (1.4), and the final limit is a result of Lemma 2.1.

(b): Again, starting from Eq. (2.2), it can be obtained that

$$\begin{aligned}
&\alpha^{-(T-2)}\sum x_{t-1}u_t \\
&= \alpha^{-(T-2)}\sum x_{t-1}^0 u_t + \alpha^{-(T-2)}\sum \frac{\left(1-\alpha^{t-1}\right)d}{1-\alpha}u_t \\
&= \alpha^{-(T-2)}\sum x_{t-1}^0 u_t - \frac{\alpha d}{1-\alpha}\sum \alpha^{-(T-t)}u_t \\
&\quad + \frac{d}{1-\alpha}\alpha^{-(T-2)}\sum u_t \\
&= \alpha^{-(T-2)}\sum x_{t-1}^0 u_t + \frac{\alpha d}{\alpha-1}y_T + o_p(1) \\
&\Rightarrow yz + \frac{\alpha d}{\alpha-1}y = y\left(z + \frac{\alpha d}{\alpha-1}\right),
\end{aligned}$$

where the third equality comes from the definition of y_T in (1.4), and the combination of the results in Lemma 2.1 and Eq. (2.4) leads to the final limit.

(c): From Model (1.1) it is easy to get $x_t - x_{t-1} = d + (\alpha-1)x_{t-1} + u_t$. Then,

$$(\alpha-1)\sum x_{t-1} = x_T - x_0 - Td - \sum u_t.$$

Hence, based on the limiting distribution derived in (a), we have

$$\begin{aligned}
(\alpha-1)\alpha^{-(T-1)}\sum x_{t-1} &= \alpha^{-(T-1)}(x_T - x_0) \\
&\quad - \alpha^{-(T-1)}Td - \alpha^{-(T-1)}\sum u_t \\
&= \alpha^{-(T-1)}x_T + o_p(1) \Rightarrow z + \frac{\alpha d}{\alpha-1}.
\end{aligned}$$

(d): Squaring both sides of Model (1.1), we get

$$x_t^2 = \alpha^2 x_{t-1}^2 + 2\alpha d x_{t-1} + 2\alpha x_{t-1}u_t + d^2 + u_t^2 + 2du_t.$$

Therefore, $x_t^2 - x_{t-1}^2 = \left(\alpha^2-1\right)x_{t-1}^2 + 2\alpha d x_{t-1} + 2\alpha x_{t-1}u_t + d^2 + u_t^2 + 2du_t$, which leads to

$$\begin{aligned}
\left(\alpha^2-1\right)\sum x_{t-1}^2 &= x_T^2 - x_0^2 - 2\alpha d\sum x_{t-1} - 2\alpha\sum x_{t-1}u_t \\
&\quad - Td^2 - \sum u_t^2 - 2d\sum u_t.
\end{aligned}$$

Based on the results reported in (a)–(c) and the assumption that $x_0 = O_p(1)$, it is straightforward to get

$$\begin{aligned}
\left(\alpha^2-1\right)\alpha^{-2(T-1)}\sum x_{t-1}^2 &= \alpha^{-2(T-1)}x_T^2 + o_p(1) \\
&\Rightarrow \left(z + \frac{\alpha d}{\alpha-1}\right)^2. \quad ■
\end{aligned}$$

Proof of Theorem 2.3. The results come immediately from Lemma 2.1 and Theorem 2.2, hence the proofs are omitted. ■

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