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Temporal aggregation and risk–return relation

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Abstract

The function form of a linear intertemporal relation between risk and return is suggested by Merton's [1973. Econometrica 41, 867–887] analytical work for instantaneous returns, whereas empirical studies have examined the nature of this relation using temporally aggregated data, i.e., daily, monthly, quarterly, or even yearly returns. Our paper carefully examines the temporal aggregation effect on the validity of the linear specification of the risk–return relation at discrete horizons, and on its implications on the reliability of the resulting inference about the risk–return relation based on different observation intervals. Surprisingly, we show that, based on the standard Heston's [1993. Review of Financial Studies 6, 327–343] dynamics, the linear relation between risk and return will not be distorted by the temporal aggregation at all. Neither will the sign of this relation be flipped by the temporal aggregation, even at the yearly horizon. This finding excludes the temporal aggregation issue as a potential source for the conflicting empirical evidence about the risk–return relation in the earlier studies.

1. Introduction

How the risk and return of the aggregate stock market are related in their time-variations is one of the most important questions in finance. Merton's (1973) analytical work suggests

a function form which linearly relates the instantaneous expected return to the conditional volatility. Empirically, either daily, monthly, quarterly, or even yearly rates of return have been used to estimate the parameter that measures such a linear risk–return relation.[1] However, the impact of temporal aggregation on the function form of the risk–return relation and on the sign and magnitude of the parameter measuring this relation has not been carefully analyzed.

The main purpose of this paper is to analytically examine the possible effects of temporal aggregation on the specification of the intertemporal risk–return relation and on the degree of stability of this relation at different observation intervals. This is important given that the existing empirical evidence is quite mixing about the risk–return relation[2] and that temporal aggregation could be a potential source for this confusing situation.

This work is particularly inspired by the finding of Longstaff (1989), who shows that, in the context of cross-sectional asset pricing, the instantaneous CAPM model no longer holds after temporal aggregation but becomes a nonlinear multifactor model for discretely-observed returns. Longstaff further shows that making inference on the validity of the instantaneous CAPM based on an analysis of the discrete-time CAPM could lead to erroneous results. A number of other studies also illustrate the significant impact of temporal aggregation on estimation. See, for example, Cartwright and Lee (1987) and Marcellino (1999). In all these papers, it is shown that when the data observation intervals are longer than the data generating intervals, the empirical analysis is usually marred by temporal aggregation effects, and further, that empirical inference will be sensitive to the degree of data aggregation.

Our analysis starts with a continuous-time return dynamics, consisting of a return equation capturing the linear instantaneous risk–return relation argued by Merton (1973, 1980) and a conditional volatility equation following the widely-used Heston's (1993) stochastic volatility process.

We derive a close-form solution to the function form relating risk to return for any given discrete horizon. Surprisingly, in contrast to those applications in the above-mentioned papers (e.g., the cross-sectional asset pricing of Longstaff, 1989), our result shows that the linear risk–return relation will not be distorted by the temporal aggregation at all, however long the observation intervals are.

Moreover, by formulating the parameter measuring the temporally aggregated risk–return relation, in terms of the parameter measuring the instantaneous risk–return relation as well as the parameters related to the volatility process, we show that when the volatility parameters capture the key features in the data, such as the high degree of persistence in volatility and the significant leverage effect, the risk–return relation will retain the same sign and similar magnitude for the usual horizons used for empirical analysis. Again, differing from the temporal aggregation effect in other areas, this result shows that the nature of the risk–return relation should be rather insensitive to the chosen discrete horizons.

These results are the main contribution of our study to the literature because they show that the temporal aggregation is unlikely to be the cause of the conflicting evidence about the risk–return relation reported in the existing studies.

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[1] For example, Bali and Peng (2005) examine the daily horizon, Glosten et al. (1993) look at the monthly horizon, Ludvigson and Ng (2005) study the quarterly horizon, and Harrison and Zhang (1999) use the yearly horizon.

[2] For example, French et al. (1987), Ghysels et al. (2005), Guo and Whitelaw (2006), and Ludvigson and Ng (2005) find a positive relation between the expected return and conditional volatility. In contrast, Campbell (1987), Breen et al. (1989), Glosten et al. (1993), Whitelaw (1994), and Brandt and Kang (2004) document a negative relation.

A paper closely related to ours is Bollerslev and Zhou (2006). Using a return dynamics exactly identical to the one used in our study, Bollerslev and Zhou examine the empirical linkages between returns and integrated and implied volatilities. Their focus is however quite different from ours. One result of their study concerns the volatility feedback effect. In the instantaneous dynamics, the volatility feedback effect and the risk–return relation are captured by the same parameter. These two concepts, however, differ in the temporally aggregated specification. Regarding the temporally aggregated returns, the volatility feedback effect is measured by the slope coefficient of the regression of the returns on the integrated (and hence ex post) volatility, as in Bollerslev and Zhou (2006), whereas the risk–return relation is by definition measured by the regression of the returns on the ex ante conditional return volatility, as in our study. Bollerslev and Zhou show that the feedback effect measured by regressing the returns on the integrated volatility has a downward bias from the true feedback effect due to the leverage effect in returns. Focusing on a different issue but giving a quite surprising result relative to theirs, we show that regressing the returns on the conditional volatility is unlikely to give misleading inference about the sign of the true risk–return relation.

The organization of the paper is as follows. In Section 2, we briefly discuss the work of Longstaff (1989) to illustrate the importance of the temporal aggregation issue in asset pricing and to motivate our study of this issue in the investigation of the intertemporal risk–return relation. Section 3 theoretically analyzes a standard continuous-time model and derives its implication for the temporally aggregated risk–return relation. Section 4 examines the effect of the temporal aggregation on the sign of the risk–return relation. Section 5 concludes.

2. Motivations

2.1. Importance of temporal aggregation issue (Longstaff, 1989)

In the context of cross-sectional asset pricing model, Longstaff (1989) gives a clear demonstration that a continuous-time single-factor CAPM relation can be significantly distorted by temporal aggregation of returns. In particular, he shows that the single-factor CAPM which holds instantaneously does not hold for discretely-observed returns but becomes a multifactor model after the temporal aggregation. We will briefly present the results of his work here to motivate our discussions in the following sections. For ease of illustration, we use exactly the same notations as in his paper. Details of these notations as well as the analytical derivations can be found in Longstaff (1989).

Merton (1971, 1973) derives the continuous-time CAPM, which in the Cox et al. (1985) framework can be written as

$$\alpha_i = \lambda_0 + \lambda_1 \sigma_{iM}, \tag{1}$$

where α_i denotes the instantaneous expected return of asset i, σ_{iM} is the instantaneous covariance of the return of asset i with the market return, and λ_0 and λ_1 are constants.

Model (1) suggests that, at the instantaneous moment, the cross-sectional variation in market covariance explains all the cross-sectional variation in expected returns. Naturally, we would expect that such a linear relation between expected return and market covariance can be translated from the instantaneous horizon to discrete horizons, at least sufficiently well in the sense of approximation, yielding a temporally aggregated model

$$M_i = \gamma_0 + \gamma_1 C_i, \tag{2}$$

where M_i denotes the expected return of asset i of the horizon τ and C_i is its covariance with the market return at that horizon.[3] However, by temporally aggregating instantaneous returns to discrete returns and exactly deriving the restrictions of (1) on the cross-sections of asset returns at the discrete horizons, Longstaff (1989) shows that this is actually not the case, i.e., (2) does not hold. He shows that for the discrete horizon, at which data are practically observable, M_i is actually a nonlinear function of the variance of asset i, its covariance with the market return, and its first-order autocovariance, denoted by V_i, C_i, and A_i, respectively. This suggests that the temporal aggregation can indeed have a significant distortion on the form of the asset pricing model. That is, the asset pricing model changes from a linear form to a nonlinear form. Further, although the temporally aggregated CAPM (in terms of M_i, C_i, V_i, and A_i) is shown to be nonlinear, when the return horizon is short (e.g., one month), the nonlinear model can be approximated by a linear expression

$$M_i = \gamma_0 + \gamma_1 C_i + \gamma_2 V_i + \gamma_3 A_i, \tag{3}$$

where γ_0, γ_1, γ_2, and γ_3 are constants. Equation (3) is a multifactor model, which explicitly shows that the single-factor CAPM no longer holds for discretely-observable returns even in approximation.

One important implication of the above results is that the empirical evidence which rejects a single-factor CAPM at discrete horizons (e.g., model (2) at the monthly horizon), cannot be interpreted as evidence rejecting the continuous-time CAPM model (1). In fact, Longstaff tests the continuous-time CAPM model (1) by testing the three-factor model (3) using monthly returns, and find supporting evidence for (1). In contrast, the discrete-time CAPM, model (2), is rejected by the data.

In summary, Longstaff (1989) illustrates the importance of the temporal aggregation issue in the context of asset pricing by showing that if the horizon, over which returns are measured, differs from the implicit time frame of the original CAPM, then the familiar linear CAPM relation need not hold for the observed returns. He also shows that ignoring the temporal aggregation issue and blindly using the continuous- and discrete-time versions of a model interchangeably for empirical purpose could be a dangerous practice and could lead to spurious inferences.

2.2. A close look at the intertemporal risk–return relation

Having understood that the temporal aggregation can have significant distortions on the form of cross-sectional asset pricing models, we now turn to another important question in finance, the intertemporal risk–return relation, where the temporal aggregation issue is also involved and may cause serious concerns too as we will show in the following.

Merton (1973) employs a continuous time framework to effectively linearize the consumption and portfolio problem in a time-varying economy by taking the decision horizon as infinitely small. This technique allows him to obtain analytical solutions for the demand functions as well as the intertemporal capital asset pricing model (ICAPM). As Merton (1980) argues, under certain conditions the ICAPM suggests a function form linearly relating instantaneous market

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[3] In empirical studies, discrete- and continuous-time asset pricing models are frequently used interchangeably also because the CAPM has been directly derived both in a discrete-time setting (e.g., Sharpe, 1964; Lintner, 1965) and continuous-time setting (e.g., Merton, 1971, 1973).

expected return to instantaneous conditional variance

$$E_t(dP) = \alpha + \beta \text{Var}_t(dP), \tag{4}$$

where α and β are constant parameters and P is the log price of the market portfolio.[4]

The coefficient β measures how the expected market return, $E_t(dP)$, and the conditional volatility, $\text{Var}_t(dP)$, are related in their time-variations. If the aggregate risk aversion of investors remains the same over time, it is generally expected that the equilibrium expected return on the market is an increasing function of the conditional volatility of the market, indicating a positive value for β. However, theoretical studies (e.g., Abel, 1988; Backus and Gregory, 1992) suggest that the expected return on the market could, in equilibrium, be lower during relatively riskier times if aggregate risk aversion is time-varying, which has already been empirically documented (e.g., Campbell and Cochrane, 1999; Brandt and Wang, 2003).

Thus, it becomes an empirical question as to how the expected return and conditional volatility of the market covary over time. Following the theoretical foundation of Merton (1973) and the pioneering empirical study of Merton (1980), dozens of papers have empirically examined relation (4), particularly the sign of β. But, because data are not available on a continuous-time basis, those studies proceed by estimating its discrete-time analogue using discretely observed data[5]:

$$E_t R(t, t + \tau) = \theta + \varphi V_t R(t, t + \tau), \tag{5}$$

where $R(t, t + \tau)$ denotes the continuously compounded market return from time t to $t + \tau$, and depending on τ, $R(t, t + \tau)$ could represent daily, monthly, or quarterly returns and relation (5) could describe the intertemporal risk–return relation corresponding to different horizons. Examples include French et al. (1987), Campbell (1987), Breen et al. (1989), Glosten et al. (1993), Whitelaw (1994), Harvey (2001), and Ghysels et al. (2005).

By making inference about the sign of instantaneous risk–return relation, as measured by β in (4), with the estimation results obtained for φ in (5), those studies implicitly make two critical assumptions. First, for a given horizon τ, Eq. (5) provides a close approximation to (4). In other words, the linear risk–return relation, as described in (4), holds sufficiently well even after temporal aggregation of returns. Second, given the first assumption holds, the coefficient φ in (5) should at least share the same sign, if not the same magnitude, as the parameter β in (4). If φ and β could possibly take different signs, then inferring the sign of β based on the estimate of φ could give misleading results.

However, without theoretical justifications these two assumptions seem to be very strong because, with intuitions as well as the above Longstaff's (1989) illustration, it is reasonable to speculate that, given that (4) holds instantaneously, the discrete-time linear relation (5) will hold, at best, approximately. In other words, the risk–return relation need not be linear anymore for temporally aggregated returns. Perhaps the linear risk–return relation can be substantially distorted by the temporal aggregation when τ represents a monthly horizon for example. Perhaps

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[4] Merton (1980) argues that model (4) should be a close approximation to the equilibrium relation and therefore uses it for expected return estimation. In particular, (4) arises from the CAPM assuming a constant riskless interest rate. As Merton argues, the market return should be the dominant factor among other possible risk factors, and thus the CAPM should "provide a reasonable 'first approximation' theory for equilibrium expected returns."

[5] As we discussed in the introduction, based on (5) the risk–return relation for discretely observed data can be obtained by regressing $R_{t,t+\tau}$ on a constant term and $V_t(R_{t,t+\tau})$. Of course, $V_t(R_{t,t+\tau})$ is unobservable and has to be estimated in practice. It is important to note that this regression differs from the regression of $R_{t,t+\tau}$ on the integrated volatility discussed in Bollerslev and Zhou (2006), who mainly address the issue of measuring volatility feedback effect.

the sign of φ can be changed from the sign of β by the temporal aggregation. Perhaps the sign of φ can differ at different horizons, for example between the monthly horizon and the quarterly horizon. Excluding the last possibility will be particularly important for empirical studies because, otherwise, the inference will be sensitive to the horizon that is chosen for the study. In fact, a variety of horizons have been used in earlier studies in examining the risk–return relation. For example, Bali and Peng (2005) examine the daily horizon, Glosten et al. (1993) look at the monthly horizon, Ludvigson and Ng (2005) study the quarterly horizon, and Harrison and Zhang (1999) use the yearly horizon. But, those studies have produced conflicting evidences about the sign of the risk–return relation, which is an undesirable situation in finance.

Given that the temporal aggregation issue is clearly involved in this risk–return relation question, as shown above, and that the importance of temporal aggregation issue has been well recognized in other areas, it is important to take a closer look at the question about the risk–return relation from the perspective of the temporal aggregation because to the extent that model (5) does not sufficiently capture the dynamics implied by model (4), the practice of inferring the sign of β by estimating φ based on (5) will be misleading. Accordingly, in the following sections we systematically investigate whether the temporal aggregation issue is a potential source for the mixing evidence in the existing studies.

3. The function form of temporally aggregated risk–return relation

The above discussion motivates us to examine whether and how temporal aggregation will affect the linear risk–return relation (4) which holds instantaneously as justified by Merton (1973, 1980). We will show in this section that, surprisingly, the temporal aggregation will not distort the linear risk–return relation at all under the Heston's framework. In other words, the linear relation (5) holds exactly for any horizon τ.

Relation (4) suggests a continuous-time return dynamics

$$dP = \left(\alpha + \beta \sigma^2\right) dt + \sqrt{\sigma^2} \, dZ_P, \tag{6}$$

where α and β are constant parameters, P is the log price, σ^2 is the instantaneous volatility, and Z_P is a Wiener process. The conditional volatility follows the model of Heston (1993), which is one of the most widely-used continuous-time stochastic volatility model in finance and is also utilized in Longstaff (1989) and Bollerslev and Zhou (2006). That is,

$$d\sigma^2 = k\left(\mu - \sigma^2\right) dt + \gamma \sqrt{\sigma^2} \, dZ_\sigma, \tag{7}$$

where k, μ, and γ are parameters, Z_σ is a Wiener process, and ρ denotes the correlation between dZ_P and dZ_σ. The parameter k measures the speed of mean reversion in volatility, μ determines the unconditional long-run average of conditional volatility, and γ is the volatility of volatility, directly related to the tails of the return distribution. Equation (6) together with (7) completely describe the investment opportunity set and the distribution of future market returns.[6] This setting is also very close to the standard CIR model except that the log price instead of the simple price is used and that expected return is allowed to be time-varying to be consistent with the existing empirical evidence for return predictability.

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[6] This model is first studied in the context of option pricing by Scott (1987), and later formally analyzed by Heston (1993).

Denote the continuously compounded return from time t to $t + \tau$ by $R(t, t+\tau) \equiv P(t+\tau) - P(t)$. According to (6), the return $R(t, t+\tau)$ can be written as

$$
\begin{aligned}
R(t, t+\tau) &\equiv P(t+\tau) - P(t) \\
&= \alpha\tau + \beta \int_t^{t+\tau} \sigma^2(s)\,ds + \int_t^{t+\tau} \sqrt{\sigma^2(s)}\,dZ_P(s).
\end{aligned} \tag{8}
$$

Applying Itô's lemma to $e^{kt}\sigma^2(t)$ gives

$$
\sigma^2(s) = \mu + e^{-k(s-t)}\big(\sigma^2(t) - \mu\big) + \gamma e^{-ks} \int_t^s e^{kv}\sqrt{\sigma^2(v)}\,dZ_\sigma(v).
$$

Substituting this expression into the first integral in (8) and applying a modified version of Fubini's Theorem (Ikeda and Watanabe, 1981, Lemma 4.1, p. 116) yields

$$
\begin{aligned}
R(t, t+\tau) = {}& \alpha\tau + \beta\left[\mu\tau + \big(\sigma^2(t) - \mu\big)\frac{1 - e^{-k\tau}}{k}\right] \\
&+ \gamma\beta \int_t^{t+\tau} \sqrt{\sigma^2(v)}\frac{1 - e^{-k(t+\tau-v)}}{k}\,dZ_\sigma(v) + \int_t^{t+\tau} \sqrt{\sigma^2(s)}\,dZ_P(s).
\end{aligned}
$$

With the property of the stochastic integral, the conditional expected return and volatility can be obtained as

$$
\begin{aligned}
E_t R(t, t+\tau) &= \alpha\tau + \beta\left[\mu\tau + \big(\sigma^2(t) - \mu\big)\frac{1 - e^{-k\tau}}{k}\right], \\
\mathrm{Var}_t R(t, t+\tau) &= \left[\mu\tau + \big(\sigma^2(t) - \mu\big)\frac{1 - e^{-k\tau}}{k}\right] + \frac{\gamma^2\beta^2}{k^2}A_1 + \frac{2\gamma\beta\rho}{k}A_2,
\end{aligned}
$$

where $E_t(\cdot)$ and $\mathrm{Var}_t(\cdot)$ denote the expectation and variance conditional at the information set available at time t, which is spanned by the Wiener processes Z_P and Z_σ up to time t.

$$
A_1 = a_{11} + a_{12} \cdot \sigma^2(t), \qquad A_2 = a_{21} + a_{22} \cdot \sigma^2(t),
$$

and

$$
\begin{aligned}
a_{11} &= \mu\left(\tau + \frac{1 - e^{-2k\tau}}{2k} - 3\frac{1 - e^{-k\tau}}{k} - \frac{e^{-k\tau} - e^{-2k\tau}}{k} + 2\tau e^{-k\tau}\right), \\
a_{12} &= \frac{1 - e^{-k\tau}}{k} + \frac{e^{-k\tau} - e^{-2k\tau}}{k} - 2\tau e^{-k\tau}, \\
a_{21} &= \mu\left(\tau - 2\frac{1 - e^{-k\tau}}{k} + \tau e^{-k\tau}\right), \\
a_{22} &= \frac{1 - e^{-k\tau}}{k} - \tau e^{-k\tau}.
\end{aligned}
$$

Noting that $E_t R(t, t+\tau)$, $\mathrm{Var}_t R(t, t+\tau)$, A_1, and A_2 are all functions of $\sigma^2(t)$, we can derive, with some algebra, the exact form for the temporally aggregated risk–return relation

$$
E_t R(t, t+\tau) = \theta(\tau) + \varphi(\tau)\mathrm{Var}_t R(t, t+\tau), \tag{9}
$$

where $\theta(\tau)$ and $\varphi(\tau)$ are constants, which depend on the return horizon τ

$$\theta(\tau) = \alpha\tau + \beta\mu\tau + \beta\frac{1-e^{-k\tau}}{k}\left[\frac{\mu(\frac{1-e^{-k\tau}}{k}-\tau)-\frac{\gamma^2\beta^2}{k^2}a_{11}-\frac{2\gamma\beta\rho}{k}a_{21}}{\frac{1-e^{-k\tau}}{k}+\frac{\gamma^2\beta^2}{k^2}a_{12}+\frac{2\gamma\beta\rho}{k}a_{22}}-\mu\right],$$

$$\varphi(\tau) = \frac{\beta\frac{1-e^{-k\tau}}{k}}{\frac{1-e^{-k\tau}}{k}+\frac{\gamma^2\beta^2}{k^2}a_{12}+\frac{2\gamma\beta\rho}{k}a_{22}}.$$

Equation (9) gives a closed-form solution for the function form of the temporally aggregated risk–return relation. It suggests that the linear risk–return relation is well inherited from the instantaneous model by the temporally aggregated model. In other words, regardless of the horizon τ at which the returns are sampled, the expected return is exactly linearly related to the conditional volatility in their time-variations.

This result partially supports the conventional practice of examining the instantaneous relation (4) by estimating its discrete-time counterpart (5). It is, however, quite surprising and is in sharp contrast to the illustration of Longstaff (1989), where the temporal aggregation is shown to have substantial distortions on the form of the cross-sectional asset pricing model.

It is also important to note that although we are studying a different question from Longstaff (1989), we use a continuous-time return dynamics very similar to Longstaff (1989). So, it is unlikely that the contrasting difference about the temporal aggregation effect between our intertemporal setting and his cross-sectional setting is caused by the model specifications.

Although (9) retains the linear form of (4), $\varphi(\tau)$ differs from β and could potentially take different sign from β, in which case blindly making inference about the sign of β based on the estimation of (5) could still produce spurious results. In the next section, we will carefully examine whether this possibility could occur.

4. The sign of temporally aggregated risk–return relation

Although we have shown that the linear risk–return relation remains valid even after the temporal aggregation, it is still of concern that as the horizon increases, $\varphi(\tau)$ may change the sign from β. This possibility can be seen by noting that

$$\lim_{\tau\to 0}\varphi(\tau) = \beta$$

and

$$\lim_{\tau\to\infty}\varphi(\tau) = \frac{\beta}{1+\gamma^2\beta^2/k^2+2\gamma\beta\rho/k}.$$

As ρ is typically negative to reflect the leverage effect while k and γ are typically positive in the U.S. data, it could potentially occur that β is positive whereas $\varphi(\tau)$ is negative for sufficiently large τ, in which case inferring the sign of β based on the estimate for φ will be incorrect. We will show in the following that when the parameters take values realistic to the U.S. market, it is unlikely that $\varphi(\tau)$ will change sign from β, even at the yearly horizon which is the longest horizon used in the literature.

As the risk–return relation is of the main interest to our study, the characteristics of $\varphi(\tau)$ depending on parameter values will be the main focus of this section. To this end, we first need to determine the realistic values of the key parameters, β, k, γ, ρ, which $\varphi(\tau)$ relies on, noting that $\varphi(\tau)$ is independent of α and μ.

Table 1

This table computes the values of $\varphi(\tau)$ implied by the true positive values of β, according to (9), for a range of realistic parameter values for γ, ρ, and k determined in the literature. The return horizon τ takes values of 1/12 (one month), 3/12 (one quarter), 1 (one year), 5 (five years), 10 (ten years), and ∞ (infinity)

(A) $\gamma = 0.1$, $\rho = -0.2$, $k = 0.1$									(B) $\gamma = 0.1$, $\rho = -0.2$, $k = 0.2$								
τ	β								τ	β							
	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0		0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0
$\frac{1}{12}$	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.1	$\frac{1}{12}$	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.1
$\frac{3}{12}$	0.5	1.0	1.5	2.0	4.0	6.1	8.2	10.3	$\frac{3}{12}$	0.5	1.0	1.5	2.0	4.0	6.1	8.2	10.3
1	0.5	1.0	1.5	2.0	4.1	6.0	7.6	8.9	1	0.5	1.0	1.5	2.0	4.1	6.0	7.7	9.0
5	0.5	1.0	1.4	1.8	2.3	2.1	1.8	1.5	5	0.5	1.0	1.5	1.9	2.7	2.5	2.2	1.9
10	0.5	0.9	1.2	1.3	1.1	0.8	0.6	0.5	10	0.5	0.9	1.3	1.6	1.6	1.2	0.9	0.8
∞	0.4	0.6	0.5	0.4	0.2	0.1	0.1	0.1	∞	0.5	0.9	1.1	1.2	0.9	0.6	0.5	0.4

(C) $\gamma = 0.1$, $\rho = -0.3$, $k = 0.1$									(D) $\gamma = 0.1$, $\rho = -0.3$, $k = 0.2$								
τ	β								τ	β							
	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0		0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0
$\frac{1}{12}$	0.5	1.0	1.5	2.0	4.0	6.0	8.1	10.2	$\frac{1}{12}$	0.5	1.0	1.5	2.0	4.0	6.0	8.1	10.2
$\frac{3}{12}$	0.5	1.0	1.5	2.0	4.1	6.2	8.3	10.5	$\frac{3}{12}$	0.5	1.0	1.5	2.0	4.1	6.2	8.3	10.5
1	0.5	1.0	1.5	2.0	4.2	6.4	8.2	9.7	1	0.5	1.0	1.5	2.0	4.2	6.4	8.3	9.8
5	0.5	1.0	1.5	2.0	2.6	2.3	1.9	1.6	5	0.5	1.0	1.6	2.0	3.0	2.8	2.4	2.0
10	0.5	1.0	1.3	1.5	1.2	0.8	0.6	0.5	10	0.5	1.0	1.5	1.8	1.8	1.3	1.0	0.8
∞	0.5	0.7	0.6	0.5	0.2	0.1	0.1	0.1	∞	0.5	1.0	1.3	1.4	1.0	0.7	0.5	0.4

Merton (1973, 1980) shows that the parameter β can be interpreted as the representative agent's relative risk aversion if the relative risk aversion stays constant over time. He further estimates β to be around 1.5 using both monthly and daily data. Therefore, we use the value 1.5 as a benchmark for β. The parameters k, γ, and ρ all relate to the stochastic volatility process (7), which has been analyzed extensively in the volatility literature. The existing empirical evidence, such as the recent study of Bollerslev et al. (2006), has consistently shown a strong degree of volatility persistence and a highly significant contemporaneous leverage effect, so we take 0.1 and −0.2 as the benchmark values for k and ρ, respectively. Bollerslev et al. (2006) also estimate γ to be around 0.1, which we take as the benchmark value for γ. To examine the sensitivity of our results to the parameter values, we also study a variety of cases where these parameters take various combinations of realistic values around their benchmark values. The results are quite consistent. To save space, we only report several representative cases.

Table 1 presents the implied values of $\varphi(\tau)$ when the stochastic volatility parameters k, γ, and ρ take a range of realistic values determined in the literature. The true value of β ranges from 0.5 to 10, which covers the estimate of Merton (1980) and is in line with the estimate for investors' risk aversion in the literature. We make several observations from the results in the table. First, the results consistently show that with realistic parameter values for the stochastic volatility process, it is unlikely that $\varphi(\tau)$ will have a different sign from β at the usual horizons examined in the empirical studies, even when the horizon gets infinitely large ($\tau = \infty$). In other words, with correctly specified models, empirical studies should yield the same conclusion about the sign of the risk–return relation regardless of the horizons they choose to study. Second, at the monthly and quarterly horizons, $\varphi(\tau)$ is also very close to β in magnitude. Third, by comparing the results in panel (A) (or (C)) with those in panel (B) (or (D)), it appears that the lower the persistence in

Table 2
This table computes the values of $\varphi(\tau)$ implied by the true negative values of β, according to (9), for a range of realistic parameter values for γ, ρ, and k determined in the literature. The return horizon τ takes values of $1/12$ (one month), $3/12$ (one quarter), 1 (one year), 5 (five years), 10 (ten years), and ∞ (infinity)

(A) $\gamma = 0.1$, $\rho = -0.2$, $k = 0.1$							(B) $\gamma = 0.1$, $\rho = -0.2$, $k = 0.2$						
τ	β						τ	β					
	-1.0	-2.0	-4.0	-6.0	-8.0	-10.0		-1.0	-2.0	-4.0	-6.0	-8.0	-10.0
$\frac{1}{12}$	-0.9	-1.9	-3.9	-5.9	-7.8	-9.8	$\frac{1}{12}$	-0.9	-1.9	-3.9	-5.9	-7.8	-9.8
$\frac{3}{12}$	-0.9	-1.9	-3.9	-5.7	-7.5	-9.3	$\frac{3}{12}$	-0.9	-1.9	-3.9	-5.7	-7.6	-9.3
1	-0.9	-1.9	-3.5	-4.8	-5.8	-6.6	1	-0.9	-1.9	-3.5	-4.8	-5.9	-6.6
5	-0.8	-1.3	-1.6	-1.5	-1.3	-1.1	5	-0.8	-1.4	-1.8	-1.7	-1.6	-1.4
10	-0.7	-0.9	-0.8	-0.6	-0.5	-0.4	10	-0.7	-1.1	-1.1	-0.9	-0.7	-0.6
∞	-0.4	-0.3	-0.2	-0.1	-0.1	-0.1	∞	-0.6	-0.8	-0.6	-0.5	-0.4	-0.3

(C) $\gamma = 0.1$, $\rho = -0.3$, $k = 0.1$							(D) $\gamma = 0.1$, $\rho = -0.3$, $k = 0.2$						
τ	β						τ	β					
	-1.0	-2.0	-4.0	-6.0	-8.0	-10.0		-1.0	-2.0	-4.0	-6.0	-8.0	-10.0
$\frac{1}{12}$	-0.9	-1.9	-3.9	-5.9	-7.8	-9.7	$\frac{1}{12}$	-0.9	-1.9	-3.9	-5.9	-7.8	-9.7
$\frac{3}{12}$	-0.9	-1.9	-3.8	-5.7	-7.4	-9.1	$\frac{3}{12}$	-0.9	-1.9	-3.8	-5.7	-7.4	-9.1
1	-0.9	-1.8	-3.4	-4.6	-5.5	-6.2	1	-0.9	-1.8	-3.4	-4.6	-5.6	-6.2
5	-0.8	-1.3	-1.5	-1.4	-1.2	-1.1	5	-0.8	-1.3	-1.7	-1.6	-1.5	-1.3
10	-0.6	-0.8	-0.7	-0.6	-0.4	-0.4	10	-0.7	-1.0	-1.0	-0.8	-0.7	-0.6
∞	-0.3	-0.3	-0.2	-0.1	-0.1	-0.1	∞	-0.6	-0.7	-0.6	-0.5	-0.4	-0.3

the conditional volatility, the less distortion the temporal aggregation will have on the magnitude of $\varphi(\tau)$. It is intuitively expected that when the horizon lengthens, the distortion on the magnitude of the risk–return relation will accumulate faster if the conditional volatility is highly persistent than if the returns are independent over time. By comparing the results in panel (A) (or (B)) with those in panel (C) (or (D)), it appears that the more significant the leverage effect, the closer $\varphi(\tau)$ will be to the value of β for a given τ. If ρ is zero, $\varphi(\tau)$ is sure to be smaller than β, according to the expression of $\varphi(\tau)$ in (9), as a_{12} is positive. When ρ becomes more negative, the denominator of $\varphi(\tau)$ becomes closer to $\frac{1-e^{-k\tau}}{k}$, bringing $\varphi(\tau)$ closer to β.

The parameter β should be positive if the aggregate risk aversion remains the same over time. However, if changes in preferences or in the distribution of wealth are such that the aggregate risk aversion is lower when the market is riskier, then a higher market risk level may imply a lower expected return, leading to a negative value of β. Indeed, it has been well accepted in the literature that the aggregate risk aversion is changing over time in a counter-cyclical pattern (Brandt and Wang, 2003).[7] Therefore, both signs are actually possible for β. For this reason, Table 2 computes the implied values of $\varphi(\tau)$ for the cases where the true value of β is negative. Again, the results consistently shows that the sign of the risk–return relation will not be flipped by the temporal aggregation. Figure 1 plots the risk–return relation at the monthly horizon, as measured by $\varphi(\frac{1}{12})$, as a function of the instantaneous risk–return relation, as measured by β

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[7] In a habit formation model, the representative agent's risk aversion changes with the difference between consumption and his habit formed through past consumption. Since the consumption growth exhibits a business cycle pattern, it is reasonable to expect that periods of strong economic conditions to be associated with low or falling risk aversion while recessions are associated with high or rising risk aversion.
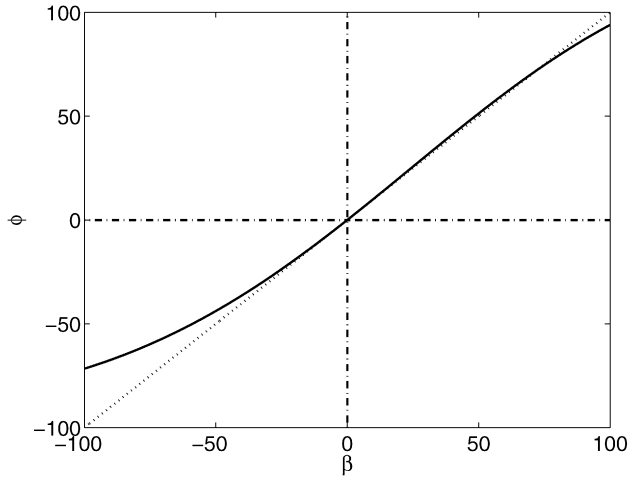
Fig. 1. This figure plots the implied risk–return relation at the monthly horizon, as measured by $\varphi(\frac{1}{12})$, as a function of the instantaneous risk–return relation, as measured by β ranging from -100 to 100. The values for other parameters are taken as $k = 0.1$, $\rho = -0.2$, $\gamma = 0.1$.

ranging from -100 to 100, while the values for other parameters are taken as $k = 0.1$, $\rho = -0.2$, $\gamma = 0.1$. Clearly, φ is a monotonic function of β and takes the same sign as β.

All the above evidence shows that, with the parameter values capturing the key characteristics documented in the volatility literature, i.e., the high degree of persistence and the evident leverage effect, the temporal aggregation is unlikely to flip the sign of the risk–return relation. This result is important as it provides theoretical justification for empirical studies to examine the risk–return relation using a variety of different discrete horizons by showing that the subsequent inference will not be sensitive to the chosen horizons.

5. Conclusion

Empirical studies have examined the linear risk–return relation of the aggregate market at daily, monthly, quarterly, and yearly horizons whereas such a linear relation is only analytically justified for instantaneous return moments by Merton (1973, 1980). In this paper, we carefully assess the impact of the temporal aggregation of returns on the empirical investigation of the risk–return relation. By explicitly deriving the implied temporally aggregated risk–return relation, we show a surprising result. That is, not only the linear function form relating risk to return remains valid for any discrete horizon, but also the parameter measuring the risk–return relation retains the same sign and similar magnitude for the horizons (e.g., month) often examined in the empirical studies. Our results justify the practice of earlier empirical studies which examine the risk–return relation at different discrete horizons, and more importantly, exclude the temporal aggregation issue as a potential source for the existing conflicting empirical evidence about the risk–return relation.

Acknowledgments

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