Online Supplement to: 'Robust Testing for Explosive Behavior with Strongly Dependent Errors'*

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November 8, 2023

This supplementary document contains all the lemmas used to prove the main results in the paper (Section 1), proofs of these lemmas and the lemmas in the main paper (Section 2), proofs and discussion of various remarks in the main paper (Section 3), additional empirical findings (Section 4), additional simulations (Sections 5 and 6), and additional analytical results (Section 7).

1 Useful lemmas

The following two lemmas are useful throughout the proofs of the main results in the paper.

Lemma 1.1 (Theorem 3.1 in Silveira (1991)) *Suppose* $u_t = \Delta_+^{-d}\epsilon_t$, $\epsilon_t \overset{iid}{\sim} (0,\sigma^2)$ $\mathbb{E}|\epsilon_1|^{2+\delta} < \infty$, *for some* $\delta > 0$ *and* $d > -0.5$. *Then, as* $n \to \infty$,

$$\frac{1}{\sigma n^{1/2+d}} \sum_{t=1}^{\lfloor nr \rfloor} u_t \Rightarrow W^H(r), \tag{A.1.1}$$

in $\mathcal{D}[0,1]$ *with the uniform metric.*

*We thank Torben Andersen (the co-editor), an Associate Editor, two referees, Yichong Zhang for useful comments. Yiu Lim Lui, The Institute for Advanced Economic Research, Dongbei University of Finance and Economics, Dalian, 116025, Liaoning, China, Email: luiyiulim@outlook.com. Peter C. B. Phillips, Cowles Foundation for Research in Economics, Yale University. Email: peter.phillips@yale.edu. Jun Yu, Department of Finance and Business Economics, Faculty of Business Administration, University of Macau, Avenida da Universidade, Macau, China. Email: junyu@um.edu.mo. Lui acknowledges financial support from the school level grant (No. DUFE202143) of Dongbei University of Finance and Economics. Phillips acknowledges research support from the NSF under Grant No. SES 18-50860 and the Kelly Foundation at the University of Auckland. Yu acknowledges research/project support from the Ministry of Education, Singapore, under its Academic Research Fund (AcRF) Tier 2 (Award Number MOE-T2EP402A20-0002).

Lemma 1.2 *Suppose the DGP is given by model (12). Let* $\tau = \lfloor nr \rfloor$ *with* $r \in (0,1]$. *Then, as* $n \to \infty$,

$$\frac{1}{n^{1+2d}} \sum_{t=1}^{\tau} y_{t-1} u_t \Rightarrow \begin{cases} \frac{\sigma^2}{2}\left[W^2(r) - r\right] & \text{if } d = 0 \\ \frac{\sigma^2}{2}\left(W^H(r)\right)^2 & \text{if } d > 0 \end{cases}, \tag{A.1.2}$$

$$\frac{1}{n^{3/2+d}} \sum_{t=1}^{\tau} y_{t-1} \Rightarrow \sigma \int_0^r W^H(s) ds, \tag{A.1.3}$$

$$\frac{1}{n^{2+2d}} \sum_{t=1}^{\tau} y_{t-1}^2 \Rightarrow \sigma^2 \int_0^r \left(W^H(s)\right)^2 ds. \tag{A.1.4}$$

Suppose the empirical regression (8) is based on $\{y_t\}_{t=1}^{\tau}$. *For* $r \in (0,1]$, *as* $n \to \infty$, *we have*

$$\tau(\hat{\rho}_\tau - 1) \Rightarrow \begin{cases} \frac{\frac{r}{2}\left((W(r))^2 - r\right)^2 - \left(\int_0^r W(s)ds\right) W(r)}{\int_0^r \left(\tilde{W}(s)\right)^2 ds} & \text{if } d = 0, \\ \frac{\frac{r}{2}\left(W^H(r)\right)^2 - \left(\int_0^r W^H(s)ds\right) W^H(r)}{\int_0^r \left(\tilde{W}^H(s)\right)^2 ds} & \text{if } d \in (0, 0.5) \end{cases}. \tag{A.1.5}$$

Furthermore, let $\tilde{\rho}_\tau = \hat{\rho}_\tau + \frac{\frac{1}{2}\sum_{t=1}^{\tau} \triangle y_t^2}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2}$. *We have*

$$\tau(\tilde{\rho}_\tau - 1) \Rightarrow \frac{\frac{r}{2}\left(W^H(r)\right)^2 - \left(\int_0^r W^H(s)ds\right) W^H(r)}{\int_0^r \left(\tilde{W}^H(s)\right)^2 ds}, \text{ for } d \in [0, 0.5). \tag{A.1.6}$$

The following lemmas are useful in proving Theorem 3.3.

Lemma 1.3 *Under the assumptions of Theorem 3.3, as* $n \to \infty$ *with* $m = n^\gamma$, *we have*

$$\sup_{1 \le t \le n} \left| \Delta_+^{1+\hat{d}} y_t - \Delta_+^{1+d} y_t \right| = O_p(m^{-1} \log n). \tag{A.1.7}$$

Lemma 1.4 *[Lemma 12 in Mikusheva (2007)] Let* $\{\epsilon_{n,j}^* : j = 1, ..., n\}$ *be a triangular array of random variables, such that for every* n, $\{\epsilon_{n,j}^*\}_{j=1}^n$ *are i.i.d. with a cumulative distribution function (CDF)* $F_n \in \mathcal{L}_r(K, M, \theta)$. *Then we can construct a process* $\eta_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nr \rfloor} \epsilon_{n,j}^*$ *and a Brownian motion* w_n *on a common probability space such that for every* $\varepsilon > 0$

$$\lim_{n \to \infty} \sup_{F_n \in \mathcal{L}_r(K,M,\theta)} \Pr\left\{ \sup_{0 \le t \le 1} |\eta_n(t) - w_n(t)| > \varepsilon n^{-\delta} \right\} = 0, \tag{A.1.8}$$

for some $\delta > 0$, *where* $\mathcal{L}_r(K, M, \theta)$ *is a class which satisfies the following three conditions:*

1. $\mu_1(F_n) = 0$;
2. $\mu_2(F_n) = \sigma_n^2$, *where* $|\sigma_n^2 - 1| \le M n^{-\theta}$, *with* $\theta > 0$;

3. $\sup_n |\mu|_r(F_n) < K$.

where $\mu_j(F)$ *and* $|\mu|_j(F)$ *are the* j*th central and absolute moments of* F*, respectively, and* M *and* K *are positive constants.*

The results in Lemma 1.4 use an expanded common probability space in which a weakly convergent sequence can be represented by a sequence that converges almost surely via the Skorohod representation (see, e.g. Pollard (1984)). Throughout the proof of Theorem 3.3, random sequences are assumed to belong to this common probability space.

The following lemma is useful in proving Theorem 4.1.

Lemma 1.5 *Let* $\tau = \lfloor nr \rfloor$ *with* $r \in (r_e, 1]$. *Then, under the local alternative model (25), as* $n \to \infty$.

1. $\frac{1}{n^{1/2+d}} y_\tau \Rightarrow \sigma \left(e^{(r-r_e)c} W^H(r_e) + \int_{r_e}^{r} e^{(r-s)c} dW^H(s) \right)$;
2. $\frac{1}{n^{3/2+d}} \sum_{t=1}^{\tau} y_{t-1} \Rightarrow \sigma A_{r,d}$;
3. $\frac{1}{n^{2+2d}} \sum_{t=1}^{\tau} y_{t-1}^2 \Rightarrow \sigma^2 B_{r,d}$;
4. $\frac{1}{n^{1+2d}} \left(\sum_{t=1}^{\tau} y_{t-1} u_t + \frac{1}{2} \sum_{t=1}^{\tau} \Delta y_t^2 \right) \Rightarrow \frac{\sigma^2}{2} C_{r,d}$;
5. $n(\tilde{\rho}_\tau - \rho_c) \Rightarrow X_c(r,d)$;
6. $n(\tilde{\rho}_\tau - 1) \Rightarrow X_c(r,d) + c$,

where

$$\begin{aligned}
A_{r,d} &= \int_0^r \left(e^{(x-r_e)c} W^H(r_e) + \int_{r_e}^{x} e^{(x-s)c} dW^H(s) \right) dx, \\
B_{r,d} &= \int_0^r \left(e^{(x-r_e)c} W^H(r_e) + \int_{r_e}^{x} e^{(x-s)c} dW^H(s) \right)^2 dx, \\
C_{r,d} &= \left(e^{(r-r_e)c} W^H(r_e) + \int_{r_e}^{r} e^{(r-s)c)} dW^H(s) \right)^2 - W^H(r_e)^2, \\
X_c(r,d) &= \frac{\frac{1}{2} C_{r,d} - \frac{1}{r} A_{r,d} W^H(r)}{B_{r,d} - \frac{1}{r} A_{r,d}^2}, \\
Y_c(r,d) &= \frac{B_{r,d} W^H(r) - \frac{1}{2} C_{r,d} A_{r,d}}{r \left(B_{r,d} - A_{r,d}^2 \right)}.
\end{aligned}$$

The lemmas below assist in proving Theorems 4.2 and 5.1.

Lemma 1.6 *Let* $B = [\tau_e, \tau_f]$ *be the bubble period,* $N_0 \in [1, \tau_e)$ *and* $N_1 = [\tau_f + 1, n]$ *be the normal market periods before and after the bubble period. Under the DGP (29), with* $t = \lfloor nr \rfloor$, *we have the following asymptotic approximations:*

1. *For* $t \in N_0$, $y_t \overset{a}{\sim} n^{1/2+d_1} \sigma W^{H_1}(r)$.
2. *For* $t \in B$, $y_t \overset{a}{\sim} \rho_n^{(t-\tau_e)} n^{1/2+d_1} \sigma W^{H_1}(r_e)$.

3. *For* $t \in N_1$, $y_{\lfloor nr \rfloor} \overset{a}{\sim} n^{1/2+d_1}\left[\sigma\left(W^{H_1}(r) - W^{H_1}(r_f)\right) + \sigma W^{H_1}(r_e)\right]$, *where* $W^H(r)$ *is a Type II fBM with the Hurst parameter* $H = 1/2 + d$.

Lemma 1.7 *For the sample average,*

1. *For* $\tau \in B$, $\frac{1}{\tau}\sum_{j=1}^{\tau} y_j \overset{a}{\sim} n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)$.
2. *For* $\tau \in N_1$, $\frac{1}{\tau}\sum_{j=1}^{\tau} y_j \overset{a}{\sim} n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)$.

Lemma 1.8 *Let* $\bar{y}_t = y_t - \frac{1}{\tau}\sum_{j=1}^{\tau} y_{j-1}$.

1. *For* $\tau \in B$, *if* $t \in N_0$,

$$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e); \tag{A.1.9}$$

if $t \in B$,

$$\bar{y}_t \overset{a}{\sim} \left(\rho_n^{(t-\tau_e)} - \frac{n^\alpha}{nrc}\rho_n^{\tau-\tau_e}\right) n^{1/2+d_1}\sigma W^{H_1}(r_e). \tag{A.1.10}$$

2. *For* $\tau \in N_1$, *if* $t \in N_0$,

$$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e), \tag{A.1.11}$$

if $t \in B$,

$$\bar{y}_t \overset{a}{\sim} \left(\rho_n^{(t-\tau_e)} - \frac{n^\alpha}{nrc}\rho_n^{\tau_f-\tau_e}\right) n^{1/2+d_1}\sigma W^{H_1}(r_e), \tag{A.1.12}$$

if $t \in N_1$,

$$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e). \tag{A.1.13}$$

Lemma 1.9 *The sample variance terms involving* $\bar{y}_t$ *behave as follows.*

1. *If* $\tau \in B$,

$$\sum_{j=1}^{\tau}\bar{y}_{j-1}^2 \overset{a}{\sim} n^{1+2d_1+\alpha}\frac{\rho_n^{2(\tau-\tau_e)}}{2c}\sigma^2 W^{H_1}(r_e)^2. \tag{A.1.14}$$

2. *If* $\tau \in N_1$,

$$\sum_{j=1}^{\tau}\bar{y}_{j-1}^2 \overset{a}{\sim} n^{1+\alpha+2d_1}\frac{\rho_n^{2(\tau_f-\tau_e)}}{2c}\sigma^2 W^{H_1}(r_e)^2. \tag{A.1.15}$$

Lemma 1.10 *The sample variances of* $\bar{y}_t$ *and* u_t *behave as follows:*

1. *For* $\tau \in B$,

$$\sum_{j=1}^{\tau}\bar{y}_{j-1}u_j = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_1)}{2}+d_1,\alpha+2d_1\right\}}\right). \tag{A.1.16}$$

2. *For* $\tau \in N_1$,

$$\sum_{j=1}^{\tau} \bar{y}_{j-1} u_j = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1, \alpha+2d_1\right\}}\right), \tag{A.1.17}$$

where

$$f(d_2) = \begin{cases} 1 & \text{if } \ d_2 \in [0, 0.5) \\ 1+\epsilon & \text{if } \ d_2 = 0.5 \\ 2d_2 & \text{if } \ d_2 > 0.5 \end{cases}, \epsilon > 0.$$

Lemma 1.11 *The sample covariances of* $\bar{y}_{j-1}$ *and* $y_j - \rho_n y_{j-1}$ *behave as follows:*

1. *For* $\tau \in B$,

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1, 2d_1+1\right\}}\right). \tag{A.1.18}$$

2. *For* $\tau \in N_1$,

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) \overset{a}{\sim} -\rho_n^{2\left(\tau_f-\tau_e\right)} n^{1+2d_1} \sigma^2 \left(W^{H_1}(r_e)\right)^2.$$

Lemma 1.12 *For* $\sum_{t=1}^{\tau} \Delta y_t^2$*, the following asymptotics apply:*

1. *When* $\tau \in B$,

$$\sum_{t=1}^{\tau} \Delta y_t^2 = O_p(n^{1+2d_1-\alpha} \rho_n^{2(\tau_f-\tau_e)}). \tag{A.1.19}$$

2. *When* $\tau \in N_1$,

$$\sum_{t=1}^{\tau} \Delta y_t^2 = O_p(n^{1+2d_1} \rho_n^{2(\tau_f-\tau_e)}). \tag{A.1.20}$$

Lemma 1.13 *For the LS estimator* $\hat{\rho}_\tau$*, the following asymptotics hold:*

1. *When* $\tau \in B$, $n\left(\tilde{\rho}_\tau - 1\right) = n^{1-\alpha} c + o_p(1) \xrightarrow{p} \infty$.
2. *When* $\tau \in N_1$, $n\left(\tilde{\rho}_\tau - 1\right) = -n^{1-\alpha} c + o_p(1) \xrightarrow{p} -\infty$.

Lemma 1.14 *Under model (29), we have the following asymptotics*

$$\hat{\Omega}_{HAR} = O_p\left(n^{2d_1} \rho_n^{2(\tau-\tau_e)}\right),$$

$$s_\tau^2 = \frac{\frac{1}{\tau}\sum_{t=1}^{\tau} \hat{u}_t^2}{\frac{1}{\tau}\sum_{t=1}^{\tau} \bar{y}_{t-1}^2} = O_p(n^{-2-\alpha}), \ \textit{for } \tau \in B.$$

2 Proofs of the lemmas

Proof of Lemma 1.2.

When $d = 0$, the error term is an i.i.d. process and the results in (A.1.2), (A.1.3), (A.1.4) and (A.1.5) are well known in the literature. Only the claims for $d > 0$ need proving. For the first claim, since $\sum_{t=1}^{\tau} y_{t-1}u_t = \frac{1}{2}\left(y_{\lfloor nr \rfloor}^2 - y_0^2 - \sum_{t=1}^{\tau} u_t^2\right)$, we have

$$n^{-1-2d}\sum_{t=1}^{\tau} y_{t-1}u_t = \frac{1}{2}\left[\left(n^{-1/2-d}y_{\lfloor nr \rfloor}\right)^2 - \frac{1}{n^{2d}}\left(\frac{1}{n}\sum_{t=1}^{\tau} u_t^2\right)\right] + o_p(1) \Rightarrow \frac{\sigma^2}{2}\left(W^H(r)\right)^2,$$

where the last step is due to $n^{-1/2-d}y_{\lfloor nr \rfloor} \Rightarrow \sigma W^H(r)$ (from Lemma 1.1) and $\frac{1}{n^{2d}}\left(\frac{1}{n}\sum_{t=1}^{\tau} u_t^2\right) \xrightarrow{p} 0$. To see why convergence in probability applies, note that for $d \in (0, 0.5)$ we have $\sum_{t=1}^{\tau} u_t^2 = O_p(n)$; when $d = 0.5$, $\sum_{t=1}^{\tau} u_t^2 = O_p(n(\ln(n))^2)$ (see Duffy and Kasparis (2021)); and when $d > 0.5$, Lemma 1.1 implies

$$\frac{u_t}{n^{1/2+d-1}} \Rightarrow \sigma W^H(r), \text{ for } t = \lfloor nr \rfloor,$$

so that $\frac{u_t}{n^{1/2+d-1}} = O_p(1)$. By continuous mapping theorem, $\frac{1}{n}\sum_{t=1}^{\tau}\left(\frac{u_t}{n^{1/2+d-1}}\right)^2 = O_p(1)$, and $\sum_{t=1}^{\tau} u_t^2 = O_p(n^{2d})$. Hence, $\frac{1}{n^{2d}}\left(\frac{1}{n}\sum_{t=1}^{\tau} u_t^2\right) \xrightarrow{p} 0$ for any $d > 0$, and we obtain (A.1.2).

For the second claim, since $\sum_{t=1}^{\tau} y_{t-1} = \sum_{t=1}^{\tau}\left(\sum_{i=1}^{t-1} u_i + y_0\right)$, applying Lemma 1.1 and continuous mapping gives

$$\frac{1}{n^{3/2+d}}\sum_{t=1}^{\tau} y_{t-1} = \frac{1}{n}\sum_{t=1}^{\tau}\left(\frac{1}{n^{1/2+d}}\sum_{i=1}^{t-1} u_i\right) + o_p(1) \Rightarrow \sigma\int_0^r W^H(s)ds.$$

Applying similar arguments, the third claim follows as

$$\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau} y_{t-1}^2 = \frac{1}{n}\sum_{t=1}^{\tau}\left(\frac{1}{n^{1/2+d}}\sum_{i=1}^{\tau-1} u_i\right)^2 + o_p(1)\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau} y_{t-1}^2 \Rightarrow \sigma^2\int_0^r \left(W^H(s)\right)^2 ds.$$

For the normalized centered LS estimator $\tau(\hat{\rho}_\tau - 1)$, we have

$$\begin{aligned}
\tau(\hat{\rho}_\tau - 1) &= \frac{\tau}{n}\frac{n^{-1-2d}}{n^{-2-2d}}\left[\frac{\sum_{t=1}^{\tau} y_{t-1}u_t - \frac{1}{\tau}\sum_{t=1}^{\tau} y_{t-1}\sum_{t=1}^{\tau} u_t}{\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{1}{\tau}\left(\sum_{t=1}^{\tau} y_{t-1}\right)^2}\right] \\
&= \frac{\tau}{n}\frac{n^{-1-2d}\sum_{t=1}^{\tau} y_{t-1}u_t - \frac{n}{\tau}n^{-3/2-d}\sum_{t=1}^{\tau} y_{t-1}n^{-1/2-d}\sum_{t=1}^{\tau} u_t}{n^{-2-2d}\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{n}{\tau}\left(n^{-3/2-d}\sum_{t=1}^{\tau} y_{t-1}\right)^2} \\
&\Rightarrow \frac{\frac{r}{2}\left(W^H(r)\right)^2 - \int_0^r W^H(s)dsW^H(r)}{\int_0^r \left(W^H(s)\right)^2 ds - \frac{1}{r}\left(\int_0^r W^H(s)ds\right)^2},
\end{aligned} \tag{A.2.21}$$

where the last result is obtained by applying (A.1.1), (A.1.3) and (A.1.4). As (A.2.21) is equivalent to (A.1.5) when $d \in (0, 0.5)$, we have proved (A.1.5).

To show (A.1.6), note that

$$\begin{aligned}
\tilde{\rho}_\tau - 1 &= \hat{\rho}_\tau - 1 + \frac{\frac{1}{2}\sum_{t=1}^{\tau} \triangle y_t^2}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2} \\
&= \frac{\sum_{t=1}^{\tau} y_{t-1}u_t - \frac{1}{\tau}\sum_{t=1}^{\tau} y_{t-1}\sum_{t=1}^{\tau} u_t}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2} + \frac{\frac{1}{2}\sum_{t=1}^{\tau} \triangle y_t^2}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2} \\
&= \frac{D_\tau}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2} - \frac{\frac{1}{\tau}\sum_{t=1}^{\tau} y_{t-1}\sum_{t=1}^{\tau} u_t}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2},
\end{aligned}$$

where $D_\tau = \sum_{t=1}^{\tau} y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^{\tau} \triangle y_t^2$. Again $\sum_{t=1}^{\tau} y_{t-1}u_t = \frac{1}{2}\left(y_\tau^2 - y_0^2\right) - \frac{1}{2}\sum_{t=1}^{\tau} u_t^2$. So when $\rho_n = 1$, $\triangle y_t = u_t$ and $\sum_{t=1}^{\tau} y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^{\tau} \triangle y_t^2 = \frac{1}{2}\left(y_\tau^2 - y_0^2\right)$. Then

$$n^{-1-2d} D_\tau \Rightarrow \frac{1}{2} W^H(r)^2. \tag{A.2.22}$$

Hence,

$$\begin{aligned}
\tau(\tilde{\rho}_\tau - 1) &= \frac{\tau}{n}\frac{n^{-1-2d}D_\tau - \frac{n}{\tau}n^{-3/2-d}\sum_{t=1}^{\tau} y_{t-1}n^{-1/2-d}\sum_{t=1}^{\tau} u_t}{n^{-2-2d}\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{n}{\tau}\left(n^{-3/2-d}\sum_{t=1}^{\tau} y_{t-1}\right)^2} \\
&\Rightarrow \frac{\frac{r}{2}\left[\left(W^H(r)\right)^2\right] - \int_0^r W^H(s)dsW^H(r)}{\int_0^r \left(W^H(s)\right)^2 ds - \frac{1}{r}\left(\int_0^r W^H(s)ds\right)^2}, \text{ for } d \in [0, 0.5).
\end{aligned}$$

■

Proof of Lemma 3.1 in the main paper.

Recall that by definition $s_\tau^2 = \frac{\frac{1}{\tau}\sum_{t=1}^{\tau} \hat{u}_t^2}{\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{1}{\tau}\left(\sum_{t=1}^{\tau} y_{t-1}\right)^2}$. As $\hat{u}_t$ $(= u_t + (1 - \hat{\rho}_\tau)y_{t-1} - \hat{\mu})$ involves $\hat{\mu}$, we first study the properties of $\hat{\mu}$. Write

$$\hat{\mu} = \frac{\sum_{t=1}^{\tau} y_{t-1}^2 \sum_{t=1}^{\tau} u_t - \sum_{t=1}^{\tau} y_{t-1}u_t \sum_{t=1}^{\tau} y_{t-1}}{\frac{\tau}{n}\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{1}{n}\left(\sum_{t=1}^{\tau} y_{t-1}\right)^2}.$$

and upon normalization we have

$$\begin{aligned}
n^{1/2-d}\hat{\mu} &= \frac{\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau} y_{t-1}^2 \frac{1}{n^{1/2+d}}\sum_{t=1}^{\tau} u_t - \frac{1}{n^{1+2d}}\sum_{t=1}^{\tau} y_{t-1}u_t \frac{1}{n^{3/2+d}}\sum_{t=1}^{\tau} y_{t-1}}{\frac{\tau}{n}\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau} y_{t-1}^2 - \left(\frac{1}{n^{3/2+d}}\sum_{t=1}^{\tau} y_{t-1}\right)^2} \\
&\Rightarrow \frac{\sigma^3 \int_0^r \left(W^H(s)\right)^2 dsW^H(r) - \frac{\sigma^3}{2}\left(W^H(r)\right)^2 \int_0^r W^H(s)ds}{r\sigma^2 \int_0^r \left(W^H(s)\right)^2 ds - \sigma^2\left(\int_0^r W^H(s)ds\right)^2},
\end{aligned} \tag{A.2.23}$$

which implies $\hat{\mu} = O_p(n^{-1/2+d}) = o_p(1)$ when $d \in (0, 0.5)$.

For the mean squared residuals $\frac{1}{\tau}\sum_{t=1}^{\tau} \hat{u}_t^2$, write

$$\frac{1}{\tau}\sum_{t=1}^{\tau} \hat{u}_t^2 = \frac{1}{\tau}\sum_{t=1}^{\tau} \left(u_t + (1 - \hat{\rho}_\tau)\, y_{t-1} - \hat{\mu}\right)^2$$

$$= \frac{1}{\tau}\sum_{t=1}^{\tau} u_t^2 + \frac{2(1-\hat{\rho}_\tau)}{\tau}\sum_{t=1}^{\tau} y_{t-1}u_t + \frac{(1-\hat{\rho}_\tau)^2}{\tau}\sum_{t=1}^{\tau} y_{t-1}^2$$
$$- 2\hat{\mu}\frac{1}{\tau}\sum_{t=1}^{\tau} u_t - 2\frac{(1-\hat{\rho}_\tau)\hat{\mu}}{\tau}\sum_{t=1}^{\tau} y_{t-1} + \frac{\hat{\mu}^2}{\tau}\sum_{t=1}^{\tau} 1.$$
$$= \frac{1}{\tau}\sum_{t=1}^{\tau} u_t^2 + \frac{2\tau(1-\hat{\rho}_\tau)}{\tau^2}\sum_{t=1}^{\tau} y_{t-1}u_t + \frac{(\tau(1-\hat{\rho}_\tau))^2}{\tau^3}\sum_{t=1}^{\tau} y_{t-1}^2 - 2\tau(1-\hat{\rho}_\tau)\hat{\mu}\left(\frac{1}{\tau^2}\sum_{t=1}^{\tau} y_{t-1}\right) + \hat{\mu}^2.$$

From Lemma 1.2, $\tau(1-\hat{\rho}_\tau) = O_p(1)$, $\tau^{-2}\sum_{t=1}^{\tau} y_{t-1}u_t = O_p\left(n^{2d-1}\right) = o_p(1)$, $\tau^{-3}\sum_{t=1}^{\tau} y_{t-1}^2 = O_p(n^{2d-1}) = o_p(1)$, $\tau^{-2}\sum_{t=1}^{\tau} y_{t-1} = O_p\left(n^{d-1/2}\right)$. From (A.2.23), $\hat{\mu}^2 = O_p\left(n^{-1+2d}\right) = o_p(1)$, and so

$$\frac{1}{\tau}\sum_{t=1}^{\tau} \hat{u}_t^2 = \frac{1}{\tau}\sum_{t=1}^{\tau} u_t^2 + o_p(1). \tag{A.2.24}$$

Applying Lemma 1.2,

$$\begin{aligned} & n^{-(2+2d)}\left[\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{1}{\tau}\left(\sum_{t=1}^{\tau} y_{t-1}\right)^2\right] \\ = \; & n^{-(2+2d)}\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{\left(n^{-3/2-d}\sum_{t=1}^{\tau} y_{t-1}\right)^2}{\frac{\tau}{n}} \\ \Rightarrow \; & \sigma^2\left(\int_0^r \left(W^H(s)\right)^2 ds - \frac{1}{r}\left(\int_0^r W^H(s)ds\right)^2\right). \end{aligned} \tag{A.2.25}$$

We can express

$$n^{-d}DF_\tau = n^{-d}\frac{\hat{\rho}-1}{s_\tau} = \frac{\frac{n}{\tau}\tau(\hat{\rho}-1)}{\left(n^{2+2d}s_\tau^2\right)^{1/2}},$$

and, from Lemma 1.2 again, we have $\frac{n}{\tau}\tau(\hat{\rho}-1) \Rightarrow \frac{1}{r}\frac{\frac{r}{2}\left(W^H(r)\right)^2 - \left(\int_0^r W^H(s)ds\right)W^H(r)}{\int_0^r\left(\tilde{W}^H(s)\right)^2 ds}$. Next, note that $s_\tau^2 = \frac{\frac{1}{\tau}\sum_{t=1}^{\tau}\hat{u}_t^2}{\sum_{t=1}^{\tau}\bar{y}_t^2}$, and from (A.2.24), $\frac{1}{\tau}\sum_{t=1}^{\tau}\hat{u}_t^2 = \frac{1}{\tau}\sum_{t=1}^{\tau}u_t^2 + o_p(1)$. By a standard law of large numbers arugment we have $\frac{1}{\tau}\sum_{t=1}^{\tau}u_t^2 \xrightarrow{p} \mathbb{E}[u_t^2] := \sigma_u^2$. Finally, since $\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau}\bar{y}_t^2 \Rightarrow \sigma^2\int_0^r\left(\tilde{W}^H(s)\right)^2 ds$, we obtain

$$\begin{aligned} n^{-d}DF_\tau \;\; & \Rightarrow \;\; \frac{1}{r}\frac{\frac{r}{2}\left(W^H(r)\right)^2 - \left(\int_0^r W^H(s)ds\right)W^H(r)}{\int_0^r\left(\tilde{W}^H(s)\right)^2 ds}\left(\frac{\sigma^2\int_0^r\left(\tilde{W}^H(s)\right)^2 ds}{\sigma_u^2}\right)^{1/2} \\ & = \;\; \frac{\frac{\sigma r}{2}\left(W^H(r)\right)^2 - \sigma\left(\int_0^r W^H(s)ds\right)W^H(r)}{r\left(\sigma_u\int_0^r\left(\tilde{W}^H(s)\right)^2 ds\right)^{1/2}}. \end{aligned}$$

■

Proof of Lemma 1.3

Let $\xi_n = \hat{d} - d$ and, noting that $\Delta_+^{1+\hat{d}} y_t = \Delta_+^{\xi_n} \epsilon_t = \epsilon_t - \xi_n \sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k} + O_p(\xi_n^2)$, we have

$$\begin{aligned}
\sup_{1\le t\le n} \left|\Delta_+^{1+\hat{d}} y_t - \epsilon_t\right| &= \sup_{1\le t\le n} \left| -\xi_n \sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k} + O_p(\xi_n^2)\right| \\
&\le \sup_{1\le t\le n} \left| -\xi_n \sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k}\right| + O_p(\xi_n^2) \\
&\le |\xi_n| \, | \sup_{1\le t\le n} \left| \sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k}\right| + O_p(\xi_n^2).
\end{aligned}$$

Since $O(\xi_n^2)$ is dependent on n but not t as ξ_n is the estimation error of the memory parameter $\hat{d} - d$. Moreover, $\xi_n = o_p(1)$. To prove Lemma 1.3, what remains to be shown is

$$\sup_{1\le t\le n} |\sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k}| = O_p(1). \tag{A.2.26}$$

To show (A.2.26), it is sufficient to show that $|\sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k}| = O_p(1)$, for any $1 \le t \le n$. By Chebyshev's inequality, we have, for any $\delta > 0$,

$$\Pr\left(\left|\sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k}\right| \ge \delta\right) \le \frac{\sigma^2 \sum_{k=1}^{t-1}\left(\frac{1}{k^2}\right)}{\delta^2} < \frac{\sigma^2 \sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)}{\delta^2} = \sigma^2 \frac{\pi^2}{6}\frac{1}{\delta^2}.$$

Thus, for any $\varepsilon > 0$, we can find $\delta = \frac{\sigma\pi}{\sqrt{6\varepsilon}}$ such that $\Pr\left(\left|\sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k}\right| \ge \delta\right) < \varepsilon$. This implies that, for any $1 \le t \le n$, $|\sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k}| = O_p(1)$, and thus we have (A.2.26). ■

Proof of Lemma 1.5

To show the first claim, backward substitution gives

$$y_{\lfloor nr\rfloor} = \rho_n^{\lfloor nr\rfloor - (\lfloor nr_e\rfloor - 1)} y_{\lfloor nr_e\rfloor - 1} + \sum_{j=\lfloor r_e\rfloor}^{\lfloor nr\rfloor} \left(1 + \frac{c}{n}\right)^{\lfloor nr\rfloor - j} u_j,$$

and $\rho_n^{\lfloor nr\rfloor - (\lfloor nr_e\rfloor - 1)} = (1 + c/n)^{\lfloor nr\rfloor - (\lfloor nr_e\rfloor - 1)} = \exp((r - r_e)c) + o(1)$. Therefore,

$$\begin{aligned}
\frac{1}{n^{1/2+d}} y_{\lfloor nr\rfloor} &= \exp((r - r_e)c) \frac{1}{n^{1/2+d}} y_{\lfloor nr_e\rfloor - 1} + \frac{1}{n^{1/2+d}} \sum_{j=\lfloor r_e\rfloor}^{\lfloor nr\rfloor} \left(1 + \frac{c}{n}\right)^{\lfloor nr\rfloor - j} u_j \\
&\Rightarrow \sigma\left(e^{(r-r_e)c} W^H(r_e) + \int_{r_e}^{r} e^{(r-s)c} dW^H(s)\right),
\end{aligned} \tag{A.2.27}$$

where the limit follows by Lemma 1.1 and the CMT as in Lui et al. (2020).

For the second and the third claims, write $\frac{1}{n^{2/3+d}}\sum_{t=1}^{\tau} y_{t-1} = \frac{1}{n}\sum_{t=1}^{\tau}\left(\frac{1}{n^{1/2+d}} y_{t-1}\right)$ and $\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau} y_{t-1}^2 = \frac{1}{n}\sum_{t=1}^{\tau}\left(\frac{1}{n^{1/2+d}} y_{t-1}\right)^2$. Application of (A.2.27) and the CMT then yields the two results.

For the fourth claim, first consider the second component $\frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2$. Since

$$\Delta y_t = \begin{cases} u_t & \text{if } t < \tau_e, \\ \frac{c}{n} y_{t-1} + u_t & \text{otherwise} \end{cases},$$

we write

$$\begin{aligned}
\sum_{t=1}^{\tau}\Delta y_t^2 &= \sum_{t=1}^{\tau_e-1} u_t^2 + \sum_{t=\tau_e}^{\tau}\left(\frac{c}{n} y_{t-1} + u_t\right)^2 \\
&= \sum_{t=1}^{\tau} u_t^2 + \frac{c^2}{n^2}\sum_{t=\tau_e}^{\tau} y_{t-1}^2 + \frac{2c}{n}\sum_{t=\tau_e}^{\tau} y_{t-1} \\
&= \sum_{t=1}^{\tau} u_t^2 + \frac{c^2}{n^2} n^{2+2d}\left(\frac{1}{n^{2+2d}}\sum_{t=\tau_e}^{\tau} y_{t-1}^2\right) + \frac{2c}{n} n^{3/2+d}\left(\frac{1}{n^{3/2+d}}\sum_{t=\tau_e}^{\tau} y_{t-1}\right).
\end{aligned} \tag{A.2.28}$$

For the term $\frac{1}{n^{2+2d}}\sum_{t=\tau_e}^{\tau} y_{t-1}^2$, as $n \to \infty$,

$$\frac{1}{n^{2+2d}}\sum_{t=\tau_e}^{\tau} y_{t-1}^2 = \frac{1}{n^{2+2d}}\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{1}{n^{2+2d}}\sum_{t=1}^{\tau_e-1} y_{t-1}^2 \Rightarrow \sigma^2 B_{r,d} - \sigma^2\int_0^{r_e}\left(W^H(s)\right)^2 ds, \tag{A.2.29}$$

where Lemma 1.5.3 and (A.1.4) are used to obtain the limit.

For the term $\frac{1}{n^{3/2+d}}\sum_{t=\tau_e}^{\tau} y_{t-1}$, similarly, using Lemma 1.5.2 and (A.1.3), we have

$$\frac{1}{n^{3/2+d}}\sum_{t=\tau_e}^{\tau} y_{t-1} \Rightarrow \sigma A_{r,d} - \sigma\int_0^{r_e} W^H(s)ds. \tag{A.2.30}$$

Combining (A.2.28), (A.2.29) and (A.2.30),

$$\sum_{t=1}^{\tau}\Delta y_t^2 = \sum_{t=1}^{\tau} u_t^2 + R_{1,n},\ \ R_{1,n} = O_p(n^{1/2+d}).$$

Upon normalization we now have

$$\begin{aligned}
&\frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau} y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2\right) \qquad \text{(A.2.31)} \\
&= \frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau} y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2 + R_{1,n}\right) \\
&= \frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\lfloor nr_e\rfloor-1} y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^{\lfloor nr_e\rfloor-1} u_t^2\right) + \frac{1}{n^{1+2d}}\left(\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_{t-1}u_t + \frac{1}{2}\sum_{t=\lfloor nr_e\rfloor}^{\tau} u_t^2\right) + \frac{R_{1,n}}{n^{1+2d}}.
\end{aligned}$$

For the first component on the right-hand side of (A.2.31), applying (A.2.22) gives

$$\frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\lfloor nr_e\rfloor-1} y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^{\lfloor nr_e\rfloor-1} u_t^2\right) \Rightarrow \frac{\sigma^2}{2}(W^H(r_e))^2.$$

For the second component, standard calculation yields

$$\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_{t-1}u_t = \frac{1}{2\rho_n}\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_t^2 - \frac{\rho_n}{2}\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_{t-1}^2 - \frac{1}{2\rho_n}\sum_{t=\lfloor nr_e\rfloor}^{\tau} u_t^2.$$

Hence,

$$\begin{aligned}\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_{t-1}u_t + \frac{1}{2}\sum_{t=\lfloor nr_e\rfloor}^{\tau} u_t^2 &= \frac{1}{2\rho_n}\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_t^2 - \frac{\rho_n}{2}\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_{t-1}^2 - \frac{1}{2\rho_n}\sum_{t=\lfloor nr_e\rfloor}^{\tau} u_t^2 + \frac{1}{2}\sum_{t=\lfloor nr_e\rfloor}^{\tau} u_t^2 \\ &= \frac{1}{2\rho_n}\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_t^2 - \frac{\rho_n}{2}\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_{t-1}^2 + \frac{1}{2}\left(1-\frac{1}{\rho_n}\right)\sum_{t=\lfloor nr_e\rfloor}^{\tau} u_t^2.\end{aligned}$$

As $\rho_n = 1 + o(1)$, we have

$$\frac{1}{n^{1+2d}}\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_{t-1}u_t = \frac{1}{n^{1+2d}}\frac{1}{2}\left[y_\tau^2 - y_{\lfloor nr_e\rfloor-1}^2\right] - \frac{1}{n^{1+2d}}\frac{1}{2}\sum_{t=\lfloor nr_e\rfloor}^{\tau} u_t^2 + o_p(1).$$

Thus,

$$\begin{aligned}&\frac{1}{n^{1+2d}}\left(\sum_{t=\lfloor nr_e\rfloor}^{\tau} y_{t-1}u_t + \frac{1}{2}\sum_{t=\lfloor nr_e\rfloor}^{\tau} u_t^2\right) \\ = &\frac{1}{n^{1+2d}}\frac{1}{2}\left[y_\tau^2 - y_{\lfloor nr_e\rfloor-1}^2\right] + o_p(1) \\ \Rightarrow &\frac{\sigma^2}{2}\left[\left(e^{(r-r_e)c}W^H(r_e) + \int_{r_e}^{r} e^{(r-s)c)}dW^H(s)\right)^2 - (W^H(r_e))^2\right]\end{aligned} \quad (A.2.32)$$

where we obtain the limit by applying Lemma 1.5.1 and Lemma 1.1. The last term $R_{1,n}/n^{1+2d}$ in (A.2.31) vanishes because $R_{1,n} = O_p(n^{1/2+d})$ as $n \to \infty$. This confirms the fourth claim.

To show the fifth claim, note that

$$\tilde{\rho}_\tau - \rho_n = \hat{\rho}_\tau - \rho_n + \frac{\frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2}{\sum_{t=1}^{\tau}\bar{y}_{t-1}^2} = \frac{\sum_{t=1}^{\tau}\bar{y}_{t-1}(y_t - \rho_n y_{t-1}) + \frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2}{\sum_{t=1}^{\tau}\bar{y}_{t-1}^2} \quad (A.2.33)$$

The numerator of (A.2.33) is

$$\sum_{t=1}^{\tau}\bar{y}_{t-1}(y_t - \rho_n y_{t-1}) + \frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2$$

$$
\begin{aligned}
&= \sum_{t=1}^{\tau_e-1} \bar{y}_{t-1}\left(y_t - \rho_n y_{t-1}\right) + \sum_{t=\tau_e}^{\tau} \bar{y}_{t-1}\left(y_t - \rho_n y_{t-1}\right) + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2 + R_{1,n} \\
&= \sum_{t=1}^{\tau_e-1} \bar{y}_{t-1}\left(y_{t-1} + u_t - \rho_c y_{t-1}\right) + \sum_{t=\tau_e}^{\tau} \bar{y}_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2 + R_{1,n} \\
&= \left(\sum_{t=1}^{\tau} \bar{y}_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2\right) - \frac{c}{n}\sum_{t=1}^{\tau_e-1} \bar{y}_{t-1} + R_{1,n}. \qquad \text{(A.2.34)}
\end{aligned}
$$

Upon normalization, the first component in (A.2.34) is

$$
\begin{aligned}
& \frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau} \bar{y}_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2\right) \\
&= \frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau} y_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2 - \frac{1}{\tau}\sum_{t=1}^{\tau} y_{t-1}\sum_{t=1}^{\tau} u_t\right) \\
&= \frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau} y_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2\right) - \frac{n}{\tau}\frac{1}{n^{3/2+d}}\sum_{t=1}^{\tau} y_{t-1}\frac{1}{n^{1/2+d}}\sum_{t=1}^{\tau} u_t \\
&\Rightarrow \frac{1}{2}\sigma^2 C_{r,d} - \frac{1}{r}\sigma^2 A_{r,d} W^H(r), \qquad \text{(A.2.35)}
\end{aligned}
$$

by virtue of (A.2.32), Lemma 1.5 and Lemma 1.1 to obtain the limit. For the second component, note that

$$
\frac{c}{n}\sum_{t=1}^{\tau_e-1} \bar{y}_{t-1} = \frac{c}{n}\sum_{t=1}^{\tau_e-1} y_{t-1} - \frac{c}{n}\frac{1}{\tau}\sum_{t=1}^{\tau_e-1}\sum_{j=1}^{\tau} y_{j-1} = \frac{c}{n}\sum_{t=1}^{\tau_e-1} y_{t-1} - \frac{c}{n}\frac{\tau_e-1}{\tau}\sum_{j=1}^{\tau} y_{j-1}.
$$

After normalization, we have

$$
\begin{aligned}
\frac{1}{n^{1/2+d}}\left(\frac{c}{n}\sum_{t=1}^{\tau_e-1} y_{t-1} - \frac{c}{n}\frac{\tau_e-1}{\tau}\sum_{j=1}^{\tau} y_{j-1}\right) &= \frac{c}{n^{3/2+d}}\sum_{t=1}^{\tau_e-1} y_{t-1} - \frac{\tau_e-1}{\tau}\frac{c}{n^{3/2+d}}\sum_{j=1}^{\tau} y_{j-1} \\
&\Rightarrow c\sigma\int_0^{r_e} W^H(s)ds - c r_e \sigma A_{r,d},
\end{aligned}
$$

where the limit follows by using Lemma 1.2 and 1.5.2.

Therefore, the first component in (A.2.34) is $O_p(n^{1+2d})$ which dominates $\frac{c}{n}\sum_{t=1}^{\tau_e-1} \bar{y}_{t-1} = O_p(n^{1/2+d})$ and the third component $R_{1,n} = O_p(n^{1/2+d})$. Consequently,

$$
\frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau} \bar{y}_{t-1}\left(y_t - \rho_n y_{t-1}\right) + \frac{1}{2}\sum_{t=1}^{\tau} \Delta y_t^2\right) \Rightarrow \frac{1}{2}\sigma^2 C_{r,d} - \frac{1}{r}\sigma^2 A_{r,d} W^H(r).
$$

For the denominator in (A.2.33), applying Lemma 1.5.2 and 1.5.3, we have

$$
\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau} \bar{y}_{t-1}^2 = \frac{1}{n^{2+2d}}\left[\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{1}{\tau}\left(\sum_{t=1}^{\tau} y_{t-1}\right)^2\right] \Rightarrow \sigma^2\left(B_{r,d} - \frac{1}{r}A_{r,d}^2\right) \qquad \text{(A.2.36)}
$$

and then

$$
\begin{aligned}
n(\tilde{\rho}_\tau - \rho_n) &= \frac{\frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau}\bar{y}_{t-1}\left(y_t-\rho_n y_{t-1}\right)+\frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2\right)}{\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau}\bar{y}_{t-1}^2} \\
&\Rightarrow \frac{\frac{1}{2}C_{r,d}-\frac{1}{r}A_{r,d}W^H(r)}{B_{r,d}-\frac{1}{r}A_{r,d}^2} := X_c(r,d)\ .
\end{aligned}
$$

For the sixth claim, since $n(\tilde{\rho}_\tau - 1) = n(\tilde{\rho}_\tau - \rho_n) + n(\rho_n - 1) = n(\tilde{\rho}_\tau - \rho_n) + c$, we have $n(\hat{\rho}_\tau - 1) \Rightarrow X_c(r,d) + c$, completing the proof of Lemma 1.5. ■

Proof of Lemma 1.6

1. From Lemma 1.1, we have $\frac{1}{n^{1/2+d_1}}y_{\lfloor nr\rfloor} \Rightarrow \sigma W^{H_1}(r)$.
2. For $t \in B$, we have

$$
y_t = \rho_n^{t-\tau_e+1}y_{\tau_e-1} + \sum_{j=0}^{t-\tau_e}\rho_n^j u_{t-j} = \rho_n^{t-\tau_e+1}y_{\tau_e-1} + \sum_{j=0}^{t-\tau_e}\rho_n^j u_{t-j,d_2}.
$$

   Pre-multiplying both terms by $\rho_n^{-(t-\tau_e)}$

$$
\rho_n^{-(t-\tau_e)}y_t = \rho_n y_{\tau_e-1} + \rho_n^{-(t-\tau_e)}\sum_{j=0}^{t-\tau_e}\rho_n^{-j}u_{t-j,d_2}, \tag{A.2.37}
$$

   and applying Cauchy–Schwarz we have

$$
\sum_{j=0}^{t-\tau_e}\rho_n^{-j}u_{t-j,d_2} \le \left(\sum_{j=0}^{t-\tau_e}\rho_n^{-2j}\right)^{1/2}\left(\sum_{j=0}^{t-\tau_e}u_{t-j,d_2}^2\right)^{1/2} \tag{A.2.38}
$$

$$
\begin{aligned}
&= \left(\sum_{j=0}^{t-\tau_e}\rho_n^{-2j}\right)^{1/2}\left(\sum_{i=\tau_e}^{t}u_{i,d_2}^2\right)^{1/2} = \left(\frac{\rho_n^2}{\rho_n^2}\frac{\rho_n^{-2(t-\tau_e)-2}-1}{\rho_n^{-2}-1}\right)^{1/2}\left(\sum_{i=\tau_e}^{t}u_{i,d_2}^2\right)^{1/2} \\
&= \left(\frac{\rho_n^{-2(t-\tau_e)}-\rho_n^2}{1-\rho_n^2}\right)^{1/2}\left(\sum_{i=\tau_e}^{t}u_{i,d_2}^2\right)^{1/2} = O_p(n^{\alpha/2})\times O_p(n_{d_2}^{1/2}),
\end{aligned} \tag{A.2.39}
$$

   where

$$
n_d = \begin{cases} n & \text{if } d\in[0,0.5) \\ n\ln(n)^2 & \text{if } d=0.5 \\ n^{2d} & \text{if } d>0.5 \end{cases}, \tag{A.2.40}
$$

   where the above orders in (A.2.40) follow from Duffy and Kasparis (2021). Since $\rho_n^{-(t-\tau_e)} = \rho_n^{-(\lfloor nr\rfloor-\lfloor nr_e\rfloor)} = \rho_n^{-n(r-r_e)+o(1)}$ and $\rho_n^{-n(r-r_e)} = \exp\left(-Kn^{1-\alpha}\right)+o(1)$, where K is a positive constant, we have $\rho_n^{-(t-\tau_e)}\sum_{j=0}^{t-\tau_e}\rho_n^{-j}u_{t-j,d_2} = o_p(1)$ for any $d_2 \ge 0$, and so

$$
\rho_n^{-(t-\tau_e)}\frac{1}{n^{1/2+d_1}}y_t \overset{a}{\sim} \frac{\rho_n}{n^{1/2+d_1}}y_{\tau_e-1} \overset{a}{\sim} \sigma W^{H_1}(r_e).
$$

3. For $t \in N_1$, we have

$$\begin{aligned} y_{\lfloor nr \rfloor} &= \sum_{k=\tau_f+1}^{\lfloor nr \rfloor} u_{k,d_1} + y^*_{\tau_f} = \sum_{k=\tau_f+1}^{\lfloor nr \rfloor} u_{k,d_1} + y_{\tau_e} + y^* \\ &= \sum_{k=1}^{\lfloor nr \rfloor} u_{k,d_1} - \sum_{k=1}^{\tau_f} u_{k,d_1} + y_{\tau_e} + y^*. \end{aligned}$$

Note that $y_{\tau_e} \stackrel{a}{\sim} n^{1/2+d_1}\sigma W^{H_1}(r_e)$, and applying Lemma 1.1 gives

$$y_{\lfloor nr \rfloor} \stackrel{a}{\sim} n^{1/2+d_1}\left[\sigma\left(W^{H_1}(r) - W^{H_1}(r_f)\right) + \sigma W^{H_1}(r_e)\right].$$

This completes the proof of Lemma 1.6. ■

Proof of Lemma 1.7 For $\tau \in B$, we have

$$\frac{1}{\tau}\sum_{j=1}^{\tau} y_j = \frac{1}{\tau}\sum_{j=1}^{\tau_e-1} y_j + \frac{1}{\tau}\sum_{j=\tau_e}^{\tau} y_j.$$

The first term is

$$\frac{1}{\tau}\sum_{j=1}^{\tau_e-1} y_j = n^{1/2+d_1}\frac{\tau_e}{\tau}\left(\frac{1}{\tau_e}\sum_{j=1}^{\tau_e-1}\frac{1}{n^{1/2+d_1}}y_j\right) \stackrel{a}{\sim} n^{1/2+d_1}\frac{r_e}{r}\sigma\int_0^{r_e} W^{H_1}(s)ds, \tag{A.2.41}$$

where Lemma 1.6.1 and the CMT are used to obtain (A.2.41). For the second term,

$$\begin{aligned} &\frac{1}{\tau}\sum_{j=\tau_e}^{\tau} y_j \stackrel{a}{\sim} \frac{1}{\tau}\sum_{j=\tau_e}^{\tau}\rho_n^{(j-\tau_e)} n^{1/2+d_1}\sigma W^{H_1}(r_e) = n^{1/2+d_1}\sigma W^{H_1}(r_e)\frac{1}{\tau}\sum_{j=\tau_e}^{\tau}\rho_n^{j-\tau_e} \\ &= \frac{n^{1/2+d_1}\sigma W^{H_1}(r_e)}{\tau}\frac{\rho_n^{\tau-\tau_e+1}-1}{\rho_n-1} = \frac{n^{1/2+d_1}\sigma W^{H_1}(r_e)\left[\left(\rho_n^{\tau-\tau_e}\rho_n\right)n^{\alpha} - n^{\alpha}\right]}{\lfloor nr \rfloor c} \\ &\stackrel{a}{\sim} n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e), \end{aligned} \tag{A.2.42}$$

where the last asymptotic equivalence follows because $\rho_n^{\tau-\tau_e}n^{\alpha}$ dominates n^{α}. Comparing (A.2.42) with (A.2.41) and since $\rho_n^{\tau-\tau_e}$ gives an exponential divergence rate, we have the results in Lemma 1.7.1.

For $\tau \in N_1$,

$$\frac{1}{\tau}\sum_{j=1}^{\tau} y_j = \frac{1}{\tau}\sum_{j=1}^{\tau_e-1} y_j + \frac{1}{\tau}\sum_{j=\tau_e}^{\tau_f} y_j + \frac{1}{\tau}\sum_{j=\tau_f+1}^{\tau} y_j. \tag{A.2.43}$$

For the first term, similar to (A.2.41) we have

$$\frac{1}{\tau}\sum_{j=1}^{\tau_e-1} y_j \stackrel{a}{\sim} n^{1/2+d_1}\frac{r_e}{r}\sigma\int_0^{r_e} W^{H_1}(s)ds.$$

For the second term, similar to (A.2.42), we have

$$\frac{1}{\tau}\sum_{j=\tau_e}^{\tau_f} y_j \overset{a}{\sim} n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e).$$

For the last term, using Lemma 1.6.3 we have

$$\frac{1}{\tau}\sum_{j=\tau_f+1}^{\tau} y_j = \frac{\tau-\tau_f}{\tau}n^{1/2+d_1}O_p(1) \overset{a}{\sim} O_p(n^{1/2+d_1}).$$

As in the proof of Lemma 1.7, the second term in (A.2.43) has the highest order. So the result in Lemma 1.7.2 follows. ■

Proof of Lemma 1.8

1. Suppose $\tau \in B$. If $t \in N_0$, from Lemma 1.6.1, $y_t = O_p(n^{1/2+d_1})$. Following Lemma 1.7.1, we obtain $\frac{1}{\tau}\sum_{j=1}^{\tau} y_{j-1} \overset{a}{\sim} n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)$. Hence, the second term has higher order and

   $$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e).$$

   If $t \in B$, from Lemma 1.6.2,

   $$\bar{y}_t \overset{a}{\sim} \rho_n^{(t-\tau_e)}n^{1/2+d_1}\sigma W^{H_1}(r_e) - n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e).$$

2. Suppose $\tau \in N_1$. If $t \in N_0$, then similar to the proof in Lemma 1.8.1 as y_t is asymptotically dominated by the latter term, we have

   $$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e).$$

   If $t \in B$,

   $$\bar{y}_t \overset{a}{\sim} \left(\rho_n^{(t-\tau_e)} - \rho_n^{\tau_f-\tau_e}\frac{n^\alpha}{nrc}\right)n^{1/2+d_1}\sigma W^{H_1}(r_e).$$

   If $t \in N_1$, components in y_t are dominated by the components in $\frac{1}{\tau}\sum_{j=1}^{\tau} y_{j-1}$. Following the proof of Lemma 1.8.1, we have

   $$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e).$$

   This completes the proof of Lemma 1.8. ■

Proof of Lemma 1.9

1. For $\tau \in B$,

   $$\sum_{j=1}^{\tau}\bar{y}_{j-1}^2 = \sum_{j=1}^{\tau_e-1}\bar{y}_{j-1}^2 + \sum_{j=\tau_e}^{\tau}\bar{y}_{j-1}^2. \qquad \text{(A.2.44)}$$

For the first term in (A.2.44), applying (A.1.9) gives

$$\sum_{j=1}^{\tau_e-1}\bar{y}_{j-1}^2 \overset{a}{\sim} \sum_{j=1}^{\tau_e-1}\left(-n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)^2$$
$$= \frac{(\tau_e-1)}{n}n^{2(\alpha+d_1)}\rho_n^{2(\tau-\tau_e)}\frac{1}{r^2c^2}\sigma^2 W^{H_1}(r_e)^2 \overset{a}{\sim} \frac{r_e}{r^2c^2}n^{2(\alpha+d_1)}\rho_n^{2(\tau-\tau_e)}\sigma^2 W^{H_1}(r_e)^2. \tag{A.2.45}$$

For the second term in (A.2.44), applying (A.1.10) gives

$$\sum_{j=\tau_e}^{\tau}\bar{y}_{j-1}^2 \overset{a}{\sim} \sum_{j=\tau_e}^{\tau}\left[\left(\rho_n^{(j-\tau_e)}-\frac{n^\alpha}{nrc}\rho_n^{\tau-\tau_e}\right)n^{1/2+d_1}\sigma W^{H_1}(r_e)\right]^2$$
$$= n^{1+2d_1}\sigma^2\left(W^{H_1}(r_e)\right)^2\sum_{j=\tau_e}^{\tau}\left(\rho_n^{(j-\tau_e)}-\frac{n^\alpha}{nrc}\rho_n^{\tau-\tau_e}\right)^2$$
$$= n^{1+2d_1}\sigma^2\left(W^{H_1}(r_e)\right)^2\sum_{j=\tau_e}^{\tau}\left(\rho_n^{2(j-\tau_e)}-2\rho_n^{(j-\tau_e)}\frac{n^\alpha}{nrc}\rho_n^{\tau-\tau_e}+\frac{n^{2\alpha}}{n^2r^2c^2}\rho_n^{2(\tau-\tau_e)}\right)$$
$$= n^{1+2d_1}\sigma^2\left(W^{H_1}(r_e)\right)^2\left[\frac{n^\alpha\rho_n^{2(\tau-\tau_e)}}{2c}-2\frac{n^{2\alpha-1}\rho_n^{2(\tau-\tau_e)}}{nrc}+\frac{r-r_e+1/n}{r^2c^2}n^{2\alpha-1}\rho_n^{2(\tau-\tau_e)}\right]$$
$$\overset{a}{\sim} n^{1+2d_1+\alpha}\sigma^2\left(W^{H_1}(r_e)\right)^2\frac{\rho_n^{2(\tau-\tau_e)}}{2c}\text{ , as }\alpha>2\alpha-1.$$

Since $1+2d_1+\alpha > 2(\alpha+d_1)$, $\sum_{j=\tau_e}^{\tau}\bar{y}_{j-1}^2$ dominates $\sum_{j=1}^{\tau_e-1}\bar{y}_{j-1}^2$ asymptotically and we have

$$\sum_{j=1}^{\tau}\bar{y}_{j-1}^2 \overset{a}{\sim} n^{1+2d_1+\alpha}\sigma^2\left(W^{H_1}(r_e)\right)^2\frac{\rho_n^{2(\tau-\tau_e)}}{2c}. \tag{A.2.46}$$

2. For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2}\bar{y}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_e-1}\bar{y}_{j-1}^2+\sum_{j=\tau_e}^{\tau_f}\bar{y}_{j-1}^2+\sum_{j=\tau_f+1}^{\tau}\bar{y}_{j-1}^2. \tag{A.2.47}$$

For the first term in (A.2.47), similar to (A.2.45), we have

$$\sum_{j=1}^{\tau_e-1}\bar{y}_{j-1}^2 \overset{a}{\sim} \frac{r_e}{r^2c^2}n^{2(\alpha+d_1)}\rho_n^{2\left(\tau_f-\tau_e\right)}\sigma^2\left(W^{H_1}(r_e)\right)^2.$$

For the second term, similar to (A.2.46), we have

$$\sum_{j=\tau_e}^{\tau_f}\bar{y}_{j-1}^2 \overset{a}{\sim} n^{1+\alpha+2d_1}\sigma^2\left(W^{H_1}(r_e)\right)^2\frac{\rho_n^{2(\tau_f-\tau_e)}}{2c}.$$

For the third term, applying (A.1.13) gives

$$\sum_{j=\tau_f+1}^{\tau} \bar{y}_{j-1}^2 \overset{a}{\sim} \sum_{j=\tau_f+1}^{\tau} \left(-n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)^2$$

$$= \frac{\tau-\tau_f}{n} n^{2(\alpha+d_1)}\rho_n^{2\left(\tau_f-\tau_e\right)}\frac{1}{r^2c^2}\sigma^2\left(W^{H_1}(r_e)\right)^2 \overset{a}{\sim} \frac{(r-r_f)}{r^2c^2} n^{2(\alpha+d_1)}\rho_n^{2\left(\tau_f-\tau_e\right)}\sigma^2\left(W^{H_1}(r_e)\right)^2.$$

As $1+\alpha+2d_1 > 2(\alpha+d_1)$, the middle term in the right hand side of (A.2.47) dominates and we have

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}^2 \overset{a}{\sim} n^{1+\alpha+2d_1}\sigma^2\left(W^{H_1}(r_e)\right)^2 \frac{\rho_n^{2\left(\tau_f-\tau_e\right)}}{2c}.$$

This completes the proof of Lemma 1.9. ■

Proof of Lemma 1.10

1. For $\tau \in B$,

$$\sum_{j=1}^{\tau_2} \bar{y}_{j-1}u_j = \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}u_{j,d_1} + \sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}u_{j,d_2}. \tag{A.2.48}$$

From (A.1.9), the first term in (A.2.48) can be written as

$$\begin{aligned}
\sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}u_{j,d_1} &\overset{a}{\sim} \sum_{j=1}^{\tau_e-1}\left(-n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)u_{j,d_1} \\
&= \left(-n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)\sum_{j=1}^{\tau_e-1} u_{j,d_1} \\
&= \frac{-n^{\alpha+d_1-1/2+d_1+1/2}}{rc}\rho_n^{\tau-\tau_e}\sigma W^{H_1}(r_e)\frac{1}{n^{1/2+d_1}}\sum_{j=1}^{\tau_e-1} u_{j,d_1} \\
&\overset{a}{\sim} -\frac{n^{\alpha+2d_1}}{rc}\rho_n^{\tau-\tau_e}\sigma^2\left(W^{H_1}(r_e)\right)^2. \qquad (A.2.49)
\end{aligned}$$

For the second term in (A.2.48),

$$\begin{aligned}
&\sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}u_{j,d_2} \\
\overset{a}{\sim}\ & \sum_{j=\tau_e}^{\tau}\left[\left(\rho_n^{(j-\tau_e)} - \rho_n^{\tau-\tau_e}\frac{n^\alpha}{nrc}\right)n^{1/2+d_1}\sigma W^{H_1}(r_e)\right]u_{j,d_2} \\
=\ & n^{1/2+d_1}\sigma W^{H_1}(r_e)\sum_{j=\tau_e}^{\tau}\left(\rho_n^{(j-\tau_e)}u_{j,d_2} - \rho_n^{\tau-\tau_e}\frac{n^\alpha}{nrc}u_{j,d_2}\right) \\
=\ & n^{1/2+d_1}\sigma W^{H_1}(r_e)\rho_n^{\tau-\tau_e}\left[\sum_{j=\tau_e}^{\tau}\rho_n^{-(\tau-j)}u_{j,d_2} - \frac{n^\alpha}{nrc}\sum_{j=\tau_e}^{\tau} u_{j,d_2}\right]
\end{aligned}$$

$$
\begin{aligned}
&= n^{1/2+d_1}\sigma W^{H_1}(r_e)\rho_n^{\tau-\tau_e}\left[O_p(n^{\alpha/2})O_p(n_{d_2}^{1/2}) - \frac{1}{rc}O_p(n^{d_2+\alpha-1/2})\right] \\
&= n^{1/2+d_1}\sigma W^{H_1}(r_e)\rho_n^{\tau-\tau_e}O_p(n^{\alpha/2})O_p(n_{d_2}^{1/2}), \qquad \text{(A.2.50)}
\end{aligned}
$$

where n_{d_2} is defined in (A.2.39). The second last equality is obtained using the approach in (A.2.39), and the last equality follows by verifying that for any $d \geq 0$, $n^{\alpha/2}n_{d_2}^{1/2}$ diverges faster than $n^{d_2+\alpha-1/2}$.

The asymptotic orders of $\sum_{j=1}^{\tau_e-1}\bar{y}_{j-1}u_{j,d_1}$ and $\sum_{j=\tau_e}^{\tau}\bar{y}_{j-1}u_{j,d_2}$ depend on the magnitude of α, d_1 and d_2. We have

$$
\sum_{j=1}^{\tau_2}\bar{y}_{j-1}u_j = O_p\left(\rho_n^{\tau-\tau_e}n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,\alpha+2d_1\right\}}\right). \qquad \text{(A.2.51)}
$$

2. For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$
\sum_{j=\tau_1}^{\tau_2}\bar{y}_{j-1}u_j = \sum_{j=\tau_1}^{\tau_e-1}\bar{y}_{j-1}u_j + \sum_{j=\tau_d}^{\tau_f}\bar{y}_{j-1}u_j + \sum_{j=\tau_f+1}^{\tau_2}\bar{y}_{j-1}u_j.
$$

As in (A.2.49), the first term is

$$
\sum_{j=\tau_1}^{\tau_e-1}\bar{y}_{j-1}u_j \overset{a}{\sim} -\frac{n^{\alpha+2d_1}}{rc}\rho_n^{\tau-\tau_e}\sigma^2\left(W^{H_1}(r_e)\right)^2.
$$

As in (A.2.50), the second term is

$$
\sum_{j=\tau_e}^{\tau_f}\bar{y}_{j-1}u_j \overset{a}{\sim} n^{1/2+d_1}\sigma W^{H_1}(r_e)\rho_n^{\tau_f-\tau_e}\times O_p(n^{\alpha/2})\times O_p(n_{d_2}^{1/2}).
$$

The third term is

$$
\begin{aligned}
\sum_{j=\tau_f+1}^{\tau}\bar{y}_{j-1}u_j &\overset{a}{\sim} \sum_{j=\tau_f+1}^{\tau}\left(-n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)u_{j,d_1} \\
&= -n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\sum_{j=\tau_f+1}^{\tau}u_{j,d_1} \\
&= -n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)n^{1/2+d_1}\left(W^{H_1}(r) - W^{H_1}(r_f)\right) \\
&\overset{a}{\sim} -n^{\alpha+2d_1}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\left(W^{H_1}(r) - W^{H_1}(r_f)\right).
\end{aligned}
$$

Then, similar to (A.2.51), we can write

$$
\sum_{j=\tau_1}^{\tau_2}\bar{y}_{j-1}u_j = O_p\left(\rho_n^{\tau-\tau_e}n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,\alpha+2d_1\right\}}\right).
$$

and the proof of Lemma 1.10 is complete. ■

Proof of Lemma 1.11

1. Separate $\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1})$ into two parts

$$\begin{aligned}
\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) &= \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) + \sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) \\
&= \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}\left[(1-\rho_n)\, y_{j-1} + u_t\right] + \sum_{j=\tau_e}^{\tau} \bar{y}_{j-1} u_t \\
&= \sum_{j=1}^{\tau} \bar{y}_{j-1} u_t - \frac{c}{n^\alpha} \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1} y_{j-1}.
\end{aligned}$$

From (A.1.16),

$$\sum_{j=1}^{\tau} \bar{y}_{j-1} u_t = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1, \alpha+2d_1\right\}}\right). \tag{A.2.52}$$

For the second term, applying (A.1.9) leads to

$$\begin{aligned}
\frac{c}{n^\alpha} \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1} y_{j-1} &\overset{a}{\sim} \frac{c}{n^\alpha} \sum_{j=1}^{\tau_e-1} \left(-n^{\alpha+d_1-1/2} \rho_n^{\tau-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e)\right) y_{j-1} \\
&= \left(-n^{d_1-1/2} \rho_n^{\tau-\tau_e} \frac{1}{r} \sigma W^{H_1}(r_e)\right) \tau_e \left(\frac{1}{\tau_e} \sum_{j=1}^{\tau_e-1} y_{j-1}\right) \\
&\overset{a}{\sim} \left(-n^{d_1-1/2} \frac{1}{r} \rho_n^{\tau-\tau_e} \sigma W^{H_1}(r_e)\right) \tau_e \left(n^{1/2+d_1} \sigma \int_0^{r_e} W^{H_1}(s) ds\right) \\
&= -n^{2d_1+1} \rho_n^{\tau-\tau_e} \frac{r_e}{r} \sigma^2 W^{H_1}(r_e) \int_0^{r_e} W^{H_1}(s) ds. \qquad \text{(A.2.53)}
\end{aligned}$$

Comparing the order in (A.2.52) and (A.2.53), as $\alpha < 1$, we can write

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1, 2d_1+1\right\}}\right).$$

2. For $\tau \in N_1$, we can express

$$\begin{aligned}
&\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) \\
= &\sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) + \sum_{j=\tau_e}^{\tau_f} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) + \bar{y}_{\tau_f}(y_{\tau_f} - \rho_n y_{\tau_f}) \\
&+ \sum_{j=\tau_f+2}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}\left[-\frac{c}{n^\alpha}y_{j-1}+u_t\right] + \sum_{j=\tau_e}^{\tau_f} \bar{y}_{j-1}u_t + \bar{y}_{\tau_f}(y_{\tau_f+1}-\rho_n y_{\tau_f}) \\
&\quad + \sum_{j=\tau_f+2}^{\tau} \bar{y}_{j-1}\left[-\frac{c}{n^\alpha}y_{j-1}+u_t\right] \\
&= \sum_{j=1}^{\tau} \bar{y}_{j-1}u_t - \frac{c}{n^\alpha}\sum_{j=1}^{\tau_e-1}\bar{y}_{j-1}y_{j-1} - \frac{c}{n^\alpha}\sum_{j=\tau_f+2}^{\tau}\bar{y}_{j-1}y_{j-1} - \rho_n\bar{y}_{\tau_f}y_{\tau_f}.
\end{aligned}
$$

For the first term, similar to (A.1.16), we have

$$
\sum_{j=1}^{\tau} \bar{y}_{j-1}u_t = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,\alpha+2d_1\right\}}\right).
$$

For the second term, following the steps in obtaining (A.2.53), we have

$$
\frac{c}{n^\alpha}\sum_{j=1}^{\tau_e-1}\bar{y}_{j-1}y_{j-1} \overset{a}{\sim} -n^{2d_1+1}\rho_n^{\tau_f-\tau_e}\frac{r_e}{r}\sigma^2 W^{H_1}\int_0^{r_e} W^{H_1}(s)ds.
$$

For the third term,

$$
\begin{aligned}
&\frac{c}{n^\alpha}\sum_{j=\tau_f+2}^{\tau}\bar{y}_{j-1}y_{j-1} \overset{a}{\sim} \frac{c}{n^\alpha}\sum_{j=\tau_f+2}^{\tau}\left(-n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)y_{j-1} \\
&= -n^{d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\sum_{j=\tau_f+2}^{\tau}y_{j-1} \\
&= -n^{d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\left(\tau-\tau_f-1\right)\frac{n^{1/2+d_1}}{\tau-\tau_f-1}\sum_{j=\tau_f+2}^{\tau}\frac{1}{n^{1/2+d_1}}y_{j-1} \\
&\overset{a}{\sim} -n^{2d_1+1}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma^2\left(W^{H_1}(r_e)\right)^2(r-r_f)\int_{r_f}^{r}W^{H_1}(s)ds.
\end{aligned}
$$

For the last term, from Lemma 1.6.2 and (A.1.12),

$$
\begin{aligned}
\rho_n\bar{y}_{\tau_f}y_{\tau_f} &\overset{a}{\sim} \bar{y}_{\tau_f}y_{\tau_f} \text{ (as } \rho_n\to 1) \\
&\overset{a}{\sim} \left(\rho_n^{\tau_f-\tau_e}-\rho_n^{\tau_f-\tau_e}\frac{n^\alpha}{nrc}\right)n^{1/2+d_1}\sigma W^{H_1}(r_e)\left(\rho_n^{\tau_f-\tau_e}n^{1/2+d_1}\sigma_{d_1}W^{H_1}(r_e)\right) \\
&= \rho_n^{2\left(\tau_f-\tau_e\right)}n^{1+2d_1}\sigma^2\left(W^{H_1}(r_e)\right)^2.
\end{aligned}
$$

Note that the last component involves $\rho_n^{2\left(\tau_f-\tau_e\right)}$ and thus dominates the previous terms. Finally, we have

$$
\sum_{j=1}^{\tau}\bar{y}_{j-1}(y_j-\rho_n y_{j-1}) \overset{a}{\sim} -\rho_n^{2\left(\tau_f-\tau_e\right)}n^{1+2d_1}\sigma^2\left(W^{H_1}(r_e)\right)^2.
$$

This completes the proof of Lemma 1.11. ■

Proof of Lemma 1.12

For Δy_t, note that it has different expressions at different periods, viz.,

$$\Delta y_t = \begin{cases} u_{t,d_1} & if \quad t < \tau_e, \\ (\rho_n - 1)y_{t-1} + u_{t,d_2} & if \quad \tau_e \leq t \leq \tau_f, \\ y_{\tau_e} + y^* + u_{\tau_f+1,d_1} - y_{\tau_f} & if \quad t = \tau_f + 1, \\ u_{t,d_1} & if \quad t > \tau_f + 1. \end{cases} . \tag{A.2.54}$$

Note that

$$\sum_{t=1}^{\tau} \Delta y_t^2 = \sum_{t=1}^{\tau_e - 1} \Delta y_t^2 + \sum_{t=\tau_e}^{\tau_f} \Delta y_t^2 + \Delta y_{\tau_f+1}^2 + \sum_{t=\tau_f+2}^{\tau} \Delta y_t^2, \tag{A.2.55}$$

and using (A.2.54), we can write

$$\sum_{t=1}^{\tau} \Delta y_t^2 \tag{A.2.56}$$

$$= \sum_{t=1}^{\tau_e-1} u_{t,d_1}^2 + \sum_{t=\tau_e}^{\tau_f} ((\rho_n - 1)y_{t-1} + u_{t,d_2})^2 + \left(y_{\tau_e} + y^* + u_{\tau_f+1,d_1} - y_{\tau_f}\right)^2 + \sum_{t=\tau_f+2}^{\tau} u_{t,d_1}^2$$

$$= \sum_{t=1}^{\tau_e-1} u_{t,d_1}^2 + \frac{c^2}{n^{2\alpha}} \sum_{t=\tau_e}^{\tau_f} y_{t-1}^2 + \frac{2c}{n^\alpha} \sum_{t=\tau_e}^{\tau_f} y_{t-1} u_{t,d_2} + \sum_{t=\tau_e}^{\tau_f} u_{t,d_2}^2 + \left(y_{\tau_e} + y^* - y_{\tau_f}\right)^2$$

$$+2\left(y_{\tau_e} + y^* - y_{\tau_f}\right) u_{\tau_f+1,d_1} + u_{\tau_f+1,d_1}^2 + \sum_{t=\tau_f+2}^{\tau} u_{t,d_1}^2 \tag{A.2.57}$$

$$= \left[\sum_{t=1}^{\tau_e-1} u_{t,d_1}^2 + \sum_{t=\tau_f+1}^{\tau} u_{t,d_1}^2 + \sum_{t=\tau_e}^{\tau_f} u_{t,d_2}^2\right] + \frac{c^2}{n^{2\alpha}} \sum_{t=\tau_e}^{\tau_f} y_{t-1}^2 + \frac{2c}{n^\alpha} \sum_{t=\tau_e}^{\tau_f} y_{t-1} u_t$$

$$+ \left(y_{\tau_e} + y^* - y_{\tau_f}\right)^2 + 2\left(y_{\tau_e} + y^* - y_{\tau_f}\right) u_{\tau_f+1}. \tag{A.2.58}$$

Now compare the stochastic orders of the components in (A.2.56). First, $\sum_{i=1}^{t} u_{i,d}^2 = O_p(n_d)$, where n_d is defined in (A.2.40). For the second term, applying Lemma 1.7, we have

$$\frac{c^2}{n^{2\alpha}} \sum_{t=\tau_e}^{\tau_f} y_{t-1}^2 \overset{a}{\sim} \frac{c^2}{n^{2\alpha}} \left(n^{1/2+d_1} \sigma W^{H_1}(r_e)\right)^2 \sum_{t=\tau_e}^{\tau_f} \rho_n^{2(t-\tau_e)}$$

$$= \frac{c^2}{n^{2\alpha}} \left(n^{1/2+d_1} \sigma W^{H_1}(r_e)\right)^2 \frac{n^\alpha \rho_n^{2(\tau_f - \tau_e)}}{2c} = \frac{c}{2} n^{1+2d_1-\alpha} \rho_n^{2(\tau_f-\tau_e)} \sigma^2 W^{H_1}(r_e)^2$$

$$= O_p(n^{1+2d_1-\alpha} \rho_n^{2(\tau_f - \tau_e)}). \tag{A.2.59}$$

Suppose $\tau \in B$. We do not have the term in (A.2.55), and (A.2.59) yields (A.1.19). For the third term, note that

$$y_{\tau_e} + y^* - y_{\tau_f} \overset{a}{\sim} n^{1/2+d_1} \sigma W^{H_1}(r_e) + O_p(1) - \rho_n^{(\tau_f - \tau_e)} n^{1/2+d_1} \sigma W^{H_1}(r_e)$$

$$= O_p\left(\rho_n^{(\tau_f-\tau_e)} n^{1/2+d_1}\right),$$

which implies $\left(y_{\tau_e}+y^*-y_{\tau_f}\right)^2 = O_p\left(\rho_n^{2(\tau_f-\tau_e)} n^{1+2d_1}\right)$; and

$$\begin{aligned} 2\left(y_{\tau_e}+y^*-y_{\tau_f}\right) u_{\tau_f+1} &= O_p\left(\rho_n^{(\tau_f-\tau_e)} n^{1/2+d_1}\right) \times O_p(1) \\ &= O_p\left(\rho_n^{(\tau_f-\tau_e)} n^{1/2+d_1}\right). \end{aligned}$$

Thus, the third term which involves $\rho_n^{2(\tau_f-\tau_e)}$ dominates the other terms as $n \to \infty$, and we have (A.1.20). This completes the proof. ■

Proof of Lemma 1.13

By definition, $\tilde{\rho}_\tau = \hat{\rho}_\tau + \frac{\frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2}$. From Lemma 1.9 and Lemma 1.12, it is clear that $\frac{1}{2}\sum_{j=1}^{\tau}\Delta y_t^2$ is at most $O_p\left(n^{1+2d_1}\rho_n^{2(\tau_f-\tau_e)}\right)$ and $\sum_{j=1}^{\tau}\bar{y}_{j-1}^2 = O_p\left(n^{1+2d_1+\alpha}\rho_n^{2(\tau_f-\tau_e)}\right)$ for $\tau \in B \cup N_1$. Hence, $\frac{\frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2} = o_p(1)$. This leaves the asymptotic properties of $\hat{\rho}_\tau$. First, focus on the centered statistics $\hat{\rho}_\tau - \rho_n = \frac{\sum_{j=1}^{\tau}\bar{y}_{j-1}(y_j-\rho_n y_{j-1})}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2}$.

1. When $\tau \in B$, applying (A.1.18) and (A.1.14) gives

$$\begin{aligned} \frac{\sum_{j=1}^{\tau}\bar{y}_{j-1}(y_j-\rho_n y_{j-1})}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2} &= \frac{O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,2d_1+1\right\}}\right)}{O_p\left(n^{1+2d_1+\alpha}\rho_n^{2(\tau-\tau_e)}\right)} \qquad (A.2.60) \\ &= \frac{O_p\left(n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,2d_1+1\right\}-1-2d_1-\alpha}\right)}{O_p\left(\rho_n^{\tau-\tau_e}\right)}. \end{aligned}$$

   Note that (A.2.60) also implies $\tilde{\rho}_\tau - \rho_n = O_p\left(n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,2d_1+1\right\}-2d_1-\alpha}\rho_n^{-(\tau-\tau_e)}\right) = o_p(1)$ as $\rho_n^{\tau-\tau_e}$ diverges exponentially. As $n(\tilde{\rho}_\tau-1) = n(\rho_n-1)+n(\tilde{\rho}_\tau-\rho_n)$, we have

$$n(\rho_n-1)+n(\tilde{\rho}_\tau-\rho_n) = n^{1-\alpha}c + o_p(1) \to \infty. \qquad (A.2.61)$$

2. When $\tau \in N_1$,

$$\begin{aligned} \frac{\sum_{j=1}^{\tau}\bar{y}_{j-1}(y_j-\rho_n y_{j-1})}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2} &\overset{a}{\sim} \frac{-\rho_n^{2\left(\tau_f-\tau_e\right)} n^{1+2d_1}\sigma^2\left(W^{H_1}(r_e)\right)^2}{n^{1+\alpha+2d_1}\frac{\rho_n^{2(\tau_f-\tau_e)}}{2c}\sigma^2 W^{H_1}(r_e)^2} \\ &= -n^{-\alpha}2c, \qquad (A.2.62) \end{aligned}$$

   and (A.2.62) similarly gives the order of $\tilde{\rho}_\tau - \rho_n$.

Then, as $n(\tilde{\rho}_\tau - 1) = n(\rho_n - 1) + n(\tilde{\rho}_\tau - \rho_n)$, we have

$$n(\rho_n - 1) + n(\tilde{\rho}_\tau - \rho_n) = n^{1-\alpha}c - n\left(n^{-\alpha}2c\right) + o_p(1) = -n^{1-\alpha}c + o_p(1) \to -\infty. \tag{A.2.63}$$

This completes the proof of Lemma 1.13. ■

Proof of Lemma 1.14

Recall from (49) that

$$\hat{\Omega}_{HAR} = \frac{1}{\tau}\sum_{i=1}^{\tau-1}\frac{1}{\tau}\sum_{j=1}^{\tau-1}\tau^2 D_\tau\left(\frac{i-j}{\tau}\right)\frac{1}{\sqrt{\tau}}\hat{S}_i\frac{1}{\sqrt{\tau}}\hat{S}_j,$$

and to find its order we need only study the limit of $\frac{1}{\sqrt{\tau}}\hat{S}_i$. Suppose $\tau \in B$, we have

$$\begin{aligned}
\frac{1}{\sqrt{n}}\hat{S}_\tau &= \frac{1}{\sqrt{n}}\sum_{i=1}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1}) = \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{\tau_e-1}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1}) + \sum_{i=\tau_e}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})\right] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}(u_{i,d_1} - (\hat{\rho}_\tau - 1)\bar{y}_{i-1}) + \frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau}(u_{i,d_2} - (\hat{\rho}_\tau - \rho_n)\bar{y}_{i-1}) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}u_{i,d_1} + \frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau}u_{i,d_2} - (\hat{\rho}_\tau - 1)\frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1} - (\hat{\rho}_\tau - \rho_n)\frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau}\bar{y}_{j-1}.
\end{aligned} \tag{A.2.64}$$

Now compare the order of the three terms in (A.2.64). It is clear that $\frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}u_{i,d_1} + \frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau}u_{i,d_2} = O_p\left(n^{\max\{d_1,d_2\}}\right)$. For the second term, note that $(\hat{\rho}_\tau - 1) \overset{a}{\sim} \frac{c}{n^\alpha}$ and

$$\begin{aligned}
\sqrt{n}\frac{1}{n}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1} &= \sqrt{n}\frac{\lfloor nr\rfloor}{n}\frac{1}{\tau}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1} = \sqrt{n}\frac{\lfloor nr\rfloor}{n}\left(\frac{1}{\tau}\sum_{i=1}^{\tau_e-1}y_{i-1} - \frac{\tau_e - 1}{\tau}\frac{1}{\tau}\sum_{i=1}^{\tau}y_i\right) \\
&= O(\sqrt{n}) \times \left(O_p\left(n^{1/2+d_1}\right) - O_p\left(n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\right)\right) = O_p\left(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}\right),
\end{aligned}$$

where we obtain the third equality by the continuous mapping theorem, Lemma 1.6, and Lemma 1.7.

This makes $(\hat{\rho}_\tau - 1)\frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1} = \frac{c}{n^\alpha}O_p\left(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}\right) = O_p\left(n^{d_1}\rho_n^{\tau-\tau_e}\right)$. For the last term in (A.2.64), note that from (A.2.61), we have $\hat{\rho}_\tau - \rho_n = O_p\left(\frac{1}{n^\alpha\rho_n^{\tau-\tau_e}}\right)$ and

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau}\bar{y}_{j-1} &= r\sqrt{n}\frac{\lfloor nr\rfloor}{nr}\frac{1}{\tau}\sum_{i=\tau_e}^{\tau}\bar{y}_{j-1} = r\sqrt{n}\frac{\lfloor nr\rfloor}{nr}\frac{1}{\tau}\sum_{i=\tau_e}^{\tau}\left(y_{j-1} - \frac{1}{\tau}\sum_{i=1}^{\tau}y_j\right) \\
&= r\sqrt{n}\frac{\lfloor nr\rfloor}{nr}\left(\frac{1}{\tau}\sum_{i=\tau_e}^{\tau}y_{j-1} - \frac{\tau - \tau_e + 1}{\tau}\frac{1}{\tau}\sum_{i=1}^{\tau}y_j\right).
\end{aligned}$$

From Lemma 1.7.1 and Lemma (A.2.42), we have

$$\frac{1}{\tau}\sum_{j=\tau_e}^{\tau} y_{j-1} = O_p\left(n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\right),$$

and

$$\frac{1}{\tau}\sum_{j=1}^{\tau} y_{j} = O_p\left(n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\right).$$

Thus,

$$\frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau} \bar{y}_{j-1} = O\left(n^{1/2}\right)O_p\left(n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\right) = O_p\left(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}\right),$$

and this implies

$$(\hat{\rho}_\tau-\rho_n)\frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau} \bar{y}_{j-1} = O_p\left(\frac{1}{n^{\alpha}\rho_n^{\tau-\tau_e}}\right)O_p\left(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}\right) = O_p\left(n^{d_1}\right). \tag{A.2.65}$$

Comparing the order of the three terms in (A.2.64), we obtain

$$\frac{1}{\sqrt{n}}\hat{S}_\tau \overset{a}{\sim} -(\hat{\rho}_\tau-1)\frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1} = O_p\left(n^{d_1}\rho_n^{\tau-\tau_e}\right). \tag{A.2.66}$$

Then (49) and (50) imply $\hat{\Omega}_{HAR} = O_p(n^{2d_1}\rho_n^{2(\tau-\tau_e)})$.

Suppose $\tau \in N_1$. In this case

$$\begin{aligned}
\frac{1}{\sqrt{n}}\hat{S}_\tau &= \frac{1}{\sqrt{n}}\sum_{i=1}^{\tau}\left(\bar{y}_i-\hat{\rho}_\tau\bar{y}_{i-1}\right) \\
&= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{\tau_e-1}\left(\bar{y}_i-\hat{\rho}_\tau\bar{y}_{i-1}\right)+\sum_{i=\tau_e}^{\tau_f}\left(\bar{y}_i-\hat{\rho}_\tau\bar{y}_{i-1}\right)+\sum_{i=\tau_f+1}^{\tau}\left(\bar{y}_i-\hat{\rho}_\tau\bar{y}_{i-1}\right)\right] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}\left(u_{i,d_1}-(\hat{\rho}_\tau-1)\bar{y}_{i-1}\right)+\frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau_f}\left(u_{i,d_2}-(\hat{\rho}_\tau-\rho_n)\bar{y}_{i-1}\right) \\
&\quad +\frac{1}{\sqrt{n}}\sum_{i=\tau_f+1}^{\tau}\left(u_{i,d_1}-(\hat{\rho}_\tau-1)\bar{y}_{i-1}\right) \\
&= \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{\tau_e-1}u_{i,d_1}+\sum_{i=\tau_e}^{\tau_f}u_{i,d_2}+\sum_{i=\tau_f+1}^{\tau}u_{i,d_1}\right)+(\hat{\rho}_\tau-1)\frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1} \\
&\quad -(\hat{\rho}_\tau-\rho_n)\frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau}\bar{y}_{j-1}+(\hat{\rho}_\tau-1)\frac{1}{\sqrt{n}}\sum_{i=\tau_f+1}^{\tau}\bar{y}_{i-1}.
\end{aligned} \tag{A.2.67}$$

The orders of the first three terms in (A.2.67) are $O_p(n^{\max\{d_1,d_2\}})$, $O_p\left(n^{d_1}\rho_n^{\tau-\tau_e}\right)$ (as in (A.2.66)), and $O_p\left(n^{d_1}\right)$ (as in (A.2.65)), respectively. For the last term, we have

$$
\begin{aligned}
(\hat{\rho}_\tau - 1)\frac{1}{\sqrt{n}}\sum_{i=\tau_f+1}^{\tau}\bar{y}_{i-1} &\overset{a}{\sim} \frac{c}{n^\alpha}\frac{1}{\sqrt{n}}\sum_{i=\tau_f+1}^{\tau}\left(-n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right) \\
&= O_p(n^{d_1}\rho_n^{\tau_f-\tau_e}),
\end{aligned}
$$

where we have applied (A.1.13) and (A.2.63) to obtain the asymptotic equivalence. As $\tau \in N_1$, $\tau_f < \tau$, eventually we have the same expression as in (A.2.66) and this implies that $\hat{\Omega}_{HAR} = O_p(n^{2d_1}\rho_n^{2(\tau-\tau_e)})$.

For $s_\tau^2 = \frac{\frac{1}{\tau}\sum_{i=1}^{\tau}\hat{u}_t^2}{\frac{1}{\tau}\sum_{i=1}^{\tau}\bar{y}_{i-1}^2}$, note that when $\tau \in B$ we can write

$$
\begin{aligned}
&\frac{1}{\tau}\sum_{i=1}^{\tau}\hat{u}_t^2 = \frac{1}{\tau}\sum_{i=1}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})^2 = \frac{1}{\tau}\sum_{i=1}^{\tau_{e-1}}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})^2 + \frac{1}{\tau}\sum_{i=\tau_e}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})^2 \\
&= \frac{1}{\tau}\sum_{i=1}^{\tau_{e-1}}(u_{i,d_1} - (\hat{\rho}_\tau - 1)\bar{y}_{i-1})^2 + \frac{1}{\tau}\sum_{i=\tau_e}^{\tau}(u_{i,d_2} - (\hat{\rho}_\tau - \rho_n)\bar{y}_{i-1})^2 \\
&= \frac{1}{\tau}\left(\sum_{i=1}^{\tau_{e-1}}u_{i,d_1}^2 + \sum_{i=\tau_e}^{\tau}u_{i,d_2}^2\right) - 2(\hat{\rho}_\tau - 1)\frac{1}{\tau}\sum_{i=1}^{\tau_{e-1}}\bar{y}_{i-1}u_{i,d_1} - 2(\hat{\rho}_\tau - \rho_n)\frac{1}{\tau}\sum_{i=\tau_e}^{\tau}\bar{y}_{i-1}u_{i,d_2} \\
&+ (\hat{\rho}_\tau - 1)^2\frac{1}{\tau}\sum_{i=1}^{\tau_{e-1}}\bar{y}_{i-1}^2 + (\hat{\rho}_\tau - \rho_n)^2\frac{1}{\tau}\sum_{i=\tau_e}^{\tau}\bar{y}_{i-1}^2. \qquad \text{(A.2.68)}
\end{aligned}
$$

As $n \to \infty$, note that $\frac{1}{\tau}\left(\sum_{i=1}^{\tau_{e-1}}u_{i,d_1}^2 + \sum_{i=\tau_e}^{\tau}u_{i,d_2}^2\right) = \left(O_p(n_{d_1}) + O_p(n_{d_2})\right)/n$, where n_d is defined in (A.2.40), and

$$
(\hat{\rho}_\tau - 1)\frac{1}{\tau}\sum_{i=1}^{\tau_{e-1}}\bar{y}_{i-1}u_i = O_p(n^{-\alpha}) \times O_p(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}) = O_p(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}),
$$

where we obtain the first equality from (A.2.49) and (A.2.61). For the fourth term in (A.2.68),

$$
(\hat{\rho}_\tau - 1)^2\frac{1}{\tau}\sum_{i=1}^{\tau}\bar{y}_{i-1}^2 = O_p(n^{-2\alpha}) \times O_p(n^{-1+2(\alpha+d_1)}\rho_n^{2(\tau-\tau_e)}) = O_p(n^{2d_1-1}\rho_n^{2(\tau-\tau_e)}),
$$

where the first equality follows from (A.2.45) and (A.2.61). For the last term in (A.2.68)

$$
\begin{aligned}
(\hat{\rho}_\tau - \rho_n)\frac{1}{\tau}\sum_{i=\tau_e}^{\tau}\bar{y}_{i-1}u_i &= O_p(n^{-\alpha}\rho_n^{-(\tau-\tau_e)}) \times O_p(\rho_n^{\tau-\tau_e}n^{\frac{1}{2}(1+\alpha)+d_1+d_1\alpha-1}) \\
&= O_p(n^{-\frac{1}{2}(1+\alpha)+d_1+d_1\alpha}),
\end{aligned}
$$

where the first equality follows from (A.2.50) and (A.2.61).

Note that the fourth term in (A.2.68) asymptotically dominates the other 4 terms. Therefore, we have $\frac{1}{\tau}\sum_{i=1}^{\tau}\hat{u}_t^2 = O_p(n^{2d_1-1}\rho_n^{2(\tau-\tau_e)})$. Combining the result from Lemma 1.9.1, where $\sum_{j=1}^{\tau}\bar{y}_{j-1}^2 = O_p(n^{1+2d_1+\alpha}\rho_n^{2(\tau-\tau_e)})$, we have

$$s_\tau^2 = \frac{O_p(n^{2d_1-1}\rho_n^{2(\tau-\tau_e)})}{O_p(n^{1+2d_1+\alpha}\rho_n^{2(\tau-\tau_e)})} = O_p(n^{-2-\alpha}),$$

which completes the proof of Lemma 1.14. ■

Proof of Lemma 6.1 in the main paper.

We sketch the proof as it is similar to Silveira (1991). Let $d_y = 1 + d$, $\pi_j^d = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}$, and write $y_t = y_{t-1} + \Delta_+^{-d}\epsilon_t = \Delta_+^{-d_y}\epsilon_t + y_0 = \sum_{j=1}^t \pi_{t-j}^{d_y}\epsilon_j + y_0$. Let $Y_n(r) = n^{1/2-d_y}\ y_{\lfloor nr \rfloor}$, and then

$$\begin{aligned} Y_n(r) &= n^{1/2-d_y}\sum_{j=1}^{\lfloor nr \rfloor}\pi_{\lfloor nr \rfloor - j}^{d_y}\sigma_j\varepsilon_j + o_p(1) \\ &= n^{1/2-d_y}\sum_{j=1}^{\phi}\pi_{\phi-j}^{d_y}\sigma_j\left(S_j - S_{j-1}\right) + o_p(1), \end{aligned}$$

where $\phi = \lfloor nr \rfloor$ and $S_j = \sum_{i=1}^j \varepsilon_i$.

Setting $V_j = \sum_{i=1}^j z_i$ and $z_t \overset{iid}{\sim} \mathcal{N}(0,1)$, we can write

$$Y_n(r) = Q_{1n}(r) + Q_{2n}(r) + Q_{3n}(r) + Q_{4n}(r) + o_p(1),$$

where

$$\begin{aligned} Q_{1n}(r) &= n^{1/2-d_y}\sum_{j=1}^{\phi-1}\frac{(\phi-j)^{d_y-1}}{\Gamma(d_y)}\sigma_j\left(V_j - V_{j-1}\right), \\ Q_{2n}(r) &= n^{1/2-d_y}\sum_{j=1}^{\phi-1}\pi_{\phi-j}^{d_y}\sigma_j\left[(S_j - S_{j-1}) - (V_j - V_{j-1})\right], \\ Q_{3n}(r) &= n^{1/2-d_y}\sum_{j=1}^{\phi-1}\left(\pi_{\phi-j}^{d_y} - \frac{(\phi-j)^{d_y-1}}{\Gamma(d_y)}\right)\sigma_j\left(V_j - V_{j-1}\right), \\ Q_{4n}(r) &= n^{1/2-d_y}\sigma_\phi\left(S_\phi - S_{\phi-1}\right). \end{aligned}$$

The ideas is to show $Q_{1n}(r) \Rightarrow W_g^H(r)$ and $Q_{2n}(r)$, $Q_{3n}(r)$ and $Q_{4n}(r)$ are all $o_p(1)$. Given the finiteness of $\sup_{s\in[0,1]} g(s)$, it is straightforward to show $Q_{2n}(r)$, $Q_{3n}(r)$ and $Q_{4n}(r)$ are $o_p(1)$ following the approach in Silveira (1991). As the proof is similar to Silveira (1991), we only illustrate the idea for one term, viz., $Q_{2n}(r) = o_p(1)$.

Applying Corollary 2.1 in Silveira (1991), an iid Gaussian sequence $z_t \overset{iid}{\sim} N(0,\sigma^2)$ can form a partial sum $\sum_{t=1}^n z_t$ such that in a suitably extended probability space, $\sum_{t=1}^n z_t$ can approximate $\sum_{t=1}^n \epsilon_t$ and

$$\left|\sum_{t=1}^n \epsilon_t - \sum_{t=1}^n z_t\right| = o\left(n^{\frac{1}{2(1+\theta)}}\right),\ \theta > 0,\ \text{almost surely.} \tag{A.2.69}$$

By virtue of Abel summation by parts, $Q_{2n}(r)$ can be written as

$$\begin{aligned} Q_{2n}(r) &= n^{1/2-d_y}\sum_{j=1}^{\phi-1}\pi_{\phi-j}^{d_y}\sigma_j\left(S_j-S_{j-1}-(V_j-V_{j-1})\right) \\ &= n^{1/2-d_y}\sum_{j=1}^{\phi-2}\left(\pi_{\phi-j}^{d_y}\sigma_j-\pi_{\phi-j-1}^{d_y}\sigma_{j-1}\right)(S_j-V_j) \qquad \text{(A.2.70)} \\ &\quad +n^{1/2-d_y}\pi_1^{d_y}\sigma_{\phi-1}\left(S_{\phi-1}-V_{\phi-1}\right). \end{aligned}$$

For the second term in (A.2.70), we have

$$\begin{aligned} n^{1/2-d_y}\pi_1^{d_y}\sigma_{\phi-1}\left(S_{\phi-1}-V_{\phi-1}\right) &= \pi_1^{d_y}\sigma_{\phi-1}n^{\frac{1}{2}-d_y+\frac{1}{2(1+\theta)}}\frac{\left(S_{\phi-1}-V_{\phi-1}\right)}{n^{\frac{1}{2(1+\theta)}}} \\ &= \pi_1^{d_y}\sigma_{\phi-1}n^{\frac{1}{2}-d_y+\frac{1}{2(1+\theta)}}o_p(1)=o_p(1) \text{ for } \theta>0, \end{aligned}$$

where the second equality is obtained using (A.2.69). For the first term in (A.2.70), we have

$$\begin{aligned} & n^{1/2-d_y}\sum_{j=1}^{\phi-2}\left(\pi_{\phi-j}^{d_y}\sigma_j-\pi_{\phi-j-1}^{d_y}\sigma_{j-1}\right)(S_j-V_j) \\ &= n^{1/2-d_y}\sum_{j=1}^{\phi-2}\left(\pi_{\phi-j}^{d_y}\sigma_j-\pi_{\phi-j-1}^{d_y}\sigma_j+\pi_{\phi-j-1}^{d_y}\sigma_j-\pi_{\phi-j-1}^{d_y}\sigma_{j-1}\right)(S_j-V_j) \\ &= n^{1/2-d_y}\sum_{j=1}^{\phi-2}\sigma_j\left(\pi_{\phi-j}^{d_y}-\pi_{\phi-j-1}^{d_y}\right)(S_j-V_j)+n^{1/2-d_y}\sum_{j=1}^{\phi-2}\pi_{\phi-j-1}^{d_y}\left(\sigma_j-\sigma_{j-1}\right)(S_j-V_j). \end{aligned} \qquad \text{(A.2.71)}$$

For the first term in (A.2.71),

$$\begin{aligned} & n^{1/2-d_y}\sum_{j=1}^{\phi-2}\sigma_j\left(\pi_{\phi-j}^{d_y}-\pi_{\phi-j-1}^{d_y}\right)(S_j-V_j) \\ \le\ & \sup_{1\le j\le\phi}|\sigma_j|\,n^{1/2-d_y}\sum_{j=1}^{\phi-2}\left|\pi_{\phi-j}^{d_y}-\pi_{\phi-j-1}^{d_y}\right||S_j-V_j| \\ =\ & \sup_{1\le j\le\phi}|\sigma_j|\left(\sum_{j=1}^{\phi-2}\left|\pi_{\phi-j}^{d_y}-\pi_{\phi-j-1}^{d_y}\right|\right)\sup_{1\le k\le n}n^{1/2-d_y}|S_k-V_k| \\ \le\ & C\left(\sum_{j=1}^{\phi-2}\left|\pi_{\phi-j}^{d_y}-\pi_{\phi-j-1}^{d_y}\right|\right)\sup_{1\le k\le n}n^{1/2-d_y}|S_k-V_k| \\ =\ & o_p(1). \end{aligned}$$

where the above convergence in probability can be obtained from the proof of Lemma 5 in Marinucci and Robinson (2000). For the second term in (A.2.71), since $\sigma_j = \sigma_{j,n} = g(t/n)$ and g is differentiable, we have $|\sigma_j - \sigma_{j-1}| \leq C/n$ for some positive C, and

$$
\begin{aligned}
& n^{1/2-d_y} \sum_{j=1}^{\phi-2} \pi_{\phi-j-1}^{d_y} (\sigma_j - \sigma_{j-1}) (S_j - V_j) \\
\leq \quad & \frac{C}{n} n^{1/2-d_y+\frac{1}{2(1+\theta)}} \sum_{j=1}^{\phi-2} \pi_{\phi-j-1}^{d_y} \left(\sup_{1\leq j\leq n} \frac{|S_j - V_j|}{n^{\frac{1}{2(1+\theta)}}} \right) \\
\leq \quad & CKn^{1/2-d_y-1+\frac{1}{2(1+\theta)}} \sum_{j=1}^{\phi-2} (\phi - j)^{d_y-1} \left(\sup_{1\leq j\leq n} \frac{|S_j - V_j|}{n^{\frac{1}{2(1+\theta)}}} \right) \\
= \quad & CKn^{1/2-d_y+\frac{1}{2(1+\theta)}-1} O(n^{d_y}) o_p(1) \\
= \quad & CKO\left(n^{-1/2+\frac{1}{2(1+\theta)}}\right) o_p(1) = o_p(1) \text{ for any } \theta > 0,
\end{aligned}
$$

where K is a positive constant. For the second equality, we use the result in Silveira (1991) that, for a sufficiently large K, $\pi_{\phi-j-1}^{d_y} \leq K(\phi - j)^{d_y-1}$. The above results imply that $Q_{2n}(r) = o_p(1)$.

For the weak convergence of $Q_{1n}(r)$, Silveira (1991) showed that for Gaussian processes $Q_{1n}(r)$ and $W_g^H(r)$, we need to verify the following conditions:

1. $\lim_{n\to\infty} \mathbb{E}[Q_{1n}(r)] = \mathbb{E}\left[W_g^H(r)\right]$.
2. $\lim_{n\to\infty} \mathbb{E}[Q_{1n}(r)Q_{1n}(s)] = \mathbb{E}\left[W_g^H(r)W_g^H(s)\right]$.
3. $\mathbb{E}[Q_{1n}(r) - Q_{1n}(q)]^2 \mathbb{E}[Q_{1n}(s) - Q_{1n}(r)]^2 \leq D|s-q|^\gamma$, for all $n \geq 1$ and $0 \leq q < r < s \leq 1$ and D and γ are some positive constants. [1]
4. $\mathbb{E}\left[W_g^H(s) - W_g^H(r)\right]^2 \leq D|s-r|^\gamma$, for $0 \leq s < r \leq 1$ and D and γ are some positive constants.

It is trivial to verify the first condition since both $Q_{1n}(r)$ and $W_g^H(r)$ are Gaussian random variables with a zero mean. For the second condition, note that

$$
\begin{aligned}
\mathbb{E}[Q_{1n}(r)Q_{1n}(s)] &= \mathbb{E}\left[n^{1/2-d_y} \sum_{j=1}^{\lfloor nr \rfloor -1} \frac{(\lfloor nr \rfloor - j)^{d_y-1}}{\Gamma(d_y)} \sigma_j z_j, n^{1/2-d_y} \sum_{j=1}^{\lfloor ns \rfloor -1} \frac{(\lfloor ns \rfloor - j)^{d_y-1}}{\Gamma(d_y)} \sigma_j z_j \right] \\
&= \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=1}^{\lfloor nr \rfloor -1} \sigma_j^2 \left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y-1} \left(\frac{\lfloor ns \rfloor - j}{n} \right)^{d_y-1}. \qquad \text{(A.2.72)}
\end{aligned}
$$

Clearly, (A.2.72) converges to $\frac{1}{\Gamma(d_y)^2} \int_0^r g(x)^2 (r-x)^{d_y-1} (s-x)^{d_y-1} ds$ by the dominated convergence theorem, and the second condition is satisfied.

For the third condition, note that

$$
Q_{1n}(r) - Q_{1n}(q)
$$

[1]This tightness criterion is also used by Akonom and Gourieroux (1987) (p.13) and Marinucci and Robinson (2000) (p.114).

$$= \frac{n^{1/2-d_y}}{\Gamma(d_y)} \left[\sum_{j=1}^{\lfloor nr \rfloor -1} (\lfloor nr \rfloor - j)^{d_y-1} \sigma_j z_j - n^{1/2-d_y} \sum_{j=1}^{\lfloor nq \rfloor -1} (\lfloor nq \rfloor - j)^{d_y-1} \sigma_j z_j \right]$$

$$= \frac{n^{1/2-d_y}}{\Gamma(d_y)} \left[\sum_{j=1}^{\lfloor nq \rfloor -1} \left((\lfloor nr \rfloor - j)^{d_y-1} - (\lfloor nq \rfloor - j)^{d_y-1} \right) \sigma_j z_j + \sum_{j=\lfloor nq \rfloor}^{\lfloor nr \rfloor -1} (\lfloor nr \rfloor - j)^{d_y-1} \sigma_j z_j \right]$$

$$= \frac{1}{\Gamma(d_y)} \frac{1}{n^{1/2}} \sum_{j=1}^{\lfloor nq \rfloor -1} \left(\left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y-1} - \left(\frac{\lfloor nq \rfloor - j}{n} \right)^{d_y-1} \right) \sigma_j z_j$$

$$+ \frac{1}{\Gamma(d_y)} \frac{1}{n^{1/2}} \sum_{j=\lfloor nq \rfloor}^{\lfloor nr \rfloor -1} \left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y-1} \sigma_j z_j.$$

Therefore, we have

$$\begin{aligned} \mathbb{E}\left[Q_{1n}(r) - Q_{1n}(q)\right]^2 &= \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=1}^{\lfloor nq \rfloor -1} \left(\left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y-1} - \left(\frac{\lfloor nq \rfloor - j}{n} \right)^{d_y-1} \right)^2 \sigma_j^2 \\ &\quad + \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=\lfloor nq \rfloor}^{\lfloor nr \rfloor -1} \left(\frac{\lfloor nr \rfloor - j}{n} \right)^{2(d_y-1)} \sigma_j^2 \\ &\leq \tilde{\sigma}^2 (\tilde{Q}_{a,n}(r) + \tilde{Q}_{b,n}(r)), \end{aligned}$$

where

$$\tilde{Q}_{a,n}(r) = \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=1}^{\lfloor nq \rfloor -1} \left(\left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y-1} - \left(\frac{\lfloor nq \rfloor - j}{n} \right)^{d_y-1} \right)^2,$$

$$\tilde{Q}_{b,n}(r) = \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=\lfloor nq \rfloor}^{\lfloor nr \rfloor -1} \left(\frac{\lfloor nr \rfloor - j}{n} \right)^{2(d_y-1)}$$

and $\tilde{\sigma}^2 = \sup_{j \in [1,n]} \sigma_j^2$. Lemma 3-A-3 in Silveira (1991) shows $\tilde{Q}_{a,n}(r) + \tilde{Q}_{b,n}(r) \leq c(r-q)^{\gamma}$ for some constants c and γ. This implies the third condition is satisfied.

For the fourth condition concerning $\mathbb{E}\left[W_g^H(s) - W_g^H(r)\right]^2$, given $\sup_s g(s)^2 < \infty$ and using the same steps as in showing the third condition, it is straightforward to see that the fourth condition holds.

Finally, since $Q_{2n}(r)$, $Q_{3n}(r)$ and $Q_{4n}(r)$ are all $o_p(1)$, the proof of Lemma 6.1 is completed using the weak convergence of $Q_{1n}(r)$. ■

3 Proofs and discussions of various remarks

Proof of Remark 3.4

As in (49), write

$$\hat{\Omega}_{HAR} = \frac{1}{\tau} \sum_{i=1}^{\tau-1} \frac{1}{\tau} \sum_{j=1}^{\tau-1} \tau^2 D_{\tau} \left(\frac{i-j}{b\tau} \right) \frac{1}{\sqrt{\tau}} S_i \frac{1}{\sqrt{\tau}} S_j,$$

and following the steps in proving Theorem 4 in Sun (2004), we can show $\lim_{n\to\infty}\tau^2 D_\tau\left(\frac{i-j}{b\tau}\right) = -\frac{1}{b^2r^2}K''\left(\frac{p-q}{br}\right)$, given $(i/n, j/n)\to(p,q)$. Combining (49), (51) and applying the CMT, we have

$$\begin{aligned}\frac{1}{n^{2d}}\hat{\Omega}_{HAR} &= \frac{1}{\tau}\sum_{i=1}^{\tau-1}\frac{1}{\tau}\sum_{j=1}^{\tau-1}D_\tau\left(\frac{i-j}{b\tau}\right)\frac{1}{n^d}\frac{1}{\sqrt{\tau}}\hat{S}_i\frac{1}{n^d}\frac{1}{\sqrt{\tau}}\hat{S}_j \\ &\Rightarrow -\frac{\sigma^2}{b^2r^3}\int_0^r\int_0^r K''\left(\frac{p-q}{br}\right)W^H(p)W^H(q)dpdq \quad \text{(A.3.73)}\end{aligned}$$

Since $s^2_{\tau,HAR} = \frac{\hat{\Omega}_{HAR}}{\sum_{t=1}^{\tau}y_{t-1}^2-\tau^{-1}\left(\sum_{t=1}^{\tau}y_{t-1}\right)^2}$, Lemma 1.2 and (A.3.73) give

$$\tau^2 s^2_{\tau,adj}\Rightarrow\frac{\int_0^r\int_0^r -K''\left(\frac{p-q}{br}\right)W^H(p)W^H(q)dpdq}{b^2r\int_0^r\left(\tilde{W}^H(s)\right)^2ds}. \qquad \text{(A.3.74)}$$

Finally, since $\widetilde{DF}_{\tau,HAR} = \frac{\tau(\tilde{\rho}_\tau-1)}{\left(\tau^2 s^2_{\tau,HAR}\right)^{1/2}}$, and using (A.1.6) and (A.3.74), standard calculation yields

$$\widetilde{DF}_{\tau,HAR}\Rightarrow\frac{\frac{br^{3/2}}{2}\left(W^H(r)\right)^2 - br^{1/2}\left(\int_0^r W^H(s)ds\right)W^H(r)}{\left(\left(\int_0^r\left(\tilde{W}^H(s)\right)^2ds\right)\int_0^r\int_0^r -K''\left(\frac{p-q}{br}\right)W^H(p)W^H(q)dpdq\right)^{1/2}}.$$

This completes the proof of Remark 3.4. ■

Discussion of Remark 4.2

Our proof of the consistency requires that Lemma 1.6.2 holds. The rest of the proof goes through in the same way as the original proof if we replace n^α by k_n. Following our original proof of expressions (A.2.37) - (A.2.40), the sufficient condition to ensure Lemma 1.6.2 is

$$\rho_n^{-2(t-\tau_e)}\left(\frac{\rho_n^{-2(t-\tau_e)}-\rho_n^2}{1-\rho_n^2}\right)n_{d_2} = \left(\frac{\rho_n^{-4(t-\tau_e)}-\rho_n^{-2(t-\tau_e)}\rho_n^2}{1-\rho_n^2}\right)n_{d_2}\to 0.$$

Note that for $\rho_n = 1+\frac{c}{k_n} = \exp(\frac{c}{k_n}) + o\left(\frac{1}{k_n}\right)$, we have

$$\left(\frac{\rho_n^{-4(t-\tau_e)}-\rho_n^{-2(t-\tau_e)}\rho_n^2}{1-\rho_n^2}\right)n_{d_2} = \frac{\exp(-2c\frac{n(r-r_e)}{k_n})}{2c}n_{d_2}k_n\left(1+o\left(\frac{1}{k_n}\right)\right).$$

If we normalize $c=1$, then the above condition is equivalent to

$$\exp\left(-2\frac{n(r-r_e)}{k_n}\right)n_{d_2}k_n\to 0, \qquad \text{(A.3.75)}$$

and taking a log transformation of (A.3.75) gives

$$-2\frac{n}{k_n}(r-r_e)+\log(n_{d_2})+\log(k_n)\to-\infty.$$

Since $\frac{n}{k_n} \to \infty$, $n_{d_2} \geq n$, and $\log(n_{d_2}) = O(\log(n))$ from (A.2.40), condition (A.3.75) is satisfied if $r > r_e$ and

$$\frac{\log(n)k_n}{n} \to 0. \tag{A.3.76}$$

Discussion of Remark 4.3

Setting $\rho_n = \rho = 1 + c$ with $c > 0$, we obtain similar results to those in the lemmas with small changes in stochastic orders. To see this, recall that when the model is mildly explosive, from Lemma 1.6.2 and Lemma 1.7, we have:

$$y_t \overset{a}{\sim} \rho_n^{t-\tau_e} n^{1/2+d_1} \sigma W^H(r_e), \text{ for } t \in B. \tag{A.3.77}$$

$$\frac{1}{\tau}\sum_{j=1}^{\tau} y_j \overset{a}{\sim} \rho_n^{\tau-\tau_e} n^{\alpha+d_1-1/2} \frac{1}{rc} \sigma W^H(r_e), \text{ for } \tau \in B. \tag{A.3.78}$$

$$\frac{1}{\tau}\sum_{j=1}^{\tau} y_j \overset{a}{\sim} \rho_n^{\tau_f-\tau_e} n^{\alpha+d_1-1/2} \frac{1}{rc} \sigma W^H(r_e), \text{ for } \tau \in N_1. \tag{A.3.79}$$

Now extend (A.3.77)-(A.3.79) to the case where $\rho = 1 + c > 1$.

First, from the inequality (A.2.39), we have

$$\begin{aligned}\sum_{j=0}^{t-\tau_e} \rho^{-j} u_{t-j,d_2} &\leq \left(\frac{\rho^2 - \rho^{-2(t-\tau_e)}}{\rho^2 - 1}\right)^{1/2} \left(\sum_{i=\tau_e}^{t} u_{i,d_2}^2\right)^{1/2} \\ &= \left(\frac{\rho^2 - o(1)}{\rho^2 - 1}\right)^{1/2} \left(\sum_{i=\tau_e}^{t} u_{i,d_2}^2\right)^{1/2} \\ &= O(1)O_p(n_{d_2}^{1/2}) = O_p(n_{d_2}^{1/2}).\end{aligned}$$

Eventually, we have

$$\rho^{-(t-\tau_e)} y_t = \rho y_{\tau_e - 1} + \rho^{-(t-\tau_e)} \sum_{j=0}^{t-\tau_e} \rho_n^{-j} u_{t-j,d_2} = \rho y_{\tau_e-1} + o_p(1).$$

Thus,

$$y_t \overset{a}{\sim} \rho^{t-\tau_e+1} n^{1/2+d_1} \sigma W^H(r_e), \text{ for } t \in B. \tag{A.3.80}$$

For $\tau \in B$, similar to (A.2.42), we can show $\frac{1}{\tau}\sum_{j=1}^{\tau} y_j \overset{a}{\sim} \frac{1}{\tau}\sum_{j=\tau_e}^{\tau} y_j$, and

$$\frac{1}{\tau}\sum_{j=\tau_e}^{\tau} y_j \overset{a}{\sim} \frac{n^{1/2+d_1}\sigma W^H(r_e)}{\tau} \frac{\rho^{\tau-\tau_e+1} - 1}{\rho - 1} \overset{a}{\sim} \rho^{\tau-\tau_e+1} \frac{n^{-1/2+d_1}\sigma W^H(r_e)}{rc}. \tag{A.3.81}$$

Similarly, for $\tau \in N_1$,

$$\frac{1}{\tau}\sum_{j=1}^{\tau} y_j \overset{a}{\sim} \rho^{\tau_f-\tau_e+1} \frac{n^{-1/2+d_1}\sigma W^H(r_e)}{rc}. \tag{A.3.82}$$

A direct comparison of (A.3.77)-(A.3.79) and (A.3.80)-(A.3.82) shows that one can simply set $\alpha = 0$ and replace $\rho_n^{t-\tau_e}$ by $\rho^{t-\tau_e+1}$ to obtain analogous results under a purely explosive alternative. Consistency of the test is then obtained when $\alpha = 0$.

4 Additional empirical results

Table 1 below supports the empirical findings in Table 1 of the main paper using the same time series but over a longer observation period. As in Table 1 of the main paper, DF_n, $\hat{d}$, and the confidence intervals for d are reported, and the results support the conclusions that (i) the time series has a strongly dependent error in view of the ELW estimates $\hat{d}$ and the associated confidence intervals for d, and (ii) there is evidence of explosive behavior in the data using right sided DF_n tests.

Table 1: Right-tailed unit root tests for the S&P 500 PD ratio

Sampling Period	DF_n	$\hat{d}$	90% CI	95% CI
(a') Jan 1871 to Feb 1880	1.32	0.24	(0.06,0.42)	(0.03,0.45)
(b') Jan 1882 to May 1887	0.71	0.31	(0.10,0.52)	(0.06,0.56)
(c') Nov 1936 to Jun 1946	0.61	0.28	(0.08,0.48)	(0.04,0.52)
(d') Aug 1947 to Nov 1955	1.42	0.23	(0.04,0.42)	(0.01,0.45)
(e') Jun 1977 to Mar 1987	1.93	0.21	(0.03,0.39)	(0.00,0.42)
(f') May 1988 to Apr 1998	3.76	0.24	(0.06,0.42)	(0.03,0.45)

5 Simulations for empirical rejection rates under various c and n

The simulation design here uses the following alternative model:

$$\begin{cases} y_t &= (y_{t-1} + u_t)\, 1\{t < \tau_e\} + (\rho_n y_{t-1} + u_t)\, 1\{\tau_e \le t \le n\}, t = 1, ..., n, \\ u_t &= \Delta_+^{-d}\epsilon_t, d = 0.25, \epsilon_t \overset{iid}{\sim} \mathcal{N}(0,1), \tau_e = \lfloor n \times 0.5 \rfloor, \\ \rho_n &= 1 + c/n^{0.7}, c \in \{1, 1.5\}, n \in \{100, 400, 700, ..., 5200\}, y_0 = 0. \end{cases}$$

The number of replications in all experiments is 2,500. Figure 1 below reports the empirical rejection rates of the right-tailed $\widetilde{DF}_{n,HAR}$ test at the 5% level. The red dotted line and dashed blue line show the empirical powers of the test under $c = 1$ and $c = 1.5$, respectively. Empirical power evidently rises with increasing c and n.

6 Simulations for the heteroskedastic model

This simulation employs the following design model

$$\begin{cases} y_t &= y_{t-1} + u_t,\ y_0 = 0, t = 1, ..., n, \\ u_t &= \Delta_+^{-d}\epsilon_t = \Delta_+^{-d} g(t)\varepsilon_t,\ \varepsilon_t \overset{iid}{\sim} \mathcal{N}(0,1), \end{cases}$$

where $d \in \{0, 0.05, 0.1, ..., 0.45\}, n \in \{100, 500\}$ and $g(t)$ takes one of the following three forms:

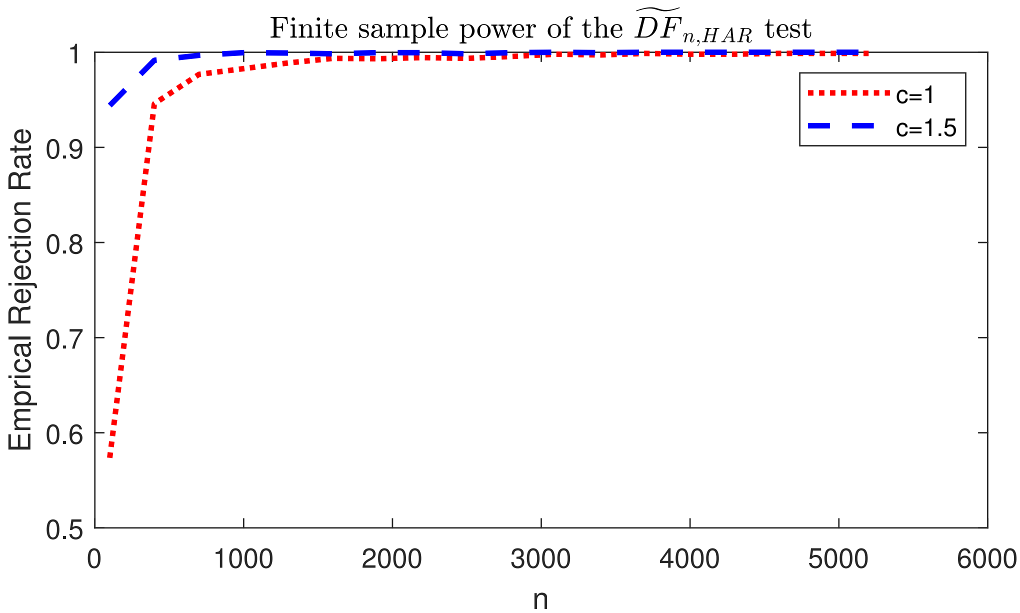

Figure 1: Empirical power of the $\widetilde{DF}_{n,HAR}$ test as a function of n.

1. $g(t) = 1 + \left(\sqrt{2} - 1\right) \frac{t}{n}$ (DGP1).
2. $g(t) = 1 + \frac{\sqrt{2}-1}{1+\exp(-\theta(t-n)/2)}$, $\theta = 0.25$ (DGP2).
3. $g(t) = 1 + \frac{\sqrt{2}-1}{1+\exp(-\theta(t-n)/2)}$, $\theta = -0.25$ (DGP3).

To estimate $g(t)$, similar to Astill et al. (2023), we choose the bandwidth ν^* to minimize

$$CV(\nu) = \frac{1}{T} \sum_{t=1}^{n} \left(\hat{g}_v^2(t) - \Delta^{\hat{d}} y_t^2 \right)^2,$$

for $\nu \leq V$, where $\hat{g}_v^2(t)$ is the estimator of $g^2(t)$ given the bandwidth ν and the tuning parameter V is set to 20. And we use the one sided Epanechnikov kernel to estimate $g^2(t)$. The number of replications in all experiments is 2,500. The empirical size of $\widetilde{DF}^x_{n,HAR}$ tests with 5% significance level is reported in Table 2 below.

7 Sup statistic

As mentioned in Remark 3.6, a version of right-tailed sup statistics can be employed. The sup statistic $\widetilde{SDF}_{HAR}(\tau_0)$ is given by

$$\widetilde{SDF}_{HAR}(\tau_0) = \sup_{\tau \in [\tau_0, n]} \frac{\tilde{\rho}_\tau - 1}{s_{\tau,HAR}},$$

and its limit theory under null and alternative hypotheses are given in the following result.

Table 2: Empirical sizes of $\widetilde{DF}^{x}_{n,HAR}$ for various d and DGPs with a nominal 5% right-tailed critical value

					$n=100$					
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$DGP1$	0.07	0.07	0.07	0.06	0.07	0.07	0.06	0.07	0.06	0.07
$DGP2$	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.07
$DGP3$	0.08	0.07	0.07	0.08	0.07	0.07	0.07	0.07	0.07	0.07
					$n=500$					
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$DGP1$	0.05	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.04
$DGP2$	0.05	0.04	0.05	0.04	0.05	0.04	0.04	0.04	0.04	0.04
$DGP3$	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05

Theorem 7.1 *Let $M=\lfloor b\tau \rfloor$ and $K_B(x)$ be the Bartlett kernel function. Under model (12), as $n\to\infty$,*

$$\widetilde{SDF}_{HAR}(\tau_0) \Rightarrow \sup_{r\in[r_0,1]} \frac{b^{1/2}\left[\frac{r}{2}(W^H(r)^2)-\left(\int_0^r W^H(s)dsW^H(r)\right)\right]}{\left[2r\left(\int_0^r \tilde{W}^H(s)^2ds\right)\left(\int_0^1 W^H(p)^2dp-\int_0^{1-b}W^H(p)W^H(p+br)dp\right)\right]^{1/2}}. \tag{A.7.83}$$

Under model (25), as $n\to\infty$,

$$\widetilde{SDF}_{HAR}(\tau_0) \Rightarrow \sup_{r\in[r_0,1]} \frac{b^{1/2}\left(\frac{r}{2}C_{r,d}-A_{r,d}W^H(r)+B_{r,d}cr-cA^2_{r,d}\right)}{\left[2r\left(B_{r,d}-\frac{1}{r}A^2_{r,d}\right)\left(\int_0^1 G_{r_e,c}(p)^2dp-\int_0^{(1-b)}G_{r_e,c}(p)G_{r_e,c}(p+br)dp\right)\right]^{1/2}}.$$

Under model (27)-(28) if $\tau_0<\tau_f$, $\widetilde{SDF}_{HAR}(\tau_0)\xrightarrow{p}\infty$, as $n\to\infty$.

Theorem 7.1 above establishes the asymptotic behavior of $\widetilde{SDF}_{HAR}(\tau_0)$ under the null hypothesis, local alternative, and mildly explosive alternative. Under the null the sup statistic has a well-defined limit. Under the local alternative, the limit of the test statistic can be used to obtain the local power function. Under the mildly explosive alternative, the divergent behavior of the test statistic implies consistency. Further, the limit distribution (A.7.83) and consistent estimation of d allow us to obtain the $\beta\times 100\%$ critical value, denoted by $scv^{\beta}_{HAR}\left(\hat{d}\right)$, for practical implementation of the test.

To investigate the empirical size of $\widetilde{SDF}_{HAR}(\tau_0)$, we perform a Monte Carlo study based on the DGP (39). Let $d\in\{0,0.05,...,0.45\}$. To calculate $\widetilde{SDF}_{HAR}(\tau_0)$, as in Section 7, we let $b=0.05$. For the minimum window, based on extensive simulations, we find that the following rule of thumb gives satisfactory size and power performance in finite samples: $r_0=0.01+4.9/\sqrt{n}$. So $r_0\approx 0.5$ if $n=100$ and $r_0\approx 0.23$ if $n=500$. For comparison we report both the empirical size of $SDF(\tau_0)$ and $\widetilde{SDF}_{HAR}(\tau_0)$ based on the 5% critical value in Table 3 below. The findings echo those in Table 2 of the main paper. Mostly importantly, $SDF(\tau_0)$ suffers severe oversizing when d is large, whereas $\widetilde{SDF}_{HAR}(\tau_0)$ has empirical size close to the nominal level.

Table 3: Empirical size of $SDF(\tau_0)$ and $\widetilde{SDF}_{HAR}(\tau_0)$

	$n = 100, r_0 = 0.50$									
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$SDF(\tau_0)$	0.04	0.08	0.15	0.23	0.32	0.41	0.50	0.56	0.63	0.68
$\widetilde{SDF}_{HAR}(\tau_0)$	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.06	0.06	0.06
	$n = 500,\ r_0 = 0.23$									
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$SDF(\tau_0)$	0.04	0.15	0.32	0.49	0.64	0.75	0.83	0.87	0.90	0.93
$\widetilde{SDF}_{HAR}(\tau_0)$	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05

To explore power of the sup test the simulations are based on model (29) with the following parameter settings: $y_0 = 100$, $n = 100$, $c = 1$, $\alpha = 0.6$, $r_e = 0.6$, $r_f \in \{0.7, 0.75, 0.85\}$ and $d \in \{0, 0.05, ..., 0.45\}$. Similar to Table 3, we report the power of $\widetilde{SDF}_{HAR}(\tau_0)$ based on the 5% critical value in Table 4. The results show that $\widetilde{SDF}_{HAR}(\tau_0)$ has good power performance in detecting explosive behavior.

Table 4: Power of $\widetilde{SDF}_{HAR}(\tau_0)$ when $n = 100, r_0 = 0.5$

d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
	$r_e = 0.6, r_f = 0.7$									
$\widetilde{SDF}_{HAR}(\tau_0)$	1.00	1.00	1.00	0.99	0.97	0.94	0.90	0.85	0.80	0.75
	$r_e = 0.6, r_f = 0.75$									
$\widetilde{SDF}_{HAR}(\tau_0)$	1.00	1.00	1.00	1.00	0.99	0.98	0.95	0.91	0.87	0.82
	$r_e = 0.6, r_f = 0.8$									
$\widetilde{SDF}_{HAR}(\tau_0)$	1.00	1.00	1.00	1.00	1.00	0.98	0.98	0.95	0.92	0.87

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