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On leverage in a stochastic volatility model

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Abstract

This paper is concerned with the specification for modelling financial leverage effect in the context of stochastic volatility (SV) models. Two alternative specifications co-exist in the literature. One is the Euler approximation to the well-known continuous time SV model with leverage effect and the other is the discrete time SV model of Jacquier et al. (J. Econometrics 122 (2004) 185). Using a Gaussian nonlinear state space form with uncorrelated measurement and transition errors, I show that it is easy to interpret the leverage effect in the conventional model whereas it is not clear how to obtain and interpret the leverage effect in the model of Jacquier et al. Empirical comparisons of these two models via Bayesian Markov chain Monte Carlo (MCMC) methods further reveal that the specification of Jacquier et al. is inferior. Simulation experiments are conducted to study the sampling properties of Bayes MCMC for the conventional model.

JEL classification: C11; C15; G12

Keywords: Bayes factors; Leverage effect; Markov chain Monte Carlo; Nonlinear state space models; Quasi maximum likelihood; Particle filter

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1. Introduction

Stochastic volatility (SV) models have gained much attention in both the option pricing and financial econometrics literature (see Ghysels et al. (1996) and Shephard (1996) for reviews of SV models and their applications). For example, Melino and Turnbull (1990) show that prices of European call options on currencies based on the basic SV models are more accurate than those based on the Black–Scholes model. Kim et al. (1998) provide evidence of better in-sample-fit of the basic SV model relative to GARCH-type models. Despite these documented advantages, it is known that the basic SV model can be too restrictive for many financial time series.

An important and well documented empirical feature in many financial time series is the financial leverage effect (Black, 1976; Christie, 1982; Engle and Ng, 1993). When such an asymmetric feature is not permitted in the SV model, option prices could be substantially biased (Hull and White, 1987). Motivated by this empirical evidence, Harvey and Shephard (1996) propose a SV model with leverage effect which is termed the asymmetric SV (ASV1 hereafter) model. This model is the Euler approximation to the continuous time asymmetric SV model widely used in the option price literature; see for example Hull and White (1987), Wiggins (1987), and Chesney and Scott (1989). Harvey and Shephard fit the model to stock data using a quasi-maximum likelihood (QML) method while Meyer and Yu (2000) fit it to an exchange rate series using a Bayesian Markov chain Monte Carlo (MCMC) method. The findings of both papers reveal overwhelming evidence of the leverage effect. Motivated by the same empirical evidence, Jacquier et al. (2004) generalized the basic SV model by incorporating an asymmetric feature which is also termed the leverage effect (ASV2 hereafter). A Bayesian MCMC approach was then developed to estimate the ASV2 model and strong evidence of "leverage effect" was found in most financial time series considered. Chan et al. (2004) extend the specification of Jacquier et al. to a multivariate setting. Unfortunately, these two specifications are not identical although both are claimed to be able to capture the leverage effect. They differ in how the correlation of two error processes is modelled.

The main purpose of this paper is to compare these two alternative specifications. The results obtained in the present paper show that the ASV2 model is inferior to the ASV1 model, judged from both theoretical and empirical view points. Firstly, the ASV2 model is not consistent with the efficient market hypothesis because the model is not a martingale difference sequence. Secondly, while the interpretation of the leverage effect using a parameter in the ASV1 model is clear, the strict interpretation of leverage is not obvious in the ASV2 model. Finally, I find the ASV2 model to be empirically inferior to the ASV1 model when S&P500 and Center for Research in Security Prices (CRSP) data are used.

To relate both SV models to the financial leverage effect, I derive a Gaussian nonlinear state space representation for each model. I then fit them to two stock indices using a Bayesian MCMC method. The choice of the MCMC method for inferences is mainly due to a result obtained by Andersen et al. (1999)

in a Monte Carlo study, where MCMC was found to be one of the most efficient tools for estimating the basic SV model. This finding is not surprising since MCMC provides a fully likelihood-based inference (Jacquier et al., 1994).

The remainder of the article is organized as follows. Section 2 compares the two asymmetric SV models from theoretical view points. Section 3 discusses methods for parameter estimation and for model comparison. The methods are then applied to actual return series in Section 4. In Section 5, I present the sampling properties of MCMC for the ASV1 model. Section 6 concludes.

2. Leverage effect and asymmetric stochastic volatility models

The relationship between volatility and price/return has been a subject under extensive study. The usual claim is that when there is bad news, which decreases the price and hence increases the debt-to-equity ratio (i.e. financial leverage), it makes the firm riskier and tends to increase future expected volatility. As a result, the leverage effect must correspond to a negative relationship between volatility and price/return. Black (1976) and Christie (1982) have found empirical evidence of this leverage effect, i.e., volatility tends to rise in response to bad news but fall in response to good news. Christie (1982) provides a theoretical explanation of leverage effect under a Modigliani/Miller economy.

Depending on how volatility is defined, various approaches have been suggested to test the leverage effect. By computing quarterly volatility from daily data, Christie (1982) postulates a parametric form to relate volatility to return, enabling a simple test of leverage effect. In the ARCH literature, often the conditional variance is specified to be a function of the size as well as the sign of return (Glosten et al., 1993; Nelson, 1991). Then the asymmetric response of volatility to return is tested by checking the significance of the relevant coefficient. In the SV literature, Harvey and Shephard (1996) relate the filtered volatility to the sign of return. In the present paper, I define the leverage effect as a negative relationship between $\mathrm{E}(\ln \sigma_{t+1}^2 | X_t)$ and X_t, where X_t is the return at period t and σ_t^2 the return volatility at period t.

In the option pricing literature, the asymmetric SV model is often formulated in terms of stochastic differential equations. The widely used asymmetric SV model specifies the following equations for the logarithmic asset price $s(t)$ and the corresponding volatility $\sigma^2(t)$,

$$\begin{aligned} \mathrm{d}s(t) &= \sigma(t)\mathrm{d}B_1(t), \\ \mathrm{d}\ln\sigma^2(t) &= \alpha + \beta\ln\sigma^2(t)\,\mathrm{d}t + \sigma_v\,\mathrm{d}B_2(t), \end{aligned} \tag{2.1}$$

where $B_1(t)$ and $B_2(t)$ are two Brownian motions, $\mathrm{corr}(\mathrm{d}B_1(t), \mathrm{d}B_2(t)) = \rho$ and $s(t) = \ln S(t)$, with $S(t)$ being the asset price. When $\rho < 0$, we have the leverage effect.

In the empirical literature, the above model is often discretized to facilitate estimation. For instance, the Euler–Maruyama approximation leads to the discrete time ASV1 model:

$$X_t = \sigma_t u_t,$$
$$\ln \sigma_{t+1}^2 = \alpha + \phi \ln \sigma_t^2 + \sigma_v v_{t+1}, \tag{2.2}$$

where $X_t = s(t+1) - s(t)$ is the continuously compounded return, $u_t = B_1(t+1) - B_1(t)$, $v_{t+1} = B_2(t+1) - B_2(t)$, $\phi = 1 + \beta$. Hence, u_t and v_t are iid N(0, 1) and $\text{corr}(u_t, v_{t+1}) = \rho$. Compared with the basic SV model, a contemporaneous dependence is allowed in the ASV1 model.[1] This ASV1 model is estimated by QML in Harvey and Shephard (1996) and by MCMC (i.e. a likelihood-based inference) in Meyer and Yu (2000).

Comparing Eq. (2.2) with Eq. (8) in Jacquier et al. (2004), I note a small but important difference. Instead of assuming $\text{corr}(u_t, v_{t+1}) = \rho$, Jacquier et al. adopt the specification of $\text{corr}(u_t, v_t) = \rho$, i.e., an inter-temporal dependence, instead of contemporaneous dependence, is permitted. One implication is, as argued in Harvey and Shephard (1996), that the ASV1 model is a martingale difference sequence whereas ASV2 is not and hence not even consistent with the efficient market hypothesis. This is obvious because for the ASV1 model, we have

$$\text{E}(X_{t+1}|X_t, \sigma_t) = \text{e}^{1/2(\alpha + \ln\ \sigma_t^2)}\text{E}(\text{e}^{1/2\sigma_v v_{t+1}})\text{E}(u_{t+1}|X_t, \sigma_t) = 0.$$

However, for the ASV2 model, we have

$$\text{E}(X_{t+1}|X_t, \sigma_t) = \text{e}^{(1/2)(\alpha + \ln\ \sigma_t^2)}\text{E}(\text{e}^{(1/2)\sigma_v v_{t+1}} u_{t+1}|X_t, \sigma_t) = \frac{1}{2}\rho\sigma_v \text{e}^{(1/2)(\alpha + \ln \sigma_t^2)}\text{e}^{(1/8)\sigma_v^2},$$

and then,

$$\text{E}(X_{t+1}|X_t) = \text{E}[\text{E}(X_{t+1}|X_t, \sigma_t)] = \frac{1}{2}\rho\sigma_v \exp\left\{\frac{2-\phi}{2-2\phi}\alpha + \frac{2-\phi^2}{8-8\phi^2}\sigma_v^2\right\}.$$

This quantity is different from zero unless ρ is zero. For example, using the empirical estimates for S&P500 (see Table 1) and CRSP (see Table 3), I find $\text{E}(X_{t+1}|X_t) = -0.035$ and -0.063. These correspond respectively to an annual return of -8.75% and -15.75%. Both figures, particularly the latter, seem economically substantial.

To fully understand the linkage of these two alternative specifications to the leverage effect, it is convenient to adopt a Gaussian nonlinear state space form with uncorrelated measurement and transition equation errors. To do this, denote $w_{t+1} \equiv (v_{t+1} - \rho u_t)/\sqrt{1-\rho^2}$, and rewrite Eq. (2.2) as

$$X_t = \sigma_t u_t,$$
$$\ln \sigma_{t+1}^2 = \alpha + \phi \ln \sigma_t^2 + \rho\sigma_v\sigma_t^{-1}X_t + \sigma_v\sqrt{1-\rho^2}w_{t+1}, \tag{2.3}$$

[1] Although we follow the notation used in much of the literature, it would be clearer and more consistent if we defined $B_2(t+1) - B_2(t)$ by v_t. If so, subsequent changes in notation are needed throughout the paper.

where w_t is iid N(0, 1) and $\mathrm{corr}(u_t, w_{t+1}) = 0$. Obviously, $\mathrm{E}(\ln \sigma_{t+1}^2 | X_t, \sigma_t) = \alpha + \phi \ln \sigma_t^2 + \rho\sigma_v\sigma_t^{-1}X_t$, which implies that

$$\mathrm{E}(\ln \sigma_{t+1}^2 | X_t) = \alpha + \frac{\alpha\phi}{1-\phi^2} + \rho\sigma_v \exp\left(-\frac{\sigma_v^4}{4(1-\phi^2)^2} + \frac{\sigma_v^2\alpha}{(1-\phi^2)(1-\phi)}\right)X_t.$$

This is a linear function in X_t and implies that if $\rho < 0$, and everything else is held constant, a fall in the stock price/return leads to an increase of $\mathrm{E}(\ln \sigma_{t+1}^2 | X_t)$ and thus the leverage effect is ensured.

Using the same approach, I rewrite Eq. (8) in Jacquier et al. (2004) in the following Gaussian nonlinear state space form:

$$\begin{aligned} X_t &= \sigma_t u_t, \\ \ln \sigma_{t+1}^2 &= \alpha + \phi \ln \sigma_t^2 + \rho\sigma_v\sigma_{t+1}^{-1}X_{t+1} + \sigma_v\sqrt{1-\rho^2}w_{t+1}, \end{aligned} \tag{2.4}$$

where w_t is iid N(0, 1) and $\mathrm{corr}(u_t, w_t) = 0$. As a result, we have $\mathrm{E}(\ln \sigma_{t+1}^2 | X_t, \sigma_t) = \alpha + \phi \ln \sigma_t^2 + \rho\sigma_v \mathrm{E}(\sigma_{t+1}^{-1}X_{t+1} | X_t, \sigma_t)$. Because σ_{t+1}^2 appears at both sides of the equation and also because of the nonlinearity in $\sigma_{t+1}^{-1}X_{t+1}$, it is not so easy, if not impossible, to obtain the relationship between $\mathrm{E}(\ln \sigma_{t+1}^2 | X_t)$ and X_t in an analytical form, and hence not clear how to interpret the leverage effect in the ASV2 model. This is in sharp contrast to the ASV1 model where the interpretation of the leverage effect is obvious.

3. Methods for estimation and model comparison

3.1. Method for estimation

Although many estimation methods have been suggested in the literature to fit the basic SV model, only a small subset was used to estimate asymmetric SV models. In this paper, a Bayesian MCMC method is my choice for estimation and inference. I refer readers to Chib (2001) for a recent survey on MCMC in a general context.

Various MCMC algorithms have been proposed to sample the parameters in the context of the basic SV model. An early example is the single-move Metropolis–Hastings (MH) algorithm developed by Jacquier et al. (1994). It has been shown in Kim et al. (1998) that for the basic SV model, a single-move algorithm is not very efficient from a simulation perspective because the components of $\{\ln \sigma_t^2\}$ are highly correlated. To achieve better simulation efficiency, Kim et al. (1998) developed several multi-move algorithms, all based on a log-squared transformation of return and an offset mixture approximation to a $\ln \chi^2$ distribution. The evidence of drastic reduction in simulation inefficiency is found when estimating the basic SV model. The algorithms are further modified in Chib et al. (2002) to successfully estimate several more complex SV specifications. However, because the multi-move algorithms developed in Kim et al. (1998) and Chib et al. (2002) rely on the log-squared transformation, such a transformation would lose the information on the dependence between the two error terms (Harvey and Shephard, 1996) and hence, these algorithms are not directly applicable to the asymmetric SV models studied here.

In the present paper, I make use of the all-purpose Bayesian software package BUGS to estimate asymmetric SV models and it does not require any transformation. Since the full conditional distributions are not log-concave for the asymmetric SV models, a Metropolis–Hastings updating step is needed. A drawback with BUGS is that the algorithm is single-move and hence cannot be simulation-efficient. However, as in Meyer and Yu (2000), I also find that the simulation inefficiency is less a problem for the asymmetric SV models than for the basic SV model (see Section 5 below). Furthermore, the results obtained from a simulation study (see Section 6 below) clearly show that BUGS produces reliable results. An advantage of using BUGS lies in its ease of implementation. For example, following Meyer and Yu (2000), the ASV1 and ASV2 models can be rewritten, respectively, by

$$h_{t+1}|h_t, \alpha, \phi, \sigma_v^2 \sim \mathrm{N}(\alpha + \phi h_t, \sigma_v^2),$$
$$X_t|h_{t+1}, h_t, \alpha, \phi, \sigma_v^2, \rho \sim \mathrm{N}\left(\frac{\rho}{\sigma_v}\mathrm{e}^{h_t/2}(h_{t+1} - \alpha - \phi h_t), \mathrm{e}^{h_t}(1-\rho^2)\right)$$

and

$$h_t|h_{t-1}, \alpha, \phi, \sigma_v^2 \sim \mathrm{N}(\alpha + \phi h_{t-1}, \sigma_v^2),$$
$$X_t|h_t, h_{t-1}, \alpha, \phi, \sigma_v^2, \rho \sim \mathrm{N}\left(\frac{\rho}{\sigma_v}\mathrm{e}^{h_t/2}(h_t - \alpha - \phi h_{t-1}), \mathrm{e}^{h_t}(1-\rho^2)\right),$$

where $h_t = \ln \sigma_t^2$. These representations permit straightforward Bayesian MCMC parameter estimation using BUGS (see Meyer and Yu (2000) for details).

Regarding the prior distributions, for the parameters ϕ and σ_v^2, I follow exactly the prior specifications of Kim et al. (1998): $\sigma_v^2 \sim$ Inverse-Gamma(2.5,0.025) which has a mean of 0.167 and a standard deviation of 0.024, and $\phi^* \sim$ Beta-distribution with parameters 20 and 1.5 which has a mean of 0.93 and a standard deviation of 0.055, where $\phi^* = (\phi + 1)/2$. Furthermore, I assume $\mu \sim$ N(0,25) where $\mu = \alpha/(1-\phi)$. The correlation parameter ρ is assumed to be uniformly distributed with support between -1 and 1, and hence is completely flat. All prior distributions are assumed to be independent.

In all cases I choose a burn-in period of 10,000 iterations and a follow-up period of 100,000.[2] The MCMC sampler is initialized by setting $\mu = 0, \phi = 0.98, \sigma_v^2 = 0.025$, and $\rho = -0.4$. As it is important to check convergence to ensure that the sample is drawn from the stationary distribution, all the results reported in this paper are based on samples which have passed the Heidelberger and Welch, convergence test (Heidelberger and Welch, 1983) for all parameters.

[2]By iterating the BUGS algorithm for 1,100,000 times with the first 100,000 iterations discarded, I find almost identical empirical results.

3.2. Methods for model comparison

The first method that I use to compare empirically the two asymmetric SV models is via Bayes factors. Specifically, I calculate the Bayes factors using the marginal likelihood approach of Chib (1995). Chib's method is only briefly summarized here but I refer readers to Chib (1995) for further details.

Define $m(y), f(y|z), \pi(z|y), \pi(z)$ to be the marginal likelihood of the model, the likelihood of the model, the posterior distribution of the parameters, and the prior distribution of the parameters, where y and z denote, respectively, the vectors of observations and parameters. Bayes' theorem implies that

$$\ln L = \ln m(y) = \ln f(y|z) + \ln \pi(z) - \ln \pi(z|y). \tag{3.5}$$

Following the suggestion in Chib (1995), I calculate the log-marginal likelihood ln L at the posterior means of parameters (say, $\bar{z}$), which hence requires evaluation of $\ln f(y|\bar{z})$, ln $\pi(\bar{z})$ and ln $\pi(\bar{z}|y)$. Calculation of ln $\pi(\bar{z})$ is trivial. An approximation to ln $\pi(\bar{z}|y)$ can be obtained by using a multivariate kernel density estimate and this was suggested in Kim et al. (1998). The difficult part in the calculation of the log-marginal likelihood value lies in the evaluation of the log-likelihood value at posterior means. This is because $\ln f(y|z)$ has no analytical form for the SV models as it is marginalized over the latent states $\{\ln \sigma_t^2\}$.

The problem of evaluating the likelihood function and more generally filtering a nonlinear non-Gaussian state space model can be solved by the particle filter method. Gordon et al. (1993), Kitagawa (1996), and Pitt and Shephard (1999) are among important contributions in this area. As applications to filtering and likelihood evaluation in SV models, Berg et al. (2004) used the particle filter algorithm proposed by Kitigawa (1996) while Kim et al. (1998) and Pitt and Shephard (1999) used a more efficient particle filter algorithm developed by Pitt and Shephard (1999). In this paper, I use Kitigawa's algorithm (1996) due to the ease in its implementation. Note that Kitigawa's algorithm is applicable to a broad class of nonlinear non-Gaussian state space models with uncorrelated measurement and transition errors.

To employ Kitagawa's algorithm, I rewrite ASV1 and ASV2 by, respectively,

$$\begin{aligned} X_t &= \sigma_t u_t, \\ \ln \sigma_{t+1}^2 &= \alpha + \phi \ln \sigma_t^2 + \rho\sigma_v\sigma_t^{-1}X_t + \sigma_v\sqrt{1-\rho^2}w_{t+1} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} X_t &= \sigma_t\left(\sqrt{1-\rho^2}\varepsilon_t + \frac{\rho}{\sigma_v}(\ln \sigma_t^2 - \alpha - \phi \ln \sigma_{t-1}^2)\right), \\ \ln \sigma_t^2 &= \alpha + \phi \ln \sigma_{t-1}^2 + \sigma_v v_t, \end{aligned} \tag{3.7}$$

where $w_{t+1} = (v_{t+1} - \rho u_t)/\sqrt{1-\rho^2}$ and $\varepsilon_t = (u_t - \rho v_t)\sqrt{1-\rho^2}$. Hence, $\mathrm{corr}(u_t, w_{t+1}) = 0$ in equation (3.6) and $\mathrm{corr}(\varepsilon_t, v_t) = 0$ in Eq. (3.7).

An alternative way for empirically comparing the two asymmetric SV models is to nest them into a single model. To do so, consider the following specification,

$$X_t = \sigma_t u_t,$$
$$\ln \sigma_{t+1}^2 = \alpha + \phi \ln \sigma_t^2 + \sigma_v(\rho_1 u_t + \rho_2 u_{t+1} + \sqrt{1-\rho_1^2-\rho_2^2} w_{t+1}), \tag{3.8}$$

where both u_t and w_t are iid N(0, 1) and $\mathrm{corr}(u_t, w_{t+1}) = 0$. Define $\rho_1 u_t + \rho_2 u_{t+1} + \sqrt{1-\rho_1^2-\rho_2^2} w_{t+1}$ by v_{t+1}. It can be seen that $\mathrm{corr}(u_t, v_t) = \rho_2$ and $\mathrm{corr}(u_t, v_{t+1}) = \rho_1$. Hence, in this model I allow correlation at both time lags, but with possibly different degrees of correlation. When $\rho_1 = 0$ we have the ASV2 model, but when $\rho_2 = 0$ we have the ASV1 model.

To make use of BUGS, I obtain the following state and observation equations for the encompassing model:

$$h_{t+1}|h_t, h_{t-1}, \alpha, \phi, \sigma_v^2 \sim \mathrm{N}(\alpha + \phi h_t + \rho_1\rho_2(h_t - \alpha - \phi h_{t-1}), \sigma_v^2(1-\rho_1^2\rho_2^2)),$$

$$X_t|h_{t+1}, h_t, h_{t-1}, \alpha, \phi, \sigma_v^2, \rho_1, \rho_2, \sim \mathrm{N}\Bigg(\frac{\mathrm{e}^{h_t/2}}{\sigma_v(1+\rho_1\rho_2)}(\rho_2(h_t - \alpha - \phi h_{t-1}) + \rho_1(h_{t+1} - \alpha - \phi h_t)), \mathrm{e}^{h_t}\left(1 - \frac{\rho_1^2+\rho_2^2}{1+\rho_1\rho_2}\right)\Bigg).$$

As to the prior distributions, I adopt the same specifications for μ, ϕ and σ_v^2 as before. For both ρ_1 and ρ_2 I assume a uniform prior with support between -1 and 1.

To evaluate the log-likelihood value at the posterior means, I rewrite the model using the following nonlinear state space form with uncorrelated errors,

$$X_t = \sigma_t\left\{\sqrt{\frac{1-\rho_1^2-\rho_2^2}{1-\rho_1^2}} e_t + \frac{\rho_2}{\sigma_v(1-\rho_1^2)}(\ln \sigma_t^2 - \alpha - \phi \ln \sigma_{t-1}^2 - \sigma_v\rho_1 X_{t-1}\sigma_{t-1}^{-1})\right\},$$
$$\ln \sigma_t^2 = \alpha + \phi \ln \sigma_{t-1}^2 + \rho_1\sigma_v X_{t-1}\sigma_{t-1}^{-1} + \sigma_v\sqrt{1-\rho^2} v_t,$$

where e_t is iid N(0, 1) and $\mathrm{corr}(e_t, v_t) = 0$.

4. Empirical results

As argued in Section 2, the ASV1 model is theoretically appealing relative to the ASV2 model. However, ASV2 is not necessarily a worse model in practice, and hence it is interesting to compare the empirical performance of these two alternative specifications. In this section, I employ two stock indices to make empirical comparisons. The first series contains 2022 daily returns of S&P500 from January 1980 to December 1987 while the second one contains 2529 daily returns of CRSP from January 1986 to December 1995.

In Table 1, I summarize the results for the S&P500 from the two asymmetric SV models, including the posterior means, standard deviations, 95% Bayes credible intervals, simulation inefficiency factors for all the parameters, and the log marginal

Table 1
Empirical results for S&P500

	ASV1				ASV2			
	Mean	SD	95% CI	Ineff	Mean	SD	95% CI	Ineff
ϕ	0.9720	0.0091	(0.9511, 0.9871)	131.46	0.9769	0.0081	(0.9587, 0.9902)	141.48
σ_v	0.1495	0.020	(0.1139, 0.1928)	206.90	0.1347	0.0183	(0.1031, 0.1759)	218.51
μ	−0.0688	0.1278	(−0.3077, 0.2012)	25.50	−0.0246	0.148	(−0.2954, 0.2942)	39.70
ρ	−0.3179	0.0855	(−0.4749, −0.1428)	96.35	−0.2559	0.0941	(−0.4384, −0.0730)	118.36
Log marg	−2794.587				−2803.071			

Note: I fit the two asymmetric SV models to 2022 S&P500 daily returns over the period from January 1980 to December 1987. SD denotes the standard deviation of posterior distribution; 95% CI denotes the 95% credible interval of posterior distribution; Ineff denotes the simulation inefficiency factor.

likelihood for both models. Although the estimate of ρ in both models is significant, it is markedly smaller in the ASV2 model. This suggests that if the leverage effect were estimated from the ASV2 model, it would be underestimated in magnitude by about 20%. Using the log marginal likelihood values, I obtain the Bayes factor of ASV1 over ASV2 which is 4836.758. This indicates decisive evidence in favor of ASV1 against ASV2.[3] Simulation inefficiency factors are of similar size to those reported in the literature when a single-move algorithm is used (see, for example, Table 1 in Kim et al., 1998). For the purpose of comparison, I also estimated the basic SV model in BUGS using the same data set and found evidence of better mixing in the asymmetric SV models. For example, in the basic SV model the inefficient factor for σ_v is 263.29 which is 27% and 21% higher than that in the two asymmetric SV models. This finding is consistent with that reached in Meyer and Yu (2000).

Table 2 reports the estimation results for the S&P500 from the encompassing model, including the posterior means, standard deviations, 95% Bayes credible intervals, simulation inefficiency factors for all the parameters, and the log marginal likelihood value. The posterior mean of ρ_1 is -0.3006 while the posterior mean of ρ_2 is -0.2211. They compare to the posterior mean of -0.3179 in the ASV1 model and the posterior mean of -0.2599 in the ASV2 model. The 95% posterior credibility interval for ρ_1 is $[-0.4718, -0.1381]$ which indicates the presence of a significant

[3]Although only results based on Chib's method are reported here, I also calculated the harmonic mean estimates of marginal likelihood proposed by Newton and Raftery (1994) and deviance information criterion (DIC) proposed by Spiegelhalter et al. (2002). Berg et al. (2004) compared the performance of Chib's method, harmonic mean estimate and DIC in the context of SV models and found that both the harmonic mean estimate and DIC, being much easier to implement than Chib's method, are effective tools for comparing SV models. With these two alternative criteria, I still find strong evidence against the ASV2 model. For example, for S&P500 data, the DIC values for the ASV1 and ASV2 models are 5441.74 and 5453.14, respectively, and the harmonic mean estimates of the two models are −2801.66 and −2832.49, respectively. All the numbers clearly support the superiority of ASV1 over ASV2.

Table 2
Empirical results of encompassing model for S&P500

	Mean	SD	95% CI	Ineff
ϕ	0.9761	0.0082	(0.9571, 0.9893)	149.25
σ_v	0.1356	0.0178	(0.1057, 0.1737)	215.98
μ	0.0308	0.1401	(−0.2153,0.3385)	40.63
ρ_1	−0.3006	0.086	(−0.4718, −0.1381)	63.68
ρ_2	−0.2211	0.078	(−0.3915, −0.087)	58.10
Log marg			−2801.720	

Note: I fit the encompassing SV model to 2022 S&P500 daily returns over the period from January 1980 to December 1987. SD denotes the standard deviation of posterior distribution; 95% CI denotes the 95% credible interval of posterior distribution; Ineff denotes the simulation inefficiency factor.

Table 3
Empirical results for CRSP

	ASV1		ASV2		Encompassing model	
	Mean	95% CI	Mean	95% CI	Mean	95% CI
ϕ	0.9565	(0.9375, 0.9755)	0.9726	(0.9557, 0.9854)	0.9602	(0.9362, 0.9787)
σ_v	0.2398	(0.1826, 0.3040)	0.1819	(0.1400, 0.2322)	0.2199	(0.1629, 0.2855)
μ	−0.441	(−0.6239, −0.2246)	−0.3218	(−0.6053, 0.0094)	−0.3756	(−0.6026, −0.1053)
ρ or ρ_1	−0.3941	(−0.5124,−0.2574)			−0.2820	(−0.4492,−0.0945)
ρ or ρ_2			−0.3632	(−0.5075, −0.2124)	−0.1845	(−0.3810, −0.0491)
Log marg	−2987.312		−2991.987		−2986.438	

Note: I fit the two asymmetric and encompassing SV models to 2529 CRSP daily returns over the period from January 1986 to December 1995. SD denotes the standard deviation of posterior distribution; 95% CI denotes the 95% credible interval of posterior distribution.

negative correlation between u_t and v_{t+1}. The 95% posterior credibility interval for ρ_2 is $[-0.3915, -0.087]$ which suggests some though weaker evidence of negative correlation between u_t and v_t. The marginal likelihood values from the encompassing model and ASV2 differs by 3.86, which suggests substantial evidence in favor of the encompassing specification against ASV2 according to Jeffrey's Bayes factor scale (see Chib et al. (2002, Section 2.3)). On the other hand, the marginal likelihood values from the conventional specification and the encompassing model differs by 1252.63 which suggests decisive evidence in favor of ASV1 against the encompassing specification. The overall ranking of three models is the ASV1 model comes first, followed by the encompassing A-SV model and then the ASV2 model.

Table 3 summarizes the results for CRSP from the two competing models and the encompassing model, including the posterior means, 95% Bayes credible intervals for all the parameters, and the log marginal likelihood values. All the main empirical results are similar to before. For example, the posterior mean of ρ is smaller in ASV2 than that in ASV1. In the encompassing model, the posterior mean of ρ_2 is much smaller than that of ρ_1 and also smaller than that in ASV2. The 95% posterior

Table 4
Simulations for MCMC estimates of the ASV1 model

	True value	Mean	RMSE
ρ	−0.6	−0.564	0.085
ϕ	0.95	0.945	0.0145
σ_v	0.26	0.254	0.037
μ	0	0.0066	0.152

Note: I fit the ASV1 models to 1000 returns simulated from the ASV1 model. Number of replications is 100. RMSE denotes the root mean square error.

credibility interval for ρ_1 indicates the presence of a significant negative correlation between u_t and v_{t+1}. The 95% posterior credibility interval for ρ_2 suggests some though weaker evidence of negative correlation between u_t and v_t. Bayes factors indicate decisive evidence in favor of ASV1 and the encompassing model against ASV2. Although the encompassing model has the largest marginal likelihood value, the evidence in favor of it against ASV1 is "not worth more than a bare mention". All the empirical results obtained from CRSP reinforce the superiority of ASV1 over ASV2.

5. Simulation results

Since the ASV2 specification is neither theoretically appealing nor empirically supported by the real data, the sampling properties of the Bayes MCMC estimator reported in Jacquier et al. (2004) are not practically relevant. Although the sampling properties of Bayes MCMC estimator for the continuous time asymmetric SV model were examined in Eraker et al. (2003), to the best of my knowledge, the sampling properties remain unknown for the ASV1 model. On the other hand, understanding the finite sample performance of Bayes MCMC estimator is important from several aspects. First, it checks the reliability of the proposed MCMC estimators for the ASV1 model, in particular for the new parameter, ρ. Second, since more estimation tools have been developed to estimate the discrete time asymmetric SV models than to the continuous time asymmetric SV model, it is interesting to compare directly the performance of Bayes MCMC estimates with other estimates in the discrete time context. In this section, sampling experiments are designed to obtain sampling properties of the proposed MCMC estimates for the ASV1 model.[4]

In the first experiment, I use a similar parameter setting to that in Jacquier et al. (2004). 100 samples of 1000 observations are simulated from ASV1.[5] Simulation

[4]The sampling properties of Bayes MCMC estimates for the SV model with the fat-tailed error distribution have been obtained in Chib et al. (2002).

[5]The number of replications is small here due to the high computational cost. However, a small number of replications seems not uncommon in the SV literature. For example, Chib et al. (2002) use only 50 replications.

Table 5
Simulations to compare MCMC and QML estimates of the ASV1 model

	True value	MCMC		QML	
		Mean	RMSE	Mean	RMSE
ρ	−0.9	−0.8815	0.0445	−0.911	0.079
ϕ	0.975	0.9732	0.00495	0.974	0.007
$\ln \sigma_v^2$	−4.605	−4.595	0.2086	−4.617	0.353

Note: I fit the ASV1 models to 1000 returns simulated from the ASV1 model. For MCMC the number of replications is 100. For QML the simulation results are obtained from Harvey and Shephard (1996) which are based on 1000 replications. RMSE denotes the root mean square error.

results such as the sample average and sample root mean square error (RMSE) are given in Table 4. The results indicate that the proposed Bayes MCMC method is quite reliable.

In the second experiment, I adopt the same parameter setting as in Harvey and Shephard (1996) and hence can compare the relative efficiency of the Bayes MCMC estimate to the QML estimate of Harvey and Shephard (1996). Table 5 reports the means and RMSEs of all the estimates. The simulation results for the QML estimates are obtained directly from Harvey and Shephard (1996). My results are computed using 100 replications whereas Harvey and Shephard's results were obtained based on 1000 replications. As expected, since MCMC is a fully likelihood-based method, it always performs better than QML. For example, relative efficiency of QML to MCMC are, in terms of the RMSE's, 0.5633, 0.7071 and 0.5909 respectively for ρ, ϕ and $\ln \sigma_v^2$.

6. Conclusions

In this article, I link the two alternative asymmetric SV models to the leverage effect. Given the definition of leverage effect, I have shown that the timing of the variables specified in Jacquier et al. (2004) is such that it is difficult, if not impossible, to interpret the leverage effect, whereas the interpretation of leverage effect is straightforward in the conventional model. Moreover, the empirical analysis clearly demonstrates that the model of Jacquier et al. is dominated by the conventional asymmetric specification. Simulations suggest that the proposed MCMC method is reliable and outperforms QML.

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