Robust Testing for Explosive Behavior with Strongly Dependent Errors*

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Abstract

A heteroskedasticity-autocorrelation robust (HAR) test statistic is proposed to test for the presence of explosive roots in financial or real asset prices when the equation errors are strongly dependent. Limit theory for the test statistic is developed and extended to heteroskedastic models. The new test has stable size properties unlike conventional test statistics that typically lead to size distortion and inconsistency in the presence of strongly dependent equation errors. The new procedure can be used to consistently time-stamp the origination and termination of an explosive episode under similar conditions of long memory errors. Simulations are conducted to assess the finite sample performance of the proposed test and estimators. An empirical application to the S&P 500 index highlights the usefulness of the proposed procedures in practical work.

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1 Introduction

The standard no-arbitrage condition for the determination of the price P_t of a financial or real asset at time t implies that

$$P_t = \frac{1}{1+R}\mathbb{E}_t\left(P_{t+1} + D_{t+1}\right), \tag{1}$$

where R, $\mathbb{E}_t$, and D_t denote the discount rate, expectation conditional on information available at time t, and fundamentals (such as the dividend for a stock or the rental income from a house) at time t. Solving (1) by forward substitution leads to the equation $P_t = P_t^F + B_t$, where

$$P_t^F = \sum_{i=1}^{\infty}\left(\frac{1}{1+R}\right)^i \mathbb{E}_t\left(D_{t+i}\right) \tag{2}$$

is the fundamental price,

$$B_t = \frac{1}{1+R}\mathbb{E}_t(B_{t+1}) \tag{3}$$

is the bubble component. B_t is unrelated to fundamentals and emerges as part of the general solution to (1), whereas P_t^F in (2) is the particular solution driven by fundamentals measured by the sum of the discounted expectations of future dividends if P_t is the price of a stock. Under the transversality condition $\lim_{T\to\infty}(1+R)^{-T}\mathbb{E}_t P_{t+T} = 0$, the general solution is $B_t = 0$ and $P_t = P_t^F$. When the transversality condition fails, $B_t \neq 0$. It is clear from (3) that, when $B_t > 0$, B_t satisfies the submartingale property:

$$\mathbb{E}_t(B_{t+1}) = (1+R)\, B_t > B_t, \tag{4}$$

since $1 + R > 1$. This submartingale behavior can be well captured by an explosive autoregressive (AR) model with an AR coefficient greater than 1. Explosive behavior in B_t ensures that P_t is an explosive process even when fundamentals P_t^F are themselves not explosive. It is this property that facilitates empirical analysis of bubbles in time series and panel data using autoregressive methods.

In practical work, many recent empirical studies have confirmed evidence of episodic explosive behavior in the price-fundamental ratio using bubble detection techniques; e.g., Phillips et al. (2015a) (hereafter PSYa) and Pedersen and Schütte (2020). A natural approach to bubble detection is to employ a right-tailed unit root test, initially employed by Diba and Grossman (1988) and subsequently used in sequential testing methods by Phillips et al. (2011) (hereafter PWY) and Phillips and Yu (2011) (hereafter PY) that provide consistent estimates of bubble initiation and termination dates. PSYa extended that work to allow for the detection of multiple bubbles by means of sequential evolving search methods for episodic bubbles in time series. Harvey et al. (2016, 2018, 2019) provided further extensions of these methods by allowing for models with heteroskedastic errors and Pedersen and Schütte (2020) emphasized the importance of treating autocorrelated errors in the small sample procedures that are inevitably involved in sequential and evolving testing algorithms. Readers are referred to Phillips and Shi (2020) and Shi and Phillips (2023) for recent overviews of these methods, including instrumental variable methods for calculating fundamentals, bootstrap methods for controlling the multiplicity issues that affect sequential testing, and algorithms for practical implementation.

The simplest model for explosive behavior testing has the following first-order AR form

$$y_t = \rho y_{t-1} + \epsilon_t,\ y_0 = O_p(1),\ \text{with } \epsilon_t \overset{iid}{\sim} (0,\sigma^2), t = 1, ..., n. \tag{5}$$

Under normal market conditions, time series y_t of asset prices typically follow random wandering behavior. Correspondingly, the common null hypothesis for such conditions is that y_t is a random walk process with $\rho = 1$. Under the alternative hypothesis of bubble behavior originating from some point of initialization in the sample, the process y_t displays explosive behavior with a fixed coefficient $\rho > 1$ or mildly explosive behavior with locally defined coefficient $\rho = 1 + c/n^\alpha$, $c > 0$, and $\alpha \in (0, 1)$, as in Phillips and Magdalinos (2007) and Magdalinos (2012).[1] Against both these alternatives, right-tailed unit root tests have finite sample power and are consistent as $n \to \infty$ (PWY). This framework provides the basis for more complex versions of tests for explosive behavior and bubbles that are better suited to the data in financial and real asset markets.

Pedersen and Schütte (2020) allowed for weakly dependent errors in their application, noting that failure to do so led to considerable size distortion in bubble testing algorithms, particularly those that use recursive sample methods. The present paper is motivated by similar concerns and extends the analysis of earlier work by considering a generating mechanism such as (5) in which the errors follow a strongly dependent process. The phenomenon of strong dependence is widespread in economic and financial time series. Cheung (1993) and Baillie et al. (1996) found empirical evidence of strong dependence in exchange rates. Christensen and Nielsen (2007), Andersen et al. (2003) and Ohanissian et al. (2008) provided evidence of strong dependence in volatilities of stock returns and exchange rate returns; and empirical studies by Gil-Alana et al. (2014) and Barros et al. (2014) showed similar evidence of strong dependence in housing prices. More recently, Chevillon and Mavroeidis (2017) utilized statistical learning methods in long memory analysis, finding strong dependence in the US monthly CPI inflation rates.

Consider the following unit root process driven by long memory errors u_t

$$\begin{cases} y_t &= y_{t-1} + u_t,\ t = 1, ..., n \\ u_t &= \Delta_+^{-d}\epsilon_t,\ d > 0,\ \epsilon_t \overset{iid}{\sim} (0,\sigma^2),\ \mathbb{E}|\epsilon_1|^{2+\delta} < \infty,\ \delta > 0 \end{cases}, \tag{6}$$

where the operator Δ_+^{-d} associated with the memory parameter d is defined by

$$\Delta_+^{-d}\epsilon_t = (1-L)^{-d}1(t \geq 1)\epsilon_t = (1-L)^{-d}\epsilon_t 1(t \geq 1) = \sum_{j=0}^{t-1} \frac{(d)_j}{j!}\epsilon_{t-j}, \tag{7}$$

[1] Mildly explosive models have distinct advantages over purely explosive systems and have become commonly used in the recent literature. First, since purely explosive models are asymptotically more explosive than a mildly explosive model, a test that is consistent in detecting mild explosiveness is also consistent in detecting pure explosiveness. Hence, the test has asymptotic power unity against both mildly explosive and purely explosive processes. Second, no central limit theory or invariance principle properties apply for estimation or testing in a purely explosive model, in contrast to midly explosive processes for which invariance principles apply that validate inference.

with $(d)_j = \Gamma(d+j)/\Gamma(d)$ and initialization at time $t = 0$. The moving average coefficients $(d)_j/j! =: c_{dj}$ in (7) are positive when $d > 0$ for all j. By standard gamma function asymptotic expansion $c_{dj} = \frac{1}{\Gamma(d)j^{1-d}}\{1 + O\left(\frac{1}{j}\right)\} \overset{a}{\sim} \frac{1}{\Gamma(d)j^{1-d}}$ as $j \to \infty$. If $d = 0$ in (6), model (6) reduces to (5) with $\rho = 1$. When $d > 0$, there is strong dependence in the sequence u_t, commonly written as $u_t \sim FI(d)$, so that u_t is fractionally integrated (FI) of order d or with long memory parameter d.[2] Since the first difference of y_t is $FI(d)$, it follows that $y_t \sim FI(d_y)$ with $d_y = 1 + d$.

Figure 1: The unbroken and dashed blue lines are time series plots of the actual and fitted monthly PD ratios (left axis), where the fitted value is obtained by an AR(1) regression with an intercept. The unbroken and dotted red lines are time series plots of residuals from a fitted least squares autoregression and exact local Whittle (ELW) estimation of the long memory parameter (right axis). See the text for further details.

Although data y_t generated by (6) follow a unit root process driven by strongly dependent errors, it is not uncommon to observe realizations that form time paths with episodes mimicking an explosive trajectory. Solving (6) gives the partial sum representation $y_t = \sum_{i=1}^{t} u_i + y_0$. Since u_t is strongly dependent with representation (7) and moving average coefficients $c_{dj} > 0$ for all j, it is evident that any large positive shock ε_{t-j} provides a sustained positive impact on y_t due to strong dependence. Since y_t is the cumulative

[2] More precisely, u_t is a Type II FI time series with fractional order d – see Marinucci and Robinson (1999) and Davidson and Hashimzade (2009) for further discussion of this terminology and Type I FI time series together with definitions of corresponding fractional Brownian motion processes.

effect of such inputs, a succession of positive shocks produces an upward trend in the process that can mimic an explosive time series.

A standard procedure for testing explosive behavior is to fit an AR model such as (5) and employ a right-tailed unit root t-test. In this event, if data is generated according to (6), it is well known that the t-statistic diverges as $n \to \infty$ (see Sowell (1990)), so that a conventional right-tailed test will inevitably lead to rejection of a unit root null when n is large. Thus, for data from a unit root process with long memory innovations such as (6), application of right-tailed tests that ignore strong dependence in the innovations when such dependence is present will lead to the mistaken conclusion of explosive behavior and spurious detection of a rational bubble in the data. That is, when the true AR parameter is unity, the test mistakenly concludes that the data is explosive.

To address this problem, we introduce heteroskedasticity-autocorrelation robust (HAR) statistics to test explosive behavior in data when the error term is strongly dependent. Unlike the conventional t-test, the new tests avoid the aforementioned spurious detection problem and have a stable size property. The tests can also be used to consistently estimate the origination date and the termination date of an explosive episode. Monte Carlo simulations are conducted to check the finite sample performance of the proposed tests and an empirical illustration of the methods to the S&P 500 index is provided.

The remainder of this paper is organized as follows. Section 2 first briefly reviews traditional right-tail unit root tests for explosiveness and procedures for date stamping explosive periods in data. This section also motivates the present paper based on findings from the S&P 500 data. Section 3 introduces the model with strongly dependent errors, proposes the new test, and derives asymptotic theory under the null. Section 4 examines asymptotic properties under explosive alternatives. New estimators of bubble origination and termination dates in recursive applications of the new statistic are given in Section 5. Section 6 discusses how to conduct tests in the presence of time-varying volatilities. Simulations exploring finite sample properties of the procedures are reported in Section 7. An empirical study using the S&P 500 index is conducted in Section 8 where the results are compared with earlier findings that employ standard test procedures. Section 9 concludes. Proofs of the main results in the paper are given in the Appendix. An Online Supplement provides useful lemmas with proofs, proofs and discussion relating to several remarks, together with additional technical and simulation results, including the development of a sup HAR statistic, asymptotic theory and finite sample analysis of the sup HAR statistic. Notation is standard with $\overset{p}{\to}, \overset{d}{\to}, \overset{as}{\to}, \Rightarrow, \overset{a}{\sim}, \lfloor \cdot \rfloor, :=$ and $=:$ denoting convergence in probability, convergence in distribution, almost sure convergence, weak convergence on the relevant probability space, asymptotic equivalence, the floor function, and definitional equality.

2 Related Literature and Motivation

Before introducing our new approach, we first briefly review right-tailed unit root tests and methods to timestamp the origination and termination of explosive episodes in time series data. Model (5) is fitted by LS regression with an intercept from the full sample giving the coefficient estimate $\hat{\rho}_n$ and associated t-statistic $DF_n = (\hat{\rho}_n - 1)/se(\hat{\rho}_n)$, where $se(\hat{\rho}_n)$ is the usual standard error of $\hat{\rho}_n$. Under the null hypothesis that $\rho = 1$, by standard methods (Phillips, 1987a) as $n \to \infty$, $DF_n \Rightarrow \int_0^1 \breve{W}(s)dW(s)/\left(\int_0^1 \breve{W}(s)^2 ds\right)^{1/2} =: DF_\infty$, where $W(s)$ is standard Brownian motion (BM) and $\breve{W}(s) = W(s) - \int_0^1 W(p)dp$ is demeaned BM. Right-tailed unit root tests are implemented by rejecting the null when DF_n exceeds its right-tailed critical value.

In practical work, potentially explosive episodes are typically investigated within sample at some point of time $\tau_e = \lfloor nr_e \rfloor$, with corresponding sample fraction $r_e \in (0,1)$. Such episodes may then end later in the sample at some time $\tau_f = \lfloor nr_f \rfloor$, with $r_e < r_f < 1$, when there is a market correction or shock that terminates exuberance. If explosive behavior emerges and collapses within sample in this way, PWY prove that $DF_n \xrightarrow{p} -\infty$, revealing that full sample right-tailed unit root tests of the type suggested in Diba and Grossman (1988) have no discriminatory power for detecting financial bubbles. Instead, PWY and PY propose a sup statistic based on recursive regressions of the form

$$y_t = \hat{\mu} + \hat{\rho}_\tau y_{t-1} + \hat{u}_t, \text{ for } t = 1, ..., \tau = \lfloor nr \rfloor, \ r > r_0 \tag{8}$$

where $\hat{\mu}_\tau$, $\hat{\rho}_\tau$, and $\hat{u}_t$ are the fitted intercept, AR coefficient, and residuals from regressions, respectively, with $\tau = \lfloor nr \rfloor > \tau_0 = \lfloor nr_0 \rfloor$ and τ_0 is an initiating sample size for the recursion for which it is assumed that $\tau_0 < \tau_e$. Subsequent regressions proceed from the initiating sample of size $\tau_0 = \lfloor nr_0 \rfloor$ until the full sample size n with $r = 1$ is reached. Using the t-statistic $DF_\tau = (\hat{\rho}_\tau - 1)/s_\tau$ based on the regression with τ observations and recursive standard error $s_\tau = \left(\frac{1}{\tau}\sum_{t=1}^{\tau} \hat{u}_t^2 / \left[\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{1}{\tau}\left(\sum_{t=1}^{\tau} y_{t-1}\right)^2\right]\right)^{1/2}$ of $\hat{\rho}_\tau$, the test statistic proposed by PWY and PY is $\sup_{\tau \in [\tau_0, n]} DF_\tau$, whose limit distribution is given by the corresponding functional

$$SDF := \sup_{\tau \in [\tau_0, n]} DF_\tau \Rightarrow \sup_{r \in [r_0, 1]} \frac{\int_0^r \tilde{W}(s)dW(s)}{\left(\int_0^r \tilde{W}(s)^2 ds\right)^{1/2}}, \text{ as } n \to \infty.$$

The null hypothesis is rejected in favor of the presence of an explosive episode in the sample if the statistic SDF exceeds the right-tailed critical value corresponding to the specified significance level.

Once evidence of an explosive episode is detected, the origination and termination dates of the episode, represented by $\tau_e = \lfloor nr_e \rfloor$ and $\tau_f = \lfloor nr_f \rfloor$ with sample fraction forms r_e and r_f, can be estimated. Suppose the generating mechanism under the alternative of an explosive episode within the sample is given by

$$\begin{cases} y_t &= y_{t-1} 1\{t < \tau_e\} + \rho_n y_{t-1} 1\{\tau_e \leq t \leq \tau_f\} \\ & \quad + \left(\sum_{k=\tau_f+1}^{t} \epsilon_k + y_{\tau_f}^*\right) 1\{t > \tau_f\} + \epsilon_t \ 1\{t \leq \tau_f\} \\ \rho_n &= 1 + \frac{c}{n^\alpha}, \quad c > 0, \quad \alpha \in (0,1), \epsilon_t \overset{iid}{\sim} (0, \sigma^2), \ \mathbb{E}|\epsilon_1|^{2+\delta} < \infty, \ \delta > 0 \end{cases}, \tag{9}$$

where $y_{\tau_f}^* = y_{\tau_e} + y^*$ with $y^* \sim O_p(1)$. Model (9) has two structural breaks. Before the first break (at $t = \tau_e$), y_t follows a unit root process. After the first break and before the second break (i.e. $\tau_e \leq t \leq \tau_f$), the process is mildly explosive with autoregressive coefficient $\rho_n = 1 + \frac{c}{n^\alpha}$ and localizing coefficient $c > 0$. At $\tau_f + 1$, the explosive period ends with a collapse in the process to $y^*_{\tau_f}$, which is assumed to be in an $O_p(1)$ neighborhood of y_{τ_e}, the value reached before the explosive episode begins. The sample fractions r_e and r_f are the true origination and termination dates of the explosive period, which may be estimated by

$$\hat{r}_e^{PWY} = \inf_{r \geq r_0} \{r : DF_\tau > cv_n\}, \tag{10}$$

$$\hat{r}_f^{PWY} = \inf_{s \geq \hat{r}_e + \frac{\gamma \ln(n)}{n}} \{s : DF_s < cv_n\}, \tag{11}$$

the latter estimate being conditional on evidence of an originating date $\hat{r}_e^{PWY}$ to the episode. In (10) and (11), the critical value cv_n increases with the sample size. If $cv_n \to \infty$ at a slower rate than $n^{1-\alpha/2}$, Phillips and Yu (2009) showed that $\hat{r}_e^{PWY} \xrightarrow{p} r_e$ and $\hat{r}_f^{PWY} \xrightarrow{p} r_f$ and the two estimates are consistent under some general regularity conditions. In empirical applications, PWY set cv_n proportional to $\ln \ln n$.

The methods reviewed above assume the errors in the AR model have weak dependence. But if the errors in (6) have long memory with memory parameter $d \in (0, 0.5)$, Sowell (1990) showed that the t-statistic diverges with n. This means that as the sample size rises, conventional right-tailed unit root tests will eventually reject a unit root null, leading to a spurious bubble conclusion.

To showcase the empirical relevance of this problem, Figure 1 plots historical data for the monthly price-dividend (PD) ratio of the S&P 500 in the unbroken blue line, following PSYa.[3] The panels shown in Figure 1 cover six periods: (a) January 1872 to February 1880; (b) June 1882 to May 1887; (c) May 1940 to February 1946; (d) June 1948 to November 1955; (e) May 1979 to March 1987 and (f) May 1989 to August 1997. Each period contains a trajectory for which there is some apparent exuberance in the S&P 500 market. Under the assumption that the generating mechanism is (5) with errors that are not strongly dependent, autoregressions with an intercept are fitted for each subperiod and Dickey-Fuller t-statistics (denoted DF_n) are calculated and reported in Table 1. The results show rejection of a unit root at the 1% level for each of subperiods, indicating strong statistical evidence for a rational bubble in each case.[4]

PSYa found evidence of rational bubbles in the S&P 500 for the following periods: the long-depression period (October 1878 to April 1880), the great crash episode (November 1928 to October 1929), the postwar boom in 1954 (January 1955 to April 1956), Black Monday in October 1987 (June 1986 to September 1987), and the dot-com bubble (November 1995 to August 2001). Our sampling periods (a), (d), (e) and (f) overlap four of the PSYa estimated rational bubble periods, re-affirming the evidence for market exuberance in these periods.[5]

[3] Monthly price-dividend ratio measurements are shown on the left axis. The figure also plots the fitted monthly price-dividend ratio (the blue dashed line), obtained from least squares (LS) autoregression with an intercept on the left axis, and the residuals obtained from that regression and the LM model with memory parameter fitted by the exact local Whittle (ELW) method (shown by the red unbroken line and the red dashed line) on the right axis.

[4] The asymptotic right-tail 95% critical value for the standard t statistic for the presence of a unit root is 0.60 (Table B.6 in Hamilton, 1994).

[5] The periods where statistical significance of a positive LM parameter d is not established are not reported here.

Table 1: Right-tailed unit root tests for the S&P 500 PD ratio, exact local Whittle estimates $\hat{d}$ of d, and corresponding confidence intervals

Sampling Period	DF_n	$\hat{d}$	90% CI	95% CI
(a) Jan 1872 to Feb 1880	1.35	0.24	(0.05,0.43)	(0.02,0.46)
(b) Jun 1882 to May 1887	0.66	0.32	(0.10,0.54)	(0.06,0.58)
(c) May 1940 to Feb 1946	1.38	0.34	(0.13,0.55)	(0.09,0.59)
(d) Jun 1948 to Nov 1955	1.70	0.29	(0.10,0.48)	(0.06,0.52)
(e) May 1979 to Mar 1987	1.73	0.21	(0.02,0.40)	(-0.01,0.43)
(f) May 1989 to Apr 1998	2.78	0.24	(0.05,0.42)	(0.02,0.46)

If the data were assumed to be fractionally integrated as in (6) the composite long memory parameter d_y could be estimated directly and the corresponding memory parameter d of the innovations could be deduced. Accordingly, the exact local Whittle (ELW) procedure (Shimotsu and Phillips, 2005) was used to estimate d_y, deduce d, and test for short memory ($d = 0$) in the innovations against strong dependence ($d > 0$).[6]

In Figure 1, the red solid line is a plot of $\{\hat{\varepsilon}_{t,LS}\}_{t=2}^{n}$ obtained from least squares (LS) autoregression and the red dotted line is a plot of $\{\hat{\varepsilon}_{t,ELW}\}_{t=2}^{n}$ obtained by ELW estimation[7] for each of the six sampling periods. Note that an exuberance trajectory can be generated either by an explosive AR model with an error sequence $\{\hat{\varepsilon}_{t,LS}\}_{t=2}^{n}$ or by a fractionally integrated time series (6) with $d = \hat{d}$ and the error sequence $\{\hat{\varepsilon}_{t,ELW}\}_{t=2}^{n}$. Table 1 reports the ELW estimate of d and its 90% and 95% confidence intervals for each subperiod.[8] In all cases, $\hat{d}$ is positive and the null hypothesis of short memory is rejected against the alternative of strong dependence at either the 5% or 10% level, supporting evidence of long memory in the innovations u_t in the sampling periods. These findings suggest that a plausible alternative model for the generating mechanism is a unit root model with strong dependent errors (6) instead of the explosive model indicated by the results of unit root testing. Hence, empirical rejection of a unit root null in favor of an explosive process may arise from the presence of strong dependence in the errors, raising the possibility of spurious inference concerning the presence of a rational bubbles.[9]

Motivated by these empirical findings and the potential implications for bubble detection with standard right-tail unit root tests, the present paper seeks to address the problem of spurious test outcomes from right-tail tests. We propose to modify standard test procedures by constructing a heteroskedasticity-autocorrelation robust (HAR) statistic which controls performance so that the test statistic does not diverge and has a well defined limit distribution under the null test but diverges and is consistent under the alternative

[6] We followed Shimotsu (2010) in the implementation of the ELW procedure taking the unknown initial condition into account. See Shimotsu (2010, Equation (9)) for details.

[7] These residuals are estimated by computing $\hat{\varepsilon}_{t,ELW} = (1-L)_{+}^{-\hat{d}}\Delta y_t$.

[8] Shimotsu and Phillips (2005) show that the asymptotic distribution of the ELW estimate of d is given by $\sqrt{m}(\hat{d}-d) \xrightarrow{d} \mathcal{N}(0, 1/4)$ using bandwidth $m = n^{\delta}$, where n is the sample size and δ is a bandwidth parameter that controls the number of periodogram ordinates near the origin that are used in estimation. The setting $\delta = 0.65$ was used in the computations reported in Table 1.

[9] An analysis based on a larger sample size is conducted in the Online Supplement, confirming strong dependence in the residuals using the ELW estimator and explosiveness in the data using DF_n tests.

of an explosive or mildly explosive root. The new HAR test has asymptotic discriminatory power in detecting explosive time series even in models driven by long memory errors. The test can be implemented in recursive algorithms to consistently timestamp origination and termination dates of episodic bubbles. The modified test statistic is constructed in the same spirit as the use of HAR statistics to perform valid testing in potentially spurious relationships (Sun, 2004; Phillips et al., 2019).

3 A New Test and Asymptotic Null Distribution

Motivated by the empirical findings in Section 1, we consider the following prototypical model

$$\begin{cases} y_t &= \rho_n y_{t-1} + u_t, \ t = 1, ..., n \\ u_t &= \Delta_+^{-d} \epsilon_t, \ d > 0, \ \epsilon_t \overset{iid}{\sim} (0, \sigma^2), \ \mathbb{E}|\epsilon_1|^{2+\delta} < \infty, \ \delta > 0 \\ y_0 &= o_p(n^{1/2+d}) \end{cases} . \tag{12}$$

Model (12) differs from (5) in that u_t can be strongly dependent. The model also differs from Sowell (1990), who used Type I FI $u_t = \Delta^{-d}\epsilon_t = \sum_{j=0}^{\infty} \frac{(d)_j}{j!}\epsilon_{t-j}$ with $d \in (0, 0.5)$ to model strong dependence, because the Type II formulation $u_t = \Delta_+^{-d}\epsilon_t$ allows for a wide range of stationary and nonstationary long range dependence, for which consistent estimation of d or d_y is possible using the ELW procedure with associated pivotal Gaussian inference, as noted in Shimotsu and Phillips (2005) and Hualde and Robinson (2011). We first consider the asymptotic behavior of the traditional Dickey-Fuller t-test when $\rho_n = 1$.

3.1 Asymptotic null distribution of DF_τ

Lemma 3.1 *Assume the true data generation process (DGP) is given by (12) with $\rho_n = 1$ and $d \in (0, 0.5)$. For any $r \in (0, 1]$ and $\tau = \lfloor nr \rfloor$ as $n \to \infty$,*

$$n^{-d} DF_\tau \Rightarrow \frac{\frac{\sigma r}{2}\left(W^H(r)\right)^2 - \sigma\left(\int_0^r W^H(s)ds\right) W^H(r)}{r\left(\sigma_u \int_0^r \left(\tilde{W}^H(s)\right)^2 ds\right)^{1/2}}. \tag{13}$$

where $\sigma_u^2 := \mathbb{E}[u_t^2]$.

Lemma 3.1 implies that $DF_\tau = O_p\left(n^d\right)$, so that the statistic DF_τ diverges with the sample size, implying rejection of the null hypothesis as $n \to \infty$ which leads to spurious inference concerning explosive behavior in the data. With Type I fractional integration and $d \in (0, 0.5)$, Theorem 4 in Sowell (1990) also showed divergence $DF_n \xrightarrow{p} \infty$. Lemma 3.1 extends that result to Type II fractional integration and the divergence rate $O_p\left(n^d\right)$ shows faster divergence for larger d holding for any $r \in (0, 1]$, so the result is relevant for subsample inference.

Remark 3.1 *To detect the presence of explosiveness, PWY and PSYa and Phillips et al. (2015b) (hereafter PSYb) proposed to use SDF and GSDF statistics defined by*

$$SDF(\tau_0) = \sup_{\tau \in [\tau_0, n]} DF_\tau \ \ \textit{and} \ \ GSDF(\tau_0) = \sup_{\tau_2 \in [\tau_0, n], \tau_1 \in [0, \tau_2 - \tau_0]} DF_{\tau_1}^{\tau_2},$$

where $\tau_0 = \lfloor nr_0 \rfloor$ is the minimum data window and $DF_{\tau_1}^{\tau_2}$ is the t statistic based on the observations from $\tau_1 = \lfloor nr_1 \rfloor$ to $\tau_2 = \lfloor nr_2 \rfloor$. As Lemma 3.1 holds uniformly for $r \in (0,1]$, under model (12) with $\rho_n = 1$ and $d \in (0, 0.5)$, we have

$$SDF(\tau_0) = O_p\left(n^d\right) \text{ and } GSDF(\tau_0) = O_p\left(n^d\right).$$

Both statistics lead to the detection of spurious explosive behavior as $n \to \infty$.

Remark 3.2 *Similar to the framework in Phillips et al. (2014), model (12) can be extended to include an asymptotically negligible intercept, which can be useful in capturing the presence of a small drift in the data. In this case,*

$$y_t = \mu_n + \rho_n y_{t-1} + u_t, \tag{14}$$

where $\mu_n = O(n^{-\theta})$ with $\theta > 1/2 - d$. It can be shown that Lemma 3.1 continues to hold in this case. The result in Lemma 3.1 also continues to hold when the ADF test or the CUSUM test of Homm and Breitung (2011) is used.

3.2 A new test statistic

The failure of the standard t statistic stems from the use of an inappropriate standard error based on the sample variance of residuals, $\frac{1}{\tau}\sum_{t=1}^{\tau} \hat{u}_t^2$, which results in the divergence of DF_τ. Instead, we use a self-normalized statistic that employs a robust standard error estimate, leading to a well defined limit distribution as $n \to \infty$ for $d \in [0, 0.5)$. Specifically, allowing for potential strong dependence in u_t, we employ the HAR estimate

$$\hat{\Omega}_{HAR} = \sum_{j=-\tau+1}^{\tau} K\left(\frac{j}{M}\right)\hat{\gamma}_j, \tag{15}$$

where $K(\cdot)$ is a kernel function with bandwidth $M = M_\tau$, and $\hat{\gamma}_j = \frac{1}{\tau}\sum_{t=j+1}^{\tau} \Delta y_t \Delta y_{t-j}$ is the j^{th} order sample autocovariance over the subsample $t = 1, ..., \tau$. Based on $\hat{\Omega}_{HAR}$, the t statistic becomes

$$DF_{\tau,HAR} = \frac{\hat{\rho}_\tau - 1}{s_{\tau,HAR}}, \tag{16}$$

in which the robust standard error is

$$s_{\tau,HAR} = \sqrt{\frac{\hat{\Omega}_{HAR}}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2}}, \quad \text{where } \bar{y}_t = y_t - \frac{1}{\tau}\sum_{t=1}^{\tau} y_{t-1}. \tag{17}$$

For HAR estimation, the bandwidth is selected by the fixed-b setting $M_\tau = b \times \tau$ with $b \in (0, 1]$ so the bandwidth is of the same order of magnitude as the subsample τ in the regression window. This approach follows Kiefer and Vogelsang (2002a,b, 2005), Bunzel et al. (2001), Vogelsang (2003) and many subsequent works that employ the fixed-b method. In the present setting, the HAR normalization of the test statistic plays the same role as in Sun (2004) in the context of potentially spurious cointegration.

Theorem 3.1 *Suppose $M_\tau = b\tau$, and $K(x) = K_B(x)$ is the Bartlett kernel with $K_B(x) = (1-|x|)\mathbf{1}(|x| \leq 1)$. Under model (12), with $\tau = \lfloor nr \rfloor$ and $r \in (0,1]$ as $n \to \infty$, $DF_{\tau=\lfloor nr\rfloor,HAR}$ has the following fixed-b asymptotic distribution,*

$$DF_{\tau,HAR} \Rightarrow \begin{cases} \dfrac{b^{1/2} r \int_0^r \tilde{W}(s)dW(s)}{\left[2\int_0^r \tilde{W}(s)^2 ds\left(\int_0^r W(p)^2dp - \int_0^{(1-b)r} W(p)W(p+br)dp\right)\right]^{1/2}} =: F^*_{r,0} & \text{for } d = 0 \\ \dfrac{b^{1/2}\left[\frac{r}{2}\left(W^H(r)\right)^2 - \left(\int_0^r W^H(s)ds\right)W^H(r)\right]}{\left[2\int_0^r \left(\tilde{W}^H(s)\right)^2 ds\left(\int_0^r W^H(p)^2dp - \int_0^{(1-b)r} W^H(p)W^H(p+br)dp\right)\right]^{1/2}} =: F_{r,d} & \text{for } d \in (0, 0.5) \end{cases}, \tag{18}$$

where $W^H(r) = \frac{1}{\Gamma(H+1/2)}\int_0^r (r-s)^{H-1/2}dW(s)$ is Type II fractional Brownian motion (fBM) with the Hurst parameter $H = 1/2 + d$ and $\tilde{W}^H(r) = W^H(r) - \frac{1}{r}\int_0^r W^H(s)ds$ is demeaned Type II fBM.[10]

In contrast to the divergence of DF_τ, Theorem 3.1 shows that $DF_{\tau,HAR}$ converges weakly to a well-defined limit distribution for any $\tau = \lfloor nr \rfloor$ whether $d = 0$ or $d > 0$. Hence, provided the DGP does not have an explosive root, $DF_{\tau,HAR}$ has well-behaved asymptotics for $d \geq 0$ and in this respect is better suited to testing. However, although the statistic is well normalized for $d \geq 0$, the limit distribution of $DF_{\tau,HAR}$ in (18) is not uniform over d. The lack of uniformity arises because the centered LS estimator $\hat{\rho}_\tau - 1$ involves a component involving $n^{-1-2d}\left(\sum_{t=1}^{\tau} u_t^2\right)$ that is asymptotically negligible when $d > 0$ but non-negligible when $d = 0$, thereby affecting the limit theory in that case and leading to a discontinuity in the limit distribution when $d \to 0$. As is evident from the form of the two limits given in (18), the denominator of $F_{r,d}$ is equivalent to the denominator of $F^*_{r,0}$ as $d \to 0$ (and $H \to 1/2$) but this is not true of the numerators. The discrepancy produces the discontinuity in the limit theory as $d \to 0$.

Simulations (not reported here) show that critical values obtained from the limit distribution of $DF_{\tau,HAR}$ do not provide satisfactory performance and lead to size distortion in testing when d is close to zero. This distortion stems from two factors. First, when $d > 0$ but is close to zero, the component $n^{-1-2d}\left(\sum_{t=1}^{\tau} u_t^2\right)$ converges in probability to zero very slowly and the limit distribution $F_{r,d}$ does not provide a good finite sample approximation to the true distribution. Second, when $d = 0$, use of $F_{r,\hat{d}}$ for an asymptotic approximation with a plug-in estimate $\hat{d} > 0$ can be a poor approximation to the correct distribution $F^*_{r,0}$ which should be used to provide critical values.

The size distortion in the use of $DF_{\tau,HAR}$ and the source of the discrepancy in the limit theory motivates the design of a modified statistic $\widetilde{DF}_{\tau,HAR}$ whose limit expression smooths over the discontinuity as $d \to 0$ and assists in delivering satisfactory size performance. The modified statistic has the form

$$\widetilde{DF}_{\tau,HAR} = \frac{\tilde{\rho}_\tau - 1}{s_{\tau,HAR}}, \tag{19}$$

where $\tilde{\rho}_\tau = \hat{\rho}_\tau + \frac{\frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2}{\sum_{t=1}^{\tau}\bar{y}_{t-1}^2}$ and $\Delta y_t = y_t - y_{t-1}$. This correction is analogous to the weak dependence correction in semiparametric unit root tests in Phillips (1987a).

[10] It can be shown that $DF_{\tau,HAR} \Rightarrow F_{r,d}$ for $d > 0.5$.

Theorem 3.2 *Under the same assumptions as Theorem 3.1, for $\tau = \lfloor nr \rfloor$ with $r \in (0, 1]$ and $n \to \infty$,*

$$\widetilde{DF}_{\tau,HAR} \Rightarrow F_{r,d}, \text{ for } d \geq 0. \tag{20}$$

where $F_{r,d}$ is defined in (18).

Theorem 3.2 shows that the limit theory given by $F_{r,d}$ provides a smooth transition to $F_{r,0}$ as $d \to 0$, replacing $F^*_{r,0}$ when $d = 0$. When $H = 1/2$ we have $W^H(r) = W(r)$ and $F_{r,d} \to F_{r,0}$ as $d \to 0$. The continuity in the limit theory is achieved by simple algebraic removal of the component $n^{-1-2d}\left(\sum_{t=1}^{\tau} u_t^2\right)$ in the centered LS estimator $\hat{\rho}_\tau - 1$. The component $n^{-1-2d}\left(\sum_{t=1}^{\tau} u_t^2\right)$ is no longer relevant in the limit theory and there is no abrupt shift in the asymptotic behavior of $\widetilde{DF}_{\tau,HAR}$ at $d = 0$.

To perform a right-tailed unit root test based on the sample $\{y_t\}_{t=1}^{\tau}$ with $\tau = \lfloor nr \rfloor$ the statistic $\widetilde{DF}_{\tau,HAR}$ can be used in conjunction with the $\beta \times 100\%$ asymptotic critical value $cv^{\beta}_{r,HAR}(d)$, for which

$$\Pr\left(F_{r,d} > cv^{\beta}_{r,HAR}(d)\right) = \beta, \text{ for } r \in (0, 1], \tag{21}$$

where $F_{r,d}$ is defined in (18). This procedure applies to the full sample statistic $\widetilde{DF}_{n,HAR}$ with limit variate $F_{1,d}$ and corresponding critical value $cv^{\beta}_{1,HAR}$ satisfying (21).

Remark 3.3 *The limit distributions given in Theorem 3.1 and Remark 3.4 below apply when the error term u_t follows a stationary ARFIMA(p, d, q) process with $d > 0$. Suppose the ARFIMA(p, d, q) process u_t is written as*

$$\begin{aligned} u_t &= \Delta_+^{-d} A(L)\epsilon_t, \\ A(L)\epsilon_t &= \sum_{j=0}^{\infty} a_{t-j}\epsilon_{t-j}, \sum_{j=0}^{\infty} |a_j| < \infty. \end{aligned}$$

Silveira (1991); Marinucci and Robinson (2000) show that, as $n \to \infty$,

$$\frac{1}{n^{1/2+d}} \sum_{t=1}^{\lfloor nr \rfloor} u_t \Rightarrow \tilde{\sigma} W^H(r), \tag{22}$$

for $r \in [0, 1]$, where $\tilde{\sigma}^2 = \left(E[\epsilon_t^2] + 2\sum_{k=2}^{\infty} E[\epsilon_1\epsilon_k]\right) A(1)$ with $A(1) = \sum_{j=0}^{\infty} a_j$. In view of the the identity

$$\frac{1}{n^{2d}} \frac{1}{n} \sum_{t=1}^{n} y_{t-1}u_t = \frac{1}{2n^{2d}}\left(y_n^2 - y_0^2\right) - \frac{1}{2n^{2d}}\left(\frac{1}{n}\sum_{t=1}^{n} u_t^2\right),$$

the second term on the right-hand side vanishes asymptotically when $d \in (0, 0.5)$. Hence, adding the ARMA component provides only a scaling effect on the variance $\tilde{\sigma}^2$ appearing in (22), and the asymptotic distribution of $DF_{\tau,HAR}$ and $\widetilde{DF}_{\tau,HAR}$ are independent of $\tilde{\sigma}^2$. Hence, the results in Theorem 3.1 and Remark 3.4 continue to apply.

Remark 3.4 *Other kernel functions* $(K_2(\cdot))$ *may be used in place of the Bartlett kernel* $K_B(\cdot)$ *and similarly lead to a fixed-b limit distribution of the corresponding statistic* $\widetilde{DF}_{\tau,HAR}$ *constructed with this kernel. For instance, suppose* $\hat{\Omega}_{HAR} = \sum_{j=-\tau+1}^{\tau} K_2\left(\frac{j}{M}\right)\hat{\gamma}_j$ *with some twice continuously differentiable positive and symmetric kernel function* $K_2(\cdot)$. *Then, for all* $d \in [0, 0.5)$ *and* $\tau = \lfloor nr \rfloor$ *with* $r \in (0, 1]$ *we have the limit theory as* $n \to \infty$,

$$\widetilde{DF}_{\tau,HAR} \Rightarrow \frac{\frac{br^{3/2}}{2}\left(W^H(r)\right)^2 - br^{1/2}\left(\int_0^r W^H(s)ds\right)W^H(r)}{\left(\left(\int_0^r \tilde{W}^H(s)^2 ds\right)\int_0^r\int_0^r -K_2''\left(\frac{p-q}{br}\right)W^H(p)W^H(q)dpdq\right)^{1/2}} =: \widetilde{F}_{r,d}, \quad (23)$$

where $K_2''(\cdot)$ *is the second derivative of* $K_2(\cdot)$.

Remark 3.5 *The preceding results are given for long memory time series formulated in terms of Type II FI and the corresponding limit theory involves Type II fBM. Similar results apply for innovations involving Type I FI time series with limit theory involving a Type I fBM process. Specifically, when* $d \in (0, 0.5)$ *we can replace* $u_t = \Delta_+^{-d}\epsilon_t$ *in (12) with* $u_t = (1-L)^{-d}\epsilon_t = \sum_{j=0}^{\infty}\frac{(d)_j}{j!}\epsilon_{t-j}$ *and* $W^H(t)$ *in (18) and (23) with*

$$B^H(t) = W^H(t) + \frac{1}{\Gamma(H+1/2)}\int_{-\infty}^{0}[(r-s)^{H-1/2} - (-s)^{H-1/2}]dW(s).$$

For discussion of Type I and Type II formulations of fBM see Marinucci and Robinson (1999, 2000); Davidson and Hashimzade (2009).

Remark 3.6 *As in PWY, a sup statistic* $\sup_{\tau\in[\tau_0,n]}\widetilde{DF}_{\tau,HAR}$ *can be constructed from recursive regression for empirical testing covering subsamples of the full sample. The limit theory can be obtained by continuous mapping in the usual way (PSYa; PSYb) and employed in practical work to identify explosive behavior in a subsample. This construction is discussed in detail in the Online Supplement.*

Both d_y and d can be consistently estimated by ELW estimation (Shimotsu and Phillips, 2005) or quasi-maximum likelihood estimation (QMLE) (Hualde and Robinson, 2011). Let $\widehat{d}$ denote the estimate of d so obtained. Critical values for testing can then be found for $F_{r,\widehat{d}}$ by simulation. Alternatively, we can tabulate critical values of $F_{r,d}$ for a set of grid points for $d \in [0, 0.5)$ and interpolation can be employed to obtain the critical value of $F_{r,\widehat{d}}$. Simulations reported in section 7 show that this plug-in approach delivers good size performance in finite samples even for $n = 100$ and performs nearly as well as the infeasible method where the true value of d is used to construct critical values. For practical implementation it is convenient to impose bounds in estimation so that $d_y \in [1, 1+\bar{d}]$ for some $0 < \bar{d} < \infty$ when performing optimization in calculating estimates of d_y. Then $\widehat{d_y} \in [1, 1+\bar{d}]$ and $\widehat{d} \in [0, \bar{d}]$. In the simulations of Section 7 we set $\bar{d} = 0.49$.

The plug-in method does not take into account the randomness in $\hat{d}$. An alternative feasible inferential procedure that does so is to use the fractional differencing bootstrap algorithm of Kapetanios et al. (2019) to obtain asymptotically correct critical values or bootstrap p-values. This approach involves the following five steps.

1. Let $e_{n,t} = \Delta_+^{1+\hat{d}} y_t$ and $\tilde{e}_{n,t} = (e_{n,t} - \bar{e}_{n,t}) / \hat{\sigma}_e$ where $\hat{d}$ is an estimate of d, $\bar{e}_{n,t}$ and $\hat{\sigma}_e$ are the sample mean and sample standard deviation of $e_{n,t}$.

2. Redraw i.i.d. samples $\left\{ e_{n,t}^* \right\}$ from the empirical distribution of $\tilde{e}_{n,t}$ with replacement.

3. Let
$$u_t^* = \hat{\sigma}_e \Delta_+^{-\hat{d}} e_{n,t}^*, y_t^* = y_{t-1}^* + u_t^*, \quad \text{with } y_0 = 0,$$
and calculate $\widetilde{DF}_{n,HAR}^*$ as in (19).

4. Repeat Steps 2-4 B times and calculate the bootstrap empirical cdf
$$\hat{F}^*(x) = \frac{1}{B} \sum_{j=1}^{B} 1(\widetilde{DF}_{n,HAR}^{*,j} \leq x).$$
Define the $\beta \times 100\%$ bootstrap critical value (bcv^β) as the $1 - \beta$ quantile of $\hat{F}^*$ and let the bootstrap p-value be
$$p^*(\widetilde{DF}_{n,HAR}) = 1 - \hat{F}^*(\widetilde{DF}_{n,HAR}). \tag{24}$$

5. Reject the unit root null hypothesis when $\widetilde{DF}_{n,HAR} > bcv^\beta$ or $p^*(\widetilde{DF}_{n,HAR}) < \beta$.

The following theorem shows that this bootstrap approach delivers the correct test size asymptotically.

Theorem 3.3 *Suppose we reject the hypothesis $\rho_n = 1$ when $p^*(\widetilde{DF}_{n,HAR})$ in (24) is less than β. Under the assumptions specified in Theorem 3.1 and if, $n^\gamma(\hat{d} - d) \xrightarrow{d} \mathcal{N}(0, V)$, with $1/4 < \gamma \leq 1/2$ and $V > 0$, then as $n \to \infty$, we have*
$$\begin{aligned} \widetilde{DF}_{n,HAR}^* &\Rightarrow F_{1,d}, \\ p^*(\widetilde{DF}_{n,HAR}) &\Rightarrow U[0,1]. \end{aligned}$$

4 Alternative Hypothesis and Asymptotic Theory

To study the asymptotic behavior of the proposed test statistic under an alternative hypothesis, we follow the literature and use two popular ways of modeling explosive departures from unity. The first alternative adopts the local to unit root (LUR) framework of Phillips (1987b) – see Harvey et al. (2016, 2018, 2019). The advantage of using the locally explosive model is that it facilitates the computation of local power. The second alternative is the mildly explosive model of Phillips and Magdalinos (2007); see PWY, PSYa, PSYb and PY. Under a mildly explosive alternative a consistent test is obtained.

4.1 Locally explosive model

We first consider the alternative hypothesis with the following locally explosive setting:

$$\begin{cases} y_t &= (y_{t-1}+u_t)\,1\{t<\tau_e\} + (\rho_n y_{t-1}+u_t)\,1\{\tau_e \le t \le n\}, t=1,...,n, \\ u_t &= \Delta_+^{-d}\epsilon_t, d\ge 0, \epsilon_t \overset{iid}{\sim} (0,\sigma^2),\ \mathbb{E}|\epsilon_1|^{2+\delta}<\infty,\ \delta>0, \tau_e=\lfloor nr_e \rfloor, \\ \rho_n &= 1+c/n, c>0, 1+c/n, c>0, \\ y_0 &= o_p\left(n^{1/2+d}\right). \end{cases} \tag{25}$$

In model (25), y_t has a unit autoregressive root generating mechanism before time τ_e and becomes locally explosive after τ_e, producing a structural break at τ_e. During both periods the errors in the AR model have strong dependence with the same memory parameter d. We now consider the asymptotic behavior of $\widetilde{DF}_{\tau,HAR}$.

Theorem 4.1 *Under model (25), for $\tau=\lfloor nr \rfloor$ with any $r>r_e$, as $n\to\infty$,*

$$\widetilde{DF}_{\tau,HAR} \Rightarrow \frac{\left(\frac{\left(\frac{1}{2}C_{r,d}-\frac{1}{r}A_{r,d}W^H(r)\right)r}{B_{r,d}-\frac{1}{r}A_{r,d}^2}+cr\right)\left(B_{r,d}-\frac{1}{r}A_{r,d}^2\right)^{1/2}}{\left[\frac{2}{b}\left(\int_0^r G_{r_e,c}(d,p)^2dp-\int_0^{(1-b)r}G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp\right)\right]^{1/2}} =: F_{r,d}^c, \tag{26}$$

where

$$\begin{aligned} A_{r,d} &:= \int_0^r\left(e^{(x-r_e)c}W^H(r_e)+\int_{r_e}^x e^{(x-s)c}dW^H(s)\right)dx, \\ B_{r,d} &:= \int_0^r\left(e^{(x-r_e)c}W^H(r_e)+\int_{r_e}^x e^{(x-s)c}dW^H(s)\right)^2dx, \\ C_{r,d} &:= \left(e^{(r-r_e)c}W^H(r_e)+\int_{r_e}^r e^{(r-s)c}dW^H(s)\right)^2, \\ G_{r_e,c}(p) &:= W^H(p)-cA_{p,d}-\int_0^{r_e}W^H(p)dp. \end{aligned}$$

The limit distribution in Theorem 4.1 depends on the non-centrality localizing scale parameter c. This parameter differentiates $F_{r,d}^c$ from $F_{r,d}$ and evidently $F_{r,d}^c=F_{r,d}$ for $c=0$ from (20). Since both $F_{r,d}^c$ and $F_{r,d}$ are $O_p(1)$, they may be used to compute local power of the proposed test.

4.2 Mildly explosive model

Next consider the alternative hypothesis with the following mildly explosive setting

$$\begin{cases} y_t &= (y_{t-1}+u_t)\,1\{t<\tau_e\}+(\rho_n y_{t-1}+u_t)\,1\{\tau_e\le t\le n\},\ y_0=o_p(n^{1/2+d_1}) \\ u_t &= \begin{cases}\Delta_+^{-d_1}\epsilon_t & \text{if } t<\tau_e, \\ \Delta_+^{-d_2}\epsilon_t & \text{if } \tau_e\le t\le n,\end{cases} \epsilon_t \overset{iid}{\sim}(0,\sigma^2), d_1,d_2\ge 0, \mathbb{E}|\epsilon_1|^{2+\delta}<\infty, \delta>0, \end{cases}, \tag{27}$$

where

$$\rho_n = 1+\frac{c}{n^\alpha},\ c>0,\ \alpha\in(0,1). \tag{28}$$

In model (27) y_t has unit root behavior before τ_e and becomes mildly explosive after τ_e, implying a structural break at τ_e. For both periods the errors in the AR model have strong dependence but with memory parameter d_1 prior to the break and memory parameter d_2 after the break. Since the localizing rate parameter $0 < \alpha < 1$, the specification (28) delivers stronger explosive behavior than the locally explosive LUR model considered earlier. Note that under the LUR model, the memory parameters have a role in determining whether the locally explosive trajectory is relevant asymptotically. Suppose that $\alpha = 1$, and $d_1 \neq d_2$, model (27) becomes an LUR model with long memory errors with memory parameters d_1 and d_2 at $t < \tau_e$ and $t \in [\tau_e, n]$, respectively. It can be shown that the persistence of y_t is solely determined by $\max(d_1, d_2)$ as $n \to \infty$, and if $d_1 > d_2$ the locally explosive part will be asymptotically dominated by the non-explosive episode. But when $\alpha < 1$, the mildly explosive coefficient is sufficient for a consistent test, as shown in the following result.

Theorem 4.2 *Under model (27) with (28), as $n \to \infty$,*

$$DF_n = O_p(n^{1-\alpha/2}) \xrightarrow{p} \infty \text{ and } \widetilde{DF}_{n,HAR} = O_p\left(n^{\frac{1-\alpha}{2}}\right) \xrightarrow{p} \infty.$$

Theorem 4.2 shows that the statistic DF_n diverges to infinity under mildly explosive alternatives. Combining with the result in Lemma 3.1, divergence of DF_n may be due to either strongly dependent errors or mildly explosive autoregression. But the modified HAR statistic $\widetilde{DF}_{n,HAR}$ diverges only under the alternative hypothesis given by (27) and (28). For any $\beta \times 100\%$ critical value $cv^{\beta}_{HAR}(d)$, we have $\Pr\left(\widetilde{DF}_{n,HAR} > cv^{\beta}_{HAR}(d)\right) \to 1$ under model (27) with condition (28), giving a consistent test for mildly explosive alternatives. Note that DF_n diverges at the same rate $n^{1-\alpha/2}$ as that obtained in PWY and PSYa under i.i.d. errors. The divergence rate of both statistics does not depend on d.

Remark 4.1 *Unlike the local alternative case where $\alpha = 1$, no assumption about d_1 and d_2 is needed. They can be identical or different under the mildly explosive setting (28).*

Remark 4.2 *Phillips and Magdalinos (2007) proposed the mildly explosive specification with $\rho_n = 1 + \frac{c}{k_n}$ where $k_n \to \infty$, $k_n/n \to 0$ and $c > 0$. This specification is more general than ours but with a mild additional condition on k_n the test remains consistent. In particular and without loss of generality let c be normalized to unity (see Phillips (2023) for discussion). Then, under the condition $\frac{\log(n)k_n}{n} \to 0$, our test is shown to be consistent in the Online Supplement.*[11]

Remark 4.3 *Setting $\alpha = 0$ in (28) leads to a purely explosive alternative, which amplifies explosive behavior. So test divergence and consistency in Theorem 4.2 continue to hold. The Online Supplement provides further discussion.*

[11] We thank an anonymous referee for suggesting this extension.

5 Dating Origination and Termination

We now discuss estimation of the origination and termination dates of an explosive period. Following PWY and PY, we consider the following model:

$$\begin{cases} y_t &= (y_{t-1}+u_t)\, 1\{t<\tau_e\} + (\rho_n y_{t-1}+u_t)\, 1\{\tau_e \le t \le \tau_f\} \\ & \quad + \left(\sum_{k=\tau_f+1}^{t} u_k + y^*_{\tau_f}\right) 1\{t>\tau_f\}, y_0 = o_p(n^{1/2+d_1}), \\ \rho_n &= 1+\frac{c}{n^{\alpha}},\ c>0,\ \alpha\in(0,1), \\ u_t &= \Delta_+^{-d_t}\epsilon_t,\ \epsilon_t \overset{iid}{\sim} (0,\sigma^2),\ \mathbb{E}|\epsilon_1|^{2+\delta}<\infty,\ \delta>0, \\ d_t &= d_1 \text{ for } t\in[1,\tau_e)\cup[\tau_f+1,n],\ d_t=d_2 \text{ for } t\in[\tau_e,\tau_f],\ \tau_e=\lfloor nr_e\rfloor,\ \tau_f=\lfloor nr_f\rfloor, \\ y^*_{\tau_f} &= y_{\tau_e}+y^*, \text{ and } y^* = O_p(1). \end{cases} \tag{29}$$

This model extends (9) by allowing for potentially strong dependence in the errors. As in (9), the notations τ_e (r_e) and τ_f (r_f) are the true temporal (fractional) origination and termination dates of the explosive period. Different from model (12) which has no break, model (29) has two breaks. Before the first break (i.e. $t<\tau_e]$), the model has a unit root in the AR coefficient. After the first break (i.e. $t\in[\tau_e,\tau_f]$), y_t is mildly explosive with the AR coefficient $\rho_n = 1+\frac{c}{n^{\alpha}}$, $c>0$. The explosive period ends at τ_f+1 and the process returns to a unit root process with a re-initialization at $y^*_{\tau_f}$ which lies in an $O_p(1)$ neighborhood of y_{τ_e}. This model also extends (9) by allowing for different memory parameters in the errors during the explosive period and non-explosive periods.

Break point estimators of r_e and r_f are defined by employing the HAR statistic $\widetilde{DF}_{n,HAR}$ in the usual criteria

$$\begin{aligned} \hat{r}_e^{HAR} &= \inf_{r\ge r_0}\{r: \widetilde{DF}_{\tau,HAR} > cv_{n,HAR}\}, \\ \hat{r}_f^{HAR} &= \inf_{r>\hat{r}_e+\gamma\ln(n)/n}\{r: \widetilde{DF}_{\tau,HAR} < cv_{n,HAR}\}. \end{aligned} \tag{30}$$

The following theorem shows that $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ deliver consistent estimates of r_e and r_f when $cv_{n,HAR}$ passes to infinity at a controlled rate.

Theorem 5.1 *Under model (29) with $\tau=\lfloor nr\rfloor$, $\widetilde{DF}_{\tau,HAR}$ has the following asymptotic behavior:*

$$\begin{aligned} \widetilde{DF}_{\tau,HAR} = O_p\left(n^{\frac{1-\alpha}{2}}\right) &\overset{p}{\to} \infty \text{ if } \tau\in[\tau_e,\tau_f], \\ \widetilde{DF}_{\tau,HAR} = O_p\left(n^{\frac{1-\alpha}{2}}\right) &\overset{p}{\to} -\infty \text{ if } \tau\in[\tau_f+1,n]. \end{aligned} \tag{31}$$

If $r_e\ge r_0$ and the critical value $cv_{n,HAR}$ satisfies the following condition

$$\frac{1}{cv_{n,HAR}} + \frac{cv_{n,HAR}}{n^{(1-\alpha)/2}} \to 0, \tag{32}$$

then, as $n\to\infty$,

$$\hat{r}_e^{HAR} \overset{p}{\to} r_e \quad \textit{and} \quad \hat{r}_f^{HAR} \overset{p}{\to} r_f.$$

Under the alternative hypothesis, consistent estimation of the origination and termination dates of an explosive period requires that the critical value $cv_{n,HAR}\to\infty$ but at a rate slower than $n^{(1-\alpha)/2}$. This is a slightly stronger control condition for consistency than that used in PWY for the model without strongly dependent errors (where the rate is required to be slower than $n^{(2-\alpha)/2}$). The difference is due to the presence of long memory. In PWY, the critical value is set to

$$\log(\log(nr))/100, r \in (0, 1] \tag{33}$$

which is close to the critical value corresponding to a 4% significance level of the DF test in their applications. In our applications, the diverging factor (33) is also used to construct $cv_{n,HAR}$, which leads to a critical value close to $cv^{0.03}_{n,HAR}(d)$ in (21), corresponding to a 3% significance level and satisfying the rate required in (32).

Remark 5.1 *Under the null hypothesis of no explosive behavior, i.e. model (12), if* $cv_{n,HAR} \to \infty$ *the probability of detecting an explosive episode in the data using* $\widetilde{DF}_{\tau,HAR}$ *goes to zero as* $n \to \infty$. *This is because* $\widetilde{DF}_{\tau,HAR} \sim O_p(1)$ *under model (12).*

Remark 5.2 *As in PWY, the procedure provides real-time estimates of* r_e *and* r_f *because the date estimates* $\hat{r}^{HAR}_e$ *and* $\hat{r}^{HAR}_f$ *only use subsamples of data observed to those points.*

6 Heteroskedastic model

This section explains how to conduct right-tailed unit root tests in the presence of unconditional heteroskedasticity. Time series models with unconditionally heteroskedastic errors were studied in Cavaliere and Taylor (2005, 2007) and Xu and Phillips (2008). More recently, Harvey et al. (2016, 2018, 2019) and Astill et al. (2023) adopted an AR model with time-varying volatilities and proposed new tests for explosive behavior in such settings. The following provides an extension of those ideas under strongly dependent errors.

Consider the model

$$\begin{cases} y_t & = & y_{t-1} + u_t,\ y_0 = o_p(n^{1/2+d}), t = 1, ..., n, \\ u_t & = & \Delta_+^{-d}\epsilon_t = \Delta_+^{-d}\sigma_{t,n}\varepsilon_t = \Delta_+^{-d} g\left(t/n\right)\varepsilon_t,\ \varepsilon_t \overset{iid}{\sim} (0,1),\ d \geq 0, \\ \mathbb{E}|\varepsilon_t|^q & < & K < \infty \text{ and } q \geq 4, \end{cases} \tag{34}$$

where g is a strictly positive, non-stochastic and continuously differentiable function on $[0, 1]$ with $\sup_s g(s) < C < \infty$. Model (34) has strongly dependent errors (captured by the parameter d) that are also unconditionally heteroskedastic (captured by the weakly trending function $\sigma_{t,n} = g(t/n)$).

Lemma 6.1 *Under model (34), as* $n \to \infty$, *we have*

$$\frac{1}{n^{1/2+d}} \sum_{t=1}^{\lfloor nr \rfloor} u_t \Rightarrow \frac{1}{\Gamma(H + 1/2)} \int_0^r g(s)(r - s)^{H-1/2} dW(s) =: W_g^H(r). \tag{35}$$

Remark 6.1 *When* $d = 0$, *Cavaliere (2005) and Cavaliere and Taylor (2005, 2007) showed that*

$$n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} u_t \Rightarrow \int_0^r g(s)dW(s),$$

and Lemma 6.1 extends that result to the case where $d > 0$ *and the limit involves a weighted functional of fBM.*

Using the functional law (35) yields the corresponding limit theory for the statistic $\widetilde{DF}_{\tau,HAR}$ in the case of strong dependence and unconditional heteroskedasticity.

Theorem 6.1 *Under model (34), as $n \to \infty$, we have*

$$\widetilde{DF}_{\tau,HAR} \Rightarrow \frac{b^{1/2}\left[\frac{r}{2}\left(W_g^H(r)\right)^2 - \left(\int_0^r W_g^H(s)ds\right)W_g^H(r)\right]}{\left[2\int_0^r \left(\tilde{W}_g^H(s)\right)^2 ds\left(\int_0^r W_g^H(p)^2 - \int_0^{(1-b)r} W_g^H(p)W_g^H(p+br)dp\right)\right]^{1/2}} =: F_{r,d}^g, \tag{36}$$

where $\tilde{W}_g^H(r) = W_g^H(r) - \frac{1}{r}\int_0^r W_g^H(s)ds$, $b = M/\tau$ where $M = M_\tau$ is the bandwidth in the kernel function used to construct the modified HAR statistic $\widetilde{DF}_{\tau,HAR}$.

The limit functional $F_{r,d}^g$ depends on the unknown quantities d and g. One approach to operationalize inference is to consistently estimate d and g and obtain critical values for the functional $F_{r,\hat{d}}^{\hat{g}}$ using these plug-in estimates. For example, we can consistently estimate $d_y = 1 + d$ directly from the given data, and hence d, under model (34) by ELW or QML estimation. Then $\{y_t\}$ can be filtered using $\hat{d}$ by calculating $\hat{u}_t = \Delta_+^{1+\hat{d}} y_t$, and the adaptive kernel method (Beare, 2004; Phillips and Xu, 2006; Xu and Phillips, 2008; Cavaliere et al., 2022; Astill et al., 2023) can be used to estimate g.

A second approach is to note that, under (34), we have

$$\widetilde{x}_p = \sum_{t=1}^{p} \Delta_+^{-d}\left(\frac{\Delta_+^{1+d} y_t}{g(t/n)}\right) = \sum_{t=1}^{p} \Delta_+^{-d}\varepsilon_t = \widetilde{x}_{p-1} + \Delta_+^{-d}\varepsilon_p.$$

Hence, based on $\hat{d}$ and $\hat{g}$, we can define

$$x_{\lfloor ns \rfloor} = \sum_{t=1}^{\lfloor ns \rfloor} \Delta_+^{-\hat{d}}\left(\frac{\Delta_+^{1+\hat{d}} y_t}{\hat{g}(t/n)}\right), \; x_0 = 0, \; \text{for } s \in [0,1], \tag{37}$$

where $\hat{g}^2(t/n) = \sum_{i=1}^{\tau} k_{ti}\hat{\epsilon}_i^2$ with $k_{ti} = \frac{K_\nu(t-i)}{\sum_{i=1}^{\tau} K_\nu(t-i)}$, $\hat{\epsilon}_t = \Delta_+^{1+\hat{d}} y_t$, $K_\nu(\cdot) = K(\frac{\cdot}{\nu})$ and $K(\cdot)$ is a kernel function with bandwidth ν. Following Astill et al. (2023), we assume that the kernel $K(\cdot)$ satisfies the conditions given in Theorem 6.2.

The following limit theory holds when the $\widetilde{DF}_{\tau,HAR}$ test is applied to $\{x_t\}_{t=1}^{\tau}$.

Theorem 6.2 *Assume $\{y_t\}_{t=1}^{n}$ is generated from model (34). Suppose the kernel function $K(\cdot)$ satisfies the following conditions: it is continuously differentiable over the interval $(0,1)$; $K(x) = 0$, for $x \le 0$ and $x \ge 1$; $\int_0^1 K dx > 0$, $\int_0^1 |K(x)|dx < \infty$, $\int_0^1 |K(x)x|dx < \infty$, and the characteristic function of K is absolutely integrable. Suppose that $n^\gamma\left(\hat{d} - d\right) \xrightarrow{d} \mathcal{N}(0,V)$ with $1/4 < \gamma \le 1/2$ and $V > 0$. Furthermore, assume the bandwidth ν satisfies $\nu \to \infty$, $\frac{\nu}{n} \to 0$ and $\frac{\nu^2}{n} \to \infty$ as $n \to \infty$. Let $\widetilde{DF}_{\tau,HAR}^x$ denote the test statistic $\widetilde{DF}_{\tau,HAR}$ applied to data $\{x_t\}_{t=1}^{\tau}$ constructed as in (37). As $n \to \infty$, we then have*

$$\widetilde{DF}_{\tau,HAR}^x \Rightarrow F_{r,d}. \tag{38}$$

Remark 6.2 *Theorem 6.2 states that one can use the same limit distribution as in Theorem 3.2 to obtain critical values for the test under the heteroskedastic model (34). It is therefore possible to extend the bootstrap procedures given earlier to accommodate the presence of error variance heterogeneity. Simulations, not reported here, were conducted to check size performance in these tests for different forms of error variance function* $g(\cdot)$. *Overall, the finite sample performance was found to be comparable to that based on the statistic* $\widetilde{DF}_{\tau,HAR}$ *for the homogeneous case where* σ^2 *is fixed — see Table 2 in the following section and Table 2 in the Online Supplement.*

7 Monte Carlo Studies

This section reports the results of simulation experiments designed (i) to explore the size and power performance of the proposed tests for the presence of explosive behavior in the data, and (ii) to study performance of the procedures for estimating the origination and termination dates in finite samples. The reported results relate to the model with homogeneous error variance.[12] Normalized partial sums of $u_t = \Delta_+^{-d}\epsilon_t$, with $\epsilon_t \overset{iid}{\sim} (0,1)$, were used to approximate the Type II fBM that appears in the limit theory.[13] This approximation allows us to simulate DF_∞, $F_{r,0}$ and $F_{r,d}$ to obtain the critical values. The number of replications in all experiments is 2,500.

To investigate the empirical size of the tests we use the following DGP,

$$\begin{cases} y_t &= y_{t-1} + u_t,\ t = 1, ..., n \\ u_t &= \Delta_+^{-d}\epsilon_t,\ \epsilon_t \overset{iid}{\sim} \mathcal{N}(0,1) \end{cases}, \tag{39}$$

with parameter settings: $d \in \{0, 0.05, 0.1, ..., 0.45\}$, $y_0 = 0$, and $n \in \{100, 500\}$.

For each parameter setting right-tailed unit tests were conducted using the statistics DF_n, $DF_{n,HAR}$ and $\widetilde{DF}_{n,HAR}$. For the standard right-tailed test based on DF_n, the null hypothesis is rejected when the statistic exceeds the 5% right-tail critical value of the corresponding asymptotic distribution or bootstrap distribution.[14]. Critical values for $DF_{n,HAR}$ and $\widetilde{DF}_{n,HAR}$ were obtained via simulations. The critical values of $DF_{n,HAR}$ were obtained from the simulated limit distribution (18) with the true value of d being replaced by the ELW estimate $\hat{d}$ (Shimotsu and Phillips, 2005). There are three critical values for our test statistic $\widetilde{DF}_{n,HAR}$. First, we assume d is known and obtain the asymptotic (infeasible) critical value from $F_{r,d}$, which provides a benchmark for calibrating the empirical size of the feasible tests. Second, we obtain the feasible asymptotic critical values from $F_{r,\hat{d}}$.[15] Finally, we obtain critical values from the bootstrap approach. The fixed-b scale parameter $b = 0.05$ was used for calculating $\hat{\Omega}_{HAR}$.[16]

[12] As indicated earlier, similar findings were obtained in the heteroskedastic case with several variance functions. These findings are reported in the Online Supplement only to save space.

[13] The sums $\frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor nr\rfloor}\epsilon_t$ and $\frac{1}{n^{1/2+d}}\sum_{t=1}^{\lfloor nr\rfloor}u_t$ are used to approximate $W(r)$ and $W^H(r)$ with $n = 5000$.

[14] The critical values for DF_n were obtained from Table B.6 in Hamilton (1994)

[15] We also estimate d by the QMLE method of Hualde and Robinson (2011). The empirical sizes are similar to those based on the ELW method.

[16] This value of b was chosen because extensive simulations showed that for any $b > 0.05$ the test delivered empirical size close to the nominal value. Notably, however, lower values of b were found to yield higher power, as is known from other applications of fixed-b methods.

Table 2: Empirical sizes of DF_n, $DF_{n,HAR}$ and $\widetilde{DF}_{n,HAR}$ for various d based on a nominal 5% right-tailed critical value

$n = 100$										
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
DF_n	0.05	0.09	0.14	0.20	0.26	0.31	0.36	0.41	0.45	0.48
$DF_{n,HAR}(\hat{d})$	0.12	0.12	0.11	0.09	0.06	0.05	0.04	0.03	0.04	0.04
$\widetilde{DF}_{n,HAR}(d)$	0.05	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
$\widetilde{DF}_{n,HAR}(\hat{d})$	0.04	0.05	0.05	0.06	0.05	0.05	0.06	0.06	0.06	0.06
$\widetilde{DF}^*_{n,HAR}(\hat{d})$	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.05	0.05	0.06
$n = 500$										
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
DF_n	0.05	0.10	0.18	0.27	0.34	0.40	0.45	0.49	0.53	0.56
$DF_{n,HAR}(\hat{d})$	0.15	0.11	0.07	0.03	0.02	0.03	0.03	0.04	0.04	0.05
$\widetilde{DF}_{n,HAR}(d)$	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
$\widetilde{DF}_{n,HAR}(\hat{d})$	0.04	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
$\widetilde{DF}^*_{n,HAR}(\hat{d})$	0.05	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.05	0.05

Table 2 reports the empirical sizes of DF_n, $DF_{n,HAR}$ and $\widetilde{DF}_{n,HAR}$ with the corresponding 5% critical values. For $\widetilde{DF}_{n,HAR}$, we report the test sizes using critical values obtained from $F_{r,d}$ (denoted $\widetilde{DF}_{n,HAR}(d)$), $F_{r,\hat{d}}$ (denoted $\widetilde{DF}_{n,HAR}(\hat{d})$), and the bootstrap method (denoted $\widetilde{DF}^*_{n,HAR}(\hat{d})$). Several observations can be made on the findings from Table 2. First, DF_n has satisfactory performance only when $d = 0$ and the test is oversized when $d > 0$. For instance, when $d = 0.3$ and $n = 500$, the test rejects the null about 40% of the time. These simulation results are consistent with the asymptotic theory in Sowell (1990) and the predictions from Lemma 3.1, which imply severe false detection of explosiveness as d increases. Second, use of $DF_{n,HAR}$ does not lead to a divergent empirical size. But, when the true value of d is equal to or close to zero, some size distortion in the feasible statistic $DF_{n,HAR}(\hat{d})$ is noticeable. Finally and most importantly, use of the modified test $\widetilde{DF}_{n,HAR}$ shows good size performance irrespective of the value of d and alternative ways of obtaining the critical value. The simulation evidence suggests that $\widetilde{DF}_{n,HAR}(\hat{d})$ with critical values obtained from $F_{r,\hat{d}}$ delivers overall good size performance in finite samples across all values of d.

Given its good size performance, the finite sample power properties of the $\widetilde{DF}_{n,HAR}$ test were explored next. The experiment was designed using model (27) with the following parameter settings: $n = 100$, $y_0 = 100$, $r_e = 0.5$, $d_1 = d_2 = d \in \{0, 0.01, 0.02, ..., 0.49\}$, $\rho_n = 1 + c/n^\alpha$, $c = 1$, and $\alpha \in \{0.50, 0.55, 0.56, ..., 1\}$, which corresponds to the autoregressive root ρ_n ranged from 1.1 to 1.01. [17]

Table 3 reports the empirical rejection rates (empirical power) under selected values of α and d. Figure 2 plots power as a function of α and d. Several finding are notable.

[17] The initial condition $y_0 = 100$ is used to ensure a positive sample path for the simulated data and produce an explosive episode that has an upward trajectory. This choice matches the real data considered later.

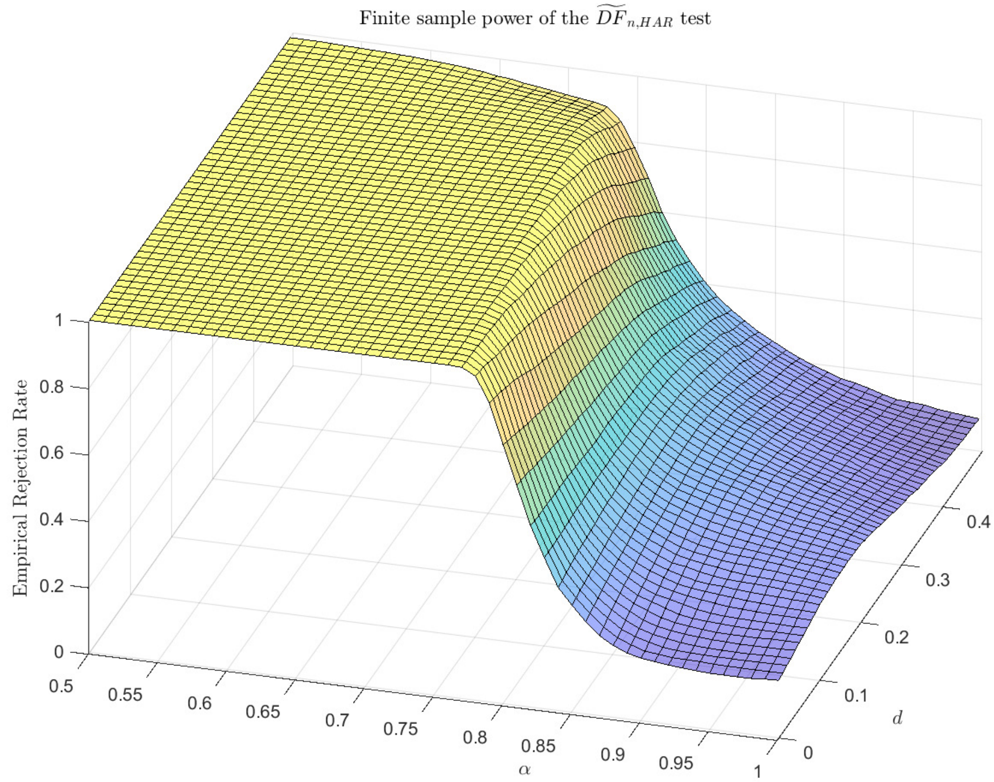

Figure 2: The empirical power of the $\widetilde{DF}_{n,HAR}$ test as a function of α and d.

First, the smaller the value of α the higher is the power. This is expected since stronger explosiveness enhances detection. A sharp contrast can be observed when $d = 0.45$ in which case the empirical rejection rate is 1 at $\alpha = 0.50$ whereas the rejection rate is 0.12 at $\alpha = 1$ for an LUR alternative. Second, when α is small, variations in memory parameter only have a small effect on the empirical rejection rates. It can be seen that for α less than or equal to 0.7, different values of d only slightly change the empirical rejection rate, whereas different values of d can materially change the empirical rejection rate as α moves closer to 1. These simulation findings show that our method is more reliable when $\alpha \leq 0.7$, and less powerful as α approaches unity. When $\alpha = 1$ the model is an LUR process, $\widetilde{DF}_{\tau,HAR}$ does not diverge, and the tests are not consistent. Similarly, when α approaches unity y_t becomes close to an LUR process and, as expected, empirical power drops.[18] Additional simulations for the empirical rejection rates under various c and n are reported in the Online Supplement.

To study the accuracy of the date detectors $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ in finite samples, we used an experimental design based on model (29) with the following parameter settings: $n =$

[18] In particular, when $\alpha = 0.85$, the detection rate drops from close to 100% to the 30% – 40% range, as seen in Table 3.

Table 3: The empirical rejection rates of $\widetilde{DF}_{n,HAR}$

d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\alpha = 0.50$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$\alpha = 0.55$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$\alpha = 0.60$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99
$\alpha = 0.65$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.98
$\alpha = 0.70$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.96
$\alpha = 0.85$	0.32	0.35	0.37	0.39	0.40	0.41	0.40	0.37	0.34	0.30

100, $y_0 = 100$, $c = 1$, $\alpha \in \{0.5, 0.55, ..., 0.7, 0.85\}$[19], $d_1 = d_2 = d \in \{0, 0.05, 0.1, ..., 0.45\}$, $\epsilon_t \overset{iid}{\sim} \mathcal{N}(0, 1)$, $y^*_{\tau_f} = y_{\tau_e}$, $r_e = 0.5$, $r_f = 0.7$, $r_0 = 0.4$, and $\gamma \ln(n)/n = 0.1$. To obtain $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$, we first calculate $\{\widetilde{DF}_{\tau,HAR}\}_{\tau=\lfloor nr_0 \rfloor}^{n}$ and then obtain $\{\hat{d}_\tau\}_{\tau=\tau_0}^{n}$ using ELW estimation based on $\{y_t\}_{t=1}^{\tau}$. The following critical values for $cv_{n,HAR}$ are employed

$$cv_{n,HAR} = cv_{n,HAR}^{0.03}\left(\hat{d}_\tau\right) + \frac{\ln(\ln(nr))}{100}, \tag{40}$$

where nr is proportional to the sample size n and $r \in (0, 1]$ is the corresponding fraction of the sample. These critical values are constructed using the 3% critical value of $\widetilde{DF}_{n,HAR}$ under $\hat{d}_\tau$ augmented with the slowly diverging factor $\frac{\ln(\ln(nr))}{100}$.[20] This factor guarantees that $cv_{n,HAR}$ satisfies condition (32) asymptotically, leading to consistent break point estimates $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$. However, in our finite sample setting $\frac{\ln(\ln(nr))}{100}$ takes values between 0.01 and 0.015 and $cv_{n,HAR}^{0.03}\left(\hat{d}_\tau\right)$ has a greater magnitude than $\frac{\ln(\ln(nr))}{100}$.

Table 4 reports the successful detection rate and the means of $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ where successful detection is obtained. The numbers in parentheses below the means are the root mean square errors of the estimates. Successful detection is defined whenever $\hat{r}_e^{HAR}$ falls into the interval $[r_e, r_f]$ (i.e. $\hat{r}_e^{HAR} \in [r_e, r_f]$). Several findings emerge from this simulation. First, when α is small, the successful detection rate is only slightly affected by changes in the memory parameter: the successful detection rate in Table 4 drops only by 0.01 when $\alpha = 0.50$ and d increases from 0 to 0.45, whereas it drops by 0.19 when $\alpha = 0.70$. Further, the estimates of $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ are less accurate when both α and d are large: the root mean square errors of $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ are 0.09 and 0.03 respectively when $d = 0.45$ and $\alpha = 0.70$, in contrast to the corresponding root mean square errors of 0.01 and 0.00 when $d = 0$ and $\alpha = 0.50$.

[19]The main simulation results cover the domain $\alpha \leq 0.7$; and, as remarked above, for values of α closer to unity such as $\alpha = 0.85$, sample paths of y_t become closer to those of an LUR process and successful detection rates (defined below) fall and can be significantly lower than 50% as apparent in the final row of Table 3. Estimates of the break points r_e and r_f are also inaccurate when α moves closer to unity.

[20]The 3% critical value is adopted here as extensive simulations suggest that it yields a higher successful detection rate (defined below) than the 5% critical value in most cases.

Table 4: Finite sample performance of $\hat{r}_e^{HAR}, \hat{r}_f^{HAR}$ when $r_e = 0.5, r_f = 0.7$

d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\alpha = 0.50$										
Detect. Rate	0.90	0.90	0.91	0.91	0.91	0.91	0.91	0.91	0.90	0.90
$\hat{r}_e^{HAR}$	0.50	0.51	0.51	0.51	0.51	0.51	0.52	0.52	0.53	0.53
	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.03)	(0.03)	(0.04)	(0.05)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)
$\alpha = 0.55$										
Detect. Rate	0.90	0.90	0.91	0.91	0.91	0.91	0.91	0.90	0.90	0.88
$\hat{r}_e^{HAR}$	0.51	0.51	0.51	0.51	0.52	0.52	0.52	0.53	0.53	0.54
	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)	(0.03)	(0.04)	(0.04)	(0.05)	(0.06)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.02)
$\alpha = 0.60$										
Detect. Rate	0.90	0.90	0.91	0.91	0.91	0.91	0.91	0.90	0.88	0.85
$\hat{r}_e^{HAR}$	0.51	0.51	0.51	0.52	0.52	0.53	0.53	0.54	0.54	0.55
	(0.01)	(0.02)	(0.02)	(0.02)	(0.03)	(0.04)	(0.05)	(0.05)	(0.06)	(0.06)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.71
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)
$\alpha = 0.65$										
Detect. Rate	0.90	0.90	0.91	0.91	0.91	0.91	0.90	0.88	0.85	0.82
$\hat{r}_e^{HAR}$	0.51	0.52	0.52	0.52	0.53	0.53	0.54	0.55	0.55	0.56
	(0.02)	(0.02)	(0.03)	(0.03)	(0.04)	(0.05)	(0.06)	(0.07)	(0.07)	(0.08)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.71	0.71	0.71
	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)
$\alpha = 0.70$										
Detect. Rate	0.90	0.90	0.91	0.90	0.90	0.90	0.88	0.83	0.77	0.71
$\hat{r}_e^{HAR}$	0.52	0.52	0.53	0.53	0.54	0.55	0.55	0.56	0.56	0.57
	(0.02)	(0.03)	(0.04)	(0.04)	(0.05)	(0.06)	(0.07)	(0.08)	(0.08)	(0.09)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.71	0.701	0.71	0.71
	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.02)	(0.02)	(0.03)	(0.03)	(0.03)

8 Empirical Application

To highlight the usefulness of the proposed test and date-stamping strategy we conduct an empirical study using the same time series as in Table 1. We calculate the $\widetilde{DF}_{n,HAR}$ statistic and use 10%, 5% and 1% critical values when performing the right-tailed unit root test. Since these data are price-dividend ratios which take account of fundamental values, explosive behavior in the time series is indicative of a rational bubble.

Table 5 reports the HAR test statistic $\widetilde{DF}_{n,HAR}$ together with 10%, 5%, and 1% critical values computed for the six different sample periods. In Table 1 it was found that standard testing using the DF_n statistic exceeded the 5% critical value for each sample period, indicating strong evidence for the presence of bubbles. Table 5 updates the analysis by using the new HAR statistic to allow for the possible presence of strong dependence in the data. The results show that for the sample period (b) the test fails to reject a unit root null at the 10% level; for period (c) the test rejects the null at the 10% level but fails to reject a unit root null at the 5% level; and for periods (a), (d), (e) and (f), the test rejects the null at the 5% level. Thus, using the conventional 5% level the four periods (a), (d), (e) and (f) show significant evidence of being bubble episodes in the S&P stock

Table 5: Empirical results for the S&P 500 with $\widetilde{DF}_{n,HAR}$ and critical values

Sampling Period	$\hat{d}$	$\widetilde{DF}_{n,HAR}$	$cv^{10\%}_{HAR}\left(\hat{d}\right)$	$cv^{5\%}_{HAR}\left(\hat{d}\right)$	$cv^{1\%}_{HAR}\left(\hat{d}\right)$
(a) Jan 1872 to Feb 1880	0.24	1.25	0.70	0.92	1.30
(b) Jun 1882 to May 1887	0.32	0.62	0.76	0.97	1.36
(c) May 1940 to Feb 1946	0.34	0.89	0.77	0.98	1.38
(d) Jun 1948 to Nov 1955	0.29	1.54	0.74	0.94	1.33
(e) May 1979 to Mar 1987	0.21	1.28	0.67	0.90	1.26
(f) May 1989 to Aug 1997	0.24	1.18	0.70	0.92	1.30

market. Taking into account the findings for the other periods, it is clear that allowing for the presence of strong dependence does change the outcomes, giving statistical evidence only for the existence of explosive behavior in periods (a), (d), (e) and (f). However, these results continue to support the presence of stock market bubble behavior, including the internet bubble of the late 1990s even in the presence of strong dependence.

The bubble dating methodology was used to estimate the origination and termination dates r_e and r_f in sample periods (a), (d), (e) and (f) where bubble behavior was evident in the data. For this implementation 48 monthly observations were used to initialize estimation, the minimum explosive episode duration was 4 months, and the statistic $\widetilde{DF}_{\tau,HAR}$ and critical value $cv^{3\%}_{n,HAR}\left(\hat{d}_\tau\right)$ were computed recursively, as in PSYa.

Table 6: Empirical results for bubble origination and termination ($\hat{r}^{HAR}_e$, $\hat{r}^{HAR}_f$)

Sampling period	$\hat{r}^{HAR}_e$	$\hat{r}^{HAR}_f$	Duration
(a) Jan 1872 to May 1880	Oct 1879	Apr 1880	6 months
(d) Jun 1948 to Feb 1957	Dec 1954	Feb 1956	14 months
(e) May 1979 to Jan 1988	Feb 1987	Sep 1987	7 months
(f) May 1989 to Jan 1998	Feb 1997	Nov 1997	9 months

Table 6 reports the estimates $\hat{r}^{HAR}_e$ and $\hat{r}^{HAR}_f$ and associated bubble duration (in months) for episodes (a), (d), (e) and (f). The following conclusions can be drawn from these results. First, in episode (a) a rational bubble is found to originate in October 1879 and collapse in April 1880, lasting six months. Second, in episode (d), the bubble lasts for fourteen months from December 1954 to February 1956. Third, in episode (e), the explosive period begins in February 1987 and ends in September 1987, lasting seven months. Finally, in episode (f), the bubble starts in February 1997 and ends in November 1997, lasting night months.

These findings coincide with those of PSYa in rational bubble identification. In particular, the explosive episodes in the Great Depression, postwar boom, before Black Monday, and the dotcom bubble period are also found using our estimation method. However, while explosive behavior is detected using our methods, the episode durations are often shorter than those obtained by PSYa. In PSYa the explosive episodes in the Great Depression, postwar boom, Black Monday, and dotcom bubble periods were estimated to last for 18 months, 15 months, 15 months and 87 months, respectively. The explosive episodes identified by our method last for 6 months, 14 months, 7 months and 9 months, respectively. The presence of strong dependence in the data therefore does affect bubble duration. Nonetheless, the most striking overall result is that the empirical findings in PSYa of several major bubble episodes in the historical S&P 500 data are sustained using methods that are robust to data dependence, including possible long memory in the data.

9 Conclusion

This paper introduces a new right-tailed test and new dating algorithm to detect the presence of explosive episodes in time series data. The approach is motivated by showing empirical evidence of strong dependence in the errors of the autoregressive model employed for estimation and inference. Strongly dependent errors lead to divergent unit root test statistics, thereby leading to potential spurious detection of explosive behavior in traditional right-tailed unit root test statistics. To avert problems of spurious detection, this paper proposes a robust approach to inference using an appropriately self-normalized HAR statistic that accommodates potential strong dependence in the errors. Recursive implementation of this procedure enables consistent estimation of the origination and termination dates of explosive episodes in the data. The proposed test and asymptotics are extended to models with unconditional heteroskedasticity, thereby accommodating features that are known to be relevant in practice, particularly in financial data. Simulations show reliable finite sample performances of the new method in terms of both size and power. An empirical application corroborates the robustness of earlier findings on certain bubble episodes in historical S&P 500 data but leads typically to shorter duration periods of financial exuberance.

This paper has not addressed the complex additional issue of possible multiple bubble episodes in the same time series sample. However, the procedures developed here can be extended to allow for such multiple periods and break points in the data in precisely the same way as PSYa and PSYb. This extension simply involves replacing the use of the DF_τ statistic by $\widetilde{DF}_{\tau,HAR}$ in the PSY algorithm and imposing the conditions used here for consistency in the presence of strong dependence. We expect that when modified in this way the algorithm will retain validity for multiple bubble detection using the robust statistic $\widetilde{DF}_{\tau,HAR}$. This investigation is left for future study.

10 Appendix

The main results rely on several lemmas which are given, with proofs, in the Online Supplement.

10.1 Proofs of Theorem 3.1 and Theorem 3.2

Write

$$DF_{\tau,HAR} = \frac{\hat{\rho}_\tau - 1}{s_{\tau,HAR}} = \frac{\tau\left(\hat{\rho}_\tau - 1\right)}{\left(\tau^2 s_{\tau,HAR}^2\right)^{1/2}}, \tag{41}$$

$$\widetilde{DF}_{\tau,HAR} = \frac{\tau\left(\tilde{\rho}_\tau - 1\right)}{\left(\tau^2 s^2_{\tau,HAR}\right)^{1/2}}. \tag{42}$$

and to show the limit we first study the denominator in (41) and (42). Note that $s^2_{\tau,HAR} = \frac{\hat{\Omega}_{HAR}}{\sum_{t=1}^{\tau}\bar{y}^2_{t-1}}$. For $\hat{\Omega}_{HAR}$, letting $K_{i,j} = K\left(\frac{i-j}{b\tau}\right)$ and $S_t = \sum_{i=1}^{t}\Delta y_i$, we have

$$\begin{aligned}
\hat{\Omega}_{HAR} &= \sum_{j=-\tau+1}^{\tau} K\left(\frac{j}{b\tau}\right)\hat{\gamma}_j = \frac{1}{\tau}\sum_{i=1}^{\tau}\sum_{i=1}^{\tau}\Delta y_i K_{i,j}\Delta y_j \\
&= \frac{1}{\tau}\sum_{i=1}^{\tau-1}\frac{1}{\tau}\sum_{j=1}^{\tau-1}\tau^2\left[(K_{i,j} - K_{i,j+1}) - (K_{i+1,j} - K_{i+1,j+1})\right]\frac{1}{\sqrt{\tau}}\hat{S}_i\frac{1}{\sqrt{\tau}}\hat{S}_j \\
&= \frac{1}{\tau}\sum_{i=1}^{\tau-1}\frac{1}{\tau}\sum_{j=1}^{\tau-1}\tau^2 D_\tau\left(\frac{i-j}{b\tau}\right)\frac{1}{\sqrt{\tau}}S_i\frac{1}{\sqrt{\tau}}S_j,
\end{aligned} \tag{43}$$

where $D_\tau\left(\frac{i-j}{b\tau}\right) = (K_{i,j} - K_{i,j+1}) - (K_{i+1,j} - K_{i+1,j+1})$. The last equality follows from Equation (A.1) in Kiefer and Vogelsang (2002b).

Straightforward calculations show that

$$D_\tau\left(\frac{i-j}{b\tau}\right) = \begin{cases} \frac{2}{b\tau} & \text{if } |i-j| = 0 \\ -\frac{1}{b\tau} & \text{if } |i-j| = \lfloor b\tau \rfloor \\ 0 & \text{otherwise} \end{cases},$$

which implies

$$\begin{aligned}
\hat{\Omega}_{HAR} &= \sum_{i=1}^{\tau-1}\sum_{j=1}^{\tau-1} D_\tau\left(\frac{i-j}{b\tau}\right)\frac{1}{\sqrt{\tau}}S_i\frac{1}{\sqrt{\tau}}S_j \\
&= \frac{2}{b\tau}\sum_{i=1}^{\tau-1}\left(\frac{1}{\sqrt{\tau}}S_i\right)^2 - \frac{2}{b\tau}\sum_{i=1}^{\tau-\lfloor b\tau\rfloor-1}\left(\frac{1}{\sqrt{\tau}}S_i\right)\left(\frac{1}{\sqrt{\tau}}S_{i+\lfloor b\tau\rfloor}\right) \\
&= \frac{2}{b}\frac{n}{\lfloor nr\rfloor}\frac{1}{n}\sum_{i=1}^{\tau-1}\left(\frac{1}{\sqrt{\tau}}S_i\right)^2 - \frac{2}{b}\frac{n}{\lfloor nr\rfloor}\frac{1}{n}\sum_{i=1}^{\tau-\lfloor b\tau\rfloor-1}\left(\frac{1}{\sqrt{\tau}}S_i\right)\left(\frac{1}{\sqrt{\tau}}S_{i+\lfloor b\tau\rfloor}\right).
\end{aligned} \tag{44}$$

Thus, with $i = \lfloor np \rfloor$ and under the assumption $\rho_n = 1$, we have $S_i = \sum_{j=1}^{i}\Delta y_j = \sum_{j=1}^{\lfloor np\rfloor} u_j$. This implies that

$$\frac{1}{n^d}\frac{1}{\sqrt{\tau}}S_{\lfloor np\rfloor} = \left(\frac{n}{\tau}\right)^{1/2}\frac{1}{n^{1/2+d}}\sum_{t=1}^{\lfloor np\rfloor}u_i \Rightarrow \frac{\sigma}{r^{1/2}}W^H(p). \tag{45}$$

Therefore

$$\frac{1}{n^{2d}}\hat{\Omega}_{HAC} = \frac{2n}{b\tau}\frac{1}{n}\sum_{i=1}^{\tau-1}\left(\frac{1}{n^d}\frac{1}{\sqrt{\tau}}S_{\lfloor np\rfloor}\right)^2 - \frac{2n}{b\tau}\frac{1}{n}\sum_{i=1}^{\tau-\lfloor b\tau\rfloor-1}\left(\frac{1}{n^d}\frac{1}{\sqrt{\tau}}S_i\right)\left(\frac{1}{n^d}\frac{1}{\sqrt{\tau}}S_{i+\lfloor b\tau\rfloor}\right)$$

$$\Rightarrow \frac{2}{br}\int_0^r \left(\frac{\sigma}{r^{1/2}}W^H(p)\right)^2 dp - \frac{2}{br}\int_0^{(1-b)r} \frac{\sigma^2}{r}W^H(p)W^H(p+br)dp \tag{46}$$

$$= \frac{2\sigma^2}{br^2}\left(\int_0^r \left(W^H(p)\right)^2 dp - \int_0^{(1-b)r} W^H(p)W^H(p+br)dp\right), \tag{47}$$

where we have applied (45) and continuous mapping to obtain the limit (46).

Combining (A.1.3) in the Online Supplement and (46), upon normalization we have

$$\tau^2 s^2_{\tau,HAR} = \left(\frac{\tau}{n}\right)^2 \frac{\frac{1}{n^{2d}}\hat{\Omega}_{HAR}}{\frac{1}{n^{2+2d}}\left(\sum_{t=1}^{\tau} y_{t-1}^2 - \tau^{-1}\left(\sum_{t=1}^{\tau} y_{t-1}\right)^2\right)}$$

$$\Rightarrow \frac{2\left(\int_0^r W^H(p)^2 dp - \int_0^{(1-b)r} W^H(p)W^H(p+br)dp\right)}{b\int_0^r \left(\tilde{W}^H(s)\right)^2 ds}. \tag{48}$$

We now proceed to obtain the limit of $DF_{\tau,HAR}$. When $d=0$, we have

$$DF_{\tau,HAR} = \frac{\tau\left(\hat{\rho}_\tau - 1\right)}{\left(\tau^2 s^2_{\tau,HAR}\right)^{1/2}} = \frac{\lfloor nr \rfloor}{nr} r \frac{n\left(\hat{\rho}_\tau - 1\right)}{\left(\tau^2 s^2_{\tau,HAR}\right)^{1/2}}$$

$$\Rightarrow \frac{r\int_0^r \tilde{W}(s)dW(s)}{\int_0^r \left(\tilde{W}(s)\right)^2 ds}\left(\frac{b\int_0^r \tilde{W}(s)^2 ds}{2\left(\int_0^r (W(p))^2 dp - \int_0^{(1-b)r} W(p)W(p+br)dp\right)}\right)^{1/2}$$

$$= \frac{b^{1/2} r\int_0^r \tilde{W}(s)dW(s)}{\left[2\int_0^r \left(\tilde{W}(s)\right)^2 ds\left(\int_0^r (W(p))^2 dp - \int_0^{(1-b)r} W(p)W^H(p+br)dp\right)\right]^{1/2}},$$

where the standard result $n\left(\hat{\rho}_\tau - 1\right) \Rightarrow \int_0^r \tilde{W}(s)dW(s)/\int_0^r \left(\tilde{W}(s)\right)^2 ds$ and (48) are used with $H=1/2$.

For $d \in (0, 0.5)$, similarly write

$$DF_{\tau,HAR} = \frac{\tau\left(\hat{\rho}_\tau - 1\right)}{\left(\tau^2 s^2_{\tau,HAR}\right)^{1/2}}$$

$$\Rightarrow \frac{\frac{r}{2}\left(W^H(r)\right)^2 - \left(\int_0^r W^H(s)ds\right)W^H(r)}{\int_0^r \left(\tilde{W}^H(s)\right)^2 ds}\left(\frac{b\int_0^r \tilde{W}^H(s)^2 ds}{2\left(\int_0^r \left(W^H(p)\right)^2 dp - \int_0^{(1-b)r} W^H(p)W^H(p+br)dp\right)}\right)^{1/2}$$

$$= \frac{\frac{rb^{1/2}}{2}\left(W^H(r)\right)^2 - b^{1/2}\left(\int_0^r W^H(s)ds\right)W^H(r)}{\left[2\int_0^r \left(\tilde{W}^H(s)\right)^2 ds\left(\int_0^r \left(W^H(p)\right)^2 dp - \int_0^{(1-b)r} W^H(p)W^H(p+br)dp\right)\right]^{1/2}}, \tag{49}$$

where the limit is obtained using (A.1.5) in the Online Supplement and (48).

For $\widetilde{DF}_{\tau,HAR}$, using (A.1.6) in the Online Supplement and (48), we have

$$\widetilde{DF}_{\tau,HAR} = \frac{\tau\left(\tilde{\rho}_\tau - 1\right)}{\left(\tau^2 s^2_{\tau,HAR}\right)^{1/2}}$$

$$\Rightarrow \quad \frac{\frac{rb^{1/2}}{2}\left(W^H(r)\right)^2 - b^{1/2}\left(\int_0^r W^H(s)ds\right)W^H(r)}{\left[2\int_0^r\left(\tilde{W}^H(s)\right)^2 ds\left(\int_0^r W^H(p)^2dp - \int_0^{(1-b)r} W^H(p)W^H(p+br)dp\right)\right]^{1/2}},$$

which completes the proof of Theorems 3.1 and 3.2. ■

10.2 Proof of Theorem 3.3

In this proof, random sequences are assumed to belong to an expanded common probability space in which a weakly convergent sequence can be represented by a sequence that converges almost surely via the Skorohod representation (see, e.g. Pollard (1984)).

We first show that the bootstrap residuals fall into the class of $\mathcal{L}_r(K, M, \theta)$ in Lemma 1.4 of the Online Supplement and verify the three conditions in the lemma. The first condition is satisfied because the residuals are centered. For the third condition, note that

$$e_{n,t} = \Delta_+^{1+\hat{d}} y_t = \Delta_+^{1+d} y_t + R_n = \varepsilon_t + R_n,$$

where $R_n = O_p(m^{-1}\log n)$, and the second equality is established from Lemma 1.3 in the Online Supplement. Further,

$$\frac{1}{n}\sum_{t=1}^n |e_{n,t}|^r = \frac{1}{n}\sum_{t=1}^n |\varepsilon_t + R_n|^r \leq C_r\frac{1}{n}\sum_{t=1}^n |\varepsilon_t|^r + C_r\frac{1}{n}\sum_{t=1}^n |R_n|^r. \tag{50}$$

Note that the first term $\frac{1}{n}\sum_{t=1}^n |\varepsilon_t|^r$ is bounded almost surely by virtue of the strong law of large numbers, and with Lemma 1.3 in the Online Supplement, the second term converges almost surely to zero via the Skorohod representation theorem. This verifies the third condition in Lemma 1.4 in the Online Supplement.

For the second condition, since $\bar{e}_{n,t} \equiv \frac{1}{n}\sum_{t=1}^n e_{n,t} \overset{as}{\to} 0$ and $\hat{\sigma}_e^2 \equiv \frac{1}{n}\sum_{t=1}^n (\varepsilon_t + R_n)^2 \overset{as}{\to} \sigma^2$, therefore

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^n {e_{n,t}^*}^2 - 1 &= \frac{1}{n}\sum_{t=1}^n \left(\frac{e_{n,t} - \bar{e}_{n,t}}{\hat{\sigma}_e}\right)^2 - 1 = \frac{1}{\hat{\sigma}_e^2}\left(\frac{1}{n}\sum_{t=1}^n \left(e_{n,t}^2 - 2e_{n,t}\bar{e}_{n,t} + \bar{e}_{n,t}^2\right)\right) - 1 \\
&= \frac{1}{\hat{\sigma}_e^2}\left(\frac{1}{n}\sum_{t=1}^n e_{n,t}^2 - 2\bar{e}_{n,t}\frac{1}{n}\sum_{t=1}^n e_{n,t} + \bar{e}_{n,t}^2\right) - 1 \overset{as}{\to} 1 - 1 = 0.
\end{aligned}$$

Given that $\mathbb{E}[|\varepsilon_t|^{2+\delta}] < \infty$, the sample variance estimator has a non-trivial convergence rate. Therefore, there exist a positive θ which allows us to verify the second condition and apply the approximation in (A.1.8) of the Online Supplement.

Note that $y_t^* = y_{t-1}^* + u_t^*$ with $u_t^* = \Delta_+^{-\hat{d}} e_{n,t}^*$ and so $y_t^* = \Delta_+^{-(1+\hat{d})} e_{n,t}^*$. Let $\pi_j^d = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}$ and applying a similar argument to (A.1.7) in the Online Supplement we have

$$\begin{aligned}
y_t^* &= \hat{\sigma}_e \Delta_+^{-(1+\hat{d})} e_{n,t}^* = \hat{\sigma}_e \Delta_+^{-(1+d)} e_{n,t}^* + O_p(m^{-1}) \\
&= \hat{\sigma}_e \Delta_+^{-d_y} e_{n,t}^* + O_p(m^{-1}) = \hat{\sigma}_e \sum_{j=1}^t \pi_{t-j}^{d_y} e_{n,j}^* + O_p(m^{-1}).
\end{aligned}$$

Set $\phi = \lfloor nr \rfloor$, $S_j^* = \sum_{t=1}^{j} e_{n,t}^*$, $Y_{\lfloor nr \rfloor}^* = n^{1/2-d_y} y_{\lfloor nr \rfloor}^*$ and write

$$Y_{\lfloor nr \rfloor}^* = n^{1/2-d_y} \hat{\sigma}_e \sum_{j=1}^{\lfloor nr \rfloor} \pi_{t-j}^{d_y} e_{n,j}^* + o_p(1) = n^{1/2-d_y} \hat{\sigma}_e \sum_{j=1}^{\phi} \pi_{t-j}^{d_y} \left(S_j^* - S_{j-1}^* \right) + o_p(1).$$

Following Silveira (1991), letting $V_j = \sum_{i=1}^{j} z_i$ and $z_t \overset{iid}{\sim} \mathcal{N}(0,1)$, we have

$$Y_n^*(r) = Q_{1n}(r) + Q_{2n}(r) + Q_{3n}(r) + Q_{4n}(r) + o_p(1),$$

where

$$\begin{aligned} Q_{1n}(r) &= \hat{\sigma}_e \left(n^{1/2-d_y} \sum_{j=1}^{\phi-1} \frac{(\phi - j)^{d_y - 1}}{\Gamma(d_y)} (V_j - V_{j-1}) \right), \\ Q_{2n}(r) &= \hat{\sigma}_e n^{1/2-d_y} \sum_{j=1}^{\phi-1} \pi_{\phi-j}^{d_y} \left[\left(S_j^* - S_{j-1}^* \right) - (V_j - V_{j-1}) \right], \\ Q_{3n}(r) &= \hat{\sigma}_e n^{1/2-d_y} \sum_{j=1}^{\phi-1} \left(\pi_{\phi-j}^{d_y} - \frac{(\phi - j)^{d_y - 1}}{\Gamma(d_y)} \right) (V_j - V_{j-1}), \\ Q_{4n}(r) &= \hat{\sigma}_e n^{1/2-d_y} \left(S_\phi^* - S_{\phi-1}^* \right). \end{aligned}$$

Silveira (1991) shows that

$$n^{1/2-d_y} \sum_{j=1}^{\phi-1} \frac{(\phi - j)^{d_y - 1}}{\Gamma(d_y)} (V_j - V_{j-1}) \Rightarrow W^H(r),$$

$$n^{1/2-d_y} \sum_{j=1}^{\phi-1} \left(\pi_{\phi-j}^{d_y} - \frac{(\phi - j)^{d_y - 1}}{\Gamma(d_y)} \right) (V_j - V_{j-1}) \xrightarrow{p} 0.$$

We can also show $Q_{4n}(r) = o_p(1)$ by applying Donsker's theorem for martingale difference arrays (MDAs) as in Theorem 27.14 of Davidson (1994). Coupled with $\hat{\sigma}_e \xrightarrow{p} \sigma$, we find that $Q_{1n}(r) \Rightarrow \sigma W^H(r)$, $Q_{3n}(r) = o_p(1)$, and $Q_{4n}(r) = o_p(1)$.

To show $Q_{2n}(r) = o_p(1)$, note that

$$\begin{aligned} &\sup_r |Q_{2n}(r)| \leq \hat{\sigma}_e \sup_r \sum_{j=1}^{\phi-1} |\pi_{\phi-j}^{d_y-1}| \sup_{j \leq n} n^{1/2-d_y} |S_j - V_j| \\ &= \hat{\sigma}_e \frac{1}{n^d} \sup_r \sum_{j=1}^{\phi-1} |\pi_{\phi-j}^{d_y-1}| \sup_{j \leq n} \left| \frac{S_j - V_j}{n^{1/2}} \right| \leq C \hat{\sigma}_e \frac{1}{n^d} \sum_{j=1}^{n-1} (n-j)^{d_y-2} \sup_{j \leq n} \left| \frac{S_j - V_j}{n^{1/2}} \right| \\ &= C \hat{\sigma}_e \frac{1}{n^d} \sum_{j=1}^{n-1} j^{d_y-2} \times O_p(n^{-\delta}), \end{aligned} \tag{51}$$

where C is a constant and $S_j = \sum_{t=1}^{j} e_t$ with $e_t \sim_{i.i.d.} \mathcal{N}(0,1)$. Note that $\sup_r \sum_{j=1}^{\phi-1} |\pi_{\phi-j}^{d_y-1}| \leq \sum_{j=1}^{n-1} (n-j)^{d_y-2}$ is obtained by applying Lemma 3-A-2 in Silveira (1991) and the last equality is due to Lemma 1.4 in the Online Supplement.

If $d=0$, $\sum_{j=1}^{n-1} j^{d_y-2} = \sum_{j=1}^{n-1} \frac{1}{j}$ diverges at the $\log n$ rate and is dominated by $O_p(n^{-\delta})$. If $d>0$, $\sum_{j=1}^{n-1} j^{d_y-2}$ diverges at the n^d rate and this divergence is neutralized by the factor $\frac{1}{n^d}$, so that the whole term in (51) is of order $O_p(n^{-\delta})$ in this case. We deduce that $Q_{2n}(r) = o_p(1)$ and $\frac{1}{n^{1/2+d}} y^*_{\lfloor nr \rfloor} \Rightarrow \sigma W^H(r)$. Then by repeated applications of the continuous mapping theorem (CMT) and analysis analogous to Lemma 1.2 in the Online Supplement and Theorem 3.2, we obtain $\widetilde{DF}_{\tau,HAR} \Rightarrow F_{r,d}$.

This result implies that the CDF of $\widetilde{DF}_{n,HAR}$ converges to the CDF of $F_{1,d}$ uniformly in probability. Therefore, $p^*(\widetilde{DF}_{n,HAR}) \Rightarrow U[0,1]$ under the null hypothesis and the proof of Theorem 3.3 is completed. ■

10.3 Proof of Theorem 4.1

From (44)

$$\hat{\Omega}_{HAR} = \frac{2}{b}\frac{n}{\lfloor nr \rfloor}\frac{1}{n}\sum_{i=1}^{\tau-1}\left(\frac{1}{\sqrt{\tau}}S_i\right)^2 - \frac{2}{b}\frac{n}{\lfloor nr \rfloor}\frac{1}{n}\sum_{i=1}^{\tau-\lfloor b\tau \rfloor-1}\left(\frac{1}{\sqrt{\tau}}S_i\right)\left(\frac{1}{\sqrt{\tau}}S_{i+\lfloor b\tau \rfloor}\right), \tag{52}$$

where $S_{\lfloor np \rfloor} = \sum_{i=1}^{\lfloor np \rfloor} \Delta y_i$. Write the partial sum $S_{\lfloor np \rfloor} = \sum_{i=1}^{\lfloor np \rfloor} \Delta y_i$ as

$$\begin{aligned} S_{\lfloor np \rfloor} &= \sum_{i=1}^{\tau_e-1} \Delta y_i + \sum_{i=\tau_e}^{\lfloor np \rfloor} \Delta y_i = \sum_{i=1}^{\tau_e-1} u_i + \frac{c}{n}\sum_{i=\tau_e}^{\lfloor np \rfloor} y_{i-1} + \sum_{i=\tau_e}^{\lfloor np \rfloor} u_i \\ &= \sum_{i=1}^{\lfloor np \rfloor} u_i + \frac{c}{n}\sum_{i=1}^{\lfloor np \rfloor} y_{i-1} - \frac{c}{n}\sum_{i=1}^{\tau_e-1} y_{i-1}. \end{aligned}$$

Upon normalization, we have

$$\begin{aligned} \frac{1}{n^{1/2+d}}\hat{S}_{\lfloor np \rfloor} &= \frac{1}{n^{1/2+d}}\sum_{i=1}^{\lfloor np \rfloor} u_i + \frac{c}{n^{3/2+d}}\sum_{i=1}^{\lfloor np \rfloor} y_{i-1} - \frac{c}{n^{3/2+d}}\sum_{i=1}^{\tau_e-1} y_{i-1} \\ &\Rightarrow \sigma\left(W^H(p) + cA_{p,d} - \int_0^{r_e} W^H(p)dp\right) := \sigma G_{r_e,c}(p). \end{aligned} \tag{53}$$

Thus, combining (52) and (53), as $n \to \infty$,

$$\frac{1}{n^{2d}}\hat{\Omega}_{HAR} \Rightarrow \frac{2\sigma^2}{br^2}\left(\int_0^r G_{r_e,c}(d,p)^2 dp - \int_0^{(1-b)r} G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp\right).$$

With Lemma 1.5 in the Online Supplement,

$$\begin{aligned} \tau^2 s^2_{\tau,HAR} &= \left(\frac{\tau}{n}\right)^2 \frac{\frac{1}{n^{2d}}\hat{\Omega}_{HAR}}{\frac{1}{n^{2+2d}}\left(\sum_{t=1}^{\tau} y_{t-1}^2 - \tau^{-1}\left(\sum_{t=1}^{\tau} y_{t-1}\right)^2\right)} \\ &\Rightarrow \frac{\frac{2}{b}\left(\int_0^r G_{r_e,c}(d,p)^2 dp - \int_0^{(1-b)r} G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp\right)}{B_{r,d} - \frac{1}{r}A^2_{r,d}}. \end{aligned} \tag{54}$$

Hence,

$$\begin{aligned}
\widetilde{DF}_{\tau,HAR} &= \frac{\lfloor nr \rfloor}{nr} \frac{nr\left(\tilde{\rho}_{\tau}-1\right)}{\left(\tau^2 s^2_{\tau,HAR}\right)^{1/2}} \\
&\Rightarrow \frac{\frac{\left(\frac{1}{2}C_{r,d}-\frac{1}{r}A_{r,d}W^H(r)\right)r}{B_{r,d}-\frac{1}{r}A^2_{r,d}}+cr}{\sqrt{\frac{\frac{2}{b}\left(\int_0^r G_{r_e,c}(d,p)^2dp-\int_0^{(1-b)r}G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp\right)}{B_{r,d}-\frac{1}{r}A^2_{r,d}}}} \\
&= \frac{\left(\frac{\left(\frac{1}{2}C_{r,d}-\frac{1}{r}A_{r,d}W^H(r)\right)r}{B_{r,d}-\frac{1}{r}A^2_{r,d}}+cr\right)\left(B_{r,d}-\frac{1}{r}A^2_{r,d}\right)^{1/2}}{\left[\frac{2}{b}\left(\int_0^r G_{r_e,c}(d,p)^2dp-\int_0^{(1-b)r}G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp\right)\right]^{1/2}},
\end{aligned}$$

where the limit holds by applying Lemma 1.5.6 in the Online Supplement and (54) since $\frac{\lfloor nr \rfloor}{nr} \to 1$. This completes the proof of Theorem 4.1. ■

10.4 Proofs of Theorems 4.2 and 5.1

These proofs are similar and are combined. Since the error u_t involves two memory parameters in non-explosive periods and the explosive period (viz., d_1 and d_2), we write u_t as $u_{t,d}$ when u_t is an $FI(d)$ process. Let $B = [\tau_e, \tau_f]$ be the bubble period and $N_0 \in [1, \tau_e)$ and $N_1 = [\tau_f + 1, n]$ be the normal market periods before and after the bubble period.

Recall $DF_\tau = \frac{\hat{\rho}_\tau - 1.}{s_\tau}$ and suppose that $\tau \in B$. Applying Lemma 1.9.1, 1.13.1 and 1.14 in the Online Supplement, we obtain

$$\frac{\hat{\rho}_\tau - 1}{s_\tau} = O_p(n^{1+\alpha/2})\frac{c}{n^\alpha} = O_p(n^{1-\alpha/2}). \tag{55}$$

This proves the first claim in Theorem 4.2.

Note that $\widetilde{DF}_{\tau,HAR} = \left(\frac{\sum_{i=1}^{\tau} \bar{y}^2_{i-1}}{\hat{\Omega}_{HAR}}\right)^{1/2}(\tilde{\rho}_\tau - 1)$. Suppose that $\tau \in B$. As in showing (55), we find that

$$\begin{aligned}
\left(\frac{\sum_{i=1}^{\tau} \bar{y}^2_{i-1}}{\hat{\Omega}_{HAR}}\right)^{1/2}(\tilde{\rho}_\tau - 1) &= O_p\left(\frac{n^{1+\alpha+2d_1}\rho_n^{2(\tau-\tau_e)}}{n^{2d_1}\rho_n^{2(\tau-\tau_e)}}\right)^{1/2}\frac{c}{n^\alpha} \\
&= O_p\left(n^{\frac{1-\alpha}{2}}\right) \to \infty,
\end{aligned}$$

which gives the second claim of Theorem 4.2.

Suppose that $\tau \in N_1$. Applying the results in Lemma 1.9.1, 1.13.2 and 1.14 in the Online Supplement, we have

$$\left(\frac{\sum_{i=1}^{\tau} \bar{y}^2_{i-1}}{\hat{\Omega}_{HAR}}\right)^{1/2}(\tilde{\rho}_\tau - 1) = O_p\left(\frac{n^{1+\alpha+2d_1}\rho_n^{2(\tau_f-\tau_e)}}{n^{2d_1}\rho_n^{2(\tau-\tau_e)}}\right)^{1/2}\left(-\frac{c}{n^\alpha}\right) = -O_p\left(n^{\frac{1-\alpha}{2}}\right) \to -\infty.$$

To show $\hat{r}_e^{HAR} \xrightarrow{p} r_e$ and $\hat{r}_f^{HAR} \xrightarrow{p} r_f$, note that if $\tau \in N_0$,

$$\lim_{n\to\infty} \Pr(\widetilde{DF}_{\tau,HAR} > cv_{n,HAR}) = \Pr\left(F_{r,d} > \infty\right) = 0.$$

If $\tau \in B$, $\lim_{n\to\infty} \Pr(\widetilde{DF}_{\tau,HAR} > cv_{n,HAR}) = 1$, given that $\frac{cv_{n,HAR}}{n^{(1-\alpha)/2}} \to 0$. If $\tau \in N_1$, $\lim_{n\to\infty} \Pr(\widetilde{DF}_{\tau,HAR} > cv_{n,HAR}) = 0$, as $\widetilde{DF}_{\tau,HAR} = -O_p\left(n^{\frac{1-\alpha}{2}}\right)$. It follows that, for any $\eta, \vartheta > 0$, we have

$$\Pr(\hat{r}_e^{HAR} > r_e + \eta) \to 0, \text{ and } \Pr(\hat{r}_f^{HAR} < r_f + \vartheta) \to 0,$$

due to the fact that $\Pr(\widetilde{DF}_{(\tau_e+\alpha_\eta/n),HAR} > r_e + \eta) \to 1$ for all $0 < \alpha_\eta < \eta$ and $\Pr(\widetilde{DF}_{(\tau_f-\alpha_\vartheta/n),HAR} > cv_{n,HAR}) \to 1$ for all $0 < \alpha_\vartheta < \vartheta$. As η and ϑ are arbitrary and $\Pr(\hat{r}_e^{HAR} < r_e) \to 0$ and $\Pr(\hat{r}_f^{HAR} > r_f) \to 0$, we deduce that $\Pr(\left|\hat{r}_e^{HAR} - r_e\right| > \eta) \to 0$ and $\Pr(\left|\hat{r}_f^{HAR} - r_f\right| > \vartheta) \to 0$ as $n \to \infty$, provided that

$$\frac{1}{cv_{n,HAR}} + \frac{cv_{n,HAR}}{n^{(1-\alpha)/2}} \to 0.$$

This completes the proof of Theorem 5.1. ■

10.5 Proof of Theorem 6.2

We shall only prove that under the assumptions in Theorem 6.2, we have

$$\frac{1}{n^{1/2+d}} x_{\lfloor ns \rfloor} \Rightarrow W^H(s), \tag{56}$$

as when (56) holds we can use the steps in proving Theorem 3.1 to establish the claim in Theorem 6.2. We first show the following two results which will be useful in establishing (56).

Letting $m = n^\gamma$, we have

$$\sup_{1\le t\le n} \left|\Delta_+^{1+\hat{d}} y_t - \Delta_+^{1+d} y_t\right| = O_p(m^{-1}\ln n), \tag{57}$$

and

$$\max_{1\le t\le n} \left|\widehat{g}^2(t/n) - g^2(t/n)\right| = o_p(1). \tag{58}$$

Set $\xi_n = \hat{d} - d$ and $z_t = \Delta_+^{1+d} y_t = g(t/n)\varepsilon_t$. To show (57), note that

$$\Delta_+^{1+\hat{d}} y_t = \Delta_+^{\hat{d}-d}\left(\Delta_+^{1+d} y_t\right) = \Delta_+^{\xi_n} z_t, \text{ and}$$

$$\Delta_+^{\xi_n} z_t = \sum_{k=0}^{t-1} \binom{\xi_n}{k} (-L)^k z_t = z_t - \xi_n \left(\sum_{k=1}^{t-1} \frac{z_{t-k}}{k}\right) + O_p(\xi_n^2). \tag{59}$$

Therefore

$$\begin{aligned} \sup_{1\le t\le n} \left|\Delta_+^{1+\hat{d}} y_t - \Delta_+^{1+d} y_t\right| &= \sup_{1\le t\le n} \left|-\xi_n \left(\sum_{k=1}^{t-1} \frac{z_{t-k}}{k}\right) + O_p(\xi_n^2)\right| \\ &\le |\xi_n| \sup_{1\le t\le n} \left|\sum_{k=1}^{t-1} \frac{z_{t-k}}{k}\right| + O_p(\xi_n^2). \end{aligned} \tag{60}$$

Note that ξ_n is dependent on n but not t. Also, for any $1 \le t \le n$, by Chebyshev's inequality, we have, for any $\delta > 0$ and some positive constant $C < \infty$,

$$Pr\left(\left|\sum_{k=1}^{t-1} \frac{z_{t-k}}{k}\right| \geq \delta\right) \leq \sup_{1 \leq t \leq n} g\left(\frac{t}{n}\right)^2 \sigma^2 \frac{\sum_{k=1}^{t-1} \frac{1}{k^2}}{\delta^2} < C \frac{\sum_{t=1}^{\infty} \frac{1}{k^2}}{\delta^2} = C \frac{\pi}{6} \frac{1}{\delta^2},$$

so that $\left|\sum_{k=1}^{t-1} \frac{z_{t-k}}{k}\right| = O_p(1)$ for all $1 \le t \le n$, and then (60) and $\xi_n \xrightarrow{p} 0$ give (57).

To show (58), note that

$$\begin{aligned} \widehat{g}^2\left(\frac{t}{n}\right) &= \sum_{j=1}^{\tau} k_{tj} \left(\Delta_+^{1+\hat{d}} y_j\right)^2 = \sum_{j=1}^{\tau} k_{tj} \left[g(j/n)\varepsilon_j + R_n\right]^2 \\ &= \sum_{j=1}^{\tau} k_{tj} g^2(j/n)\varepsilon_j^2 + 2R_n \sum_{j=1}^{\tau} k_{tj} g(j/n)\varepsilon_j + R_n^2, \end{aligned} \tag{61}$$

where R_n is $O_p(m^{-1} \ln n)$, as indicated by (57).

We now show that the second term in (61) is $o_p(1)$. Note that

$$\sum_{j=1}^{\tau} k_{tj} g(j/n)\varepsilon_j = \frac{\sum_{j=1}^{\tau} K\left(\frac{t-j}{\nu}\right) g(j/n)\varepsilon_j}{\sum_{i=1}^{\tau} K\left(\frac{t-i}{\nu}\right)}. \tag{62}$$

First consider the numerator of (62). As in Theorem 2.8 of Pagan and Ullah (2006), write

$$\begin{aligned} \sum_{j=1}^{\tau} K\left(\frac{t-j}{\nu}\right) g(j/n)\varepsilon_j &= \frac{1}{2\pi} \sum_{j=1}^{\tau} \int \exp\left(-iv\left(\frac{t-j}{\nu}\right)\right) g(j/n)\varepsilon_j \phi(v) dv \\ &= \frac{1}{2\pi} \int \sum_{j=1}^{\tau} \exp\left(\frac{ivj}{\nu}\right) g(j/n)\varepsilon_j \phi(v) \exp\left(\frac{-ivt}{\nu}\right) dv \\ &= \frac{\nu}{2\pi} \int \sum_{j=1}^{\tau} \exp(ixj)\, g(j/n)\varepsilon_j \phi(\nu x) \exp(-ixt)\, dx, \end{aligned}$$

where $\phi(\cdot)$ is the characteristic function of K and we let $v = \nu x$ to obtain the third equality. Thus,

$$\begin{aligned} \max_{t<\tau} \left|\sum_{j=1}^{\tau} K\left(\frac{t-j}{\nu}\right) g(j/n)\varepsilon_j\right| &= \max_{t<\tau} \left|\frac{\nu}{2\pi} \int \sum_{j=1}^{\tau} \exp(ixj)\, g(j/n)\varepsilon_j \phi(\nu x) \exp(-ixt)\, dx\right| \\ &\leq \frac{\nu}{2\pi} \int \left|\sum_{j=1}^{\tau} \exp(ixj)\, g(j/n)\varepsilon_j\right| \left(\max_{t<\tau} |\exp(-ixt)|\right) |\phi(\nu x)|\, dx \\ &\leq \frac{\nu}{2\pi} \int \left|\sum_{j=1}^{\tau} \exp(ixj)\, g(j/n)\varepsilon_j\right| |\phi(\nu x)|\, dx. \end{aligned}$$

Note that

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{\tau} \exp(ixj)\, g(j/n)\varepsilon_j = \frac{1}{\sqrt{n}} \sum_{j=1}^{\tau} \cos(xj) g(j/n)\varepsilon_j + i \frac{1}{\sqrt{n}} \sum_{j=1}^{\tau} \sin(xj) g(j/n)\varepsilon_j$$

$$= \quad O_p(1),$$

which implies $\sum_{j=1}^{\tau} \exp(ixj)\, g(j/n)\varepsilon_j = O_p(\sqrt{n})$. Therefore,

$$\max_{t<\tau} \left| \sum_{j=1}^{\tau} K\left(\frac{t-j}{\nu}\right) g(j/n)\varepsilon_j \right| \leq \frac{\nu}{2\pi} O_p(\sqrt{n}) \int |\phi(\nu x)|\, dx = O_p(\sqrt{n}).$$

Note that the denominator in (62) is $O(\nu)$. Hence, the second term in (61) is

$$R_n \sum_{j=1}^{t} k_{tj} g(j/n)\varepsilon_j = O_p(m^{-1}\ln n)\frac{O_p(\sqrt{n})}{O(\nu)} = O_p\left(\frac{n^{1/2-\gamma}}{\nu}\ln n\right) = O_p\left(\left(\frac{n}{\nu^2}\right)^{1/2}\frac{\ln n}{n^\gamma}\right) = o_p(1).$$

For the first term in (61), given the rate condition of ν and the kernel function of $K(\cdot)$, Lemma 1 in Astill et al. (2023) shows $\max_t \left|\sum_{i=1}^{\tau} k_{ti} g^2(i/n)\varepsilon_i^2 - g^2(t/n)\right| = o_p(1)$. This implies that

$$\max_{1\leq t\leq n} \left|\widehat{g}^2(t/n) - g^2(t/n)\right| = o_p(1). \tag{63}$$

Since $\Delta_+^{1+\hat{d}} y_t = g(t/n)\varepsilon_t + O_p(m^{-1})$, letting $R_n = O_p(m^{-1}\ln n)$ we have

$$\begin{aligned}
&\frac{1}{n^{1/2+d}} \sum_{s=1}^{t} \Delta_+^{-\hat{d}} \left(\frac{\Delta_+^{1+\hat{d}} y_s}{\widehat{g}(s/n)}\right) \\
= \;& \frac{1}{n^{1/2+d}} \sum_{s=1}^{t} \Delta_+^{-\hat{d}} \left[\frac{g(s/n)\varepsilon_s}{\widehat{g}(s/n)} + \frac{R_n}{\widehat{g}(s/n)}\right] \\
= \;& \frac{1}{n^{1/2+d}} \sum_{s=1}^{t} \Delta_+^{-\hat{d}} \frac{g(s/n)}{\widehat{g}(s/n)}\varepsilon_s + \frac{1}{n^{1/2+d}} \sum_{s=1}^{t} \Delta_+^{-\hat{d}} \frac{R_n}{\widehat{g}(s/n)}.
\end{aligned} \tag{64}$$

Equation (63) implies that second term of (64) is $O_p(n^{-1/2-\gamma})$. By Stirling's approximation, for large enough S there is a constant C such that, as $n \to \infty$,

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} R_n \leq \frac{1}{n^{1/2+d}} C + R_n \frac{C}{n^{1/2+d}} \sum_{s=S}^{\lfloor nr \rfloor} s^{d-1} \leq R_n \frac{C}{n^{1/2+d}} \int_0^n s^{d-1} ds = \frac{R_n C}{n^{1/2}} \to 0.$$

Further, (63) implies that $\widehat{g}^2(t/n) = O_p(1)$ for all t. Therefore, the second term of (64) is $O_p(n^{-1/2-\gamma})$.

For the first term in (64), let $t = \lfloor nr \rfloor$and note that

$$\begin{aligned}
&\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \frac{g(s/n)}{\widehat{g}(s/n)}\varepsilon_s \\
= \;& \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \left[\left(\frac{g(s/n) - \widehat{g}(s/n)}{\widehat{g}(s/n)}\right) + 1\right]\varepsilon_s \\
= \;& \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \varepsilon_s + \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \left(\frac{g(s/n) - \widehat{g}(s/n)}{\widehat{g}(s/n)}\right)\varepsilon_s.
\end{aligned} \tag{65}$$

We shall prove

$$\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-\hat{d}}\varepsilon_s \Rightarrow W^H(r) \text{ and} \tag{66}$$

$$\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-\hat{d}}\left(\frac{g(s/n)-\widehat{g}(s/n)}{\widehat{g}(s/n)}\right)\varepsilon_s = o_p(1). \tag{67}$$

Since

$$\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-\hat{d}}\varepsilon_s = \frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-\hat{d}+d}\left(\Delta_+^{-d}\varepsilon_s\right),$$

using the same technique as in (59), we have

$$\begin{aligned}
&\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-\hat{d}+d}\left(\Delta_+^{-d}\varepsilon_s\right)\\
=\ &\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\left(\Delta_+^{-d}\varepsilon_s+\xi_n\left(\sum_{j=1}^{s-1}\frac{\Delta_+^{-d}\varepsilon_{s-j}}{j}\right)\right)+\frac{\lfloor nr\rfloor}{n^{1/2+d}}O_p(\xi_n^2)\\
=\ &\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}\varepsilon_s+\frac{\xi_n}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\sum_{j=1}^{s-1}\frac{\Delta_+^{-d}\varepsilon_{s-j}}{j}+o_p(1),
\end{aligned} \tag{68}$$

where $\frac{\lfloor nr\rfloor}{n^{1/2+d}}O_p(\xi_n^2) = O_p(n^{-(1/2+d-1+2\gamma)}) = o_p(1)$ because $\gamma > 1/4$.

For the first term in (68), by Lemma 1.1,

$$\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}\varepsilon_s \Rightarrow W^H(r).$$

For the second term in (68), suppose that $d \geq 0$. In particular, to simplify notation let $u_s = \Delta_+^{-d}\varepsilon_s$. Then

$$\begin{aligned}
&\frac{\xi_n}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\sum_{j=1}^{s-1}\frac{\Delta_+^{-d}\varepsilon_{s-j}}{j} = \frac{\xi_n}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\sum_{j=1}^{s-1}\frac{u_{s-j}}{j} = \frac{\xi_n}{n^{1/2+d}}\sum_{j=1}^{\lfloor nr\rfloor-1}\frac{1}{j}\sum_{s=1}^{\lfloor nr\rfloor-j}u_s\\
&= \xi_n\sum_{j=1}^{\lfloor nr\rfloor-1}\frac{1}{j}\left(\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor-j}u_s\right) = \xi_n\sum_{j=1}^{\lfloor nr\rfloor-1}\frac{1}{j}O_p(1) = \xi_n\sum_{j=1}^{\lfloor nr\rfloor-1}O_p\left(\frac{1}{j}\right)\\
&= O_p\left(\xi_n\ln n\right) = o_p(1).
\end{aligned}$$

Therefore, $\frac{\xi_n}{n^{1/2}}\sum_{s=1}^{\lfloor nr\rfloor}\sum_{j=0}^{s-1}\frac{\varepsilon_{s-j}}{j} = o_p(1)$ and (66) is established.

To show (67), note that

$$\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-\hat{d}}\left(\frac{g(s/n)-\widehat{g}(s/n)}{\widehat{g}(s/n)}\right)\varepsilon_s = \frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}\left(\frac{g(s/n)-\widehat{g}(s/n)}{\widehat{g}(s/n)}\right)\varepsilon_s + o_p(1),$$

and that

$$\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}\left(\frac{g(s/n)-\widehat{g}(s/n)}{\widehat{g}(s/n)}\right)\varepsilon_s=\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\sum_{j=0}^{s-1}\frac{(d)_j}{j!}\left(\frac{g(j/n)-\widehat{g}(j/n)}{\widehat{g}(j/n)}\right)\epsilon_{s-j}$$

$$\le \max_j\left|\frac{g(j/n)-\widehat{g}(j/n)}{\widehat{g}(j/n)}\right|\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\sum_{j=0}^{s-1}\frac{(d)_j}{j!}||\epsilon_{s-j}|$$

$$=\max_j\left|\frac{g(j/n)-\widehat{g}(j/n)}{\widehat{g}(j/n)}\right|\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}|\epsilon_s|$$

$$=o_p(1)\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}|\epsilon_s|$$

$$=o_p(1)\left[\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}\left(|\varepsilon_s|-\mathbb{E}\,|\varepsilon_s|\right)+\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}\mathbb{E}\,|\varepsilon_s|\right]. \tag{69}$$

By Lemma 1.1,

$$\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}\left(|\varepsilon_s|-\mathbb{E}\,|\varepsilon_s|\right)=O_p(1).$$

And for $\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-d}\mathbb{E}\,|\varepsilon_s|$ in (69), there exists a bound for which $\frac{C}{n^{1/2+d}}\frac{1}{\Gamma(d)}\sum_{s=1}^{\lfloor nr\rfloor}s^{d-1}<\frac{C}{n^{1/2+d}}\frac{1}{\Gamma(d)}n^d\to 0$, with a positive constant C. This implies $\frac{1}{n^{1/2+d}}\sum_{s=1}^{\lfloor nr\rfloor}\Delta_+^{-\hat{d}}\left(\frac{g(s/n)-\widehat{g}(s/n)}{\widehat{g}(s/n)}\right)\varepsilon_s=o_p(1)$.

Combining (65), (66) and (67), we have $\frac{1}{n^{1/2+d}}x_{\lfloor ns\rfloor}\Rightarrow W^H(s)$. The limit theorem in (38) follows in a straightforward way.

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