Supplement to Two Papers on Multiple Bubbles*†

Peter C. B. Phillips
Yale University, University of Auckland,
University of Southampton & Singapore Management University

Shu-Ping Shi
The Australian National University

Jun Yu
Singapore Management University

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This paper provides a supplement to two companion papers by the authors: "Testing for Multiple Bubbles: Historical Episodes of Exuberance and Collapse in the S&P 500" (PSY1 hereafter); and "Testing for Multiple Bubbles: Limit Theory of Real Time Detectors" (PSY2 hereafter). Section 1 supplements the empirical application of PSY1 by examining the robustness of the bubble identification and dating results to the choice of the minimum window size parameter used in the rolling regression framework of PSY. Section 2 provides proofs of supplementary lemmas that facilitate analysis of the multiple bubble case, derives the limit behaviour of the recursive unit root and BDF test statistics discussed in PSY2 in a model with two bubble episodes, and gives complete proofs for Theorem 4-9 in PSY2 which describe the consistency properties of the PWY, PSY and sequential PWY dating procedures.

1 Empirical Supplement to PSY1

The minimum window size used in this section contains 36 observations (3 years of monthly data), which is approximately 2% of the sample of 1680 observations, compared with the (approximate) 5% of the sample used in the empirical application of PSY1. Importantly, as r_0

*(i) "Testing for Multiple Bubbles: Historical Episodes of Exuberance and Collapse in the S&P 500"; (ii) "Testing for Multiple Bubbles: Limit Theory of Real Time Detectors".

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approaches zero, the tests tend to suffer from increased size distortion. Extensive simulations conducted by the authors led to the recommendation in PSY1 of the sample size dependent choice rule $r_0 = 0.01 + 1.8/\sqrt{T}$ for use in applications. The exercise conducted here analyzes the impact of using a smaller value of r_0 in the empirics. The discussion below follows that of PSY1, details the new findings and compares the results with those given in PSY1, showing that the main episodes of exuberance and collapse that are detected in PSY1 are robust to the choice of a smaller value of r_0.

We first apply the summary SADF and GSADF tests to the price-dividend ratio. Table 1 presents critical values for these two tests obtained by simulation with $2,000$ replications and sample size $1,680$. From Table 1, the SADF and GSADF statistics for the full data series are 3.30 and 4.21, obtained from subsamples 1987M01-2000M07 and 1976M04-1999M06, respectively. Both exceed their respective 1% right-tail critical values (i.e. $3.30 > 2.17$ and $4.21 > 3.31$), giving strong evidence that the S&P 500 price-dividend ratio had explosive subperiods. We conclude from both summary tests that there is evidence of bubbles in the S&P 500 stock market data. These calculations used a transient dynamic lag order $k = 0$. The findings are robust to other choices. For example, the same conclusion applies when $k = 3$, where the SADF and GSADF tests for the full data series are 2.16 and 3.88 with corresponding 5% critical values of 1.70 and 3.40. These results corroborate closely with those reported in PSY1 using the rule $r_0 = 0.01 + 1.8/\sqrt{T}$.

Table 1: The SADF test and the GSADF test of the S&P500 price-dividend ratio

	Test Stat.	Finite Sample Critical Values		
		90%	95%	99%
SADF	3.30	1.45	1.70	2.17
GSADF	4.21	2.55	2.80	3.31

Note: Critical values of both tests are obtained from Monte Carlo simulation with $2,000$ replications (sample size 1,680). The smallest window has 36 observations.

Next, we conduct a (pseudo) real-time bubble monitoring exercise for the S&P 500 stock market using PSY, PWY, sequential PWY, and CUSUM dating strategies. With a training period of 36 observations, we monitor the time series behavior of the price-dividend ratio for the market from January 1874 until the end of the sample period.[1]

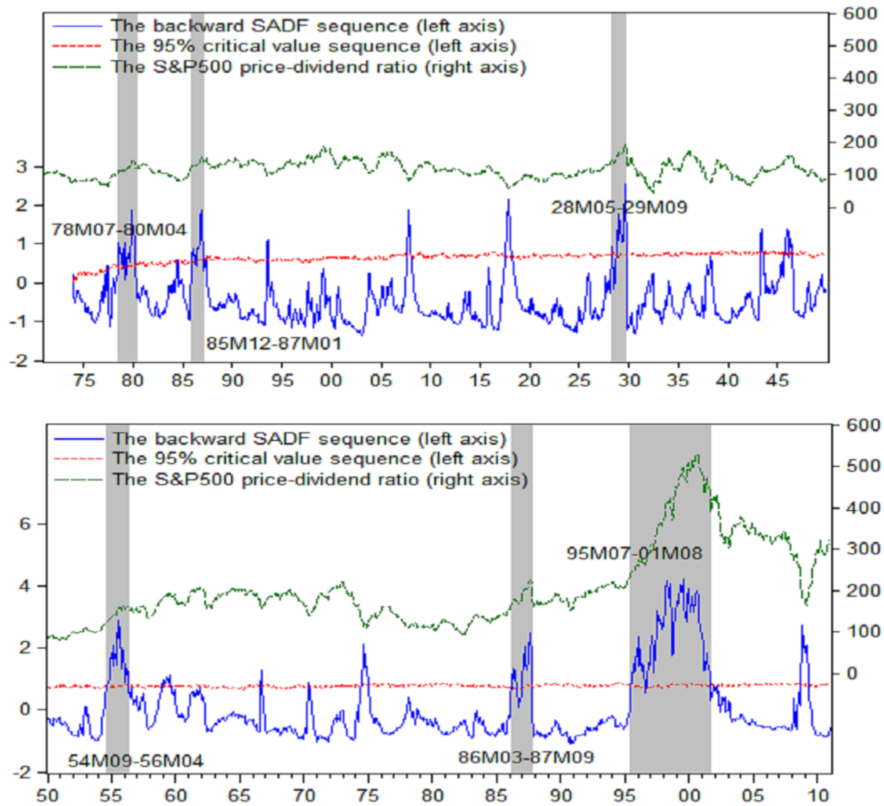

Figure 1: Date-stamping bubble periods in the S&P 500 price-dividend ratio: the GSADF test.

For the PSY real-time dating strategy, we compared the backward SADF statistic with the 95% SADF critical value (obtained from Monte Carlo simulations with $2,000$ replications) for each observation of interest. The top panel of Figure 1 displays the results for this date-stamping exercise over the period from January 1874 to December 1949 and the bottom panel displays results over the rest of the sample period. We focus attention on episodes with duration longer than one year (i.e. $\delta = 3.7$). The identified periods of exuberance in the market include *the so-called post long-depression period (1878M07-1880M04)*, the stock market expansion in the

[1]We have also applied the PSY procedure to the logarithm of the real S&P 500 price index (instead of the price-dividend ratio) and considered minimum window sizes of 48 and 60 observations (equivalent to 4 and 5 years). There were only minor discrepancies in the test results from these robustness exercises.

late 1880s (1885M12-1887M01), *the great crash episode* (1928M05-1929M09), *the postwar boom in 1954* (1954M09-1956M04), *black Monday in October 1987* (1986M03-1987M09), and *the dot-com bubble* (1995M07-2001M08). With regard to the *dot-com bubble*, the PSY strategy detects mildly explosive market behavior 5 years before the market crashes. These results again align closely with the empirical findings in PSY1 using the choice rule $r_0 = 0.01 + 1.8/\sqrt{T}$.

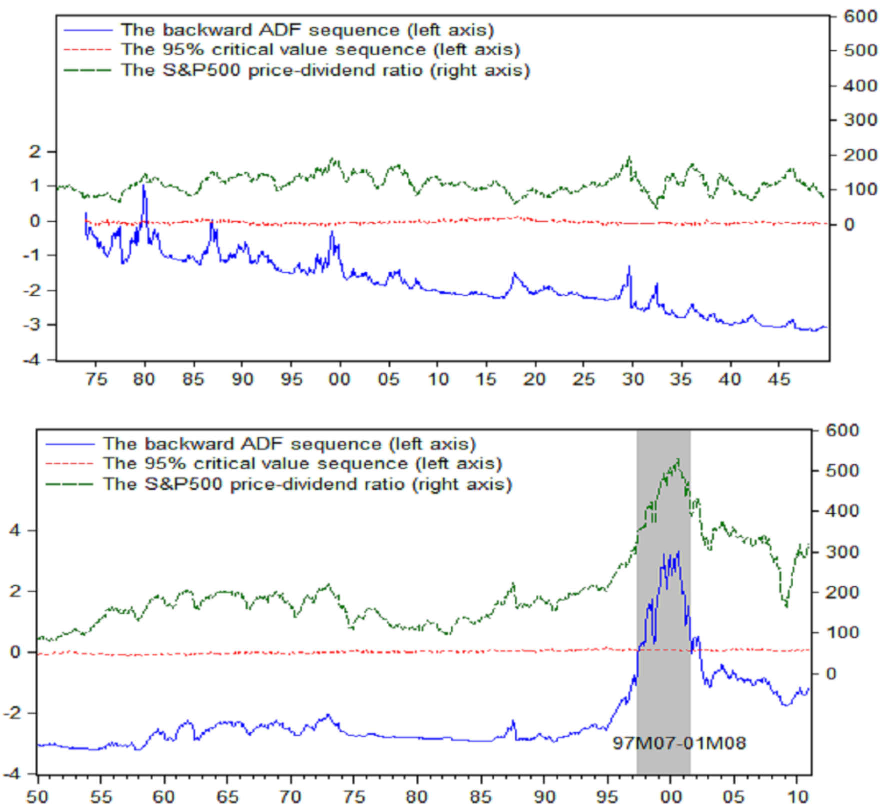

Figure 2: Date-stamping bubble periods in the S&P 500 price-dividend ratio: the SADF test.

The PSY strategy also identifies several short crisis periods, including *the banking panic of* 1907 (1907M09-1908M02), *the 1917 stock market crash* (1917M08-1918M04), *the 1974 stock market crash* (1974M07-M12), and *the subprime mortgage crisis* (2008M10-2009M04). These shorter episodes all relate to market downturns rather than expansionary bubble periods. There are several reasons why bubble tests can identify crashes. These are mainly associated with the use of a small minimum window width r_0, especially in the presence of rapid changes in the data, conditional heteroskedasticity or large (possibly nonstationary) residual variation during periods of turbulence.

Figure 2 plots the ADF statistic sequence against the 95% ADF critical value sequence. We can see that the PWY strategy (based on the SADF test) identifies only a single explosive period with duration longer than one year – *the dot-com bubble* (1997M07-2001M08).

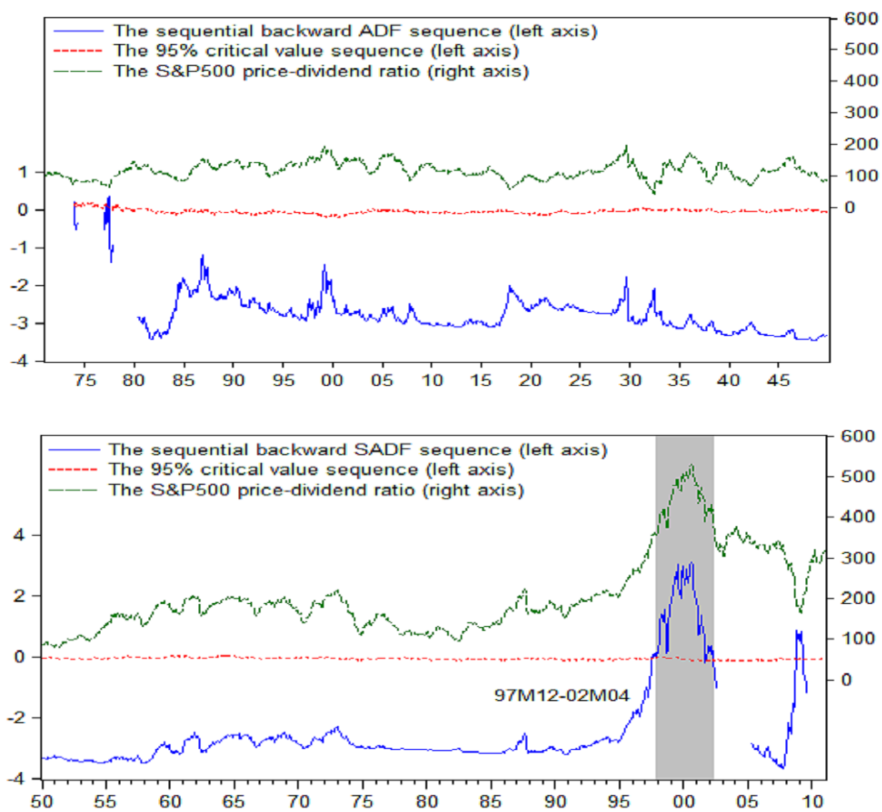

Figure 3: Date-stamping bubble periods in the S&P 500 price-dividend ratio: the sequential PWY strategy.

Empirical results from the sequential PWY procedure are shown in Figure 3 which plots the ADF statistic sequence against the 95% ADF critical value sequence (as for the PWY dating strategy). The sequential ADF plot has two breaks in the Figure, each corresponding to the reinitialization of the test procedure following a collapse. Findings from the sequential PWY test indicate a single bubble with duration longer than one year – the *dot-com bubble* from 1997M12 to 2002M04.[2]

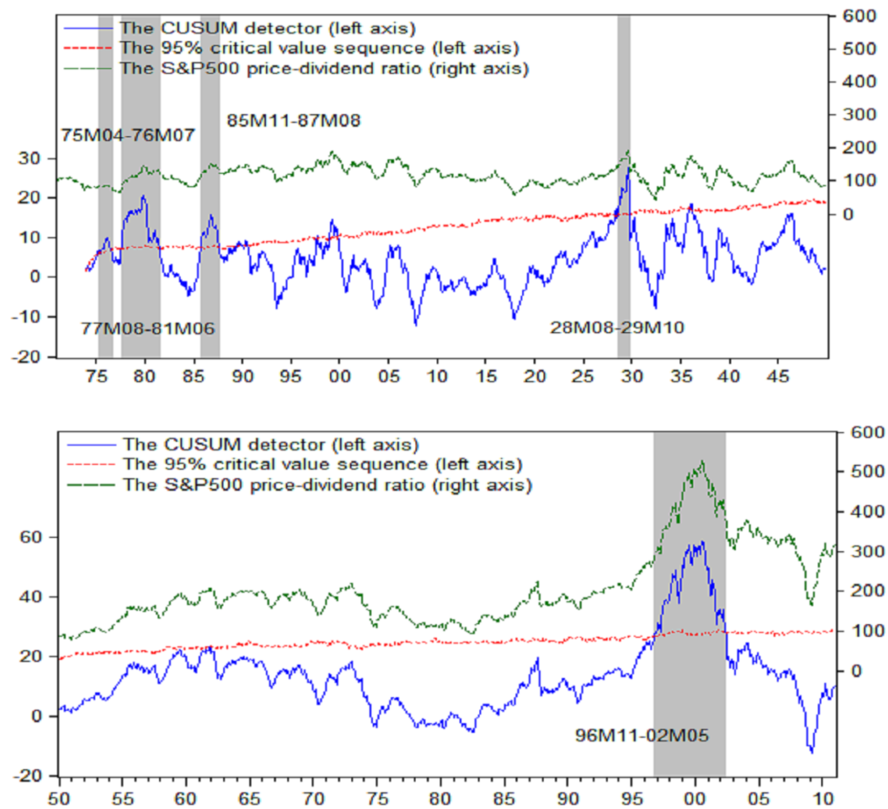

Figure 4: Date-stamping bubble periods in the S&P 500 price-dividend ratio: the CUSUM monitoring procedure.

For further comparison, we applied the CUSUM monitoring procedure to the detrended S&P 500 price-dividend ratio (i.e. to the residuals from the regression of y_t on a constant and a linear time trend). To be consistent with the SADF and GSADF dating strategies, we chose a training sample of 36 months. Figure 4 plots the CUSUM detector sequence against the 95% critical

[2]If the transient dynamic lag order is $k = 3$, the backward SADF strategy identifies two additional episodes (namely, 1945M12-1946M07 and 1969M11-1970M12). The PWY and sequential PWY strategies identify the same bubble episodes but with slight changes in dates.

value sequence. The critical value sequence is obtained from Monte Carlo simulation (through application of the CUSUM detector to data simulated from a pure random walk) with 2,000 replications.

As is evident in Figure 4, the CUSUM test identifies four bubble episodes for periods before 1900 (with three longer than one year). For the post-1900 sample, the procedure detects only *the great crash* (28M08-29M10) and *the dot-com bubble* (96M11-02M05) episodes. It does not provide any prior warning for or acknowledgment of the great crash in 1929 or the black Monday episode in October 1987, among other events that are identified by the GSADF dating strategy. These results closely match the findings in PSY1 for the post 1900 data. For the period before 1900, the CUSUM test here identifies three episodes in addition to the *post long-depression* episode found in PSY1 using CUSUM.

2 Technical Supplement to PSY2

2.1 Notation and lemmas

- The two bubble periods are $B_1 = [\tau_{1e}, \tau_{1f}]$ and $B_2 = [\tau_{2e}, \tau_{2f}]$, where $\tau_{1e} = \lfloor Tr_{1e} \rfloor$, $\tau_{1f} = \lfloor Tr_{1f} \rfloor$, $\tau_{2e} = \lfloor Tr_{2e} \rfloor$ and $\tau_{2f} = \lfloor Tr_{2f} \rfloor$.
- The normal periods are $N_0 = [1, \tau_{1e})$, $N_1 = (\tau_{1f}, \tau_{2e})$, $N_2 = (\tau_{2f}, \tau]$, where $\tau = \lfloor Tr \rfloor$ is the last observation of the sample.

We use the data generating process

$$X_t = \begin{cases} X_{t-1} + \varepsilon_t & \text{for } t \in N_0 \\ \delta_T X_{t-1} + \varepsilon_t & \text{for } t \in B_i \text{ with } i = 1, 2 \\ X^*_{\tau_{if}} + \sum_{k=\tau_{if}+1}^{t} \varepsilon_k & \text{for } t \in N_i \text{ with } i = 1, 2 \end{cases}, \tag{1}$$

where $\delta_T = 1 + cT^{-\alpha}$ with $c > 0$ and $\alpha \in (0, 1)$, $\varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$ and $X^*_{\tau_{if}} = X_{\tau_{ie}} + X^*$ with $X^* = O_p(1)$ for $i = 1, 2$.

Under the stated conditions, partial sums of ε_t satisfy the functional law

$$T^{-1/2} \sum_{t=1}^{\lfloor T \cdot \rfloor} \varepsilon_t \Rightarrow B(\cdot) := \sigma W(\cdot), \tag{2}$$

where W is standard Brownian motion. We start by proving the following lemmas which give limit theory for the components in recursive unit root statistics under successive stochastic trend and bubble regimes. We follow the approach developed in the unpublished working paper by Phillips and Yu (2009). The results given here require some very detailed and in some cases complex calculations to obtain the limits and orders of magnitude of components of the recursive unit root statistics in the various detection algorithms. While these results are specific to the bubble model context under study, the methods will be useful in other recursive regression contexts. With minor modifications, the results continue to hold under correspondingly general conditions on the innovations ε_t for which the weak convergence (2) applies as well as the limit theory for mildly explosive processes given in Phillips and Magdalinos (2007a, 2007b). To keep this supplement to manageable length we do not go into the details of those extensions here.

Lemma S1. *Under the data generating process,*

(1) *For* $t \in N_0$, $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} B(p)$.

(2) *For* $t \in B_i$ *with* $i = 1, 2$, $X_{t=\lfloor Tp \rfloor} = \delta_T^{t-\tau_{ie}} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{1/2} \delta_T^{t-\tau_{ie}} B(r_{ie})$.

(3) *For* $t \in N_i$ *with* $i = 1, 2$, $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} [B(p) - B(r_{if}) + B(r_{ie})]$.

Proof. (1) For $t \in N_0$, X_t is a unit root process. We know that $T^{-1/2} X_{t=\lfloor Tp \rfloor} \xrightarrow{L} B(p)$ as $T \to \infty$.

(2) For $t \in B_i$ with $i = 1, 2$, the data generating process

$$X_t = \delta_T X_{t-1} + \varepsilon_t = \delta_T^{t-\tau_{ie}+1} X_{\tau_{ie}-1} + \sum_{j=0}^{t-\tau_{ie}} \delta_T^j \varepsilon_{t-j}.$$

Based on Phillips and Magdalinos (2007a, lemma 4.2), we know that for $\alpha < 1$,

$$T^{-\alpha/2} \sum_{j=0}^{t-\tau_{ie}} \delta_T^{-(t-\tau_{ie})+j} \varepsilon_{t-j} \xrightarrow{L} X_c \equiv N\left(0, \sigma^2/2c\right)$$

as $t - \tau_{ie} \to \infty$. Furthermore, we know that $T^{-1/2} X_{\tau_{ie}-1} \xrightarrow{L} B(r_{ie})$ and $\delta_T \to 1$ as $T \to \infty$. Therefore,

$$\frac{\delta_T^{-(t-\tau_{ie})}}{T^{1/2}} X_t = \delta_T T^{-1/2} X_{\tau_{ie}-1} + T^{-1/2} \sum_{j=0}^{t-\tau_{ie}} \delta_T^{-(t-\tau_{ie})+j} \varepsilon_{t-j}$$

$$= \delta_T T^{-1/2} X_{\tau_{ie}-1} + T^{-(1-\alpha)/2} T^{-\alpha/2} \sum_{j=0}^{t-\tau_{ie}} \delta_T^{-(t-\tau_{ie})+j} \varepsilon_{t-j} \xrightarrow{L} B(r_{ie}).$$

This implies that $\delta_T^{t-\tau_{ie}+1} X_{\tau_{ie}-1}$ has a higher order than $\sum_{j=0}^{t-\tau_{ie}} \delta_T^j \varepsilon_{t-j}$ and hence

$$X_t = \delta_T^{t-\tau_{ie}} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{1/2} \delta_T^{t-\tau_{ie}} B(r_{ie}).$$

(3) For $t \in N_i$ with $i = 1, 2$,

$$X_t = \sum_{k=\tau_{if}+1}^{t} \varepsilon_k + X_{\tau_{if}}^* = \sum_{k=\tau_{if}+1}^{t} \varepsilon_k + X_{\tau_{ie}} + X^* \sim_a T^{1/2} \left[B(p) - B(r_{if}) + B(r_{ie})\right]$$

due to the fact that $X_{\tau_{ie}} \sim_a T^{1/2} B(r_{ie})$, $\sum_{k=\tau_{if}+1}^{t} \varepsilon_k \sim_a T^{1/2} \left[B(p) - B(r_{if})\right]$ and $X^* = O_p(1)$. □

Lemma S2. *Under the data generating process,*

(1) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_2 - \tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_2 - \tau_{ie}} \frac{1}{r_w c} B(r_{ie}).$$

(2) *For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{if} - \tau_1}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{if} - \tau_1} \frac{1}{r_w c} B(r_{ie}).$$

(3) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = X_{\tau_{ie}} \frac{T^\alpha \delta_T^{\tau_{if} - \tau_{ie}}}{\tau_w c} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{if} - \tau_{ie}} \frac{1}{r_w c} B(r_{ie}).$$

(4) *For $\tau_1 \in N_0$ and $\tau_2 \in N_2$, if $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$*

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{1f} - \tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{1f} - \tau_{1e}} \frac{1}{r_w c} B(r_{1e})$$

and if $\tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e}$

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{2f} - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{2f} - \tau_{2e}} \frac{1}{r_w c} B(r_{2e}).$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$, *if* $\tau_{1f} - \tau_1 > \tau_2 - \tau_{2e}$

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\{1+o_p(1)\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\frac{1}{r_w c}B(r_{1e});$$

if $\tau_{1f} - \tau_1 \leq \tau_2 - \tau_{2e}$

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\{1+o_p(1)\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c}B(r_{2e}).$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$, *if* $\tau_{1f} - \tau_1 > \tau_{2f} - \tau_{2e}$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\{1+o_p(1)\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\frac{1}{r_w c}B(r_1)$$

and if $\tau_{1f} - \tau_1 \leq \tau_{2f} - \tau_{2e}$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\{1+o_p(1)\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\frac{1}{r_w c}B(r_{2e}).$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$, *if* $\tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e}$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\{1+o_p(1)\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\frac{1}{r_w c}B(r_{1e})$$

and if $\tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e}$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\{1+o_p(1)\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c}B(r_{2e}).$$

Proof. (1) For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{ie}-1} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{ie}}^{\tau_2} X_j.$$

We know that

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{ie}-1} X_j = T^{1/2}\frac{\tau_{ie}-\tau_1}{\tau_w}\left(\frac{1}{\tau_{ie}-\tau_1}\sum_{j=\tau_1}^{\tau_{ie}-1}\frac{X_j}{\sqrt{T}}\right) \sim_a T^{1/2}\frac{r_{ie}-r_1}{r_w}\int_{r_1}^{r_{ie}} B(s)\,ds, \qquad (3)$$

$$\begin{aligned}
\frac{1}{\tau_w}\sum_{j=\tau_{ie}}^{\tau_2} X_j &= \frac{X_{\tau_{ie}}}{\tau_w}\sum_{j=\tau_{ie}}^{\tau_2}\delta_T^{j-\tau_{ie}}\left\{1+o_p(1)\right\} \text{ from Lemma S1}\\
&= \frac{X_{\tau_{ie}}}{\tau_w}\frac{\delta_T^{\tau_2-\tau_{ie}+1}-1}{\delta_T-1}\left\{1+o_p(1)\right\}\\
&= X_{\tau_{ie}}\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}+c\delta_T^{\tau_2-\tau_{ie}}-T^{\alpha}}{\tau_w c}\left\{1+o_p(1)\right\}\\
&= X_{\tau_{ie}}\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c}\left\{1+o_p(1)\right\} \qquad (4)\\
&\sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{ie}}\frac{1}{r_w c}B(r_{ie}). \qquad (5)
\end{aligned}$$

Since

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{ie}}}{T^{1/2}} = \frac{\delta_T^{\tau_2-\tau_{ie}}}{T^{1-\alpha}} = \frac{e^{c(r_2-r_{ie})T^{1-\alpha}}}{T^{1-\alpha}} > 1,$$

$\frac{1}{\tau}\sum_{j=\tau_{ie}}^{\tau_2} X_j$ has a higher order than $\frac{1}{\tau}\sum_{j=\tau_1}^{\tau_{ie}-1} X_j$ and hence

$$\begin{aligned}
\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_{ie}}^{\tau_2} X_j\left\{1+o_p(1)\right\} &= \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c}X_{\tau_{ie}}\left\{1+o_p(1)\right\}\\
&\sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{ie}}\frac{1}{r_w c}B(r_{ie}).
\end{aligned}$$

(2) For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{if}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{if}+1}^{\tau_2} X_j.$$

We know that

$$\begin{aligned}
\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{if}} X_j &= \frac{X_{\tau_{ie}}}{\tau_w}\sum_{j=\tau_1}^{\tau_{if}}\delta_T^{j-\tau_{ie}}\left\{1+o_p(1)\right\} \text{ from Lemma S1}\\
&= \frac{X_{\tau_{ie}}}{\tau_w}\frac{\delta_T^{\tau_{if}-\tau_1+1}-1}{\delta_T-1}\left\{1+o_p(1)\right\}\\
&= \frac{X_{\tau_{ie}}}{\tau_w}\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}+c\delta_T^{\tau_{if}-\tau_1}-T^{\alpha}}{c}\left\{1+o_p(1)\right\}\\
&= \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}X_{\tau_{ie}}\left\{1+o_p(1)\right\}\\
&\sim_a T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_1}\frac{1}{r_w c}B(r_{ie}), \qquad (6)
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{\tau_w}\sum_{j=\tau_{if}+1}^{\tau_2} X_j \\
&= \frac{1}{\tau_w}\sum_{j=\tau_{if}+1}^{\tau_2}\left[\sum_{k=\tau_{if}+1}^{j}\varepsilon_k + X_{\tau_{ie}}\right] \qquad (7)\\
&= T^{1/2}\frac{\tau_2-\tau_{if}}{\tau_w}\left[\frac{1}{\tau_2-\tau_{if}}\sum_{j=\tau_{if}+1}^{\tau_2}\left(\frac{1}{\sqrt{T}}\sum_{k=\tau_{if}+1}^{j}\varepsilon_k\right)\right] + \frac{\tau_2-\tau_{if}}{\tau_w}T^{1/2}\left(\frac{1}{\sqrt{T}}X_{\tau_{ie}}\right)\\
&\sim_a T^{1/2}\frac{r_2-r_{if}}{r_w}\int_{r_{if}}^{r_2}\left[B(s)-B(r_{if})\right]ds + T^{1/2}\frac{r_2-r_{if}}{r_w}B(r_{ie})\\
&= T^{1/2}\frac{r_2-r_{if}}{r_w}\left\{\int_{r_{if}}^{r_2}\left[B(s)-B(r_{if})\right]ds - B(r_{ie})\right\}. \qquad (8)
\end{aligned}
$$

Since

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_1}}{T^{1/2}} = \frac{\delta_T^{\tau_{if}-\tau_1}}{T^{1-\alpha}} = \frac{e^{c\left(r_{if}-r_1\right)T^{1-\alpha}}}{T^{1-\alpha}} > 1,$$

$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{if}} X_j$ has a higher order than $\frac{1}{\tau_w}\sum_{j=\tau_{if}+1}^{\tau_2} X_j$ and hence

$$
\begin{aligned}
\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{if}} X_j\left\{1+o_p(1)\right\} &= \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}X_{\tau_{ie}}\left\{1+o_p(1)\right\}\\
&\sim_a T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_1}\frac{1}{r_w c}B(r_{ie}).
\end{aligned}
$$

(3) For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{ie}-1} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{ie}}^{\tau_{if}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{if}+1}^{\tau_2} X_j.$$

We know that

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{ie}-1} X_j \sim_a T^{1/2}\frac{r_{ie}-r_1}{r_w}\int_{r_1}^{r_{ie}} B(s)\,ds \text{ from (3)},$$

$$
\begin{aligned}
\frac{1}{\tau_w}\sum_{j=\tau_{ie}}^{\tau_{if}} X_j &= \frac{X_{\tau_{ie}}}{\tau_w}\sum_{j=\tau_{ie}}^{\tau_{if}}\delta_T^{j-\tau_{ie}}\left\{1+o_p(1)\right\} \text{ from Lemma S1}\\
&= \frac{X_{\tau_{ie}}}{\tau_w}\frac{\delta_T^{\tau_{if}-\tau_{ie}+1}-1}{\delta_T-1}\left\{1+o_p(1)\right\}\\
&= \frac{X_{\tau_{ie}}}{\tau_w c}\left[T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}} + c\delta_T^{\tau_{if}-\tau_{ie}} - T^{\alpha}\right]\left\{1+o_p(1)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} \\
&\sim_a T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_{ie}} \frac{1}{r_w c} B(r_{ie}), \qquad (9)
\end{aligned}
$$

$$
\frac{1}{\tau_w}\sum_{j=\tau_{if}+1}^{\tau_2} X_j \sim_a T^{1/2}\frac{r_2 - r_{if}}{r_w}\left\{\int_{r_{if}}^{r_2} [B(s) - B(r_{if})]\, ds - B(r_{ie})\right\} \text{ from (8).}
$$

Since

$$
\frac{T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_{ie}}}{T^{1/2}} = \frac{e^{c(r_{if}-r_{ie})T^{1-\alpha}}}{T^{1-\alpha}} > 1,
$$

$\frac{1}{\tau_w}\sum_{j=\tau_{ie}}^{\tau_{if}} X_j$ dominates $\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{ie}-1} X_j$ and $\frac{1}{\tau_w}\sum_{j=\tau_{if}+1}^{\tau_2} X_j$ and hence

$$
\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_{ie}} \frac{1}{c r_w} B(r_{ie}).
$$

(4) For $\tau_1 \in N_0$ and $\tau_2 \in N_2$,

$$
\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{1e}-1} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{1e}}^{\tau_{1f}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{2e}}^{\tau_{2f}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{2f}+1}^{\tau_2} X_j.
$$

We know that

$$
\begin{aligned}
&\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{1e}-1} X_j \sim_a T^{1/2}\frac{r_{1e} - r_1}{r_w}\int_{r_1}^{r_{1e}} B(s)\, ds \text{ from (3),} \\
&\frac{1}{\tau_w}\sum_{j=\tau_{1e}}^{\tau_{1f}} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}} \frac{1}{r_w c} B(r_{1e}) \text{ from (9) ,} \\
&\frac{1}{\tau_w}\sum_{j=\tau_{1f}+1}^{\tau_{2e-1}} X_j \sim_a T^{1/2}\frac{r_{2e} - r_{1f}}{r_w}\left\{\int_{r_{1f}}^{r_{2e}} [B(s) - B(r_{1f})]\, ds - B(r_{1e})\right\} \text{ from (8),} \\
&\frac{1}{\tau_w}\sum_{j=\tau_{2e}}^{\tau_{2f}} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}} \frac{1}{r_w c} B(r_{2e}) \text{ from (9),} \\
&\frac{1}{\tau_w}\sum_{j=\tau_{2f}+1}^{\tau_2} X_j \sim_a T^{1/2}\frac{r_2 - r_{2f}}{r_w}\left\{\int_{r_{2f}}^{r_2} [B(s) - B(r_{2f})]\, ds - B(r_{2e})\right\} \text{ from (8).}
\end{aligned}
$$

Since

$$
\frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}}{T^{1/2}} = \frac{e^{c(r_{1f}-r_{1e})T^{1-\alpha}}}{T^{1-\alpha}} > 1,
$$

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}}{T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}} = \delta_T^{\left(\tau_{1f}-\tau_{1e}\right)-\left(\tau_{2f}-\tau_{2e}\right)},$$

we have

$$\begin{aligned}\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j &= \begin{cases} \frac{1}{\tau_w}\sum_{j=\tau_{1e}}^{\tau_{1f}} X_j \left\{1+o_p(1)\right\} & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ \frac{1}{\tau_w}\sum_{j=\tau_{2e}}^{\tau_{2f}} X_j \left\{1+o_p(1)\right\} & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases} \\ &= \begin{cases} \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}} \frac{1}{r_w c} B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}} \frac{1}{r_w c} B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases}.\end{aligned}$$

(5) For $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{1f}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{1f}+1}^{\tau_{2e-1}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{2e}}^{\tau_2} X_j.$$

We know that

$$\begin{aligned}&\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{1f}} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\frac{1}{r_w c}B\left(r_1\right) \text{ from 6,} \\ &\frac{1}{\tau_w}\sum_{j=\tau_{1f}+1}^{\tau_{2e-1}} X_j \sim_a T^{1/2}\frac{r_{2e}-r_{1f}}{r_w}\left\{\int_{r_{1f}}^{r_{2e}}\left[B(s)-B\left(r_{1f}\right)\right]ds - B\left(r_{1e}\right)\right\} \text{ from (8),} \\ &\frac{1}{\tau_w}\sum_{j=\tau_{2e}}^{\tau_2} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c}B\left(r_{2e}\right) \text{ from (5).}\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}}{T^{1/2}} &= \frac{\delta_T^{\tau_2-\tau_{2e}}}{T^{1-\alpha}} = \frac{e^{c\left(r_2-r_{2e}\right)T^{1-\alpha}}}{T^{1-\alpha}} > 1, \\ \frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}}{T^{1/2}} &= \frac{\delta_T^{\tau_{1f}-\tau_1}}{T^{1-\alpha}} = \frac{e^{c\left(r_{1f}-r_1\right)T^{1-\alpha}}}{T^{1-\alpha}} > 1 \\ \frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}}{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}} &= \delta_T^{\left(\tau_2-\tau_{2e}\right)-\left(\tau_{1f}-\tau_1\right)}.\end{aligned}$$

Therefore,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \begin{cases} \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \left\{1+o_p(1)\right\} & \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\frac{1}{r_w c}B\left(r_{1e}\right) \quad \text{if } \tau_{1f}-\tau_1 > \tau_2-\tau_{2e} \\ \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1+o_p(1)\right\} & \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c}B\left(r_{2e}\right) \quad \text{if } \tau_{1f}-\tau_1 \le \tau_2-\tau_{2e} \end{cases}$$

(6) For $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{1f}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{1f}+1}^{\tau_{2e-1}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{2e}}^{\tau_{2f}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{2f+1}}^{\tau_2} X_j.$$

We know that

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{1f}} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\frac{1}{r_w c}B\left(r_1\right) \text{ from (6),}$$
$$\frac{1}{\tau_w}\sum_{j=\tau_{1f}+1}^{\tau_{2e-1}} X_j \sim_a T^{1/2}\frac{r_{2e}-r_{1f}}{r_w}\left\{\int_{r_{1f}}^{r_{2e}}\left[B\left(s\right)-B\left(r_{1f}\right)\right]ds - B\left(r_{1e}\right)\right\} \text{ from (8),}$$
$$\frac{1}{\tau_w}\sum_{j=\tau_{2e}}^{\tau_{2f}} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\frac{1}{r_w c}B\left(r_{2e}\right) \text{ from (9),}$$
$$\frac{1}{\tau_w}\sum_{j=\tau_{2f+1}}^{\tau_2} X_j \sim_a T^{1/2}\frac{r_2-r_{2f}}{r_w}\left\{\int_{r_{2f}}^{r_2}\left[B\left(s\right)-B\left(r_{2f}\right)\right]ds - B\left(r_{1e}\right)\right\} \text{ from (8).}$$

Furthermore, as $T \to \infty$

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}}{T^{1/2}} = \frac{\delta_T^{\tau_{2f}-\tau_{2e}}}{T^{1-\alpha}} > 1,$$
$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}}{T^{1/2}} = \frac{\delta_T^{\tau_{1f}-\tau_1}}{T^{1-\alpha}} > 1,$$
$$\frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}}{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}} = \delta_T^{\left(\tau_{2f}-\tau_{2e}\right)-\left(\tau_{1f}-\tau_1\right)}.$$

Therefore,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \begin{cases} \frac{T^\alpha\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}X_{\tau_{1e}}\left\{1+o_p\left(1\right)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\frac{1}{r_w c}B\left(r_1\right) & \text{if } \tau_{1f}-\tau_1 > \tau_{2f}-\tau_{2e} \\ \frac{T^\alpha\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\left\{1+o_p\left(1\right)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\frac{1}{r_w c}B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_1 \le \tau_{2f}-\tau_{2e} \end{cases}.$$

(7) For $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{1e}-1} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{1e}}^{\tau_{1f}} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} X_j + \frac{1}{\tau_w}\sum_{j=\tau_{2e}}^{\tau_2} X_j.$$

We know that

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_{1e}-1} X_j \sim_a T^{1/2}\frac{r_{1e}-r_1}{r_w}\int_{r_1}^{r_{1e}} B\left(s\right)ds \text{ from (3),}$$

$$\frac{1}{\tau_w}\sum_{j=\tau_{1e}}^{\tau_{1f}} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\frac{1}{r_w c}B\left(r_{1e}\right) \text{ from (9)},$$
$$\frac{1}{\tau_w}\sum_{j=\tau_{1f}+1}^{\tau_{2e-1}} X_j \sim_a T^{1/2}\frac{r_{2e}-r_{1f}}{r_w}\left\{\int_{r_{1f}}^{r_{2e}}\left[B\left(s\right)-B\left(r_{1f}\right)\right]ds - B\left(r_{1e}\right)\right\} \text{ from (8)}$$
$$\frac{1}{\tau_w}\sum_{j=\tau_{2e}}^{\tau_2} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c}B\left(r_{2e}\right) \text{ from (5)}$$

Also, as $T\to\infty$

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}}{T^{1/2}} = \frac{e^{c\left(r_{1f}-r_{1e}\right)T^{1-\alpha}}}{T^{1-\alpha}} > 1,$$
$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}}{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}} = \frac{\delta_T^{\left(\tau_{1f}-\tau_{1e}\right)-\left(\tau_2-\tau_{2e}\right)}}{T^{1/2}}.$$

Therefore,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \begin{cases} \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}X_{\tau_{1e}}\left\{1+o_p\left(1\right)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\frac{1}{r_w c}B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\left\{1+o_p\left(1\right)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c}B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases}.$$

□

Lemma S3. *Define the centered quantity* $\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j$.

(1) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c}X_{\tau_{ie}}\left\{1+o_p\left(1\right)\right\} & \textit{if } t\in N_{i-1} \\ \left[\delta_T^{t-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c}\right]X_{\tau_{ie}}\left\{1+o_p\left(1\right)\right\} & \textit{if } t\in B_i \end{cases}.$$

(2) *For* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}\right]X_{\tau_{ie}}\left\{1+o_p\left(1\right)\right\} & \textit{if } t\in B_i \\ -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}X_{\tau_{ie}}\left\{1+o_p\left(1\right)\right\} & \textit{if } t\in N_i \end{cases}.$$

(3) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}X_{\tau_{ie}}\left\{1+o_p\left(1\right)\right\} & \textit{if } t\in N_{i-1}\cup N_i \\ \left[\delta_T^{t-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}\right]X_{\tau_{ie}}\left\{1+o_p\left(1\right)\right\} & \textit{if } t\in B_i \end{cases}.$$

(4) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$, *if* $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \left\{1 + o_p\left(1\right)\right\} & \text{if } t \in N_i \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1 + o_p\left(1\right)\right\} & \text{if } t \in B_i,\ i = 1, 2, \end{cases}$$

and if $\tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e}$

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p\left(1\right)\right\} & \text{if } t \in N_i \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p\left(1\right)\right\} & \text{if } t \in B_i,\ i = 1, 2, \end{cases}.$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$, *if* $\tau_{1f} - \tau_1 > \tau_2 - \tau_{2e}$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1 + o_p\left(1\right)\right\} & \text{if } t \in B_i,\ i = 1, 2, \\ -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \left\{1 + o_p\left(1\right)\right\} & \text{if } t \in N_1 \end{cases}$$

and if $\tau_{1f} - \tau_1 \leq \tau_2 - \tau_{2e}$

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p\left(1\right)\right\} & \text{if } t \in B_i,\ i = 1, 2, \\ -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p\left(1\right)\right\} & \text{if } t \in N_1 \end{cases}.$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$, *if* $\tau_{1f} - \tau_1 > \tau_{2f} - \tau_{2e}$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1 + o_p\left(1\right)\right\} & \text{if } t \in B_i,\ i = 1, 2, \\ -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \left\{1 + o_p\left(1\right)\right\} & \text{if } t \in N_i,\ i = 1, 2, \end{cases}$$

and if $\tau_{1f} - \tau_1 \leq \tau_{2f} - \tau_{2e}$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p\left(1\right)\right\} & \text{if } t \in B_i,\ i = 1, 2, \\ -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p\left(1\right)\right\} & \text{if } t \in N_i,\ i = 1, 2, \end{cases}.$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$, *if* $\tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e}$

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \left\{1 + o_p\left(1\right)\right\} & \text{if } t \in N_i,\ i = 1, 2, \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1 + o_p\left(1\right)\right\} & \text{if } t \in B_i,\ i = 1, 2, \end{cases}$$

and if $\tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e}$

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \{1 + o_p(1)\} & \text{if } t \in N_i,\ i = 1, 2, \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right] \{1 + o_p(1)\} & \text{if } t \in B_i,\ i = 1, 2, \end{cases}.$$

Proof. (1) Suppose $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$. When $t \in N_{i-1}$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\}, \tag{10}$$

where the second term dominates the first term due to the fact that

$$T^{-1/2} X_{t=\lfloor Tp \rfloor} \xrightarrow{L} B(p) \text{ from Lemma S1}$$
$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha-1/2} \delta_T^{\tau_2-\tau_{ie}} \frac{1}{r_w c} B(r_{ie}) \text{ from Lemma S2}$$

and

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{ie}}}{T^{1/2}} = \frac{e^{c(r_2-r_{ie})T^{1-\alpha}}}{T^{1-\alpha}} > 1;$$

When $t \in B_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c}\right] X_{\tau_{ie}} \{1 + o_p(1)\}.$$

(2) Suppose $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$. When $t \in B_i$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}\right] X_{\tau_{ie}} \{1 + o_p(1)\};$$

When $t \in N_i$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\},$$

where the second term dominates the first term due to the fact that

$$X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} [B(p) - B(r_{if}) + B(r_{ie})] \text{ from Lemma S1}$$
$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha-1/2} \delta_T^{\tau_{if}-\tau_1} \frac{1}{r_w c} B(r_{ie}) \text{ from Lemma S2}$$

and

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_1}}{T^{1/2}} = \frac{\delta_T^{\tau_{if}-\tau_1}}{T^{1-\alpha}} = \frac{e^{c\left(r_{if}-r_1\right)T^{1-\alpha}}}{T^{1-\alpha}} > 1.$$

(3) Suppose $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$. When $t \in N_{i-1}$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^\alpha \delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\},$$

where the second term dominates the first term due to the fact that

$$X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} B(p) \text{ from Lemma S1}$$
$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha-1/2} \delta_T^{\tau_{if}-\tau_{ie}} \frac{1}{r_w c} B(r_{ie}) \{1 + o_p(1)\} \text{ from Lemma S2}$$

and

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_{ie}}}{T^{1/2}} > 1.$$

When $t \in B_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}\right] X_{\tau_{ie}} \{1 + o_p(1)\}.$$

When $t \in N_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^\alpha \delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\},$$

due to the fact that

$$X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} \left[B(p) - B(r_{if}) + B(r_{ie})\right] \text{ from Lemma S1}$$

(4) Suppose $\tau_1 \in N_0$, $\tau_2 \in N_2$ and $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$. When $t \in N_0$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\},$$

where the second term dominates the first term due to the fact that

$$X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} B(p) \text{ from Lemma S1}$$

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\frac{1}{cr_w}B\left(r_{1e}\right) \text{ from Lemma S2}$$

and

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}}{T^{1/2}} > 1.$$

When $t \in B_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1 + o_p\left(1\right)\right\}.$$

When $t \in N_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\left\{1 + o_p\left(1\right)\right\},$$

due to the fact that

$$X_{t=\lfloor Tp\rfloor} \sim_a T^{1/2}\left[B\left(p\right) - B\left(r_{if}\right) + B\left(r_{ie}\right)\right] \text{ from Lemma S1.}$$

Suppose $\tau_1 \in N_0$, $\tau_2 \in N_2$ and $\tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e}$. When $t \in N_0$,

$$\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\left\{1 + o_p\left(1\right)\right\},$$

where the second term dominates the first term due to the fact that

$$X_{t=\lfloor Tp\rfloor} \sim_a T^{1/2}B\left(p\right) \text{ from Lemma S1}$$
$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\frac{1}{r_w c}B\left(r_{2e}\right) \text{ from Lemma S2}$$

and

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}}{T^{1/2}} > 1.$$

When $t \in B_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p\left(1\right)\right\}.$$

When $t \in N_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha} \delta_T^{\tau_{2f} - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p\left(1\right)\right\},$$

due to the fact that

$$X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} \left[B\left(p\right) - B\left(r_{if}\right) + B\left(r_{ie}\right)\right] \text{ from Lemma S1.}$$

(5) Suppose $\tau_1 \in B_1$ and $\tau_2 \in B_2$. If $\tau_{1f} - \tau_1 > \tau_2 - \tau_{2e}$, when $t \in B_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t - \tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha} \delta_T^{\tau_{1f} - \tau_1}}{\tau_w c} X_{\tau_{1e}}\right] \left\{1 + o_p\left(1\right)\right\}.$$

When $t \in N_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha} \delta_T^{\tau_{1f} - \tau_1}}{\tau_w c} X_{\tau_{1e}} \left\{1 + o_p\left(1\right)\right\},$$

where the second term dominates the first term due to the fact that

$$X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} \left[B\left(p\right) - B\left(r_{1f}\right) + B\left(r_{1e}\right)\right] \text{ from Lemma S1}$$
$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{1f} - \tau_1} \frac{1}{r_w c} B\left(r_{1e}\right) \text{ from Lemma S2}$$

and

$$\frac{T^{\alpha - 1/2} \delta_T^{\tau_{1f} - \tau_1}}{T^{1/2}} > 1.$$

If $\tau_{1f} - \tau_1 \leq \tau_2 - \tau_{2e}$, when $t \in B_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t - \tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha} \delta_T^{\tau_2 - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right] \left\{1 + o_p\left(1\right)\right\}.$$

When $t \in N_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha} \delta_T^{\tau_2 - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p\left(1\right)\right\},$$

where the second term dominates the first term due to the fact that

$$X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} \left[B\left(p\right) - B\left(r_{1f}\right) + B\left(r_{1e}\right)\right] \text{ from Lemma S1}$$

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c}B\left(r_{2e}\right) \text{ from Lemma S2}$$

and

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}}{T^{1/2}} > 1.$$

(6) Suppose $\tau_1 \in B_1$ and $\tau_2 \in N_2$. Suppose $\tau_{1f}-\tau_1 > \tau_{2f}-\tau_{2e}$. When $t \in B_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1+o_p\left(1\right)\right\},$$

When $t \in N_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\left\{1+o_p\left(1\right)\right\},$$

where the second term dominates the first term due to the fact that

$$X_{t=\lfloor Tp\rfloor} \sim_a T^{1/2}\left[B\left(p\right)-B\left(r_{if}\right)+B\left(r_{ie}\right)\right] \text{ from Lemma S1}$$

and

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}}{T^{1/2}} > 1.$$

Suppose $\tau_{1f}-\tau_1 \leq \tau_{2f}-\tau_{2e}$. When $t \in B_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1+o_p\left(1\right)\right\},$$

When $t \in N_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\left\{1+o_p\left(1\right)\right\},$$

where the second term dominates the first term due to the fact that

$$X_{t=\lfloor Tp\rfloor} \sim_a T^{1/2}\left[B\left(p\right)-B\left(r_{if}\right)+B\left(r_{ie}\right)\right] \text{ from Lemma S1}$$

and

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}}{T^{1/2}} > 1.$$

(7) Suppose $\tau_1 \in N_0$ and $\tau_2 \in B_2$ and $\tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e}$. When $t \in N_0$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\},$$

where the second term dominates the first term due to the fact that

$$\begin{aligned} &X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} B(p) \text{ from Lemma S1} \\ &\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha-1/2} \delta_T^{\tau_{1f}-\tau_{1e}} \frac{1}{r_w c} B(r_{1e}) \text{ from Lemma S2} \end{aligned}$$

and

$$\frac{T^{\alpha-1/2} \delta_T^{\tau_{1f}-\tau_{1e}}}{T^{1/2}} > 1.$$

When $t \in B_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \right] \{1 + o_p(1)\}.$$

When $t \in N_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\},$$

since $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} [B(p) - B(r_{if}) + B(r_{ie})]$ (from Lemma S1). Suppose $\tau_1 \in N_0$ and $\tau_2 \in B_2$ and $\tau_{1f} - \tau_{1e} \le \tau_2 - \tau_{2e}$. When $t \in N_0$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \{1 + o_p(1)\},$$

where the second term dominates the first term due to the fact that

$$\begin{aligned} &X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} B(p) \text{ from Lemma S1} \\ &\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha-1/2} \delta_T^{\tau_2-\tau_{2e}} \frac{1}{r_w c} B(r_{2e}) \text{ from Lemma S2} \end{aligned}$$

and

$$\frac{T^{\alpha-1/2} \delta_T^{\tau_2-\tau_{2e}}}{T^{1/2}} > 1.$$

When $t \in B_i$ with $i = 1, 2$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_2 - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \right] \{1 + o_p(1)\}.$$

When $t \in N_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^\alpha \delta_T^{\tau_2 - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \{1 + o_p(1)\},$$

since $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} [B(p) - B(r_{if}) + B(r_{ie})]$ (from Lemma S1). $\square$

Lemma S4. *The sample variance of $\tilde{X}_t$ has the following limit forms:*

(1) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_2 - \tau_{ie})}}{2c} X_{\tau_{ie}}^2 \{1 + o_p(1)\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(\tau_2 - \tau_{ie})}}{2c} B(r_{ie})^2.$$

(2) *For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2\left(\tau_{if} - \tau_{ie}\right)}}{2c} X_{\tau_{ie}}^2 \{1 + o_p(1)\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{if} - \tau_{ie}\right)}}{2c} B(r_{ie})^2.$$

(3) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2\left(\tau_{if} - \tau_{ie}\right)}}{2c} X_{\tau_{ie}}^2 \{1 + o_p(1)\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{if} - \tau_{ie}\right)}}{2c} B(r_{ie})^2.$$

(4) *For $\tau_1 \in N_0$ and $\tau_2 \in N_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2\left(\tau_{1f} - \tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \{1 + o_p(1)\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{1f} - \tau_{1e}\right)}}{2c} B(r_{1e})^2 & \text{if } \tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e} \\ \frac{T^\alpha \delta_T^{2\left(\tau_{2f} - \tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 \{1 + o_p(1)\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{2f} - \tau_{2e}\right)}}{2c} B(r_{2e})^2 & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e} \end{cases}.$$

(5) *For $\tau_1 \in B_1$ and $\tau_2 \in B_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2\left(\tau_{1f} - \tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \{1 + o_p(1)\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{1f} - \tau_{1e}\right)}}{2c} B(r_{1e})^2 & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ \frac{T^\alpha \delta_T^{2(\tau_2 - \tau_{2e})}}{2c} X_{\tau_{2e}}^2 \{1 + o_p(1)\} \sim_a T^{\alpha+1} \delta_T^{2(\tau_2 - \tau_{2e})} \frac{1}{2c} B(r_{2e})^2 & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e} \end{cases}.$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B\left(r_{1e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ \frac{T^{\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} B\left(r_{2e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} \leq \tau_{2f}-\tau_{2e} \end{cases}.$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B\left(r_{1e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} B\left(r_{2e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases}.$$

Proof. (1) For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_{ie}} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}^2 \left\{1+o_p(1)\right\}.$$

The first term is

$$\begin{aligned} \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_1}^{\tau_{ie}-1} \frac{T^{2\alpha}\delta_T^{2(\tau_2-\tau_{ie})}}{\tau_w^2 c^2} X_{\tau_{ie}}^2 \left\{1+o_p(1)\right\} \text{ from Lemma S3} \\ &= \frac{\tau_{ie}-\tau_1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_2-\tau_{ie})} X_{\tau_{ie}}^2 \left\{1+o_p(1)\right\} \\ &\sim_a \frac{r_{ie}-r_1}{r_w^2 c} T^{2\alpha}\delta_T^{2(\tau_2-\tau_{ie})} B\left(r_{ie}\right)^2. \end{aligned}$$

Given that

$$\begin{aligned} \sum_{j=\tau_{ie}}^{\tau_2} \delta_T^{2(j-1-\tau_{ie})} &= \frac{\delta_T^{2(\tau_2-\tau_{ie})} - \delta_T^{-2}}{\delta_T^2 - 1} = \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_{ie})}}{2c} \left\{1+o_p(1)\right\} \\ \sum_{j=\tau_{ie}}^{\tau_2} \delta_T^{j-1-\tau_{ie}} &= \frac{\delta_T^{\tau_2-\tau_{ie}} - \delta_T^{-1}}{\delta_T - 1} = \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{c} \left\{1+o_p(1)\right\}, \end{aligned}$$

the second term

$$\begin{aligned} &\sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}^2 \\ &= \sum_{j=\tau_{ie}}^{\tau_2} \left[\delta_T^{j-1-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c}\right]^2 X_{\tau_{ie}}^2 \left\{1+o_p(1)\right\} \end{aligned}$$

$$= \sum_{j=\tau e}^{\tau_2} \left[\delta_T^{2(j-1-\tau_{ie})} - 2\delta_T^{j-1-\tau_{ie}} \frac{T^\alpha \delta_T^{\tau_2-\tau_{ie}}}{\tau_w c} + \frac{T^{2\alpha} \delta_T^{2(\tau_2-\tau_{ie})}}{\tau_w^2 c^2} \right] X_{\tau_{ie}}^2 \{1 + o_p(1)\}$$

$$= \left[\frac{T^\alpha \delta_T^{2(\tau_2-\tau_{ie})}}{2c} - 2\frac{T^{2\alpha-1} \delta_T^{2(\tau_2-\tau_{ie})}}{r_w c^2} + \frac{r_2 - r_{ie} + \frac{1}{T}}{r_w^2 c^2} T^{2\alpha-1} \delta_T^{2(\tau_2-\tau_{ie})} \right] X_{\tau_{ie}}^2 \{1 + o_p(1)\}$$

$$= \frac{T^\alpha \delta_T^{2(\tau_2-\tau_{ie})}}{2c} X_{\tau_{ie}}^2 \{1 + o_p(1)\}$$

$$\sim_a \frac{T^{1+\alpha} \delta_T^{2(\tau_2-\tau_{ie})}}{2c} B(r_{ie})^2 .$$

Since $1 + \alpha > 2\alpha$, $\sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}^2$ dominates $\sum_{j=\tau_1}^{\tau_{ie}} \tilde{X}_{j-1}^2$. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_2-\tau_{ie})}}{2c} X_{\tau_{ie}}^2 \{1 + o_p(1)\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(\tau_2-\tau_{ie})}}{2c} B(r_{ie})^2 .$$

(2) For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_{if}} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}^2 .$$

Given that

$$\sum_{j=\tau_1}^{\tau_{if}} \delta_T^{2(j-1-\tau_{ie})} = \frac{T^\alpha \left[\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)} - \delta_T^{2(\tau_1-\tau_{ie}-1)} \right]}{2c + c^2 T^{-\alpha}} = \frac{T^\alpha \delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} \{1 + o_p(1)\}$$

$$\sum_{j=\tau_1}^{\tau_{if}} \delta_T^{j-1-\tau_{ie}} = \frac{T^\alpha \left[\delta_T^{\tau_{if}-\tau_{ie}} - \delta_T^{\tau_1-\tau_{ie}-1} \right]}{c} = \frac{T^\alpha \delta_T^{\tau_{if}-\tau_{ie}}}{c} \{1 + o_p(1)\} ,$$

the first term is

$$\sum_{j=\tau_1}^{\tau_{if}} \tilde{X}_{j-1}^2$$

$$= \sum_{j=\tau_1}^{\tau_{if}} \left[\delta_T^{j-1-\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{if}-\tau_1}}{\tau_w c} \right]^2 X_{\tau_{ie}}^2 \{1 + o_p(1)\}$$

$$= \left[\frac{T^\alpha \delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} - 2\frac{T^{2\alpha-1} \delta_T^{\tau_{if}-\tau_{ie}+\tau_{if}-\tau_1}}{r_w c^2} + \frac{r_{if} - r_1}{r_w^2 c^2} T^{2\alpha-1} \delta_T^{2\left(\tau_{if}-\tau_1\right)} \right] X_{\tau_{ie}}^2 \{1 + o_p(1)\}$$

$$= \frac{T^\alpha \delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} X_{\tau_{ie}}^2 \{1 + o_p(1)\} \text{ (since } \alpha > 2\alpha - 1 \text{ and } \tau_{if} - \tau_{ie} > \tau_{if} - \tau_1 \text{)}$$

$$\sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} B\left(r_{ie}\right)^2.$$

The second term is

$$\begin{aligned}
\sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_{if}+1}^{\tau_2} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_1\right)}}{\tau_w^2 c^2} X_{\tau_{ie}}^2 \left\{1+o_p(1)\right\} \\
&= \frac{\tau_2-\tau_{if}}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_1\right)} X_{\tau_{ie}}^2 \left\{1+o_p(1)\right\} \\
&\sim_a \frac{r_2-r_{if}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_1\right)} B\left(r_{ie}\right)^2.
\end{aligned}$$

Since

$$\frac{T^{\alpha+1}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_1\right)}} = T^{1-\alpha}\delta_T^{2\left[\left(\tau_{if}-\tau_{ie}\right)-\left(\tau_{if}-\tau_1\right)\right]} > 1,$$

$\sum_{j=\tau_1}^{\tau_{if}} \tilde{X}_{j-1}^2$ dominates $\sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}^2$. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} X_{\tau_{ie}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} B\left(r_{ie}\right)^2.$$

(3) For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{ie}}^{\tau_{if}} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}^2.$$

The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_1}^{\tau_{ie}-1} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{\tau_w^2 c^2} X_{\tau_{ie}}^2 \left\{1+o_p(1)\right\} \\
&= \frac{\tau_{ie}-\tau_1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)} X_{\tau_{ie}}^2 \left\{1+o_p(1)\right\} \\
&\sim_a \frac{r_{ie}-r_1}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)} B\left(r_{ie}\right)^2.
\end{aligned}$$

Given that

$$\sum_{j=\tau_{ie}}^{\tau_{if}} \delta_T^{2(j-1-\tau_{ie})} = \frac{\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)} - \delta_T^{-2}}{\delta_T^2 - 1} = \frac{T^{\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} \left\{1+o_p(1)\right\}$$

$$\sum_{j=\tau_{ie}}^{\tau_{if}} \delta_T^{j-1-\tau_{ie}} = \frac{\delta_T^{\tau_{if}-\tau_{ie}} - \delta_T^{-1}}{\delta_T - 1} = \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{c}\left\{1+o_p\left(1\right)\right\},$$

the second term

$$\begin{aligned}
&\sum_{j=\tau_{ie}}^{\tau_{if}} \tilde{X}_{j-1}^2 \\
&= \sum_{j=\tau_{ie}}^{\tau_{if}} \left[\delta_T^{j-1-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}\right]^2 X_{\tau_{ie}}^2 \left\{1+o_p\left(1\right)\right\} \\
&= \left[\frac{T^{\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} - 2\frac{T^{2\alpha-1}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{r_w c^2} + \frac{r_{if}-r_{ie}+\frac{1}{T}}{r_w^2 c^2}T^{2\alpha-1}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}\right] X_{\tau_{ie}}^2\left\{1+o_p\left(1\right)\right\} \\
&= \frac{T^{\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} X_{\tau_{ie}}^2\left\{1+o_p\left(1\right)\right\} \ \text{(since } \alpha > 2\alpha-1) \\
&\sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} B\left(r_{ie}\right)^2.
\end{aligned}$$

The third term is

$$\begin{aligned}
\sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_{if}+1}^{\tau_2} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{\tau_w^2 c^2} X_{\tau_{ie}}^2\left\{1+o_p\left(1\right)\right\} \\
&= \frac{\tau_2-\tau_{if}}{\tau_w^2 c^2}T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)} X_{\tau_{ie}}^2\left\{1+o_p\left(1\right)\right\} \\
&\sim_a \frac{r_2-r_{if}}{r_w^2 c^2}T^{2\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)} B\left(r_{ie}\right)^2.
\end{aligned}$$

Since $1+\alpha > 2\alpha$, $\sum_{j=\tau_{ie}}^{\tau_{if}} \tilde{X}_{j-1}^2$ dominates the other two terms. Therefore,

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_{ie}}^{\tau_{if}} \tilde{X}_{j-1}^2\left\{1+o_p\left(1\right)\right\} &= \frac{T^{\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} X_{\tau_{ie}}^2\left\{1+o_p\left(1\right)\right\} \\
&\sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{2c} B\left(r_{ie}\right)^2.
\end{aligned}$$

(4) For $\tau_1 \in N_0$ and $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}^2.$$

Suppose $\tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}$. The first term is

$$\begin{aligned}\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}^2 &= \sum_{j=\tau_1}^{\tau_{1e}-1}\frac{T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{\tau_w^2c^2}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\\ &= \frac{\tau_{1e}-\tau_1}{\tau_w^2c^2}T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\\ &\sim_a \frac{r_{1e}-r_1}{r_w^2c^2}T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}B\left(r_{1e}\right)^2.\end{aligned}$$

The second term is

$$\begin{aligned}&\sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}^2\\ &= \sum_{j=\tau_{1e}}^{\tau_{1f}}\left[\delta_T^{j-1-\tau_{1e}}X_{\tau_{1e}}-\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_wc}X_{\tau_{1e}}\right]^2\left\{1+o_p\left(1\right)\right\}\\ &= \left[\frac{T^\alpha\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}-2\frac{T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{r_wc^2}+\frac{r_{1f}-r_{1e}}{r_w^2c^2}T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}\right]X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\\ &= \frac{T^\alpha\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\ \text{(since }\alpha>2\alpha-1\text{)}\\ &\sim_a \frac{T^{1+\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}B\left(r_{1e}\right)^2.\end{aligned}$$

The third term is

$$\begin{aligned}&\sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}_{j-1}^2\\ &= \sum_{j=\tau_{2e}}^{\tau_{2f}}\left[\delta_T^{j-1-\tau_{2e}}X_{\tau_{2e}}-\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_wc}X_{\tau_{1e}}\right]^2\left\{1+o_p\left(1\right)\right\}\\ &= \left[\frac{T^\alpha\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c}X_{\tau_{2e}}^2-2\frac{T^{2\alpha-1}\delta_T^{\left(\tau_{2f}-\tau_{2e}\right)+\left(\tau_{1f}-\tau_{1e}\right)}}{r_wc^2}X_{\tau_{2e}}X_{\tau_{1e}}+\frac{r_{2f}-r_{2e}}{r_w^2c^2}T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}X_{\tau_{1e}}^2\right]\left\{1+o_p\left(1\right)\right\}\\ &= \frac{r_{2f}-r_{2e}}{r_w^2c^2}T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\\ &\sim_a \frac{r_{2f}-r_{2e}}{r_w^2c^2}T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}B\left(r_{1e}\right)^2.\end{aligned}$$

The fourth term is

$$\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}^2 = \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{\tau_w^2 c^2} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\}$$
$$= \frac{\tau_{2e}-\tau_{1f}-1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\}$$
$$\sim_a \frac{r_{2e}-r_{1f}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2.$$

The fifth term is

$$\sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_{2f}+1}^{\tau_2} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{\tau_w^2 c^2} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\}$$
$$= \frac{\tau_2-\tau_{2f}}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\}$$
$$\sim_a \frac{r_2-r_{2f}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2.$$

Since $1+\alpha>2\alpha$, $\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}^2$ dominates the other terms. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}^2 \tilde{X}_{j-1}^2 \left\{1+o_p\left(1\right)\right\} = \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\}$$
$$\sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B\left(r_{1e}\right)^2.$$

Suppose $\tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}$. The first term is

$$\sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_{1e}-1} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{\tau_w^2 c^2} X_{\tau_{2e}}^2 \left\{1+o_p\left(1\right)\right\}$$
$$= \frac{\tau_{1e}-\tau_1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} X_{\tau_{2e}}^2 \left\{1+o_p\left(1\right)\right\}$$
$$\sim_a \frac{r_{1e}-r_1}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} B\left(r_{2e}\right)^2.$$

The second term is

$$\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}^2$$

$$
\begin{aligned}
&= \sum_{j=\tau_{1e}}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \right]^2 \{1+o_p(1)\} \\
&= \left[\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 - 2\frac{T^{2\alpha-1}\delta_T^{\left(\tau_{1f}-\tau_{1e}\right)+\left(\tau_{2f}-\tau_{2e}\right)}}{r_w c^2} X_{\tau_{1e}} X_{\tau_{2e}} \right. \\
&\left. + \frac{r_{1f}-r_{1e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} X_{\tau_{2e}}^2 \right] \{1+o_p(1)\} \\
&= \frac{r_{1f}-r_{1e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} X_{\tau_{2e}}^2 \{1+o_p(1)\} \\
&\sim_a \frac{r_{1f}-r_{1e}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} B\left(r_{2e}\right)^2 .
\end{aligned}
$$

The third term is

$$
\begin{aligned}
&\sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}^2 \\
&= \sum_{j=\tau_{2e}}^{\tau_{2f}} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \right]^2 \{1+o_p(1)\} \\
&= \left[\frac{T^{\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 - 2\frac{T^{2\alpha-1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{r_w c^2} X_{\tau_{2e}}^2 \right. \\
&\left. + \frac{r_{2f}-r_{2e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} X_{\tau_{2e}}^2 \right] \{1+o_p(1)\} \\
&= \frac{T^{\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 \{1+o_p(1)\} \;\; (\text{since } \alpha > 2\alpha-1) \\
&\sim_a \frac{T^{1+\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} B\left(r_{2e}\right)^2 .
\end{aligned}
$$

The fourth term is

$$
\begin{aligned}
\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{\tau_w^2 c^2} X_{\tau_{1e}}^2 \{1+o_p(1)\} \\
&= \frac{\tau_{2e}-\tau_{1f}-1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} X_{\tau_{1e}}^2 \{1+o_p(1)\} \\
&\sim_a \frac{r_{2e}-r_{1f}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2 .
\end{aligned}
$$

The fifth term is

$$\begin{aligned}\sum_{j=\tau_{2f}+1}^{\tau_2}\tilde{X}_{j-1}^2 &= \sum_{j=\tau_{2f}+1}^{\tau_2}\frac{T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{\tau_w^2c^2}X_{\tau_{2e}}^2\left\{1+o_p\left(1\right)\right\}\\ &= \frac{\tau_2-\tau_{2f}}{\tau_w^2c^2}T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}X_{\tau_{2e}}^2\left\{1+o_p\left(1\right)\right\}\\ &\sim_a \frac{r_2-r_{2f}}{r_w^2c^2}T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}B\left(r_{2e}\right)^2.\end{aligned}$$

Since $1+\alpha>2\alpha$, $\sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}_{j-1}^2$ dominates the other terms. Therefore,

$$\begin{aligned}\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2 &= \sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}_{j-1}^2\tilde{X}_{j-1}^2\left\{1+o_p\left(1\right)\right\} = \frac{T^{\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c}X_{\tau_{2e}}^2\left\{1+o_p\left(1\right)\right\}\\ &\sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c}B\left(r_{2e}\right)^2.\end{aligned}$$

Thus, when $\tau_1\in N_0$ and $\tau_2\in N_2$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2=\begin{cases}\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\sim_a\frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}\\ \frac{T^{\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c}X_{\tau_{2e}}^2\left\{1+o_p\left(1\right)\right\}\sim_a\frac{T^{\alpha+1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c}B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\end{cases}.$$

(5) For $\tau_1\in B_1$ and $\tau_2\in B_2$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2=\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}_{j-1}^2+\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\tilde{X}_{j-1}^2+\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}^2.$$

Suppose $\tau_{1f}-\tau_1>\tau_2-\tau_{2e}$. The first term is

$$\begin{aligned}&\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}_{j-1}^2\\ &= \sum_{j=\tau_1}^{\tau_{1f}}\left[\delta_T^{j-1-\tau_{1e}}X_{\tau_{1e}}-\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_wc}X_{\tau_{1e}}\right]^2\left\{1+o_p\left(1\right)\right\}\\ &= \left[\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}-2\frac{T^{2\alpha-1}\delta_T^{\left(\tau_{1f}-\tau_1\right)+\left(\tau_{1f}-\tau_{1e}\right)}}{r_wc^2}+\frac{r_{1f}-r_1}{r_w^2c^2}T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}\right]X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\\ &= \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\sim_a\frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}B\left(r_{1e}\right)^2\end{aligned}$$

due to the fact that

$$\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^{2\alpha-1}\delta_T^{\left(\tau_{1f}-\tau_1\right)+\left(\tau_{1f}-\tau_{1e}\right)}} = T^{1-\alpha}\delta_T^{\left(\tau_{1f}-\tau_{1e}\right)-\left(\tau_{1f}-\tau_1\right)} > 1,$$

$$\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}} = T^{1-\alpha}\delta_T^{2\left[\left(\tau_{1f}-\tau_{1e}\right)-\left(\tau_{1f}-\tau_1\right)\right]} > 1.$$

The second term is

$$\begin{aligned}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\tilde{X}_{j-1}^2 &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\frac{T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}}{\tau_w^2c^2}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\\ &= \frac{\tau_{2e}-\tau_{1f}-1}{\tau_w^2c^2}T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\\ &\sim_a T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}\frac{r_{2e}-r_{1f}}{r_w^2c^2}B\left(r_{1e}\right)^2.\end{aligned}$$

The third term is

$$\begin{aligned}&\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}^2\\ &= \sum_{j=\tau_{2e}}^{\tau_2}\left[\delta_T^{j-1-\tau_{2e}}X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}X_{\tau_{1e}}\right]^2\left\{1+o_p\left(1\right)\right\}\\ &= \left[\frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{2c}X_{\tau_{2e}}^2 - 2\frac{T^{2\alpha-1}\delta_T^{\left(\tau_{1f}-\tau_1\right)+(\tau_2-\tau_{2e})}}{r_wc^2}X_{\tau_{1e}}X_{\tau_{2e}}\right.\\ &\left.+\frac{r_2-r_{2e}}{r_w^2c^2}T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}X_{\tau_{1e}}^2\right]\left\{1+o_p\left(1\right)\right\}\\ &= \frac{r_2-r_{2e}}{r_w^2c^2}T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\\ &\sim_a \frac{r_2-r_{2e}}{r_w^2c^2}T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}B\left(r_{1e}\right)^2\end{aligned}$$

due to the fact that

$$\frac{T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}}{T^{2\alpha-1}\delta_T^{\left(\tau_{1f}-\tau_1\right)+(\tau_2-\tau_{2e})}} = T^{1-\alpha}\delta_T^{\left(\tau_{1f}-\tau_1\right)-(\tau_2-\tau_{2e})} > 1,$$

$$\frac{T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}}{T^{\alpha}\delta_T^{2(\tau_2-\tau_{2e})}} = \frac{\delta_T^{2\left[\left(\tau_{1f}-\tau_1\right)-(\tau_2-\tau_{2e})\right]}}{T^{1-\alpha}} > 1.$$

Furthermore, since

$$\frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}}=T^{1-\alpha}\delta_T^{2\left[\left(\tau_{1f}-\tau_{1e}\right)-\left(\tau_{1f}-\tau_1\right)\right]}>1,$$

so the first term dominates the other two terms and hence

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2=\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\sim_a\frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}B\left(r_{1e}\right)^2.$$

Suppose $\tau_{1f}-\tau_1\leq\tau_2-\tau_{2e}$. The first term is

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}_{j-1}^2\\
&=\sum_{j=\tau_1}^{\tau_{1f}}\left[\delta_T^{j-1-\tau_{1e}}X_{\tau_{1e}}-\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\right]^2\left\{1+o_p\left(1\right)\right\}\\
&=\left[\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}X_{\tau_{1e}}^2-2\frac{T^{2\alpha-1}\delta_T^{\left(\tau_2-\tau_{2e}\right)+\left(\tau_{1f}-\tau_{1e}\right)}}{r_w c^2}X_{\tau_{1e}}X_{\tau_{2e}}\right.\\
&\left.+\frac{r_{1f}-r_1}{r_w^2c^2}T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_{2e})}X_{\tau_{2e}}^2\right]\left\{1+o_p\left(1\right)\right\}\\
&=\begin{cases}\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}X_{\tau_{1e}}^2\left\{1+o_p\left(1\right)\right\}\sim_a\frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c}B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e}\\ \frac{r_{1f}-r_1}{r_w^2c^2}T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_{2e})}X_{\tau_{2e}}^2\sim_a\frac{r_{1f}-r_1}{r_w^2c^2}T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})}B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_2-\tau_{2e}\end{cases}
\end{aligned}$$

The second term is

$$\begin{aligned}
\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\tilde{X}_{j-1}^2&=\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\frac{T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{\tau_w^2c^2}X_{\tau_{2e}}^2\left\{1+o_p\left(1\right)\right\}\\
&=\frac{\tau_{2e}-\tau_{1f}-1}{\tau_w^2c^2}T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})}X_{\tau_{2e}}^2\left\{1+o_p\left(1\right)\right\}\\
&\sim_a T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})}\frac{r_{2e}-r_{1f}}{r_w^2c^2}B\left(r_{2e}\right)^2.
\end{aligned}$$

The third term is

$$\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}^2$$

$$
\begin{aligned}
&= \sum_{j=\tau_{2e}}^{\tau_2} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \right]^2 \{1+o_p(1)\} \\
&= \left[\frac{T^\alpha \delta_T^{2(\tau_2-\tau_{2e})}}{2c} - 2\frac{T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_{2e})}}{r_w c^2} + \frac{r_2-r_{2e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_{2e})} \right] X_{\tau_{2e}}^2 \{1+o_p(1)\} \\
&= \frac{T^\alpha \delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \{1+o_p(1)\} \ \text{(since } \alpha > 2\alpha-1) \\
&\sim_a T^{\alpha+1}\delta_T^{2(\tau_2-\tau_{2e})} \frac{1}{2c} B(r_{2e})^2 .
\end{aligned}
$$

Therefore,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \{1+o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B(r_{1e})^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ \frac{T^\alpha \delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \{1+o_p(1)\} \sim_a T^{\alpha+1}\delta_T^{2(\tau_2-\tau_{2e})} \frac{1}{2c} B(r_{2e})^2 & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases} .
$$

Thus, when $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \{1+o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B(r_{1e})^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ \frac{T^\alpha \delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \{1+o_p(1)\} \sim_a T^{\alpha+1}\delta_T^{2(\tau_2-\tau_{2e})} \frac{1}{2c} B(r_{2e})^2 & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases} .
$$

(6) For $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}^2 .
$$

Suppose $\tau_{1f} - \tau_1 > \tau_{2f} - \tau_{2e}$. The first term is

$$
\begin{aligned}
&\sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}^2 \\
&= \sum_{j=\tau_1}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \right]^2 \{1+o_p(1)\} \\
&= \left[\frac{T^\alpha \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} - 2\frac{T^{2\alpha-1}\delta_T^{\left(\tau_{1f}-\tau_1\right)+\left(\tau_{1f}-\tau_{1e}\right)}}{r_w c^2} + \frac{r_{1f}-r_1}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_1\right)} \right] X_{\tau_{1e}}^2 \{1+o_p(1)\} \\
&= \frac{T^\alpha \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \{1+o_p(1)\} \\
&\sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B(r_{1e})^2 .
\end{aligned}
$$

The second term is

$$\begin{aligned}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}}{\tau_w^2 c^2} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\} \\ &= \frac{\tau_{2e}-\tau_{1f}-1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\} \\ &\sim_a T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)} \frac{r_{2e}-r_{1f}}{r_w^2 c^2} B\left(r_{1e}\right)^2 .\end{aligned}$$

The third term is

$$\begin{aligned}&\sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}^2 \\ &= \sum_{j=\tau_{2e}}^{\tau_{2f}} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right]^2 \left\{1+o_p\left(1\right)\right\} \\ &= \left[\frac{T^\alpha \delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 - 2\frac{T^{2\alpha-1}\delta_T^{\left(\tau_{2f}-\tau_{2e}\right)+\left(\tau_{1f}-\tau_1\right)}}{r_w c^2} X_{\tau_{1e}} X_{\tau_{2e}} + \frac{r_{2f}-r_{2e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_1\right)} X_{\tau_{1e}}^2\right] \left\{1+o_p\left(1\right)\right\} \\ &= \frac{r_{2f}-r_{2e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_1\right)} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\} \\ &\sim_a \frac{r_{2f}-r_{2e}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)} B\left(r_{1e}\right)^2 .\end{aligned}$$

The fourth term is

$$\sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_{2f}+1}^{\tau_2} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)}}{\tau_w^2 c^2} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\} \sim_a \frac{r_2-r_{2f}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_1\right)} B\left(r_{1e}\right)^2 .$$

The first term dominates the rest of the terms and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1+o_p\left(1\right)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B\left(r_{1e}\right)^2 .$$

Suppose $\tau_{1f}-\tau_1 \leq \tau_{2f}-\tau_{2e}$. The first term is

$$\begin{aligned}&\sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}^2 \\ &= \sum_{j=\tau_1}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]^2 \left\{1+o_p\left(1\right)\right\}\end{aligned}$$

$$= \left[\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 - 2\frac{T^{2\alpha-1}\delta_T^{\left(\tau_{1f}-\tau_{1e}\right)+\left(\tau_{2f}-\tau_{2e}\right)}}{r_w c^2} X_{\tau_{1e}} X_{\tau_{2e}} + \frac{r_{1f}-r_1}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} X_{\tau_{2e}}^2 \right] \{1+o_p(1)\}$$

$$= \begin{cases} \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \{1+o_p(1)\} & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ \frac{r_{1f}-r_1}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} X_{\tau_{2e}}^2 \{1+o_p(1)\} & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases}$$

$$\sim_a \begin{cases} T^{1+\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} \frac{1}{2c} B(r_{1e})^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} \frac{r_{1f}-r_1}{r_w^2 c^2} B(r_{2e})^2 & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases}.$$

The second term is

$$\begin{aligned} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{\tau_w^2 c^2} X_{\tau_{2e}}^2 \{1+o_p(1)\} \\ &= \frac{\tau_{2e}-\tau_{1f}-1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} X_{\tau_{2e}}^2 \{1+o_p(1)\} \\ &\sim_a T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} \frac{r_{2e}-r_{1f}}{r_w^2 c^2} B(r_{2e})^2 . \end{aligned}$$

The third term is

$$\begin{aligned} &\sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}^2 \\ &= \sum_{j=\tau_{2e}}^{\tau_{2f}} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \right]^2 \{1+o_p(1)\} \\ &= \left[\frac{T^{\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 - 2\frac{T^{2\alpha-1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{r_w c^2} X_{\tau_{2e}}^2 + \frac{r_{if}-r_1}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} X_{\tau_{2e}}^2 \right] \{1+o_p(1)\} \\ &= \frac{T^{\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 \{1+o_p(1)\} \ \ (\text{since } \alpha > 2\alpha-1) \\ &\sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} B(r_{2e})^2 . \end{aligned}$$

The fourth term is

$$\sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_{2f}+1}^{\tau_2} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{\tau_w^2 c^2} X_{\tau_{2e}}^2 \{1+o_p(1)\} \sim_a \frac{r_2-r_{2f}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} B(r_{2e})^2 .$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \sim_a T^{1+\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} \frac{1}{2c} B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ \frac{T^{\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_{2f}-\tau_{2e} \end{cases}$$

Thus, when $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ \frac{T^{\alpha}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_{2f}-\tau_{2e} \end{cases}.$$

(7) For $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}^2 + \sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}^2.$$

Suppose $\tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e}$. The first term is

$$\begin{aligned} \sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_1}^{\tau_{1e}-1} \frac{T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{\tau_w^2 c^2} X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \\ &= \frac{\tau_{1e}-\tau_1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \\ &\sim_a \frac{r_{1e}-r_1}{r_w^2 c^2} T^{2\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2. \end{aligned}$$

The second term is

$$\begin{aligned} &\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}^2 \\ &= \sum_{j=\tau_{1e}}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}\right]^2 X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \\ &= \left[\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} - 2\frac{T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{r_w c^2} + \frac{r_{1f}-r_{1e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}\right] X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \\ &= \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \ \text{(since } \alpha > 2\alpha-1\text{)} \\ &\sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B\left(r_{1e}\right)^2. \end{aligned}$$

The third term is

$$\begin{aligned}
&\sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1}^2 \\
&= \sum_{j=\tau_{2e}}^{\tau_2} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \right]^2 \{1+o_p(1)\} \\
&= \left[\frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 - 2\frac{T^{2\alpha-1}\delta_T^{(\tau_2-\tau_{2e})+(\tau_{1f}-\tau_{1e})}}{r_w c^2} X_{\tau_{1e}} X_{\tau_{2e}} \right. \\
&\left. + \frac{r_2-r_{2e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2(\tau_{1f}-\tau_{1e})} X_{\tau_{1e}}^2 \right] \{1+o_p(1)\} \\
&= \frac{r_2-r_{2e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2(\tau_{1f}-\tau_{1e})} X_{\tau_{1e}}^2 \{1+o_p(1)\} \sim_a \frac{r_2-r_{2e}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_{1f}-\tau_{1e})} B(r_{1e})^2 .
\end{aligned}$$

The fourth term is

$$\begin{aligned}
\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \frac{T^{2\alpha}\delta_T^{2(\tau_{1f}-\tau_{1e})}}{\tau_w^2 c^2} X_{\tau_{1e}}^2 \{1+o_p(1)\} \\
&= \frac{\tau_{2e}-\tau_{1f}-1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_{1f}-\tau_{1e})} X_{\tau_{1e}}^2 \{1+o_p(1)\} \\
&\sim_a \frac{r_{2e}-r_{1f}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_{1f}-\tau_{1e})} B(r_{1e})^2 .
\end{aligned}$$

Since $1+\alpha > 2\alpha$, $\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}^2$ dominates the other terms and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha}\delta_T^{2(\tau_{1f}-\tau_{1e})}}{2c} X_{\tau_{1e}}^2 \{1+o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_{1f}-\tau_{1e})}}{2c} B(r_{1e})^2 .$$

Suppose $\tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e}$. The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_1}^{\tau_{1e}-1} \frac{T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{\tau_w^2 c^2} X_{\tau_{2e}}^2 \{1+o_p(1)\} \\
&= \frac{\tau_{1e}-\tau_1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})} X_{\tau_{2e}}^2 \{1+o_p(1)\} \\
&\sim_a \frac{r_{1e}-r_1}{r_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})} B(r_{2e})^2 .
\end{aligned}$$

The second term is

$$\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}^2$$

$$= \sum_{j=\tau_{1e}}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \right]^2 \{1 + o_p(1)\}$$

$$= \left[\frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 - 2\frac{T^{2\alpha-1}\delta_T^{\left(\tau_{1f}-\tau_{1e}\right)+(\tau_2-\tau_{2e})}}{r_w c^2} X_{\tau_{1e}} X_{\tau_{2e}} + \frac{r_{1f}-r_{1e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_{2e})} X_{\tau_{2e}}^2 \right] \{1 + o_p(1)\}$$

$$= \frac{r_{1f}-r_{1e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_{2e})} X_{\tau_{2e}}^2 \{1 + o_p(1)\}$$

$$\sim_a \frac{r_{1f}-r_{1e}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})} B(r_{2e})^2 .$$

The third term is

$$\sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1}^2$$

$$= \sum_{j=\tau_{2e}}^{\tau_2} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \right]^2 \{1 + o_p(1)\}$$

$$= \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 - 2\frac{T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_{2e})}}{r_w c^2} X_{\tau_{2e}}^2 + \frac{r_2-r_{2e}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_{2e})} X_{\tau_{2e}}^2 \{1 + o_p(1)\}$$

$$= \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \{1 + o_p(1)\}$$

$$\sim_a \frac{T^{1+\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 B(r_{2e})^2 .$$

The fourth term is

$$\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}^2 = \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \frac{T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{\tau_w^2 c^2} X_{\tau_{2e}}^2 \{1 + o_p(1)\}$$

$$= \frac{\tau_{2e}-\tau_{1f}-1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})} X_{\tau_{2e}}^2 \{1 + o_p(1)\}$$

$$\sim_a \frac{r_{2e}-r_{1f}}{r_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_2-\tau_{2e})} B(r_{2e})^2 .$$

Since $1+\alpha > 2\alpha$, $\sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1}^2$ dominates the other terms. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \{1 + o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} B(r_{2e})^2 .$$

Thus, when $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \left\{1+o_p(1)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_2-\tau_{2e})}}{2c} B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases}.$$

□

Lemma S5. *The sample covariance of $\tilde{X}_t$ and ε_t has the following limit forms:*

(1) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{ie}} X_c B\left(r_{ie}\right).$$

(2) *For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B\left(r_{ie}\right).$$

(3) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B\left(r_{ie}\right).$$

(4) *For $\tau_1 \in N_0$ and $\tau_2 \in N_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ T^{(1+\alpha)/2}\delta_T^{\tau_{2f}-\tau_{2e}} X_c B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_{2f}-\tau_{2e} \end{cases}.$$

(5) *For $\tau_1 \in B_1$ and $\tau_2 \in B_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{2e}} X_c B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases}.$$

(6) *For $\tau_1 \in B_1$ and $\tau_2 \in N_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ T^{(1+\alpha)/2}\delta_T^{\tau_{2f}-\tau_{2e}} X_c B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_{2f}-\tau_{2e} \end{cases}.$$

(7) *For $\tau_1 \in N_0$ and $\tau_2 \in B_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{2e}} X_c B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases}.$$

Proof. (1) For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_1}^{\tau_{ie}-1} -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}}\varepsilon_j \left\{1 + o_p(1)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \sum_{j=\tau_1}^{\tau_{ie}-1} \varepsilon_j \left\{1 + o_p(1)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{r_w c} \left(T^{-1/2} X_{\tau_{ie}}\right) \left(T^{-1/2} \sum_{j=\tau_1}^{\tau_{ie}-1} \varepsilon_j\right) \left\{1 + o_p(1)\right\} \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{r_w c} B(r_{ie}) \left[B(r_{ie}) - B(r_1)\right].
\end{aligned}$$

The second term is

$$\begin{aligned}
&\sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_{ie}}^{\tau_2} \left[\delta_T^{j-1-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c}\right] X_{\tau_{ie}}\varepsilon_j \left\{1 + o_p(1)\right\} \\
&= \left[\sum_{j=\tau_{ie}}^{\tau_2} \delta_T^{j-1-\tau_{ie}}\varepsilon_j - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c} \sum_{j=\tau_{ie}}^{\tau_2} \varepsilon_j\right] X_{\tau_{ie}} \left\{1 + o_p(1)\right\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_2-\tau_{ie}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{ie}}^{\tau_2} \delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) - \frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{ie}}}{r_w c} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{ie}}^{\tau_2} \varepsilon_j\right)\right] X_{\tau_{ie}} \left\{1 + o_p(1)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_2-\tau_{ie}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{ie}}^{\tau_2} \delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) X_{\tau_{ie}} \left\{1 + o_p(1)\right\} \;\; (\text{since } \alpha/2 > \alpha - 1/2) \\
&= T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{ie}} \left(T^{-\alpha/2} \sum_{j=\tau_{ie}}^{\tau_2} \delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) \left(T^{-1/2} X_{\tau_{ie}}\right) \left\{1 + o_p(1)\right\} \\
&\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{ie}} X_c B(r_{ie}).
\end{aligned}$$

Since $(\alpha+1)/2 > \alpha$, $\sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$ dominates $\sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}\varepsilon_j$. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \{1+o_p(1)\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{ie}} X_c B(r_{ie}).$$

(2) For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_1}^{\tau_{if}} \left[\delta_T^{j-1-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}\right] X_{\tau_{ie}}\varepsilon_j \{1+o_p(1)\} \\
&= \left[\sum_{j=\tau_1}^{\tau_{if}} \delta_T^{j-1-\tau_{ie}}\varepsilon_j - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}\sum_{j=\tau_1}^{\tau_{if}} \varepsilon_j\right] X_{\tau_{ie}} \{1+o_p(1)\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_{if}-\tau_{ie}}\left(\frac{1}{T^{\alpha/2}}\sum_{j=\tau_1}^{\tau_{if}} \delta_T^{-\left(\tau_{if}-j+1\right)}\varepsilon_j\right) - \frac{T^{\alpha+1/2}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}\left(\frac{1}{\sqrt{T}}\sum_{j=\tau_1}^{\tau_{if}} \varepsilon_j\right)\right] X_{\tau_{ie}} \{1+o_p(1)\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_{if}-\tau_{ie}}\left(\frac{1}{T^{\alpha/2}}\sum_{j=\tau_1}^{\tau_{if}} \delta_T^{-\left(\tau_{if}-j+1\right)}\varepsilon_j\right) - \frac{T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_1}}{r_w c}\left(\frac{1}{\sqrt{T}}\sum_{j=\tau_1}^{\tau_{if}} \varepsilon_j\right)\right] X_{\tau_{ie}} \{1+o_p(1)\} \\
&= T^{\alpha/2}\delta_T^{\tau_{if}-\tau_{ie}}\left(T^{-\alpha/2}\sum_{j=\tau_1}^{\tau_{if}} \delta_T^{-\left(\tau_{if}-j+1\right)}\varepsilon_j\right) X_{\tau_{ie}} \{1+o_p(1)\} \\
&\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B(r_{ie}).
\end{aligned}$$

The second term is

$$\begin{aligned}
\sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_{if}+1}^{\tau_2} -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c} X_{\tau_{ie}}\varepsilon_j \{1+o_p(1)\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c} X_{\tau_{ie}} \sum_{j=\tau_{if}+1}^{\tau_2} \varepsilon_j \{1+o_p(1)\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{r_w c}\left(T^{-1/2}X_{\tau_{ie}}\right)\left(T^{-1/2}\sum_{j=\tau_{if}+1}^{\tau_2} \varepsilon_j\right) \{1+o_p(1)\}
\end{aligned}$$

$$\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{r_w c} B(r_{ie})[B(r_2) - B(r_{if})].$$

Since $(\alpha+1)/2 > \alpha$, $\sum_{j=\tau_1}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j$ dominates $\sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$, so

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j \{1 + o_p(1)\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B(r_{ie}).$$

(3) For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{ie}}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_1}^{\tau_{ie}-1} -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}}\varepsilon_j \{1 + o_p(1)\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \sum_{j=\tau_1}^{\tau_{ie}-1} \varepsilon_j \{1 + o_p(1)\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{r_w c} \left(T^{-1/2} X_{\tau_{ie}}\right) \left(T^{-1/2} \sum_{j=\tau_1}^{\tau_{ie}-1} \varepsilon_j\right) \{1 + o_p(1)\} \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{r_w c} B(r_{ie})[B(r_{ie}) - B(r_1)].
\end{aligned}$$

The second term is

$$\begin{aligned}
&\sum_{j=\tau_{ie}}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_{ie}}^{\tau_{if}} \left[\delta_T^{j-1-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}\right] X_{\tau_{ie}}\varepsilon_j \{1 + o_p(1)\} \\
&= \left[\sum_{j=\tau_{ie}}^{\tau_{if}} \delta_T^{j-1-\tau_{ie}}\varepsilon_j - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} \sum_{j=\tau_{ie}}^{\tau_{if}} \varepsilon_j\right] X_{\tau_{ie}} \{1 + o_p(1)\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_{if}-\tau_{ie}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{ie}}^{\tau_{if}} \delta_T^{-(\tau_{if}-j+1)}\varepsilon_j\right) - \frac{T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_{ie}}}{r_w c} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{ie}}^{\tau_{if}} \varepsilon_j\right)\right] X_{\tau_{ie}} \{1 + o_p(1)\} \\
&= T^{\alpha/2+1/2}\delta_T^{\tau_{if}-\tau_{ie}} \left(T^{-\alpha/2} \sum_{j=\tau_{ie}}^{\tau_{if}} \delta_T^{-(\tau_{if}-j+1)}\varepsilon_j\right) \left(T^{-1/2} X_{\tau_{ie}}\right) \{1 + o_p(1)\}
\end{aligned}$$

$$\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B\left(r_{ie}\right).$$

The third term is

$$\begin{aligned}
\sum_{j=\tau_{if}+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_{if}+1}^{\tau_2} -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}}\varepsilon_j \left\{1+o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau e}}{\tau_w c} X_{\tau_{ie}} \sum_{j=\tau_{if}+1}^{\tau_2} \varepsilon_j \left\{1+o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{r_w c} \left(T^{-1/2} X_{\tau_{ie}}\right) \left(T^{-1/2} \sum_{j=\tau_{if}+1}^{\tau_2} \varepsilon_j\right) \left\{1+o_p\left(1\right)\right\} \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{r_w c} B\left(r_{ie}\right)\left[B\left(r_2\right)-B\left(r_{if}\right)\right].
\end{aligned}$$

Since $\left(\alpha+1\right)/2 > \alpha$, $\sum_{j=\tau_{ie}}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j$ dominates the other two terms, so

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_{ie}}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j \left\{1+o_p\left(1\right)\right\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B\left(r_{ie}\right).$$

(4) For $\tau_1 \in N_0$ and $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

Suppose $\tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e}$. The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_1}^{\tau_{1e}-1} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\varepsilon_j \left\{1+o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \sum_{j=\tau_1}^{\tau_{1e}-1} \varepsilon_j \left\{1+o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} \left(T^{-1/2} X_{\tau_{1e}}\right) \left(T^{-1/2} \sum_{j=\tau_1}^{\tau_{1e}-1} \varepsilon_j\right) \left\{1+o_p\left(1\right)\right\} \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} B\left(r_{1e}\right)\left[B\left(r_{1e}\right)-B\left(r_1\right)\right].
\end{aligned}$$

The second term is

$$\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j$$

$$
\begin{aligned}
&= \sum_{j=\tau_{1e}}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \right] \varepsilon_j \left\{1+o_p(1)\right\} \\
&= \left[\sum_{j=\tau_{1e}}^{\tau_{1f}} \delta_T^{j-1-\tau_{1e}} \varepsilon_j X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j \right] \left\{1+o_p(1)\right\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_{1f}-\tau_{1e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \delta_T^{-\left(\tau_{1f}-j+1\right)} \varepsilon_j \right) X_{\tau_{1e}} \right. \\
&\left. - \frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} X_{\tau_{1e}} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j \right) \right] \left\{1+o_p(1)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_{1f}-\tau_{1e}} \left(T^{-\alpha/2} \sum_{j=\tau_{1e}}^{\tau_{1f}} \delta_T^{-\left(\tau_{1f}-j+1\right)} \varepsilon_j \right) X_{\tau_{1e}} \ (\text{since } \alpha/2 > \alpha - 1/2) \\
&\sim_a T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right).
\end{aligned}
$$

The third term is

$$
\begin{aligned}
&\sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_{2e}}^{\tau_{2f}} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \right] \varepsilon_j \left\{1+o_p(1)\right\} \\
&= \left[\sum_{j=\tau_{2e}}^{\tau_{2f}} \delta_T^{j-1-\tau_{2e}} \varepsilon_j X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j \right] \left\{1+o_p(1)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_{2f}-\tau_{2e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{2e}}^{\tau_{2f}} \delta_T^{-\left(\tau_{2f}-j+1\right)} \varepsilon_j \right) X_{\tau_{2e}} \left\{1+o_p(1)\right\} \\
&- \frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j \right) X_{\tau_{1e}} \left\{1+o_p(1)\right\} \\
&= -\frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} \left(T^{-1/2} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j \right) X_{\tau_{1e}} \sim_a -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} B\left(r_{1e}\right)\left[B\left(r_{1f}\right) - B\left(r_{1e}\right)\right].
\end{aligned}
$$

The fourth term is

$$
\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\varepsilon_j \left\{1+o_p(1)\right\}
$$

$$= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j \{1 + o_p(1)\}$$

$$= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} \left(T^{-1/2} X_{\tau_{1e}}\right) \left(T^{-1/2} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j\right) \{1 + o_p(1)\}$$

$$\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} B(r_{1e}) \left[B(r_{2e}) - B(r_{1f})\right].$$

The fifth term is

$$\sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_{2f}+1}^{\tau_2} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \varepsilon_j \{1 + o_p(1)\}$$

$$= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \sum_{j=\tau_{2f}+1}^{\tau_2} \varepsilon_j \{1 + o_p(1)\}$$

$$= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} \left(T^{-1/2} X_{\tau_{1e}}\right) \left(T^{-1/2} \sum_{j=\tau_{2f}+1}^{\tau_2} \varepsilon_j\right) \{1 + o_p(1)\}$$

$$\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} B(r_{1e}) \left[B(r_2) - B(r_{2f})\right].$$

Since $(\alpha+1)/2 > \alpha$, $\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j$ dominates the other terms, so

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j \{1 + o_p(1)\} \sim_a T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}).$$

Suppose $\tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e}$. The first term is

$$\sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_{1e}-1} -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \varepsilon_j \{1 + o_p(1)\}$$

$$= -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \sum_{j=\tau_1}^{\tau_{1e}-1} \varepsilon_j \{1 + o_p(1)\}$$

$$= -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} \left(T^{-1/2} X_{\tau_{2e}}\right) \left(T^{-1/2} \sum_{j=\tau_1}^{\tau_{1e}-1} \varepsilon_j\right) \{1 + o_p(1)\}$$

$$\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} B(r_{2e}) \left[B(r_{1e}) - B(r_1)\right].$$

The second term is

$$
\begin{aligned}
&\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_{1e}}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right] \varepsilon_j \left\{1+o_p\left(1\right)\right\} \\
&= \left[\sum_{j=\tau_{1e}}^{\tau_{1f}} \delta_T^{j-1-\tau_{1e}} \varepsilon_j X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j\right] \left\{1+o_p\left(1\right)\right\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_{1f}-\tau_{1e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \delta_T^{-\left(\tau_{1f}-j+1\right)}\varepsilon_j\right) X_{\tau_{1e}} - \frac{T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} X_{\tau_{2e}} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j\right)\right] \left\{1+o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} X_{\tau_{2e}} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j\right) \left\{1+o_p\left(1\right)\right\} \;\; (\text{since } \alpha/2 > \alpha - 1/2) \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} B\left(r_{2e}\right)\left[B\left(r_{1f}\right) - B\left(r_{1e}\right)\right].
\end{aligned}
$$

The third term is

$$
\begin{aligned}
&\sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_{2e}}^{\tau_{2f}} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right] \varepsilon_j \left\{1+o_p\left(1\right)\right\} \\
&= \left[\sum_{j=\tau_{2e}}^{\tau_{2f}} \delta_T^{j-1-\tau_{2e}} \varepsilon_j X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \sum_{j=\tau_{2e}}^{\tau_{2f}} \varepsilon_j\right] \left\{1+o_p\left(1\right)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_{2f}-\tau_{2e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{2e}}^{\tau_{2f}} \delta_T^{-\left(\tau_{2f}-j+1\right)}\varepsilon_j\right) X_{\tau_{2e}} \left\{1+o_p\left(1\right)\right\} \\
&- \frac{T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{2e}}^{\tau_{2f}} \varepsilon_j\right) X_{\tau_{2e}} \left\{1+o_p\left(1\right)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_{2f}-\tau_{2e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{2e}}^{\tau_{2f}} \delta_T^{-\left(\tau_{2f}-j+1\right)}\varepsilon_j\right) X_{\tau_{2e}} \left\{1+o_p\left(1\right)\right\} \sim_a T^{(1+\alpha)/2}\delta_T^{\tau_{2f}-\tau_{2e}} X_c B\left(r_{2e}\right).
\end{aligned}
$$

The fourth term is

$$\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\tilde{X}_{j-1}\varepsilon_j=\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}-\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\varepsilon_j\left\{1+o_p\left(1\right)\right\}$$
$$=-\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\varepsilon_j\left\{1+o_p\left(1\right)\right\}$$
$$=-\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c}\left(T^{-1/2}X_{\tau_{2e}}\right)\left(T^{-1/2}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\varepsilon_j\right)\left\{1+o_p\left(1\right)\right\}$$
$$\sim_a-\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c}B\left(r_{2e}\right)\left[B\left(r_{2e}\right)-B\left(r_{1f}\right)\right].$$

The fifth term is

$$\sum_{j=\tau_{2f}+1}^{\tau_{2}}\tilde{X}_{j-1}\varepsilon_j=\sum_{j=\tau_{2f}+1}^{\tau_{2}}-\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\varepsilon_j\left\{1+o_p\left(1\right)\right\}$$
$$=-\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\sum_{j=\tau_{2f}+1}^{\tau_{2}}\varepsilon_j\left\{1+o_p\left(1\right)\right\}$$
$$=-\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c}\left(T^{-1/2}X_{\tau_{2e}}\right)\left(T^{-1/2}\sum_{j=\tau_{2f}+1}^{\tau_{2}}\varepsilon_j\right)\left\{1+o_p\left(1\right)\right\}$$
$$\sim_a-\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}B\left(r_{2e}\right)\left[B\left(r_{2}\right)-B\left(r_{2f}\right)\right].$$

Since $\left(\alpha+1\right)/2>\alpha$, $\sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}_{j-1}\varepsilon_j$ dominates the other terms, so

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j=\sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}_{j-1}\varepsilon_j\left\{1+o_p\left(1\right)\right\}\sim_a T^{(1+\alpha)/2}\delta_T^{\tau_{2f}-\tau_{2e}}X_cB\left(r_{2e}\right).$$

Thus, when $\tau_1\in N_0$ and $\tau_2\in N_2$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j\sim_a\begin{cases}T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}}X_cB\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}\\ T^{(1+\alpha)/2}\delta_T^{\tau_{2f}-\tau_{2e}}X_cB\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\end{cases}.$$

(5) For $\tau_1\in B_1$ and $\tau_2\in B_2$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j=\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j.$$

Suppose $\tau_{1f} - \tau_1 > \tau_2 - \tau_{2e}$. The first term is

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_1}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}\right] X_{\tau_{1e}}\varepsilon_j \left\{1+o_p(1)\right\} \\
&= \left[\sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{j-1-\tau_{1e}}\varepsilon_j - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}\sum_{j=\tau_1}^{\tau_{1f}}\varepsilon_j\right] X_{\tau_{1e}} \left\{1+o_p(1)\right\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_{1f}-\tau_{1e}}\left(\frac{1}{T^{\alpha/2}}\sum_{j=\tau_1}^{\tau_{1f}}\delta_T^{-\left(\tau_{1f}-j+1\right)}\varepsilon_j\right) - \frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}}{r_w c}\left(\frac{1}{\sqrt{T}}\sum_{j=\tau_1}^{\tau_{1f}}\varepsilon_j\right)\right] X_{\tau_{1e}} \left\{1+o_p(1)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_{1f}-\tau_{1e}}\left(T^{-\alpha/2}\sum_{j=\tau_1}^{\tau_{1f}}\delta_T^{-\left(\tau_{1f}-j+1\right)}\varepsilon_j\right) X_{\tau_{1e}} \left\{1+o_p(1)\right\} \\
&\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right).
\end{aligned}$$

The second term is

$$\begin{aligned}
\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\varepsilon_j \left\{1+o_p(1)\right\} \\
&= -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\varepsilon_j \left\{1+o_p(1)\right\} \\
&= -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{r_w c}\left(T^{-1/2}X_{\tau_{1e}}\right)\left(T^{-1/2}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\varepsilon_j\right) \left\{1+o_p(1)\right\} \\
&\sim_a -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{r_w c} B\left(r_{1e}\right)\left[B\left(r_{2e}\right) - B\left(r_{1f}\right)\right].
\end{aligned}$$

The third term is

$$\begin{aligned}
&\sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_{2e}}^{\tau_2}\left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right]\varepsilon_j \left\{1+o_p(1)\right\} \\
&= \left[X_{\tau_{2e}}\sum_{j=\tau_{2e}}^{\tau_2}\delta_T^{j-1-\tau_{2e}}\varepsilon_j - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\sum_{j=\tau_{2e}}^{\tau_2}\varepsilon_j\right]\left\{1+o_p(1)\right\}
\end{aligned}$$

$$= \left[T^{\alpha/2} \delta_T^{\tau_2 - \tau_{2e}} X_{\tau_{2e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{2e}}^{\tau_2} \delta_T^{-(\tau_2 - j + 1)} \varepsilon_j \right) \right.$$

$$\left. - \frac{T^{\alpha-1/2} \delta_T^{\tau_{1f} - \tau_1}}{r_w c} X_{\tau_{1e}} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{2e}}^{\tau_2} \varepsilon_j \right) \right] \{1 + o_p(1)\}$$

$$= - \frac{T^{\alpha-1/2} \delta_T^{\tau_{1f} - \tau_1}}{r_w c} X_{\tau_{1e}} \left(T^{-1/2} \sum_{j=\tau_{2e}}^{\tau_2} \varepsilon_j \right) \{1 + o_p(1)\}$$

$$\sim_a - \frac{T^{\alpha} \delta_T^{\tau_{1f} - \tau_1}}{r_w c} B(r_{1e}) [B(r_2) - B(r_{2e})].$$

The first term dominates the other two terms and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1} \varepsilon_j \{1 + o_p(1)\} \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_{1f} - \tau_{1e}} X_c B(r_{1e}).$$

Suppose $\tau_{1f} - \tau_1 \leq \tau_2 - \tau_{2e}$. The first term is

$$\sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1} \varepsilon_j$$

$$= \sum_{j=\tau_1}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha} \delta_T^{\tau_2 - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \right] \varepsilon_j \{1 + o_p(1)\}$$

$$= \left[\sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{j-1-\tau_{1e}} \varepsilon_j X_{\tau_{1e}} - \frac{T^{\alpha} \delta_T^{\tau_2 - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \sum_{j=\tau_1}^{\tau_{1f}} \varepsilon_j \right] \{1 + o_p(1)\}$$

$$= \left[T^{\alpha/2} \delta_T^{\tau_{1f} - \tau_{1e}} \left(T^{-\alpha/2} \sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{-\left(\tau_{1f} - j + 1\right)} \varepsilon_j \right) X_{\tau_{1e}} - \frac{T^{\alpha-1/2} \delta_T^{\tau_2 - \tau_{2e}}}{r_w c} X_{\tau_{2e}} \left(T^{-1/2} \sum_{j=\tau_1}^{\tau_{1f}} \varepsilon_j \right) \right] \{1 + o_p(1)\}$$

$$= \begin{cases} T^{\alpha/2} \delta_T^{\tau_{1f} - \tau_{1e}} \left(T^{-\alpha/2} \sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{-\left(\tau_{1f} - j + 1\right)} \varepsilon_j \right) X_{\tau_{1e}} \{1 + o_p(1)\} & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ -\frac{T^{\alpha-1/2} \delta_T^{\tau_2 - \tau_{2e}}}{r_w c} X_{\tau_{2e}} \left(T^{-1/2} \sum_{j=\tau_1}^{\tau_{1f}} \varepsilon_j \right) & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e} \end{cases}$$

$$\sim_a \begin{cases} T^{(\alpha+1)/2} \delta_T^{\tau_{1f} - \tau_{1e}} X_c B(r_{1e}) & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ -T^{\alpha} \delta_T^{\tau_2 - \tau_{2e}} \frac{1}{r_w c} B(r_{2e}) [B(r_{1f}) - B(r_1)] & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e} \end{cases}.$$

The second term is

$$\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} - \frac{T^{\alpha} \delta_T^{\tau_2 - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \varepsilon_j \{1 + o_p(1)\}$$

$$
\begin{aligned}
&= -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j \left\{1+o_p(1)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{r_w c} \left(T^{-1/2}X_{\tau_{2e}}\right)\left(T^{-1/2}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j\right) \left\{1+o_p(1)\right\} \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{r_w c} B(r_{2e})\left[B(r_{2e}) - B(r_{1f})\right].
\end{aligned}
$$

The third term is

$$
\begin{aligned}
&\sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_{2e}}^{\tau_2} \left[\delta_T^{j-1-\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c}\right] X_{\tau_{2e}}\varepsilon_j \left\{1+o_p(1)\right\} \\
&= \left[\sum_{j=\tau_{2e}}^{\tau_2} \delta_T^{j-1-\tau_{2e}}\varepsilon_j - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c}\sum_{j=\tau_{2e}}^{\tau_2}\varepsilon_j\right] X_{\tau_{2e}} \left\{1+o_p(1)\right\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_2-\tau_{2e}}\left(\frac{1}{T^{\alpha/2}}\sum_{j=\tau_{2e}}^{\tau_2}\delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) - \frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}}{r_w c}\left(\frac{1}{\sqrt{T}}\sum_{j=\tau_{2e}}^{\tau_2}\varepsilon_j\right)\right] X_{\tau_{2e}} \left\{1+o_p(1)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_2-\tau_{2e}}\left(T^{-\alpha/2}\sum_{j=\tau_{2e}}^{\tau_2}\delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) X_{\tau_{2e}} \left\{1+o_p(1)\right\} \\
&\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{2e}} X_c B(r_{2e}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j &= \begin{cases} \sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j \left\{1+o_p(1)\right\} & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ T^{\alpha/2}\delta_T^{\tau_2-\tau_{2e}}\left(T^{-\alpha/2}\sum_{j=\tau_{2e}}^{\tau_2}\delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) X_{\tau_{2e}}\left\{1+o_p(1)\right\} & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases} \\
&\sim_a \begin{cases} T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{2e}} X_c B(r_{2e}). & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases}
\end{aligned}
$$

Thus, when $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{2e}} X_c B(r_{2e}). & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases}
$$

(6) For $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

Suppose $\tau_{1f} - \tau_1 > \tau_{2f} - \tau_{2e}$. The first term is

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_1}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \right] \varepsilon_j \left\{1 + o_p(1)\right\} \\
&= \left[\sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{j-1-\tau_{1e}} \varepsilon_j X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \sum_{j=\tau_1}^{\tau_{1f}} \varepsilon_j \right] \left\{1 + o_p(1)\right\} \\
&= \left[T^{\alpha/2} \delta_T^{\tau_{1f}-\tau_{1e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{-\left(\tau_{1f}-j+1\right)} \varepsilon_j \right) X_{\tau_{1e}} - \frac{T^{\alpha-1/2} \delta_T^{\tau_{1f}-\tau_1}}{r_w c} X_{\tau_{1e}} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_1}^{\tau_{1f}} \varepsilon_j \right) \right] \left\{1 + o_p(1)\right\} \\
&= T^{\alpha/2} \delta_T^{\tau_{1f}-\tau_{1e}} \left(T^{-\alpha/2} \sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{-\left(\tau_{1f}-j+1\right)} \varepsilon_j \right) X_{\tau_{1e}} \left\{1 + o_p(1)\right\} \;\; (\text{since } \alpha/2 > \alpha - 1/2) \\
&\sim_a T^{(1+\alpha)/2} \delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right).
\end{aligned}$$

The second term is

$$\begin{aligned}
\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \varepsilon_j \left\{1 + o_p(1)\right\} \\
&= -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j \left\{1 + o_p(1)\right\} \\
&= -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} \left(T^{-1/2} X_{\tau_{1e}} \right) \left(T^{-1/2} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j \right) \left\{1 + o_p(1)\right\} \\
&\sim_a -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} B\left(r_{1e}\right) \left[B\left(r_{2e}\right) - B\left(r_{1f}\right) \right].
\end{aligned}$$

The third term is

$$\sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}\varepsilon_j$$

$$
\begin{aligned}
&= \sum_{j=\tau_{2e}}^{\tau_{2f}} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \right] \varepsilon_j \left\{ 1 + o_p(1) \right\} \\
&= \left[\sum_{j=\tau_{2e}}^{\tau_2} \delta_T^{j-1-\tau_{2e}} \varepsilon_j X_{\tau_{2e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} \sum_{j=\tau_{2e}}^{\tau_2} \varepsilon_j X_{\tau_{1e}} \right] \left\{ 1 + o_p(1) \right\} \\
&= \left[T^{\alpha/2} \delta_T^{\tau_{2f}-\tau_{2e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{2e}}^{\tau_{2f}} \delta_T^{-\left(\tau_{2f}-j+1\right)} \varepsilon_j \right) X_{\tau_{2e}} \right. \\
&\left. - \frac{T^{\alpha-1/2} \delta_T^{\tau_{1f}-\tau_1}}{r_w c} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{2e}}^{\tau_{2f}} \varepsilon_j \right) X_{\tau_{1e}} \right] \left\{ 1 + o_p(1) \right\} \\
&= - \frac{T^{\alpha-1/2} \delta_T^{\tau_{1f}-\tau_1}}{r_w c} \left(T^{-1/2} \sum_{j=\tau_{2e}}^{\tau_{2f}} \varepsilon_j \right) X_{\tau_{1e}} \left\{ 1 + o_p(1) \right\} \\
&\sim_a - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{r_w c} \left[B(r_{2f}) - B(r_{2e}) \right] B(r_{1e}) .
\end{aligned}
$$

The fourth term is

$$
\begin{aligned}
\sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j &= \sum_{j=\tau_{2f}+1}^{\tau_2} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \varepsilon_j \left\{ 1 + o_p(1) \right\} \\
&= - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \sum_{j=\tau_{2f}+1}^{\tau_2} \varepsilon_j \left\{ 1 + o_p(1) \right\} \\
&= - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} \left(T^{-1/2} X_{\tau_{1e}} \right) \left(T^{-1/2} \sum_{j=\tau_{2f}+1}^{\tau_2} \varepsilon_j \right) \left\{ 1 + o_p(1) \right\} \\
&\sim_a - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} B(r_{1e}) \left[B(r_2) - B(r_{2f}) \right] .
\end{aligned}
$$

Therefore, $\sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1} \varepsilon_j$ dominates the rest of terms,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1} \varepsilon_j \left\{ 1 + o_p(1) \right\} \sim_a T^{(1+\alpha)/2} \delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}) .
$$

Suppose $\tau_{1f} - \tau_1 \leq \tau_{2f} - \tau_{2e}$. The first term is

$$
\sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1} \varepsilon_j
$$

$$= \sum_{j=\tau_1}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \right] \varepsilon_j \{1 + o_p(1)\}$$

$$= \left[\sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{j-1-\tau_{1e}} \varepsilon_j X_{\tau_{1e}} - \frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \sum_{j=\tau_1}^{\tau_{1f}} \varepsilon_j \right] \{1 + o_p(1)\}$$

$$= \left[T^{\alpha/2} \delta_T^{\tau_{1f}-\tau_{1e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{-\left(\tau_{1f}-j+1\right)} \varepsilon_j \right) X_{\tau_{1e}} - \frac{T^{\alpha-1/2} \delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} X_{\tau_{2e}} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_1}^{\tau_{1f}} \varepsilon_j \right) \right] \{1 + o_p(1)\}$$

$$= \begin{cases} T^{\alpha/2} \delta_T^{\tau_{1f}-\tau_{1e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_1}^{\tau_{1f}} \delta_T^{-\left(\tau_{1f}-j+1\right)} \varepsilon_j \right) X_{\tau_{1e}} \{1 + o_p(1)\} & \text{if } \tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e} \\ -\frac{T^{\alpha-1/2} \delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} X_{\tau_{2e}} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_1}^{\tau_{1f}} \varepsilon_j \right) & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e} \end{cases}$$

$$\sim_a \begin{cases} T^{(1+\alpha)/2} \delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}) & \text{if } \tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e} \\ -T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}} \frac{1}{r_w c} B(r_{2e}) \left[B(r_{1f}) - B(r_1) \right] & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e} \end{cases}$$

The second term is

$$\begin{aligned} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1} \varepsilon_j &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} -\frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \varepsilon_j \{1 + o_p(1)\} \\ &= -\frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j \{1 + o_p(1)\} \\ &= -\frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} \left(T^{-1/2} X_{\tau_{2e}} \right) \left(T^{-1/2} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j \right) \{1 + o_p(1)\} \\ &\sim_a -\frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} B(r_{2e}) \left[B(r_{2e}) - B(r_{1f}) \right]. \end{aligned}$$

The third term is

$$\sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1} \varepsilon_j$$

$$= \sum_{j=\tau_{2e}}^{\tau_{2f}} \left[\delta_T^{j-1-\tau_{2e}} - \frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} \right] X_{\tau_{2e}} \varepsilon_j \{1 + o_p(1)\}$$

$$= \left[\sum_{j=\tau_{2e}}^{\tau_2} \delta_T^{j-1-\tau_{2e}} \varepsilon_j - \frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} \sum_{j=\tau_{2e}}^{\tau_2} \varepsilon_j \right] X_{\tau_{2e}} \{1 + o_p(1)\}$$

$$= \left[T^{\alpha/2} \delta_T^{\tau_{2f}-\tau_{2e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{2e}}^{\tau_{2f}} \delta_T^{-\left(\tau_{2f}-j+1\right)} \varepsilon_j \right) - \frac{T^{\alpha-1/2} \delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{2e}}^{\tau_{2f}} \varepsilon_j \right) \right] X_{\tau_{2e}} \{1 + o_p(1)\}$$

$$= T^{\alpha/2}\delta_T^{\tau_{2f}-\tau_{2e}}\left(T^{-\alpha/2}\sum_{j=\tau_{2e}}^{\tau_{2f}}\delta_T^{-\left(\tau_{2f}-j+1\right)}\varepsilon_j\right)X_{\tau_{2e}}\left\{1+o_p\left(1\right)\right\}$$
$$\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{2f}-\tau_{2e}}X_cB\left(r_{2e}\right).$$

The fourth term is

$$\begin{aligned}
\sum_{j=\tau_{2f}+1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_{2f}+1}^{\tau_2} -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\varepsilon_j\left\{1+o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\sum_{j=\tau_{2f}+1}^{\tau_2}\varepsilon_j\left\{1+o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}\left(T^{-1/2}X_{\tau_{2e}}\right)\left(T^{-1/2}\sum_{j=\tau_{2f}+1}^{\tau_2}\varepsilon_j\right)\left\{1+o_p\left(1\right)\right\} \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}B\left(r_{2e}\right)\left[B\left(r_2\right)-B\left(r_{2f}\right)\right].
\end{aligned}$$

Therefore, the first term dominates the other terms and hence

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \left\{\begin{array}{ll} T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}}X_cB\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_{2f}-\tau_{2e}}X_cB\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\end{array}\right. .$$

Thus, when $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \left\{\begin{array}{ll} T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}}X_cB\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_{2f}-\tau_{2e}}X_cB\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\end{array}\right. .$$

(7) For $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j.$$

Suppose $\tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e}$. The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_1}^{\tau_{1e}-1} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}X_{\tau_{1e}}\varepsilon_j\left\{1+o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}X_{\tau_{1e}}\sum_{j=\tau_1}^{\tau_{1e}-1}\varepsilon_j\left\{1+o_p\left(1\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c}\left(T^{-1/2}X_{\tau_{1e}}\right)\left(\frac{1}{\sqrt{T}}\sum_{j=\tau_1}^{\tau_{1e}-1}\varepsilon_j\right)\{1+o_p(1)\} \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c}B(r_{1e})\left[B(r_{1e})-B(r_1)\right].
\end{aligned}
$$

The second term is

$$
\begin{aligned}
&\sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_{1e}}^{\tau_{1f}}\left[\delta_T^{j-1-\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}\right]X_{\tau_{1e}}\varepsilon_j\{1+o_p(1)\} \\
&= \left[\sum_{j=\tau_{1e}}^{\tau_{1f}}\delta_T^{j-1-\tau_{1e}}\varepsilon_j - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}\sum_{j=\tau_{1e}}^{\tau_{1f}}\varepsilon_j\right]X_{\tau_{1e}}\{1+o_p(1)\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_{1f}-\tau_{1e}}\left(\frac{1}{T^{\alpha/2}}\sum_{j=\tau_{1e}}^{\tau_{1f}}\delta_T^{-\left(\tau_{1f}-j+1\right)}\varepsilon_j\right) - \frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c}\left(\frac{1}{\sqrt{T}}\sum_{j=\tau_{1e}}^{\tau_{1f}}\varepsilon_j\right)\right]X_{\tau_{1e}}\{1+o_p(1)\} \\
&= T^{\alpha/2}\delta_T^{\tau_{1f}-\tau_{1e}}\left(T^{-\alpha/2}\sum_{j=\tau_{1e}}^{\tau_{1f}}\delta_T^{-\left(\tau_{1f}-j+1\right)}\varepsilon_j\right)X_{\tau_{1e}}\{1+o_p(1)\}\ \ (\text{since } \alpha/2>\alpha-1/2) \\
&\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}}X_c B(r_{1e}).
\end{aligned}
$$

The third term is

$$
\begin{aligned}
\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}X_{\tau_{1e}}\varepsilon_j\{1+o_p(1)\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}X_{\tau_{1e}}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\varepsilon_j\{1+o_p(1)\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}\left(T^{-1/2}X_{\tau_{1e}}\right)\left(T^{-1/2}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1}\varepsilon_j\right)\{1+o_p(1)\} \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}B(r_{1e})\left[B(r_{2e})-B(r_{1f})\right].
\end{aligned}
$$

The fourth term is

$$
\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j
$$

$$
\begin{aligned}
&= \sum_{j=\tau_{2e}}^{\tau_2} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^{\alpha} \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{1e} \right] \varepsilon_j \left\{ 1 + o_p(1) \right\} \\
&= \left[X_{\tau_{2e}} \sum_{j=\tau_{2e}}^{\tau_2} \delta_T^{j-1-\tau_{2e}} \varepsilon_j - \frac{T^{\alpha} \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{1e} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j \right] \left\{ 1 + o_p(1) \right\} \\
&= \left[T^{\alpha/2} \delta_T^{\tau_2-\tau_{2e}} X_{\tau_{2e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{2e}}^{\tau_2} \delta_T^{-(\tau_2-j+1)} \varepsilon_j \right) \right. \\
&\left. - \frac{T^{\alpha-1/2} \delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} X_{1e} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j \right) \right] \left\{ 1 + o_p(1) \right\} \\
&= \frac{T^{\alpha-1/2} \delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} X_{1e} \left(T^{-1/2} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j \right) \\
&\sim_a \frac{T^{\alpha} \delta_T^{\tau_{1f}-\tau_{1e}}}{r_w c} B(r_{1e}) \left[B(r_{1f}) - B(r_{1e}) \right]
\end{aligned}
$$

due to the fact that

$$
\frac{T^{\alpha/2} \delta_T^{\tau_2-\tau_{2e}}}{T^{\alpha-1/2} \delta_T^{\tau_{1f}-\tau_{1e}}} = \frac{T^{(1-\alpha)/2}}{\delta_T^{(\tau_{1f}-\tau_{1e})-(\tau_2-\tau_{2e})}} = \frac{T^{(1-\alpha)/2}}{e^{c\left[(r_{1f}-r_{1e})-(r_2-r_{2e})\right]T^{1-\alpha}}} < 1.
$$

Since $(\alpha+1)/2 > \alpha$, $\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1} \varepsilon_j$ dominates the other terms. Therefore,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1} \varepsilon_j \left\{ 1 + o_p(1) \right\} \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}).
$$

Suppose $\tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e}$. The first term is

$$
\begin{aligned}
\sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1} \varepsilon_j &= \sum_{j=\tau_1}^{\tau_{1e}-1} -\frac{T^{\alpha} \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \varepsilon_j \left\{ 1 + o_p(1) \right\} \\
&= -\frac{T^{\alpha} \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \sum_{j=\tau_1}^{\tau_{1e}-1} \varepsilon_j \left\{ 1 + o_p(1) \right\} \\
&= -\frac{T^{\alpha} \delta_T^{\tau_2-\tau_{2e}}}{r_w c} \left(T^{-1/2} X_{\tau_{2e}} \right) \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_1}^{\tau_{1e}-1} \varepsilon_j \right) \left\{ 1 + o_p(1) \right\} \\
&\sim_a -\frac{T^{\alpha} \delta_T^{\tau_2-\tau_{2e}}}{r_w c} B(r_{2e}) \left[B(r_{1e}) - B(r_1) \right].
\end{aligned}
$$

The second term is

$$\sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j$$
$$= \sum_{j=\tau_{1e}}^{\tau_{1f}} \left[\delta_T^{j-1-\tau_{1e}} X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right] \varepsilon_j \{1+o_p(1)\}$$
$$= \left[\sum_{j=\tau_{1e}}^{\tau_{1f}} \delta_T^{j-1-\tau_{1e}} \varepsilon_j X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j X_{\tau_{2e}}\right] \{1+o_p(1)\}$$
$$= \left[T^{\alpha/2} \delta_T^{\tau_{1f}-\tau_{1e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \delta_T^{-(\tau_{1f}-j+1)} \varepsilon_j\right) X_{\tau_{1e}} - \frac{T^{\alpha-1/2} \delta_T^{\tau_2-\tau_{2e}}}{r_w c} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j\right) X_{\tau_{2e}}\right] \{1+o_p(1)\}$$
$$= -\frac{T^{\alpha-1/2} \delta_T^{\tau_2-\tau_{2e}}}{r_w c} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j\right) X_{\tau_{2e}} \{1+o_p(1)\} \;\; (\text{since } \alpha/2 > \alpha - 1/2)$$
$$\sim_a -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{r_w c} \left[B(r_{1f}) - B(r_{1e})\right] B(r_{2e}).$$

The third term is

$$\begin{aligned}
\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \varepsilon_j \{1+o_p(1)\} \\
&= -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j \{1+o_p(1)\} \\
&= -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} \left(T^{-1/2} X_{\tau_{2e}}\right) \left(T^{-1/2} \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \varepsilon_j\right) \{1+o_p(1)\} \\
&\sim_a -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} B(r_{2e}) \left[B(r_{2e}) - B(r_{1f})\right].
\end{aligned}$$

The fourth term is

$$\begin{aligned}
&\sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_{2e}}^{\tau_2} \left[\delta_T^{j-1-\tau_{2e}} X_{\tau_{2e}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{2e}\right] \varepsilon_j \{1+o_p(1)\} \\
&= \left[X_{\tau_{2e}} \sum_{j=\tau_{2e}}^{\tau_2} \delta_T^{j-1-\tau_{2e}} \varepsilon_j - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{2e} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j\right] \{1+o_p(1)\}
\end{aligned}$$

$$
\begin{aligned}
&= \left[T^{\alpha/2} \delta_T^{\tau_2 - \tau_{2e}} X_{\tau_{2e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{2e}}^{\tau_2} \delta_T^{-(\tau_2 - j+1)} \varepsilon_j \right) \right. \\
&\left. - \frac{T^{\alpha - 1/2} \delta_T^{\tau_2 - \tau_{2e}}}{r_w c} X_{\tau_{2e}} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_{1e}}^{\tau_{1f}} \varepsilon_j \right) \right] \{1 + o_p(1)\} \\
&= T^{\alpha/2} \delta_T^{\tau_2 - \tau_{2e}} X_{\tau_{2e}} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_{2e}}^{\tau_2} \delta_T^{-(\tau_2 - j+1)} \varepsilon_j \right) \\
&\sim_a T^{(1+\alpha)/2} \delta_T^{\tau_2 - \tau_{2e}} B(r_{2e}) X_c.
\end{aligned}
$$

Since $(\alpha + 1)/2 > \alpha$, $\sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j$ dominates the other terms. Therefore,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j \{1 + o_p(1)\} \sim_a T^{(1+\alpha)/2} \delta_T^{\tau_2 - \tau_{2e}} B(r_{2e}) X_c.
$$

Thus, when $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j \sim_a \begin{cases} T^{(\alpha+1)/2} \delta_T^{\tau_{1f} - \tau_{1e}} X_c B(r_{1e}) & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ T^{(\alpha+1)/2} \delta_T^{\tau_2 - \tau_{2e}} X_c B(r_{2e}) & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e} \end{cases}.
$$

□

Lemma S6. *The sample covariance of $\tilde{X}_{j-1}$ and $X_j - \delta_T X_{j-1}$ has the following limit forms:*

(1) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} (X_j - \delta_T X_{j-1}) \sim_a \frac{r_{ie} - r_1}{r_w} T \delta_T^{\tau_2 - \tau_{ie}} B(r_{ie}) \int_{r_1}^{r_{ie}} B(s) \, ds.
$$

(2) *For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} (X_j - \delta_T X_{j-1}) \sim_a -T \delta_T^{2(\tau_{if} - \tau_{ie})} B(r_{ie})^2.
$$

(3) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} (X_j - \delta_T X_{j-1}) \sim_a -T \delta_T^{2(\tau_{if} - \tau_{ie})} B(r_{ie})^2.
$$

(4) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \sim_a \begin{cases} -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ -T\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} B\left(r_{2e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} \leq \tau_{2f}-\tau_{2e} \end{cases}.$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \sim_a \begin{cases} -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ T^\alpha \delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}} \frac{1}{r_w c} B\left(r_{2e}\right) B\left(r_{1e}\right) & \textit{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases}.$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \sim_a \begin{cases} -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ -T\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} B\left(r_{2e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} \leq \tau_{2f}-\tau_{2e} \end{cases}.$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \sim_a \begin{cases} -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2 & \textit{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ T^\alpha \delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}} \frac{1}{r_w c} B\left(r_{2e}\right) B\left(r_{1e}\right) & \textit{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases}.$$

Proof. (1) For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$\begin{aligned} \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) &= \sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}\left(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1}\right) \\ &= \sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1}\right) \\ &= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1} X_{j-1}. \end{aligned} \tag{11}$$

The first term of (11)

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_2-\tau_{ie}} X_c B\left(r_{ie}\right).$$

The second term is

$$\begin{aligned} &\frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1} X_{j-1} \\ &= \frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_{ie}-1} -\frac{T^\alpha \delta_T^{\tau_2-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} X_{j-1} \left\{1 + o_p(1)\right\} \end{aligned}$$

$$= -\frac{\delta_T^{\tau_2-\tau_{ie}}}{\tau_w} X_{\tau_{ie}} \sum_{j=\tau_1}^{\tau_{ie}-1} X_{j-1} \{1 + o_p(1)\}$$

$$= -\frac{\tau_{ie}-\tau_1}{\tau_w} T\delta_T^{\tau_2-\tau_{ie}} \left(T^{-1/2} X_{\tau_{ie}}\right) \left[\frac{1}{\tau_{ie}-\tau_1} \sum_{j=\tau_1}^{\tau_{ie}-1} \left(T^{-1/2} X_{j-1}\right)\right] \{1 + o_p(1)\}$$

$$\sim_a -\frac{r_{ie}-r_1}{r_w} T\delta_T^{\tau_2-\tau_{ie}} B(r_{ie}) \int_{r_1}^{r_{ie}} B(s)\, ds.$$

Since $(\alpha+1)/2 < 1$, $\frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1} X_{j-1}$ dominates $\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j$. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) = -\frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1} X_{j-1} \{1 + o_p(1)\}$$

$$\sim_a \frac{r_{ie}-r_1}{r_w} T\delta_T^{\tau_2-\tau_{ie}} B(r_{ie}) \int_{r_1}^{r_{ie}} B(s)\, ds$$

(2) For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) = \sum_{j=\tau_1}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j + \tilde{X}_{\tau_{if}}\left(X_{\tau_{if}+1} - \delta_T X_{\tau_{if}}\right) + \sum_{j=\tau_{if}+2}^{\tau_2} \tilde{X}_{j-1}(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1})$$

$$= \sum_{j=\tau_1}^{\tau_{if}} \tilde{X}_{j-1}\varepsilon_j + \tilde{X}_{\tau_{if}}\left(X_{\tau_{ie}} + X^* + \varepsilon_{\tau_{if}+1} - \delta_T X_{\tau_{if}}\right) + \sum_{j=\tau_{if}+2}^{\tau_2} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1}\right)$$

$$= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \delta_T \tilde{X}_{\tau_{if}} X_{\tau_{if}} - \frac{c}{T^\alpha} \sum_{j=\tau_{if}+2}^{\tau_2} \tilde{X}_{j-1} X_{j-1}.$$

The first term is

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_{if}-\tau_{ie}} X_c B(r_{ie}).$$

The second term is

$$\begin{aligned} \delta_T \tilde{X}_{\tau_{if}} X_{\tau_{if}} &= \delta_T \left[\delta_T^{\tau_{if}-\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{if}-\tau_1}}{\tau_w c}\right] X_{\tau_{ie}} X_{\tau_{if}} \{1 + o_p(1)\} \\ &= \delta_T^{\tau_{if}-\tau_{ie}+1} X_{\tau_{ie}} X_{\tau_{if}} \{1 + o_p(1)\} \sim_a T\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)} B(r_{ie})^2 \end{aligned}$$

due to the fact that

$$\frac{\delta_T^{\tau_{if}-\tau_{ie}}}{T^{\alpha-1}\delta_T^{\tau_{if}-\tau_1}} = T^{1-\alpha}\delta_T^{\tau_1-\tau_{ie}} > 1.$$

The third term is

$$\begin{aligned}
&\frac{c}{T^{\alpha}}\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}_{j-1}X_{j-1}\\
&=\frac{c}{T^{\alpha}}\sum_{j=\tau_{if}+2}^{\tau_2}-\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}X_{\tau_{ie}}X_{j-1}\left\{1+o_p\left(1\right)\right\}\\
&=-\frac{\delta_T^{\tau_{if}-\tau_1}}{\tau_w}X_{\tau_{ie}}\sum_{j=\tau_{if}+2}^{\tau_2}X_{j-1}\left\{1+o_p\left(1\right)\right\}\\
&=-\frac{\tau_2-\tau_{if}-1}{\tau_w}T\delta_T^{\tau_{if}-\tau_1}\left(T^{-1/2}X_{\tau_{ie}}\right)\left(\frac{1}{\tau_2-\tau_{if}-1}\sum_{j=\tau_{if}+2}^{\tau_2}T^{-1/2}X_{j-1}\right)\left\{1+o_p\left(1\right)\right\}\\
&\sim_a-\frac{r_2-r_{if}}{r_w}T\delta_T^{\tau_{if}-\tau_1}B\left(r_{ie}\right)\int_{r_{if}}^{r_2}B\left(s\right)ds.
\end{aligned}$$

The second term dominates the other terms and hence

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right)=-\delta_T\tilde{X}_{\tau_{if}}X_{\tau_{if}}\left\{1+o_p\left(1\right)\right\}\sim_a-T\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}B\left(r_{ie}\right)^2.$$

(3) For $\tau_1\in N_{i-1}$ and $\tau_2\in N_i$ with $i=1,2$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right)\\
&=\sum_{j=\tau_{ie}}^{\tau_{if}}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=1}^{\tau_{ie}-1}\tilde{X}_{j-1}\left(X_j-X_{j-1}+X_{j-1}-\delta_T X_{j-1}\right)\\
&+\tilde{X}_{\tau_{if}}\left(X_{\tau_{if}+1}-\delta_T X_{\tau_{if}}\right)+\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}_{j-1}\left(X_j-X_{j-1}+X_{j-1}-\delta_T X_{j-1}\right)\\
&=\sum_{j=\tau_1}^{\tau_{if}}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=1}^{\tau_{ie}-1}\tilde{X}_{j-1}\left(\varepsilon_j-\frac{c}{T^{\alpha}}X_{j-1}\right)-\delta_T\tilde{X}_{\tau_{if}}X_{\tau_{if}}+\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}_{j-1}\left(\varepsilon_j-\frac{c}{T^{\alpha}}X_{j-1}\right)\\
&=\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j-\frac{c}{T^{\alpha}}\sum_{j=1}^{\tau_{ie}-1}\tilde{X}_{j-1}X_{j-1}-\delta_T\tilde{X}_{\tau_{if}}X_{\tau_{if}}-\frac{c}{T^{\alpha}}\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}_{j-1}X_{j-1}.
\end{aligned}$$

The first term is

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}}X_cB\left(r_{ie}\right).$$

The second term is

$$\begin{aligned}
&\frac{c}{T^{\alpha}}\sum_{j=\tau_1}^{\tau_{ie}-1}\tilde{X}_{j-1}X_{j-1}\\
&=\frac{c}{T^{\alpha}}\sum_{j=\tau_1}^{\tau_{ie}-1}-\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}X_{\tau_{ie}}X_{j-1}\left\{1+o_p\left(1\right)\right\}\\
&=-\frac{\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w}X_{\tau_{ie}}\sum_{j=\tau_1}^{\tau_{ie}-1}X_{j-1}\left\{1+o_p\left(1\right)\right\}\\
&=-\frac{\tau_{ie}-\tau_1}{\tau_w}T\delta_T^{\tau_{if}-\tau_{ie}}\left(T^{-1/2}X_{\tau_{ie}}\right)\left(\frac{1}{\tau_{ie}-\tau_1}\sum_{j=\tau_1}^{\tau_{ie}-1}T^{-1/2}X_{j-1}\right)\left\{1+o_p\left(1\right)\right\}\\
&\sim_a -\frac{r_{ie}-r_1}{r_w}T\delta_T^{\tau_{if}-\tau_{ie}}B\left(r_{ie}\right)\int_{r_1}^{r_{ie}}B\left(s\right)ds.
\end{aligned}$$

The third term is

$$\begin{aligned}
\delta_T\tilde{X}_{\tau_{if}}X_{\tau_{if}} &= \delta_T\left[\delta_T^{\tau_{if}-\tau_{ie}}-\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}\right]X_{\tau_{ie}}X_{\tau_{if}}\left\{1+o_p\left(1\right)\right\}\\
&= \delta_T^{\tau_{if}-\tau_{ie}+1}X_{\tau_{ie}}X_{\tau_{if}}\left\{1+o_p\left(1\right)\right\}\sim_a T\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}B\left(r_{ie}\right)^2.
\end{aligned}$$

The fourth term is

$$\begin{aligned}
&\frac{c}{T^{\alpha}}\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}_{j-1}X_{j-1}\\
&=\frac{c}{T^{\alpha}}\sum_{j=\tau_{if}+2}^{\tau_2}-\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}X_{\tau_{ie}}X_{j-1}\left\{1+o_p\left(1\right)\right\}\\
&=-\frac{\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w}X_{\tau_{ie}}\sum_{j=\tau_{if}+2}^{\tau_2}X_{j-1}\left\{1+o_p\left(1\right)\right\}\\
&=-\frac{\tau_2-\tau_{if}+1}{\tau_w}T\delta_T^{\tau_{if}-\tau_{ie}}\left(T^{-1/2}X_{\tau_{ie}}\right)\left(\frac{1}{\tau_2-\tau_{if}-1}\sum_{j=\tau_{if}+1}^{\tau_2}T^{-1/2}X_{j-1}\right)\left\{1+o_p\left(1\right)\right\}\\
&\sim_a -\frac{r_2-r_{if}}{r_w}T\delta_T^{\tau_{if}-\tau_{ie}}B\left(r_{ie}\right)\int_{r_{if}}^{r_2}B\left(s\right)ds.
\end{aligned}$$

The third term dominates the other terms. Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right)=-\delta_T\tilde{X}_{\tau_{if}}X_{\tau_{if}}\left\{1+o_p\left(1\right)\right\}\sim_a -T\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}B\left(r_{ie}\right)^2.$$

(4) For $\tau_1 \in N_0$ and $\tau_2 \in N_2$,

$$
\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \\
&= \sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}\left(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1}\right) \\
&+ \tilde{X}_{\tau_{1f}}\left(X_{\tau_{1f}+1} - \delta_T X_{\tau_{1f}}\right) + \tilde{X}_{\tau_{2f}}\left(X_{\tau_{2f}+1} - \delta_T X_{\tau_{2f}}\right) \\
&+ \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1}\left(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1}\right) + \sum_{j=\tau_{2f}+2}^{\tau_2} \tilde{X}_{j-1}\left(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1}\right) \\
&= \sum_{j=\tau_{1e}}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha}X_{j-1}\right) \\
&- \delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} - \delta_T \tilde{X}_{\tau_{2f}} X_{\tau_{2f}} + \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha}X_{j-1}\right) + \sum_{j=\tau_{2f}+2}^{\tau_2} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha}X_{j-1}\right) \\
&= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \frac{c}{T^\alpha}\sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}X_{j-1} - \delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} - \delta_T \tilde{X}_{\tau_{2f}} X_{\tau_{2f}} - \frac{c}{T^\alpha}\sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1}X_{j-1} - \frac{c}{T^\alpha}\sum_{j=\tau_{2f}+1}^{\tau_2} \tilde{X}_{j-1}X_{j-1}
\end{aligned}
$$

Suppose $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$. The first term is

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right).
$$

The second term is

$$
\begin{aligned}
&\frac{c}{T^\alpha}\sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1}X_{j-1} \\
&= \frac{c}{T^\alpha}\sum_{j=\tau_1}^{\tau_{1e}-1} -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} X_{j-1}\left\{1 + o_p(1)\right\} \\
&= -\frac{\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w} X_{\tau_{1e}} \sum_{j=\tau_1}^{\tau_{1e}-1} X_{j-1}\left\{1 + o_p(1)\right\} \\
&= -\frac{\tau_{1e}-\tau_1}{\tau_w} T\delta_T^{\tau_{1f}-\tau_{1e}}\left(T^{-1/2}X_{\tau_{1e}}\right)\left(\frac{1}{\tau_{1e}-\tau_1}\sum_{j=\tau_1}^{\tau_{1e}-1} T^{-1/2}X_{j-1}\right)\left\{1 + o_p(1)\right\} \\
&\sim_a -\frac{r_{1e}-r_1}{r_w} T\delta_T^{\tau_{1f}-\tau_{1e}} B\left(r_{1e}\right)\int_{r_1}^{r_{1e}} B(s)\,ds.
\end{aligned}
$$

The third and fourth terms are

$$\begin{aligned}\delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} &= \delta_T \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \right] X_{\tau_{1f}} \left\{1 + o_p\left(1\right)\right\} \\ &= \delta_T^{\tau_{1f}-\tau_{1e}+1} X_{\tau_{1e}} X_{\tau_{1f}} \left\{1 + o_p\left(1\right)\right\} \sim_a T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2 .\end{aligned}$$

and

$$\begin{aligned}\delta_T \tilde{X}_{\tau_{2f}} X_{\tau_{2f}} &= \delta_T \left[\delta_T^{\tau_{2f}-\tau_{2e}} X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \right] X_{\tau_{2f}} \left\{1 + o_p\left(1\right)\right\} \\ &= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}+1}}{\tau_w c} X_{\tau_{1e}} X_{\tau_{2f}} \left\{1 + o_p\left(1\right)\right\} \sim_a -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}+\tau_{2f}-\tau_{2e}}}{r_w c} B\left(r_{1e}\right) B\left(r_{2e}\right).\end{aligned}$$

due to the fact that

$$\frac{\delta_T^{\tau_{2f}-\tau_{2e}}}{T^{\alpha-1}\delta_T^{\tau_{1f}-\tau_{1e}}} = \frac{T^{1-\alpha}}{e^{c\left[\left(r_{1f}-r_{1e}\right)-\left(r_{2f}-r_{2e}\right)\right]T^{1-\alpha}}} < 1.$$

The fifth term is

$$\begin{aligned}&\frac{c}{T^{\alpha}} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1} X_{j-1} \\ &= \frac{c}{T^{\alpha}} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} X_{j-1} \left\{1 + o_p\left(1\right)\right\} \\ &= -\frac{\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w} X_{\tau_{1e}} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} X_{j-1} \left\{1 + o_p\left(1\right)\right\} \\ &= -\frac{\tau_{2e}-\tau_{1f}-1}{\tau_w} T\delta_T^{\tau_{1f}-\tau_{1e}} \left(T^{-1/2} X_{\tau_{1e}}\right) \left(\frac{1}{\tau_{2e}-\tau_{1f}-1} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} T^{-1/2} X_{j-1}\right) \left\{1 + o_p\left(1\right)\right\} \\ &\sim_a -\frac{r_{2e}-r_{1f}}{r_w} T\delta_T^{\tau_{1f}-\tau_{1e}} B\left(r_{1e}\right) \int_{r_{1f}}^{r_{2e}} B\left(s\right) ds.\end{aligned}$$

The sixth term is

$$\begin{aligned}&\frac{c}{T^{\alpha}} \sum_{j=\tau_{2f}+2}^{\tau_2} \tilde{X}_{j-1} X_{j-1} \\ &= \frac{c}{T^{\alpha}} \sum_{j=\tau_{2f}+2}^{\tau_2} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} X_{j-1} \left\{1 + o_p\left(1\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w} X_{\tau_{1e}} \sum_{j=\tau_{2f}+2}^{\tau_2} X_{j-1} \left\{1 + o_p(1)\right\} \\
&= -\frac{\tau_2 - \tau_{2f} - 1}{\tau_w} T \delta_T^{\tau_{1f}-\tau_{1e}} \left(T^{-1/2} X_{\tau_{1e}}\right) \left(\frac{1}{\tau_2 - \tau_{2f} - 1} \sum_{j=\tau_{2f}+2}^{\tau_2} T^{-1/2} X_{j-1}\right) \left\{1 + o_p(1)\right\} \\
&\sim_a -\frac{r_2 - r_{2f}}{r_w} T \delta_T^{\tau_{1f}-\tau_{1e}} B(r_{1e}) \int_{r_{2f}}^{r_2} B(s)\, ds.
\end{aligned}
$$

Since

$$
\frac{T \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^{\alpha} \delta_T^{\tau_{1f}-\tau_{1e}+\tau_{2f}-\tau_{2e}}} = T^{1-\alpha} \delta_T^{\left(\tau_{1f}-\tau_{1e}\right)-\left(\tau_{2f}-\tau_{2e}\right)} > 1,
$$

The third term dominates the other terms and hence

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1}\right) = -\delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} \left\{1 + o_p(1)\right\} \sim_a -T \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B(r_{1e})^2 .
$$

Suppose $\tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e}$. The first term is

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j \sim_a T^{(1+\alpha)/2} \delta_T^{\tau_{2f}-\tau_{2e}} X_c B(r_{2e}) .
$$

The second term is

$$
\begin{aligned}
&\frac{c}{T^{\alpha}} \sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1} X_{j-1} \\
&= \frac{c}{T^{\alpha}} \sum_{j=\tau_1}^{\tau_{1e}-1} -\frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} X_{j-1} \left\{1 + o_p(1)\right\} \\
&= -\frac{\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w} X_{\tau_{2e}} \sum_{j=\tau_1}^{\tau_{1e}-1} X_{j-1} \left\{1 + o_p(1)\right\} \\
&= -\frac{\tau_{1e} - \tau_1}{\tau_w} T \delta_T^{\tau_{2f}-\tau_{2e}} \left(T^{-1/2} X_{\tau_{2e}}\right) \left(\frac{1}{\tau_{1e} - \tau_1} \sum_{j=\tau_1}^{\tau_{1e}-1} T^{-1/2} X_{j-1}\right) \left\{1 + o_p(1)\right\} \\
&\sim_a -\frac{r_{1e} - r_1}{r_w} T \delta_T^{\tau_{2f}-\tau_{2e}} B(r_{2e}) \int_{r_1}^{r_{1e}} B(s)\, ds.
\end{aligned}
$$

The third and fourth terms are

$$
\delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} \quad = \quad \delta_T \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha} \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right] X_{\tau_{1f}} \left\{1 + o_p(1)\right\}
$$

$$= \quad -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}X_{\tau_{1f}}\left\{1+o_p\left(1\right)\right\} \sim_a -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}+\tau_{2f}-\tau_{2e}}}{r_w c}B\left(r_{2e}\right)B\left(r_{1e}\right).$$

and

$$\begin{aligned}\delta_T \tilde{X}_{\tau_{2f}}X_{\tau_{2f}} &= \delta_T\left[\delta_T^{\tau_{2f}-\tau_{2e}}X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\right]X_{\tau_{2f}}\left\{1+o_p\left(1\right)\right\} \\ &= \delta_T^{\tau_{2f}-\tau_{2e}+1}X_{\tau_{2e}}X_{\tau_{2f}}\left\{1+o_p\left(1\right)\right\} \sim_a T\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}B\left(r_{2e}\right)^2.\end{aligned}$$

The fifth term is

$$\begin{aligned}&\frac{c}{T^{\alpha}}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}X_{j-1} \\ &= \frac{c}{T^{\alpha}}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}X_{j-1}\left\{1+o_p\left(1\right)\right\} \\ &= -\frac{\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w}X_{\tau_{2e}}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}X_{j-1}\left\{1+o_p\left(1\right)\right\} \\ &= -\frac{\tau_{2e}-\tau_{1f}-1}{\tau_w}T\delta_T^{\tau_{2f}-\tau_{2e}}\left(T^{-1/2}X_{\tau_{2e}}\right)\left(\frac{1}{\tau_{2e}-\tau_{1f}-1}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}T^{-1/2}X_{j-1}\right)\left\{1+o_p\left(1\right)\right\} \\ &\sim_a -\frac{r_{2e}-r_{1f}}{r_w}T\delta_T^{\tau_{2f}-\tau_{2e}}B\left(r_{2e}\right)\int_{r_{1f}}^{r_{2e}}B\left(s\right)ds.\end{aligned}$$

The sixth term is

$$\begin{aligned}&\frac{c}{T^{\alpha}}\sum_{j=\tau_{2f}+2}^{\tau_2}\tilde{X}_{j-1}X_{j-1} \\ &= \frac{c}{T^{\alpha}}\sum_{j=\tau_{2f}+2}^{\tau_2} -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}X_{j-1}\left\{1+o_p\left(1\right)\right\} \\ &= -\frac{\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w}X_{\tau_{2e}}\sum_{j=\tau_{2f}+2}^{\tau_2}X_{j-1}\left\{1+o_p\left(1\right)\right\} \\ &= -\frac{\tau_2-\tau_{2f}-1}{\tau_w}T\delta_T^{\tau_{2f}-\tau_{2e}}\left(T^{-1/2}X_{\tau_{2e}}\right)\left(\frac{1}{\tau_2-\tau_{2f}-1}\sum_{j=\tau_{2f}+2}^{\tau_2}T^{-1/2}X_{j-1}\right)\left\{1+o_p\left(1\right)\right\} \\ &\sim_a -\frac{r_2-r_{2f}}{r_w}T\delta_T^{\tau_{2f}-\tau_{2e}}B\left(r_{2e}\right)\int_{r_{2f}}^{r_2}B\left(s\right)ds.\end{aligned}$$

The fourth term dominates the other terms and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) = -\delta_T \tilde{X}_{\tau_{2f}} X_{\tau_{2f}} \left\{1 + o_p(1)\right\} \sim_a -T\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} B\left(r_{2e}\right)^2 .$$

(5) For $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \\
&= \sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j + \tilde{X}_{\tau_{1f}}\left(X_{\tau_{1f}+1} - \delta_T X_{\tau_{1f}}\right) + \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1}\left(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1}\right) + \sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j - \delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} + \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1}\right) + \sum_{j=\tau_{2e}}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} - \frac{c}{T^\alpha} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1} X_{j-1}.
\end{aligned}$$

The first term is

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \left\{ \begin{array}{ll} T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2 - \tau_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{2e}} X_c B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2 - \tau_{2e} \end{array} \right. .$$

Suppose $\tau_{1f} - \tau_1 > \tau_2 - \tau_{2e}$. The second tem is

$$\begin{aligned}
\delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} &= \delta_T \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right] X_{\tau_{1f}} \left\{1 + o_p(1)\right\} \\
&= \delta_T^{\tau_{1f}-\tau_{1e}+1} X_{\tau_{1e}} X_{\tau_{1f}} \left\{1 + o_p(1)\right\} \sim_a T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2 .
\end{aligned}$$

The third term is

$$\begin{aligned}
&\frac{c}{T^\alpha} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1} X_{j-1} \\
&= \frac{c}{T^\alpha} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} X_{j-1} \left\{1 + o_p(1)\right\} \\
&= -\frac{\delta_T^{\tau_{1f}-\tau_1}}{\tau_w} X_{\tau_{1e}} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} X_{j-1} \left\{1 + o_p(1)\right\}
\end{aligned}$$

$$= -\frac{\tau_{2e}-\tau_{1f}-1}{\tau_w} T\delta_T^{\tau_{1f}-\tau_1}\left(T^{-1/2}X_{\tau_{1e}}\right)\left(\frac{1}{\tau_{2e}-\tau_{1f}-1}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} T^{-1/2}X_{j-1}\right)\{1+o_p(1)\}$$

$$\sim_a -\frac{r_{2e}-r_{1f}}{r_w} T\delta_T^{\tau_{1f}-\tau_1} B(r_{1e})\int_{r_{1f}}^{r_{2e}} B(s)\,ds.$$

The second term dominates the other terms and hence

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}(X_j-\delta_T X_{j-1}) \sim_a -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B(r_{1e})^2 .$$

Suppose $\tau_{1f}-\tau_1 \leq \tau_2-\tau_{2e}$. The second term is

$$\begin{aligned}
\delta_T\tilde{X}_{\tau_{1f}}X_{\tau_{1f}} &= \delta_T\left[\delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}} - \frac{T^\alpha\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\right]X_{\tau_{1f}}\{1+o_p(1)\} \\
&= \begin{cases} \delta_T^{\tau_{1f}-\tau_{1e}+1}X_{\tau_{1e}}X_{\tau_{1f}}\{1+o_p(1)\} & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ -\frac{T^\alpha\delta_T^{\tau_2-\tau_{2e}+1}}{\tau_w c}X_{\tau_{2e}}X_{\tau_{1f}}\{1+o_p(1)\} & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases} \\
&\sim_a \begin{cases} T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}B(r_{1e})^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ -\frac{T^\alpha\delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}}}{r_w c}B(r_{2e})B(r_{1e}) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases} .
\end{aligned}$$

The third tem is

$$\frac{c}{T^\alpha}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}X_{j-1}$$

$$= \frac{c}{T^\alpha}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} -\frac{T^\alpha\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}X_{j-1}\{1+o_p(1)\}$$

$$= -\frac{\delta_T^{\tau_2-\tau_{2e}}}{\tau_w}X_{\tau_{2e}}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}X_{j-1}\{1+o_p(1)\}$$

$$= -\frac{\tau_{2e}-\tau_{1f}-1}{\tau_w} T\delta_T^{\tau_2-\tau_{2e}}\left(T^{-1/2}X_{\tau_{2e}}\right)\left(\frac{1}{\tau_{2e}-\tau_{1f}-1}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} T^{-1/2}X_{j-1}\right)\{1+o_p(1)\}$$

$$\sim_a -\frac{r_{2e}-r_{1f}}{r_w} T\delta_T^{\tau_2-\tau_{2e}} B(r_{2e})\int_{r_{1f}}^{r_{2e}} B(s)\,ds.$$

The second term dominates the other two terms and hence

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}(X_j-\delta_T X_{j-1}) \sim_a \begin{cases} -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}B(r_{1e})^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ \frac{T^\alpha\delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}}}{r_w c}B(r_{2e})B(r_{1e}) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases} .$$

Thus, when $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a \begin{cases} -T\delta_T^{2(\tau_{1f}-\tau_{1e})} B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_2 - \tau_{2e} \\ \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}}}{r_w c} B\left(r_{2e}\right) B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2 - \tau_{2e} \end{cases} .$$

(6) For $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \\
&= \sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j + \tilde{X}_{\tau_{1f}} \left(X_{\tau_{1f}+1} - \delta_T X_{\tau_{1f}} \right) + \sum_{j=\tau_{1f}+1}^{\tau_{2e}-1} \tilde{X}_{j-1} \left(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1} \right) + \sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}\varepsilon_j \\
&+ \tilde{X}_{\tau_{2f}} \left(X_{\tau_{2f}+1} - \delta_T X_{\tau_{2f}} \right) + \sum_{j=\tau_{2f}+2}^{\tau_2} \tilde{X}_{j-1} \left(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1} \right) \\
&= \sum_{j=\tau_1}^{\tau_{1f}} \tilde{X}_{j-1}\varepsilon_j - \delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} + \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1} \left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1} \right) + \sum_{j=\tau_{2e}}^{\tau_{2f}} \tilde{X}_{j-1}\varepsilon_j \\
&- \delta_T \tilde{X}_{\tau_{2f}} X_{\tau_{2f}} + \sum_{j=\tau_{2f}+2}^{\tau_2} \tilde{X}_{j-1} \left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1} \right) \\
&= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} - \frac{c}{T^\alpha} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1} X_{j-1} - \delta_T \tilde{X}_{\tau_{2f}} X_{\tau_{2f}} - \frac{c}{T^\alpha} \sum_{j=\tau_{2f}+2}^{\tau_2} \tilde{X}_{j-1} X_{j-1}.
\end{aligned}$$

The first term is (from Lemma S5)

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2} \delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f} - \tau_{2e} \\ T^{(1+\alpha)/2} \delta_T^{\tau_{2f}-\tau_{2e}} X_c B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_{2f} - \tau_{2e} \end{cases} .$$

Suppose $\tau_{1f} - \tau_1 > \tau_{2f} - \tau_{2e}$. The second term is

$$\begin{aligned}
\delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} &= \delta_T \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \right] X_{\tau_{1f}} \left\{ 1 + o_p(1) \right\} \\
&= \delta_T^{\tau_{1f}-\tau_{1e}+1} X_{\tau_{1e}} X_{\tau_{1f}} \left\{ 1 + o_p(1) \right\} \sim_a T\delta_T^{2(\tau_{1f}-\tau_{1e})} B\left(r_{1e}\right)^2 .
\end{aligned}$$

The third tem is

$$\frac{c}{T^\alpha} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1} X_{j-1} = \frac{c}{T^\alpha} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} X_{j-1} \left\{ 1 + o_p(1) \right\}$$

$$\sim_a -\frac{r_{2e}-r_{1f}}{r_w} T\delta_T^{\tau_{1f}-\tau_1} B\left(r_{1e}\right)\int_{r_{1f}}^{r_{2e}} B\left(s\right)ds.$$

The fourth term is

$$\begin{aligned}\delta_T\tilde{X}_{\tau_{2f}}X_{\tau_{2f}} &= \delta_T\left[\delta_T^{\tau_{2f}-\tau_{2e}}X_{\tau_{2e}}-\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}X_{\tau_{1e}}\right]X_{\tau_{2f}}\left\{1+o_p\left(1\right)\right\}\\ &= -\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_1+1}}{\tau_w c}X_{\tau_{1e}}X_{\tau_{2f}}\left\{1+o_p\left(1\right)\right\}\sim_a -\frac{T^\alpha\delta_T^{\left(\tau_{1f}-\tau_1\right)+\left(\tau_{2f}-\tau_{2e}\right)}}{r_w c}B\left(r_{1e}\right)B\left(r_{2e}\right).\end{aligned}$$

The fifth term is

$$\begin{aligned}\frac{c}{T^\alpha}\sum_{j=\tau_{2f}+1}^{\tau_2}\tilde{X}_{j-1}X_{j-1} &= \frac{c}{T^\alpha}\sum_{j=\tau_{2f}+1}^{\tau_2}-\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}X_{\tau_{1e}}X_{j-1}\left\{1+o_p\left(1\right)\right\}\\ &\sim_a -\frac{r_2-r_{2f}}{r_w}T\delta_T^{\tau_{1f}-\tau_1}B\left(r_{1e}\right)\int_{r_{2f}}^{r_2}B\left(s\right)ds.\end{aligned}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right)=-\delta_T\tilde{X}_{\tau_{1f}}X_{\tau_{1f}}\sim_a -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}B\left(r_{1e}\right)^2.$$

Suppose $\tau_{1f}-\tau_1\leq\tau_{2f}-\tau_{2e}$. The second term is

$$\begin{aligned}\delta_T\tilde{X}_{\tau_{1f}}X_{\tau_{1f}} &= \delta_T\left[\delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}}-\frac{T^\alpha\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\right]X_{\tau_{1f}}\left\{1+o_p\left(1\right)\right\}\\ &= \begin{cases}\delta_T^{\tau_{1f}-\tau_{1e}+1}X_{\tau_{1e}}X_{\tau_{1f}}\left\{1+o_p\left(1\right)\right\}\sim_a T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}\\ -T^\alpha\delta_T^{\tau_{2f}-\tau_{2e}+1}\frac{1}{\tau_w c}X_{\tau_{2e}}X_{\tau_{1f}}\sim_a T\delta_T^{\tau_{2f}-\tau_{2e}+\tau_{1f}-\tau_{1e}}\frac{1}{r_w c}B\left(r_{1e}\right)B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\end{cases}.\end{aligned}$$

The third term is

$$\begin{aligned}\frac{c}{T^\alpha}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}X_{j-1} &= \frac{c}{T^\alpha}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}-\frac{T^\alpha\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}X_{j-1}\left\{1+o_p\left(1\right)\right\}\\ &\sim_a -\frac{r_{2e}-r_{1f}}{r_w}T\delta_T^{\tau_{2f}-\tau_{2e}}B\left(r_{2e}\right)\int_{r_{1f}}^{r_{2e}}B\left(s\right)ds.\end{aligned}$$

The fourth term is

$$\delta_T\tilde{X}_{\tau_{2f}}X_{\tau_{2f}} = \delta_T\left[\delta_T^{\tau_{2f}-\tau_{2e}}X_{\tau_{2e}}-\frac{T^\alpha\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}}\right]X_{\tau_{2f}}\left\{1+o_p\left(1\right)\right\}$$

$$= \quad \delta_T^{\tau_{2f}-\tau_{2e}+1} X_{\tau_{2e}} X_{\tau_{2f}} \left\{1+o_p(1)\right\} \sim_a T\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} B\left(r_{2e}\right)^2.$$

The fifth term is

$$\frac{c}{T^\alpha}\sum_{j=\tau_{2f}+2}^{\tau_2} \tilde{X}_{j-1}X_{j-1} = \frac{c}{T^\alpha}\sum_{j=\tau_{2f}+2}^{\tau_2} -\frac{T^\alpha \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} X_{j-1}\left\{1+o_p(1)\right\}$$
$$\sim_a -\frac{r_2-r_{2f}}{r_w} T\delta_T^{\tau_{2f}-\tau_{2e}} B\left(r_{2e}\right)\int_{r_{2f}}^{r_2} B(s)\,ds.$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right) = \begin{cases} -\delta_T\tilde{X}_{\tau_{1f}}X_{\tau_{1f}} \sim_a -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ -\delta_T\tilde{X}_{\tau_{2f}}X_{\tau_{2f}} \sim_a -T\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases}.$$

Thus, when $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right) \sim_a \begin{cases} -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ -T\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases}.$$

(7) For $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right) \\
&= \sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}\left(X_j-X_{j-1}+X_{j-1}-\delta_T X_{j-1}\right) + \sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}\varepsilon_j + \tilde{X}_{\tau_{1f}}\left(X_{\tau_{1f}+1}-\delta_T X_{\tau_{1f}}\right) \\
&+ \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}\left(X_j-X_{j-1}+X_{j-1}-\delta_T X_{j-1}\right) + \sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha}X_{j-1}\right) + \sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}\varepsilon_j - \delta_T\tilde{X}_{\tau_{1f}}X_{\tau_{1f}} \\
&+ \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha}X_{j-1}\right) + \sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j - \frac{c}{T^\alpha}\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}X_{j-1} - \delta_T\tilde{X}_{\tau_{1f}}X_{\tau_{1f}} - \frac{c}{T^\alpha}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}X_{j-1}.
\end{aligned}$$

The first term is

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}}X_c B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{2e}}X_c B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases}.$$

Suppose $\tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e}$. The second term is

$$\begin{aligned}
&\frac{c}{T^{\alpha}} \sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1} X_{j-1} \\
&= \frac{c}{T^{\alpha}} \sum_{j=\tau_1}^{\tau_{1e}-1} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w} X_{\tau_{1e}} \sum_{j=\tau_1}^{\tau_{1e}-1} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\tau_{1e} - \tau_1}{\tau_w} T \delta_T^{\tau_{1f}-\tau_{1e}} \left(T^{-1/2} X_{\tau_{1e}}\right) \left(\frac{1}{\tau_{1e} - \tau_1} \sum_{j=\tau_1}^{\tau_{1e}-1} T^{-1/2} X_{j-1} \right) \{1 + o_p(1)\} \\
&\sim_a -\frac{r_{1e} - r_1}{r_w} T \delta_T^{\tau_{1f}-\tau_{1e}} B(r_{1e}) \int_{r_1}^{r_{1e}} B(s)\, ds.
\end{aligned}$$

The third term is

$$\begin{aligned}
\delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} &= \delta_T \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \right] X_{\tau_{1f}} \{1 + o_p(1)\} \\
&= \delta_T^{\tau_{1f}-\tau_{1e+1}} X_{\tau_{1e}} X_{\tau_{1f}} \{1 + o_p(1)\} \sim_a T \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B(r_{1e})^2 .
\end{aligned}$$

The fourth term is

$$\begin{aligned}
&\frac{c}{T^{\alpha}} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1} X_{j-1} \\
&= \frac{c}{T^{\alpha}} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w} X_{\tau_{1e}} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\tau_{2e} - \tau_{1f} - 1}{\tau_w} T \delta_T^{\tau_{1f}-\tau_{1e}} \left(T^{-1/2} X_{\tau_{1e}}\right) \left(\frac{1}{\tau_{2e} - \tau_{1f} - 1} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} T^{-1/2} X_{j-1} \right) \{1 + o_p(1)\} \\
&\sim_a -\frac{r_{2e} - r_{1f}}{r_w} T \delta_T^{\tau_{1f}-\tau_{1e}} B(r_{1e}) \int_{r_{1f}}^{r_{2e}} B(s)\, ds.
\end{aligned}$$

The third term dominates the other terms and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1}\right) = -\delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} \sim_a -T \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B(r_{1e})^2 .$$

Suppose $\tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e}$. The second term is

$$\begin{aligned}
&\frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_{1e}-1} \tilde{X}_{j-1} X_{j-1} \\
&= \frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_{1e}-1} -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\delta_T^{\tau_2-\tau_{2e}}}{\tau_w} X_{\tau_{2e}} \sum_{j=\tau_1}^{\tau_{1e}-1} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\tau_{1e} - \tau_1}{\tau_w} T \delta_T^{\tau_2-\tau_{2e}} \left(T^{-1/2} X_{\tau_{2e}}\right) \left(\frac{1}{\tau_{1e} - \tau_1} \sum_{j=\tau_1}^{\tau_{1e}-1} T^{-1/2} X_{j-1}\right) \{1 + o_p(1)\} \\
&\sim_a -\frac{r_{1e} - r_1}{r_w} T \delta_T^{\tau_2-\tau_{2e}} B(r_{2e}) \int_{r_1}^{r_{1e}} B(s)\, ds.
\end{aligned}$$

The third term is

$$\begin{aligned}
\delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} &= \delta_T \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right] X_{\tau_{1f}} \{1 + o_p(1)\} \\
&= -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}+1}}{\tau_w c} X_{\tau_{2e}} X_{\tau_{1f}} \{1 + o_p(1)\} \sim_a -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}}}{r_w c} B(r_{2e}) B(r_{1e}).
\end{aligned}$$

The fourth term is

$$\begin{aligned}
&\frac{c}{T^\alpha} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} \tilde{X}_{j-1} X_{j-1} \\
&= \frac{c}{T^\alpha} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\delta_T^{\tau_2-\tau_{2e}}}{\tau_w} X_{\tau_{2e}} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\tau_{2e} - \tau_{1f} - 1}{\tau_w} T \delta_T^{\tau_2-\tau_{2e}} \left(T^{-1/2} X_{\tau_{2e}}\right) \left(\frac{1}{\tau_{2e} - \tau_{1f} - 1} \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1} T^{-1/2} X_{j-1}\right) \{1 + o_p(1)\} \\
&\sim_a -\frac{r_{2e} - r_{1f}}{r_w} T \delta_T^{\tau_2-\tau_{2e}} B(r_{2e}) \int_{r_{1f}}^{r_{2e}} B(s)\, ds.
\end{aligned}$$

The third term dominates the other terms and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} (X_j - \delta_T X_{j-1}) = -\delta_T \tilde{X}_{\tau_{1f}} X_{\tau_{1f}} \sim_a \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}}}{r_w c} B(r_{2e}) B(r_{1e}).$$

Thus, when $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \sim_a \begin{cases} -T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_2 - \tau_{2e} \\ T^{\alpha}\delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}} \frac{1}{r_w c} B\left(r_{2e}\right) B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2 - \tau_{2e} \end{cases}.$$

□

2.2 Test asymptotics and Proofs of Theorems 4-9

The fitted regression model for the recursive unit root tests is

$$X_t = \hat{\alpha}_{r_1,r_2} + \hat{\rho}_{r_1,r_2} X_{t-1} + \hat{\varepsilon}_t,$$

where the intercept $\hat{\alpha}_{r_1,r_2}$ and slope coefficient $\hat{\rho}_{r_1,r_2}$ are obtained using data over the subperiod $[r_1, r_2]$.

Remark 1. *Based on Lemma S4 and Lemma S6, we can obtain the limit distribution of $\hat{\rho}_{r_1,r_2} - \delta_T$ using*

$$\hat{\rho}_{r_1,r_2} - \delta_T = \frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right)}{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2}.$$

(1) *When $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a T^{-\alpha}\delta_T^{-\left(\tau_2-\tau_{ie}\right)} \frac{\frac{r_{ie}-r_1}{r_w}\int_{r_1}^{r_{ie}} B(s)\,ds}{B\left(r_{ie}\right)};$$

(2) *when $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a -2T^{-\alpha}c;$$

(3) *when $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a -2T^{-\alpha}c;$$

(4) *when $\tau_1 \in N_0$ and $\tau_2 \in N_2$,*

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a -2T^{-\alpha}c;$$

(5) *when $\tau_1 \in B_1$ and $\tau_2 \in B_2$,*

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a \begin{cases} -2T^{-\alpha}c & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ T^{-1}\delta_T^{-\left(\tau_2-\tau_{2e}\right)+\left(\tau_{1f}-\tau_{1e}\right)} \frac{2B\left(r_{1e}\right)}{r_w B\left(r_{2e}\right)} & \text{if } \tau_{1f}-\tau_{1e} \leq \tau_2-\tau_{2e} \end{cases};$$

(6) *when* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$,

$$\hat{\hat{\delta}}_T - \delta_T \sim_a -2T^{-\alpha}c;$$

(7) *when* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\hat{\rho}_{r_1,r_2} - \delta_T = \begin{cases} -2T^{-\alpha}c & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ T^{-1}\delta_T^{-(\tau_2-\tau_{2e})+\left(\tau_{1f}-\tau_{1e}\right)} \frac{2B(r_{1e})}{r_w B(r_{2e})} & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e} \end{cases}.$$

Remark 2. *The asymptotic distributions of the recursive unit root coefficient Z-statistic in the various cases are as follows.* (1) *When* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$\begin{aligned} DF^z_{r_1,r_2} &= \tau_w\left(\hat{\rho}_{r_1,r_2} - 1\right) = \tau_w\left(\delta_T - 1\right) + \tau_w\left(\hat{\rho}_{r_1,r_2} - \delta_T\right) \\ &= \tau_w\left(\delta_T - 1\right) + o_p\left(r_w \frac{T^{1-\alpha}}{\delta_T^{\tau_2-\tau_{ie}}}\right) \\ &= \frac{\tau_w c}{T^\alpha} + o_p\left(r_w \frac{T^{1-\alpha}}{e^{c(r_2-r_{ie})T^{1-\alpha}}}\right) \\ &= r_w c T^{1-\alpha} + o_p(1) \to \infty. \end{aligned}$$

(2) *When* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\begin{aligned} DF^z_{r_1,r_2} &= \tau_w\left(\hat{\rho}_{r_1,r_2} - 1\right) = \tau_w\left(\delta_T - 1\right) + \tau_w\left(\hat{\rho}_{r_1,r_2} - \delta_T\right) \\ &= c r_w T^{1-\alpha} + o_p\left(r_w T^{1-\alpha}\right) \\ &= -c r_w T^{1-\alpha} \to -\infty. \end{aligned}$$

(3) *When* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\begin{aligned} DF^z_{r_1,r_2} &= \tau_w\left(\hat{\rho}_{r_1,r_2} - 1\right) = \tau_w\left(\delta_T - 1\right) + \tau_w\left(\hat{\rho}_{r_1,r_2} - \delta_T\right) \\ &= c r_w T^{1-\alpha} + o_p\left(r_w T^{1-\alpha}\right) \\ &= -c r_w T^{1-\alpha} \to -\infty. \end{aligned}$$

(4) *When* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$,

$$\begin{aligned} DF^z_{r_1,r_2} &= \tau_w\left(\hat{\rho}_{r_1,r_2} - 1\right) = \tau_w\left(\delta_T - 1\right) + \tau_w\left(\hat{\rho}_{r_1,r_2} - \delta_T\right) \\ &= c r_w T^{1-\alpha} + o_p\left(r_w T^{1-\alpha}\right) \end{aligned}$$

$$= -cr_w T^{1-\alpha} \to -\infty.$$

(5) *When* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$,

$$\begin{aligned}
DF^z_{r_1,r_2} &= \tau_w\left(\hat{\rho}_{r_1,r_2} - 1\right) = \tau_w\left(\delta_T - 1\right) + \tau_w\left(\hat{\rho}_{r_1,r_2} - \delta_T\right) \\
&= \begin{cases} cr_w T^{1-\alpha} + o_p\left(r_w T^{1-\alpha}\right) & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ cr_w T^{1-\alpha} + o_p\left(\frac{r_w}{\delta_T^{(\tau_2-\tau_{2e})-\left(\tau_{1f}-\tau_{1e}\right)}}\right) & \text{if } \tau_{1f} - \tau_{1e} \le \tau_2 - \tau_{2e} \end{cases} \\
&= \begin{cases} -cr_w T^{1-\alpha} \to -\infty & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ cr_w T^{1-\alpha} \to \infty & \text{if } \tau_{1f} - \tau_{1e} \le \tau_2 - \tau_{2e} \end{cases} .
\end{aligned}$$

(6) *When* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$,

$$\begin{aligned}
DF^z_{r_1,r_2} &= \tau_w\left(\hat{\rho}_{r_1,r_2} - 1\right) = \tau_w\left(\delta_T - 1\right) + \tau_w\left(\hat{\rho}_{r_1,r_2} - \delta_T\right) \\
&= cr_w T^{1-\alpha} + o_p\left(r_w T^{1-\alpha}\right) \\
&= -cr_w T^{1-\alpha} \to -\infty.
\end{aligned}$$

(7) *When* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\begin{aligned}
DF^z_{r_1,r_2} &= \tau_w\left(\hat{\rho}_{r_1,r_2} - 1\right) = \tau_w\left(\delta_T - 1\right) + \tau_w\left(\hat{\rho}_{r_1,r_2} - \delta_T\right) \\
&= \begin{cases} cr_w T^{1-\alpha} + o_p\left(r_w T^{1-\alpha}\right) & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ cr_w T^{1-\alpha} + o_p\left(\frac{r_w}{\delta_T^{(\tau_2-\tau_{2e})-\left(\tau_{1f}-\tau_{1e}\right)}}\right) & \text{if } \tau_{1f} - \tau_{1e} \le \tau_2 - \tau_{2e} \end{cases} \\
&= \begin{cases} -cr_w T^{1-\alpha} \to -\infty & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ cr_w T^{1-\alpha} \to \infty & \text{if } \tau_{1f} - \tau_{1e} \le \tau_2 - \tau_{2e} \end{cases} .
\end{aligned}$$

To obtain the asymptotic distributions of the unit root t-statistic, we first need to estimate the standard error of the regression equation over $[r_1, r_2]$, which is

$$\hat{\sigma}_{r_1 r_2} = \left\{ \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j - \hat{\rho}_{r_1,r_2} \tilde{X}_{j-1} \right)^2 \right\}^{1/2} .$$

Lemma S7. (1) *When* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$\hat{\sigma}^2_{r_1 r_2} \sim_a T^{-1} \delta_T^{2(\tau_2 - \tau_{ie})} \frac{r_{ie} - r_1}{c^{-1} r_w^3} B\left(r_{ie}\right)^2 .$$

(2) *When* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \frac{1}{r_w} \delta_T^{2\left(\tau_{if} - \tau_{ie}\right)} B\left(r_{ie}\right)^2 .$$

(3) *When* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \frac{\delta_T^{2\left(\tau_{if} - \tau_{ie}\right)}}{r_w} B\left(r_{ie}\right)^2 .$$

(4) *When* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$,

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \begin{cases} r_w^{-1} \delta_T^{2\left(\tau_{1f} - \tau_{1e}\right)} B\left(r_{1e}\right)^2 & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ r_w^{-1} \delta_T^{2\left(\tau_{2f} - \tau_{2e}\right)} B\left(r_{2e}\right)^2 & \text{if } r_{1f} - r_{1e} \leq r_{2f} - r_{2e} \end{cases} .$$

(5) *When* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$,

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \begin{cases} \delta_T^{2\left(\tau_{1f} - \tau_{1e}\right)} r_w^{-1} B\left(r_{1e}\right)^2 & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{-1} \delta_T^{2\left(\tau_2 - \tau_{2e}\right)} \frac{r_{2e} - r_{1f}}{r_w^3} B\left(r_{2e}\right)^2 & \text{if } r_{1f} - r_{1e} \leq r_2 - r_{2e} \end{cases} .$$

(6) *When* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$,

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \begin{cases} \delta_T^{2\left(\tau_{1f} - \tau_{1e}\right)} \frac{1}{r_w} B\left(r_{1e}\right)^2 & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ \delta_T^{2\left(\tau_{2f} - \tau_{2e}\right)} \frac{1}{r_w} B\left(r_{2e}\right)^2 & \text{if } r_{1f} - r_{1e} \leq r_{2f} - r_{2e} \end{cases} .$$

(7) *When* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \begin{cases} \delta_T^{2\left(\tau_{1f} - \tau_{1e}\right)} \frac{1}{r_w} B\left(r_{1e}\right)^2 & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{-1} \delta_T^{2\left(\tau_2 - \tau_{2e}\right)} \frac{r_{1e} - r_1 + r_{2e} - r_{1f}}{r_w^3} B\left(r_{2e}\right)^2 & \text{if } r_{1f} - r_{1e} \leq r_2 - r_{2e} \end{cases} .$$

Proof. (1)When $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$\begin{aligned} & \hat{\sigma}^2_{r_1 r_2} \\ & = \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j - \hat{\rho}_{r_1, r_2} \tilde{X}_{j-1} \right)^2 \\ & = \tau_w^{-1} \left[\sum_{j=\tau_1}^{\tau_{ie}-1} \left[\varepsilon_j - \left(\hat{\rho}_{r_1, r_2} - 1 \right) \tilde{X}_{j-1} \right]^2 + \sum_{j=\tau_{ie}}^{\tau_2} \left[\varepsilon_j - \left(\hat{\rho}_{r_1, r_2} - \delta_T \right) \tilde{X}_{j-1} \right]^2 \right] \\ & = \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \varepsilon_j^2 + \left(\hat{\rho}_{r_1, r_2} - 1 \right)^2 \tau_w{}^{-1} \sum_{j=\tau_1}^{\tau_{ie}-1} \tilde{X}_{j-1}^2 + \left(\hat{\rho}_{r_1, r_2} - \delta_T \right)^2 \tau_w^{-1} \sum_{j=\tau_{ie}}^{\tau_2} \tilde{X}_{j-1}^2 \end{aligned}$$

$$
\begin{aligned}
& - 2\left(\hat{\rho}_{r_1,r_2} - 1\right)\tau_w^{-1}\sum_{j=\tau_1}^{\tau_{ie}-1}\tilde{X}_{j-1}\varepsilon_j - 2\left(\hat{\rho}_{r_1,r_2} - \delta_T\right)\tau_w^{-1}\sum_{j=\tau_{ie}}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j \\
& = \left(\hat{\rho}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1}\sum_{j=\tau_1}^{\tau_{ie}-1}\tilde{X}_{j-1}^2 \\
& \sim_a T^{-1}\delta_T^{2(\tau_2-\tau_{ie})}\frac{r_{ie}-r_1}{c^{-1}r_w^3}B\left(r_{ie}\right)^2.
\end{aligned}
$$

Notice that in general the quantity $\sum \tilde{X}_{j-1}^2$ dominates the quantity $\sum \tilde{X}_{j-1}\varepsilon_j$. The term $\left(\hat{\rho}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1}\sum_{j=\tau_1}^{\tau_{ie}-1}\tilde{X}_{j-1}^2$ dominates the other terms due to the fact that

$$
\begin{aligned}
\left(\hat{\rho}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1}\sum_{j=\tau_1}^{\tau_{ie}-1}\tilde{X}_{j-1}^2 &= O_p\left(T^{-2\alpha}\right)O_p\left(T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_{ie})}\right) = O_p\left(T^{-1}\delta_T^{2(\tau_2-\tau_{ie})}\right) \\
\left(\hat{\rho}_{r_1,r_2} - \delta_T\right)^2 \tau_w^{-1}\sum_{j=\tau_{ie}}^{\tau_2}\tilde{X}_{j-1}^2 &= O_p\left(\frac{1}{T^{2\alpha}\delta_T^{2(\tau_2-\tau_{ie})}}\right)O_p\left(T^{\alpha}\delta_T^{2(\tau_2-\tau_{ie})}\right) = O_p\left(T^{-\alpha}\right)
\end{aligned}
$$

(2) When $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$
\begin{aligned}
& \tilde{X}_{\tau_{if}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{if}} \\
& = -\frac{\delta_T^{\tau_{if}-\tau_1}}{r_w c T^{1-\alpha}}X_{\tau_{ie}} - \tilde{X}_{\tau_{if}} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{if}} \\
& = -O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_1}\right) - O_p\left(T^{1/2}\delta_T^{\tau_{if}-\tau_{ie}}\right) - O_p\left(T^{-\alpha}\right)O_p\left(T^{1/2}\delta_T^{\tau_{if}-\tau_{ie}}\right) \\
& = -\tilde{X}_{\tau_{if}} = -\delta_T^{\tau_{if}-\tau_{ie}}X_{\tau_{ie}}\left\{1 + o_p(1)\right\},
\end{aligned}
$$

using the fact that

$$
\tilde{X}_{\tau_{if}} = \left[\delta_T^{\tau_{if}-\tau_{ie}} - \frac{\delta_T^{\tau_{if}-\tau_1}}{r_w c T^{1-\alpha}}\right]X_{\tau_{ie}}\left\{1 + o_p(1)\right\} = \delta_T^{\tau_{if}-\tau_{ie}}X_{\tau_{ie}}\left\{1 + o_p(1)\right\}.
$$

Therefore,

$$
\begin{aligned}
& \hat{\sigma}_{r_1 r_2}^2 \\
& = \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j - \hat{\rho}_{r_1,r_2}\tilde{X}_{j-1}\right)^2 \\
& = \tau_w^{-1}\left\{\sum_{j=\tau_{if}+2}^{\tau_2}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - 1\right)\tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_1}^{\tau_{if}}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - \delta_T\right)\tilde{X}_{j-1}\right]^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\tilde{X}_{\tau_{if}+1}-\hat{\rho}_{r_1,r_2}\tilde{X}^2_{\tau_{if}}-\varepsilon_{\tau_{if}+1}+\varepsilon_{\tau_{if}+1}\right]^2\Bigg\}\\
&=\tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\varepsilon_j^2+\left(\hat{\rho}_{r_1,r_2}-1\right)^2\tau_w{}^{-1}\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}^2_{j-1}+\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)^2\tau_w^{-1}\sum_{j=\tau_1}^{\tau_{if}}\tilde{X}^2_{j-1}\\
&-2\left(\hat{\rho}_{r_1,r_2}-1\right)\tau_w^{-1}\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j-2\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)\tau_w^{-1}\sum_{j=\tau_1}^{\tau_{if}}\tilde{X}_{j-1}\varepsilon_j+\tau_w^{-1}\tilde{X}^2_{\tau_{if}}\\
&=\tau_w^{-1}\tilde{X}^2_{\tau_{if}}=\tau_w^{-1}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}X^2_{\tau_{ie}}\left\{1+o_p\left(1\right)\right\}\\
&\sim_a\frac{1}{r_w}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}B\left(r_{ie}\right)^2.
\end{aligned}
$$

The term $\tau_w^{-1}\tilde{X}^2_{\tau_{if}}$ dominates the other terms due to the fact that

$$
\begin{aligned}
\left(\hat{\rho}_{r_1,r_2}-1\right)^2\tau_w{}^{-1}\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}^2_{j-1}&=O_p\left(T^{-2\alpha}\right)\left(T^{2\alpha-1}\delta_T^{2\left(\tau_{if}-\tau_1\right)}\right)=O_p\left(\frac{\delta_T^{2\left(\tau_{if}-\tau_1\right)}}{T}\right)\\
\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)^2\tau_w^{-1}\sum_{j=\tau_1}^{\tau_{if}}\tilde{X}^2_{j-1}&=O_p\left(T^{-2\alpha}\right)O_p\left(T^{\alpha}\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}\right)=O_p\left(\frac{\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{T^{\alpha}}\right)\\
\tau_w^{-1}\tilde{X}^2_{\tau_{if}}&=O_p\left(\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}\right).
\end{aligned}
$$

(3) When $\tau_1\in N_{i-1}$ and $\tau_2\in N_i$ with $i=1,2$,

$$
\begin{aligned}
&\tilde{X}_{\tau_{if}+1}-\hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{if}}-\varepsilon_{\tau_{if}+1}\\
&=-\frac{T^{\alpha-1}\delta_T^{\tau_{if}-\tau_{ie}}}{r_wc}X_{\tau_{ie}}-\tilde{X}_{\tau_{if}}-\left[\hat{\rho}_{r_1,r_2}-1\right]\tilde{X}_{\tau_{if}}\\
&=-O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{if}-\tau_{ie}}\right)-O_p\left(T^{1/2}\delta_T^{\tau_{if}-\tau_{ie}}\right)-O_p\left(T^{-\alpha}\right)O_p\left(T^{1/2}\delta_T^{\tau_{if}-\tau_{ie}}\right)\\
&=-\tilde{X}_{\tau_{if}}=-\delta_T^{\tau_{if}-\tau_{ie}}X_{\tau_{ie}}\left\{1+o_p\left(1\right)\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\hat{\sigma}^2_{r_1r_2}\\
&=\tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j-\hat{\rho}_{r_1,r_2}\tilde{X}_{j-1}\right)^2\\
&=\tau_w^{-1}\left\{\sum_{j=\tau_{if}+2}^{\tau_2}\left[\varepsilon_j-\left(\hat{\rho}_{r_1,r_2}-1\right)\tilde{X}_{j-1}\right]^2+\sum_{j=\tau_1}^{\tau_{ie}-1}\left[\varepsilon_j-\left(\hat{\rho}_{r_1,r_2}-1\right)\tilde{X}_{j-1}\right]^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=\tau_{ie}}^{\tau_{if}}\left[\varepsilon_j-\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)\tilde{X}_{j-1}\right]^2+\tilde{X}_{\tau_{if}+1}-\hat{\rho}_{r_1,r_2}\tilde{X}^2_{\tau_{if}}\Bigg\}^2\\
&=\tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\varepsilon_j^2+\left(\hat{\rho}_{r_1,r_2}-1\right)^2\tau_w{}^{-1}\left[\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}^2_{j-1}+\sum_{j=\tau_1}^{\tau_{ie}-1}\tilde{X}^2_{j-1}\right]\\
&+\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)^2\tau_w^{-1}\sum_{j=\tau_{ie}}^{\tau_{if}}\tilde{X}^2_{j-1}-2\left(\hat{\rho}_{r_1,r_2}-1\right)\tau_w^{-1}\left[\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=\tau_1}^{\tau_{ie}-1}\tilde{X}_{j-1}\varepsilon_j\right]\\
&-2\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)\tau_w^{-1}\sum_{j=\tau_{ie}}^{\tau_{if}}\tilde{X}_{j-1}\varepsilon_j+\tau_w{}^{-1}\tilde{X}_{\tau_{if}}\\
&=\tau_w^{-1}\tilde{X}_{\tau_{if}}=\frac{\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{\tau_w}X^2_{\tau_{ie}}\left\{1+o_p\left(1\right)\right\}\sim_a\frac{\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{r_w}B\left(r_{ie}\right)^2.
\end{aligned}
$$

The term $\tau_w{}^{-1}\tilde{X}^2_{\tau_{if}}$ dominates the other terms due to the fact that

$$
\begin{aligned}
\frac{\left(\hat{\rho}_{r_1,r_2}-1\right)^2}{\tau_w}\left[\sum_{j=\tau_{if}+2}^{\tau_2}\tilde{X}^2_{j-1}+\sum_{j=\tau_1}^{\tau_{ie}-1}\tilde{X}^2_{j-1}\right]&=O_p\left(\frac{\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{T}\right)\\
\frac{\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)^2}{\tau_w}\sum_{j=\tau_{ie}}^{\tau_{if}}\tilde{X}^2_{j-1}&=O_p\left(\frac{\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}}{T^\alpha}\right)\\
\tau_w^{-1}\tilde{X}^2_{\tau_{if}}&=O_p\left(\delta_T^{2\left(\tau_{if}-\tau_{ie}\right)}\right).
\end{aligned}
$$

(4) Suppose $\tau_1\in N_0$ and $\tau_2\in N_2$. When $\tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}$, we have

$$
\begin{aligned}
\tilde{X}_{\tau_{1f}}&=\left[\delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}}-\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}X_{\tau_{1e}}\right]\left\{1+o_p\left(1\right)\right\}\\
&=\delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}}\left\{1+o_p\left(1\right)\right\}\sim_a T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}B\left(r_{1e}\right);\\
\tilde{X}_{\tau_{2f}}&=\left[\delta_T^{\tau_{2f}-\tau_{2e}}X_{\tau_{2e}}-\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}X_{\tau_{1e}}\right]\left\{1+o_p\left(1\right)\right\}\\
&=-\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}X_{\tau_{1e}}\left\{1+o_p\left(1\right)\right\}\sim_a -T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\frac{1}{r_w c}B\left(r_{1e}\right).
\end{aligned}
$$

We calculate

$$
\tilde{X}_{\tau_{1f}+1}-\hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{1f}}
$$

$$
\begin{aligned}
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} - \tilde{X}_{\tau_{1f}} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}} \\
&= O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{-\alpha}\right)O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) \\
&= -\tilde{X}_{\tau_{1f}} \sim_a -T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}} B\left(r_{1e}\right);
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{X}_{\tau_{2f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{2f}} \\
&= \left[-\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} - \tilde{X}_{\tau_{2f}}\right] - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{2f}} \\
&= -\delta_T^{\tau_{2f}-\tau_{2e}} X_{\tau_{2e}}\left\{1 + o_p\left(1\right)\right\} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{2f}} \\
&= O_p\left(T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) - O_p\left(T^{-\alpha}\right)O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) \\
&= -\left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{2f}} \sim_a -T^{-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\frac{1}{r_w} B\left(r_{1e}\right).
\end{aligned}
$$

Evidently, the first residual (capturing the collapse at τ_{1f+1}) dominates the second. Suppose $\tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e}$. We have

$$
\begin{aligned}
\tilde{X}_{\tau_{1f}} &= \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\left\{1 + o_p\left(1\right)\right\} \sim -\frac{T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}}{r_w c} B\left(r_{2e}\right); \\
\tilde{X}_{\tau_{2f}} &= \left[\delta_T^{\tau_{2f}-\tau_{2e}} X_{\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p\left(1\right)\right\} \\
&= \delta_T^{\tau_{2f}-\tau_{2e}} X_{\tau_{2e}}\left\{1 + o_p\left(1\right)\right\} \sim_a T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}} B\left(r_{2e}\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\tilde{X}_{\tau_{1f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{1f}} \\
&= \left[-\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} - \tilde{X}_{\tau_{1f}}\right] - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}} \\
&= -\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}}\left\{1 + o_p\left(1\right)\right\} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}} \\
&= -O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{-\alpha}\right)O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right)
\end{aligned}
$$

$$= -\left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}}$$
$$\sim_a -T^{-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\frac{1}{r_w}B\left(r_{2e}\right);$$

and

$$\begin{aligned}
&\tilde{X}_{\tau_{2f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{2f}} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}X_{\tau_{2e}} - \tilde{X}_{\tau_{2f}} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{2f}} \\
&= -O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) - O_p\left(T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) - O_p\left(T^{-\alpha}\right)O_p\left(T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) \\
&= -\tilde{X}_{\tau_{2f}} \sim_a -T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}}B\left(r_{2e}\right).
\end{aligned}$$

Evidently in this case the second residual (measuring the collapse at τ_{2f+1}) dominates the first. Therefore,

$$\begin{aligned}
&\hat{\sigma}^2_{r_1 r_2} \\
&= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j - \hat{\rho}_{r_1,r_2}\tilde{X}_{j-1}\right)^2 \\
&= \tau_w^{-1}\Bigg\{\sum_{j=\tau_{2f}+2}^{\tau_2}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - 1\right)\tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - 1\right)\tilde{X}_{j-1}\right]^2 \\
&+ \sum_{j=\tau_1}^{\tau_{1e}-1}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - 1\right)\tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_{1e}}^{\tau_{1f}}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - \delta_T\right)\tilde{X}_{j-1}\right]^2 \\
&+ \sum_{j=\tau_{2e}}^{\tau_{2f}}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - \delta_T\right)\tilde{X}_{j-1}\right]^2 + \left[\tilde{X}_{\tau_{1f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}^2_{\tau_{1f}}\right]^2 + \left[\tilde{X}_{\tau_{2f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}^2_{\tau_{2f}}\right]^2\Bigg\} \\
&= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\varepsilon_j^2 + \left(\hat{\rho}_{r_1,r_2} - 1\right)^2\tau_w{}^{-1}\left[\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}^2_{j-1} + \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}^2_{j-1} + \sum_{j=\tau_{2f}+2}^{\tau_2}\tilde{X}^2_{j-1}\right] \\
&+ \left(\hat{\rho}_{r_1,r_2} - \delta_T\right)^2\tau_w^{-1}\left[\sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}^2_{j-1} + \sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}^2_{j-1}\right] \\
&- 2\left(\hat{\rho}_{r_1,r_2} - \delta_T\right)\tau_w^{-1}\left[\sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}_{j-1}\varepsilon_j\right]
\end{aligned}$$

$$-2\left(\hat{\rho}_{r_1,r_2}-1\right)\tau_w^{-1}\left[\sum_{j=\tau_{2f}+2}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}\varepsilon_j\right]$$

$$+\begin{cases}\tau_w^{-1}\tilde{X}_{\tau_{1f}}^2+\tau_w^{-1}\left[\hat{\rho}_{r_1,r_2}-1\right]^2\tilde{X}_{\tau_{2f}}^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}\\ \tau_w^{-1}\left[\hat{\rho}_{r_1,r_2}-1\right]^2\tilde{X}_{\tau_{1f}}^2+\tau_w^{-1}\tilde{X}_{\tau_{2f}}^2 & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\end{cases}$$

$$=\begin{cases}\tau_w^{-1}\tilde{X}_{\tau_{1f}}^2\sim_a r_w^{-1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}\\ \tau_w^{-1}\tilde{X}_{\tau_{2f}}^2\sim_a r_w^{-1}\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\end{cases}$$

This is due to the fact that

$$\frac{\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)^2}{\tau_w}\left[\sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}^2+\sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}_{j-1}^2\right]=\begin{cases}O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^\alpha}\right) & \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}\\ O_p\left(\frac{\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{T^\alpha}\right) & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\end{cases}$$

$$\frac{\left(\hat{\rho}_{r_1,r_2}-1\right)^2}{\tau_w}\left[\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}^2+\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}^2+\sum_{j=\tau_{2f}+2}^{\tau_2}\tilde{X}_{j-1}^2\right]=\begin{cases}O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T}\right) & \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}\\ O_p\left(\frac{\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{T}\right) & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\end{cases}$$

$$\begin{aligned}\frac{1}{\tau_w}\tilde{X}_{\tau_{1f}}^2 &= O_p\left(\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}\right) \quad \text{if } \tau_{1f}-\tau_{1e}>\tau_{2f}-\tau_{2e}\\ \tau_w^{-1}\tilde{X}_{\tau_{2f}}^2 &= O_p\left(\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}\right) \quad \text{if } \tau_{1f}-\tau_{1e}\leq\tau_{2f}-\tau_{2e}\ .\end{aligned}$$

(5) Suppose $\tau_1\in B_1$ and $\tau_2\in B_2$. When $\tau_{1f}-\tau_1>\tau_2-\tau_{2e}$, we have

$$\tilde{X}_{\tau_{1f}}=\left[\delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}}-\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}X_{\tau_{1e}}\right]\left\{1+o_p\left(1\right)\right\}=\delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}}\left\{1+o_p\left(1\right)\right\}\sim_a T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}B\left(r_{1e}\right)$$

and

$$\begin{aligned}&\tilde{X}_{\tau_{1f}+1}-\hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{1f}}-\varepsilon_{\tau_{1f}+1}\\ &=-\frac{T^\alpha\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}X_{\tau_{1e}}-\tilde{X}_{\tau_{1f}}-\left[\hat{\rho}_{r_1,r_2}-1\right]\tilde{X}_{\tau_{1f}}\\ &=-O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\right)-O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right)-O_p\left(T^{-\alpha}\right)O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right)\\ &=-\tilde{X}_{\tau_{1f}}\sim_a -T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}B\left(r_{1e}\right).\end{aligned}$$

When $\tau_{1f}-\tau_1\leq\tau_2-\tau_{2e}$, we have

$$\begin{aligned}
\tilde{X}_{\tau_{1f}} &= \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\{1+o_p(1)\} \\
&= \begin{cases} \delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}}\{1+o_p(1)\} & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\{1+o_p(1)\} & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases} \\
&\sim\ _a \begin{cases} T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}} B(r_{1e}) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ -T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}} \frac{1}{r_w c} B(r_{2e}) & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases}.
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{X}_{\tau_{1f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{1f}} \\
&= -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} - \tilde{X}_{\tau_{1f}} - [\hat{\rho}_{r_1,r_2}-1]\tilde{X}_{\tau_{1f}} \\
&= \begin{cases} -O_p\left(T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\right) - O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{-\alpha}\right)O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ -O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{-\alpha}\right)O_p\left(T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\right) & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases} \\
&= \begin{cases} -\tilde{X}_{\tau_{1f}} & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ -[\hat{\rho}_{r_1,r_2}-1]\tilde{X}_{\tau_{1f}} & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases} \\
&\sim_a \begin{cases} -T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}} B(r_{1e}) & \text{if } \tau_{1f}-\tau_{1e} > \tau_2-\tau_{2e} \\ -T^{-1/2}\delta_T^{\tau_2-\tau_{2e}} \frac{1}{r_w} B(r_{2e}) & \text{if } \tau_{1f}-\tau_{1e} \le \tau_2-\tau_{2e} \end{cases}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\hat{\sigma}^2_{r_1 r_2} \\
&= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j - \hat{\rho}_{r_1,r_2}\tilde{X}_{j-1}\right)^2 \\
&= \tau_w^{-1}\left\{\sum_{j=\tau_1}^{\tau_{1f}}\left[\varepsilon_j - (\hat{\rho}_{r_1,r_2}-\delta_T)\tilde{X}_{j-1}\right]^2 + \tilde{X}_{\tau_{1f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}^2_{\tau_{1f}}\right. \\
&\left. + \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\left[\varepsilon_j - (\hat{\rho}_{r_1,r_2}-1)\tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_{2e}}^{\tau_2}\left[\varepsilon_j - (\hat{\rho}_{r_1,r_2}-\delta_T)\tilde{X}_{j-1}\right]^2\right\}^2 \\
&= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\varepsilon_j^2 + (\hat{\rho}_{r_1,r_2}-\delta_T)^2\tau_w^{-1}\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}^2_{j-1} + (\hat{\rho}_{r_1,r_2}-1)^2{\tau_w}^{-1}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}^2_{j-1} \\
&+ (\hat{\rho}_{r_1,r_2}-\delta_T)^2{\tau_w}^{-1}\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}^2_{j-1} - 2(\hat{\rho}_{r_1,r_2}-\delta_T)\tau_w^{-1}\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}_{j-1}\varepsilon_j
\end{aligned}$$

$$
\begin{aligned}
&-2\left(\hat{\rho}_{r_1,r_2}-1\right)\tau_w^{-1}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}\varepsilon_j-2\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)\tau_w^{-1}\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j \\
&+\begin{cases}\tau_w^{-1}\tilde{X}_{\tau_{1f}}^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e} \\ \tau_w^{-1}\left[\hat{\rho}_{r_1,r_2}-1\right]^2\tilde{X}_{\tau_{1f}}^2 & \text{if } \tau_{1f}-\tau_{1e}\le\tau_2-\tau_{2e}\end{cases} \\
&=\begin{cases}\tau_w^{-1}\tilde{X}_{\tau_{1f}}^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e} \\ \left(\hat{\rho}_{r_1,r_2}-1\right)^2\tau_w^{-1}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}^2 & \text{if } \tau_{1f}-\tau_{1e}\le\tau_2-\tau_{2e}\end{cases} \\
&\sim_a\begin{cases}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}\frac{1}{r_w}B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e} \\ T^{-1}\delta_T^{2\left(\tau_2-\tau_{2e}\right)}\frac{r_{2e}-r_{1f}}{r_w^3}B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}\le\tau_2-\tau_{2e}\end{cases}.
\end{aligned}
$$

This is due to the fact that

$$
\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)^2{\tau_w}^{-1}\left[\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}_{j-1}^2+\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}^2\right]=\begin{cases}O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^{\alpha}}\right) & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e} \\ O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^{2-\alpha}}\right) & \text{if } \tau_{1f}-\tau_{1e}\le\tau_2-\tau_{2e}\end{cases}
$$

$$
\begin{aligned}
\left(\hat{\rho}_{r_1,r_2}-1\right)^2\tau_w^{-1}\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}^2&=\begin{cases}O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_1\right)}}{T}\right) & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e} \\ O_p\left(\frac{\delta_T^{2\left(\tau_2-\tau_{2e}\right)}}{T}\right) & \text{if } \tau_{1f}-\tau_{1e}\le\tau_2-\tau_{2e}\end{cases} \\
\tau_w^{-1}\tilde{X}_{\tau_{1f}}^2&=O_p\left(\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}\right) \quad \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e} \\
\tau_w^{-1}\left[\hat{\rho}_{r_1,r_2}-1\right]^2\tilde{X}_{\tau_{1f}}^2&=O_p\left(\frac{\delta_T^{\tau_2-\tau_{2e}}}{T^2}\right) \quad \text{if } \tau_{1f}-\tau_{1e}\le\tau_2-\tau_{2e}
\end{aligned}
$$

(6) Suppose $\tau_1\in B_1$ and $\tau_2\in N_2$. When $\tau_{1f}-\tau_1>\tau_{2f}-\tau_{2e}$, we know

$$
\begin{aligned}
\tilde{X}_{\tau_{1f}}&=\left[\delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}}-\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}X_{\tau_{1e}}\right]\left\{1+o_p(1)\right\} \\
&=\delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}}\left\{1+o_p(1)\right\}\sim_a T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}B\left(r_{1e}\right); \\
\tilde{X}_{\tau_{2f}}&=\left[\delta_T^{\tau_{2f}-\tau_{2e}}X_{\tau_{2e}}-\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}X_{\tau_{1e}}\right]\left\{1+o_p(1)\right\} \\
&=-\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c}X_{\tau_{1e}}\left\{1+o_p(1)\right\}\sim_a-\frac{T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}}{r_w c}B\left(r_{1e}\right).
\end{aligned}
$$

Then, we derive

$$
\tilde{X}_{\tau_{1f}+1}-\hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{1f}}
$$

$$
\begin{aligned}
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} - \tilde{X}_{\tau_{1f}} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}} \\
&= -O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\right) - O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{-\alpha}\right) O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) \\
&= -\tilde{X}_{\tau_{1f}} \sim_a T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}} B\left(r_{1e}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{X}_{\tau_{2f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{2f}} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} - \tilde{X}_{\tau_{2f}} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{2f}} \\
&= -\delta_T^{\tau_{2f}-\tau_{2e}} X_{\tau_{2e}} \left\{1 + o_p(1)\right\} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{2f}} \\
&= -O_p\left(T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) - O_p\left(T^{-\alpha}\right) O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\right) \\
&= -\left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{2f}} \sim_a -T^{-1/2}\delta_T^{\tau_{1f}-\tau_1}\frac{1}{r_w} B\left(r_{1e}\right)
\end{aligned}
$$

When $\tau_{1f} - \tau_1 \leq \tau_{2f} - \tau_{2e}$, we know

$$
\begin{aligned}
\tilde{X}_{\tau_{1f}} &= \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p(1)\right\} \\
&= \begin{cases} \delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} \left\{1 + o_p(1)\right\} \sim_a T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}} B\left(r_{1e}\right) & \text{if } \tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e} \\ -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p(1)\right\} \sim_a -T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}} \frac{1}{r_w c} B\left(r_{1e}\right) & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e} \end{cases}; \\
\tilde{X}_{\tau_{2f}} &= \left[\delta_T^{\tau_{2f}-\tau_{2e}} - \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c}\right] X_{\tau_{2e}} \left\{1 + o_p(1)\right\} \\
&= \delta_T^{\tau_{2f}-\tau_{2e}} X_{\tau_{2e}} \left\{1 + o_p(1)\right\} \sim_a T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}} B\left(r_{2e}\right).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\tilde{X}_{\tau_{1f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{1f}} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} - \tilde{X}_{\tau_{1f}} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}} \\
&= \begin{cases} -O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) - O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{-\alpha}\right) O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) & \text{if } \tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e} \\ -O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) - O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) - O_p\left(T^{-\alpha}\right) O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e} \end{cases}
\end{aligned}
$$

$$= \begin{cases} -\tilde{X}_{\tau_{1f}} & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ -\left[\hat{\rho}_{r_1,r_2}-1\right]\tilde{X}_{\tau_{1f}} & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases}$$
$$\sim_a \begin{cases} -T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}} B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ -T^{-1/2}\delta_T^{\tau_{2f}-\tau_{2e}} \frac{1}{r_w} B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases}$$

and

$$\begin{aligned}
&\tilde{X}_{\tau_{2f}+1}-\hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{2f}}-\varepsilon_{\tau_{2f}+1} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}-\tilde{X}_{\tau_{2f}}-\left[\hat{\rho}_{r_1,r_2}-1\right]\tilde{X}_{\tau_{2f}} \\
&= -O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right)-O_p\left(T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right)-O_p\left(T^{-\alpha}\right)O_p\left(T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\right) \\
&= -\tilde{X}_{\tau_{2f}} \sim_a T^{1/2}\delta_T^{\tau_{2f}-\tau_{2e}} B\left(r_{2e}\right).
\end{aligned}$$

Evidently, when $\tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e}$, the first collapse dominates the second, whereas when $\tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e}$ the second dominates the first. Therefore

$$\begin{aligned}
&\hat{\sigma}^2_{r_1r_2} \\
&= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j-\hat{\rho}_{r_1,r_2}\tilde{X}_{j-1}\right)^2 \\
&= \tau_w^{-1}\Bigg\{\sum_{j=\tau_1}^{\tau_{1f}}\left[\varepsilon_j-\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)\tilde{X}_{j-1}\right]^2+\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\left[\varepsilon_j-\left(\hat{\rho}_{r_1,r_2}-1\right)\tilde{X}_{j-1}\right]^2+ \\
&\sum_{j=\tau_{2e}}^{\tau_{2f}}\left[\varepsilon_j-\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)\tilde{X}_{j-1}\right]^2+\sum_{j=\tau_{2f}+2}^{\tau_2}\left[\varepsilon_j-\left(\hat{\rho}_{r_1,r_2}-1\right)\tilde{X}_{j-1}\right]^2 \\
&+\left(\tilde{X}_{\tau_{1f}+1}-\hat{\rho}_{r_1,r_2}\tilde{X}^2_{\tau_{1f}}\right)^2+\left(\tilde{X}_{\tau_{2f}+1}-\hat{\rho}_{r_1,r_2}\tilde{X}^2_{\tau_{2f}}\right)^2\Bigg\}^2 \\
&= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\varepsilon_j^2+\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)^2\left[\tau_w^{-1}\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}^2_{j-1}+\sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}^2_{j-1}\right] \\
&+\left(\hat{\rho}_{r_1,r_2}-1\right)^2\tau_w^{-1}\left[\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}^2_{j-1}+\sum_{j=\tau_{2f}+2}^{\tau_2}\tilde{X}^2_{j-1}\right] \\
&-2\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)\tau_w^{-1}\left[\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}_{j-1}\varepsilon_j\right]
\end{aligned}$$

$$
\begin{aligned}
&- 2\left(\hat{\rho}_{r_1,r_2} - 1\right)\tau_w^{-1}\left[\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_{2f}+2}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j\right] \\
&+ \begin{cases} \tau_w^{-1}\tilde{X}_{\tau_{1f}}^2\left\{1+o_p(1)\right\} & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ \tau_w^{-1}\tilde{X}_{\tau_{2f}}^2\left\{1+o_p(1)\right\} & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases} \\
&= \begin{cases} \tau_w^{-1}\tilde{X}_{\tau_{1f}}^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ \tau_w^{-1}\tilde{X}_{\tau_{2f}}^2 & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases} \\
&\sim_a \begin{cases} \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}\frac{1}{r_w}B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ \delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}\frac{1}{r_w}B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases}
\end{aligned}
$$

The term $\tau_w^{-1}X_{\tau_{1f}}^2$ dominates the other terms due to the fact that

$$
\left(\hat{\rho}_{r_1,r_2} - \delta_T\right)^2\tau_w^{-1}\left[\sum_{j=\tau_1}^{\tau_{1f}}\tilde{X}_{j-1}^2 + \sum_{j=\tau_{2e}}^{\tau_{2f}}\tilde{X}_{j-1}^2\right] = \begin{cases} O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^\alpha}\right) & \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\ O_p\left(\frac{\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{T^\alpha}\right) & \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e} \end{cases}
$$

$$
\left(\hat{\rho}_{r_1,r_2} - 1\right)^2\tau_w^{-1}\left[\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}^2 + \sum_{j=\tau_{2f}+2}^{\tau_2}\tilde{X}_{j-1}^2\right] = \begin{cases} O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_1\right)}}{T}\right) & \text{if } \tau_{1f}-\tau_1 > \tau_{2f}-\tau_{2e} \\ O_p\left(\frac{\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{T}\right) & \text{if } \tau_{1f}-\tau_1 \le \tau_{2f}-\tau_{2e} \end{cases}
$$

$$
\begin{aligned}
\tau_w^{-1}\tilde{X}_{\tau_{1f}}^2 &= O_p\left(\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}\right) \quad \text{if } \tau_{1f}-\tau_{1e} > \tau_{2f}-\tau_{2e} \\
\tau_w^{-1}\tilde{X}_{\tau_{2f}}^2 &= O_p\left(\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}\right) \quad \text{if } \tau_{1f}-\tau_{1e} \le \tau_{2f}-\tau_{2e}
\end{aligned}
$$

(7) Suppose $\tau_1 \in N_0$ and $\tau_2 \in B_2$. When $\tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e}$, we know

$$
\begin{aligned}
\tilde{X}_{\tau_{1f}} &= \left[\delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}} - \frac{T^\alpha\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c}X_{\tau_{1e}}\right]\left\{1+o_p(1)\right\} \\
&= \delta_T^{\tau_{1f}-\tau_{1e}}X_{\tau_{1e}}\left\{1+o_p(1)\right\} \\
&\sim_a T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}B\left(r_{1e}\right).
\end{aligned}
$$

Then, we have

$$
\tilde{X}_{\tau_{1f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{1f}}
$$

$$
\begin{aligned}
&= -\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} - \tilde{X}_{\tau_{1f}} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}} \\
&= -O_p\left(T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{-\alpha}\right)O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) \\
&= -\tilde{X}_{\tau_{1f}} \sim_a -T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}} B\left(r_{1e}\right).
\end{aligned}
$$

When $\tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e}$, we know

$$
\begin{aligned}
\tilde{X}_{\tau_{1f}} &= \left[\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p\left(1\right)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\left\{1 + o_p\left(1\right)\right\} \sim_a -T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c} B\left(r_{2e}\right).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\tilde{X}_{\tau_{1f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{1f}} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} - \tilde{X}_{\tau_{1f}} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}} \\
&= -\delta_T^{\tau_{1f}-\tau_{1e}} X_{\tau_{1e}}\left\{1 + o_p\left(1\right)\right\} - \left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}} \\
&= -O_p\left(T^{1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\right) - O_p\left(T^{-\alpha}\right)O_p\left(T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\right) \\
&= -\left[\hat{\rho}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_{1f}} \sim_a T^{-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w} B\left(r_{2e}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\hat{\sigma}_{r_1 r_2}^2 \\
&= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j - \hat{\rho}_{r_1,r_2}\tilde{X}_{j-1}\right)^2 \\
&= \tau_w^{-1}\left\{\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - 1\right)\tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_1}^{\tau_{1e}-1}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - 1\right)\tilde{X}_{j-1}\right]^2\right. \\
&\left. + \sum_{j=\tau_{1e}}^{\tau_{1f}}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - \delta_T\right)\tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_{2e}}^{\tau_2}\left[\varepsilon_j - \left(\hat{\rho}_{r_1,r_2} - \delta_T\right)\tilde{X}_{j-1}\right]^2 + \left[\tilde{X}_{\tau_{1f}+1} - \hat{\rho}_{r_1,r_2}\tilde{X}_{\tau_{1f}}^2\right]^2\right\} \\
&= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\varepsilon_j^2 + \left(\hat{\rho}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1}\left[\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}^2 + \sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}^2\right]
\end{aligned}
$$

$$
+\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)^2\tau_w^{-1}\left[\sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}^2+\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}^2\right]-2\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)\tau_w^{-1}\left[\sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j\right]
$$

$$
-2\left(\hat{\rho}_{r_1,r_2}-1\right)\tau_w^{-1}\left[\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}\varepsilon_j\right]
$$

$$
+\begin{cases}\tau_w^{-1}\tilde{X}_{\tau_{1f}}^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e}\\ \tau_w^{-1}\left[\hat{\rho}_{r_1,r_2}-1\right]^2\tilde{X}_{\tau_{1f}}^2 & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_2-\tau_{2e}\end{cases}
$$

$$
=\begin{cases}\tau_w^{-1}\tilde{X}_{\tau_{1f}}^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e}\\ \frac{\left(\hat{\rho}_{r_1,r_2}-1\right)^2}{\tau_w}\left[\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}^2+\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}^2\right] & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_2-\tau_{2e}\end{cases}
$$

$$
\sim_a\begin{cases}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}\frac{1}{r_w}B\left(r_{1e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e}\\ T^{-1}\delta_T^{2(\tau_2-\tau_{2e})}\frac{r_{1e}-r_1+r_{2e}-r_{1f}}{r_w^3}B\left(r_{2e}\right)^2 & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_2-\tau_{2e}\end{cases}
$$

The term $\tau_w^{-1}\tilde{X}_{\tau_{2f}}^2$ dominates the other terms due to the fact that

$$
\frac{\left(\hat{\rho}_{r_1,r_2}-\delta_T\right)^2}{\tau_w}\left[\sum_{j=\tau_{1e}}^{\tau_{1f}}\tilde{X}_{j-1}^2+\sum_{j=\tau_{2e}}^{\tau_2}\tilde{X}_{j-1}^2\right]=\begin{cases}O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^\alpha}\right) & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e}\\ O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T^{2-\alpha}}\right) & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_2-\tau_{2e}\end{cases}
$$

$$
\frac{\left(\hat{\rho}_{r_1,r_2}-1\right)^2}{\tau_w}\left[\sum_{j=\tau_1}^{\tau_{1e}-1}\tilde{X}_{j-1}^2+\sum_{j=\tau_{1f}+2}^{\tau_{2e}-1}\tilde{X}_{j-1}^2\right]=\begin{cases}O_p\left(\frac{\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{T}\right) & \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e}\\ O_p\left(\frac{\delta_T^{2(\tau_2-\tau_{2e})}}{T}\right) & \text{if } \tau_{1f}-\tau_{1e}\leq\tau_2-\tau_{2e}\end{cases}
$$

$$
\tau_w^{-1}\tilde{X}_{\tau_{1f}}^2=O_p\left(\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}\right)\ \text{if } \tau_{1f}-\tau_{1e}>\tau_2-\tau_{2e},
$$

$$
\tau_w^{-1}\left[\hat{\rho}_{r_1,r_2}-1\right]^2\tilde{X}_{\tau_{1f}}^2=O_p\left(\frac{\delta_T^{2(\tau_2-\tau_{2e})}}{T^2}\right)\ \text{if } \tau_{1f}-\tau_{1e}\leq\tau_2-\tau_{2e}
$$

□

Remark 3. *The asymptotic distributions of the unit root t-statistic*

$$
DF_{r_1,r_2}^t=\left(\frac{\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2}{\hat{\sigma}_{r_1r_2}^2}\right)^{1/2}\left(\hat{\rho}_{r_1,r_2}-1\right)
$$

may now be calculated as follows. (1) *When* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$DF^t_{r_1,r_2} \sim_a T^{1-\alpha/2} \frac{r_w^{3/2}}{\sqrt{2\,(r_{ie} - r_1)}} \to \infty;$$

(2) *when* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty;$$

(3) *when* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty;$$

(4) *when* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty;$$

(5) *when* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$,

$$DF^t_{r_1,r_2} \sim_a \begin{cases} -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ \left[\frac{c r_w^3}{2\left(r_{2c} - r_{1f}\right)}\right]^{1/2} T^{1-\alpha/2} \to \infty & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e} \end{cases};$$

(6) *when* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty;$$

(7) *when* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$DF^t_{r_1,r_2} \sim_a \begin{cases} -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty & \text{if } \tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e} \\ \left[\frac{c r_w^3}{2\left(r_{1e} - r_1 + r_{2e} - r_{1f}\right)}\right]^{1/2} T^{1-\alpha/2} \to \infty & \text{if } \tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e} \end{cases}.$$

Taken together with (11) and (12) in PSY2, these results establish the limit behavior of the unit root statistics DF_r and $BSDF_r(r_0)$ in the two cases considered in theorems 4 and 5 (see also (14) below).

2.2.1 The PWY Strategy

The originations of the bubble expansion (r_{1e}, r_{2e}) and the termination dates of the bubble collapse (r_{1f}, r_{2f}) based on the recursive PWY algorithm are identified by the crossing time events

$$\hat{r}_{1e} = \inf_{r\in[r_0,1]} \left\{ r_2 : DF_r > cv^{\beta_T} \right\} \text{ and } \hat{r}_{1f} = \inf_{r\in[\hat{r}_{1e}+r_T,1]} \left\{ r_2 : DF_r < cv^{\beta_T} \right\},$$
$$\hat{r}_{2e} = \inf_{r\in\left(\hat{r}_{1f},1\right]} \left\{ r_2 : DF_r > cv^{\beta_T} \right\} \text{ and } \hat{r}_{2f} = \inf_{r\in[\hat{r}_{2e}+r_T,1]} \left\{ r_2 : DF_r < cv^{\beta_T} \right\},$$

where $r_T = \frac{L_T}{T}$, and $L_T \to \infty$ is a slowly varying function at infinity. We know that when $\beta_T \to 0$, $cv^{\beta_T} \to \infty$. We consider two cases depending on the relative duration of the bubbles.

Case I

Suppose $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$. The asymptotic distributions of the unit root statistic under the alternative hypothesis are as follows:

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \frac{r^{3/2}}{\sqrt{2(r_{ie}-r_1)}} & \text{if } r \in B_1 \\ -T^{(1-\alpha)/2} \left(\frac{1}{2}cr\right)^{1/2} & \text{if } r \in N_1 \cup B_2 \cup N_2 \end{cases}.$$

It is obvious that if $r \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = \Pr\left\{F_{r_2}(W) = \infty\right\} = 0.$$

If $r \in B_1$, $\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_1$, $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\left\{\hat{r}_{1e} > r_{1e} + \eta\right\} \to 0 \text{ and } \Pr\left\{\hat{r}_{1f} < r_{1f} - \gamma\right\} \to 0,$$

due to the fact that $\Pr\left\{DF_{r_{1e}+a_\eta} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{DF_{r_{1f}-a_\gamma} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary, $\Pr\left\{\hat{r}_{1e} < r_{1e}\right\} \to 0$ and $\Pr\left\{\hat{r}_{1f} > r_{1f}\right\} \to 0$, we deduce that $\Pr\left\{|\hat{r}_{1e} - r_{1e}| > \eta\right\} \to 0$ and $\Pr\left\{|\hat{r}_{1f} - r_{1f}| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0. \tag{12}$$

The strategy can therefore consistently estimate both r_{1e} and r_{1f} under suitable rate conditions on cv^{β_T} satisfying (12).

But since $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$ when $r \in N_1 \cup B_2 \cup N_2$, the strategy cannot estimate r_{2e} and r_{2f} consistently when $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$. This proves Theorem 6.

Case II

Suppose $\tau_{1f} - \tau_{1e} \leq \tau_{2f} - \tau_{2e}$. The asymptotic distributions of the recursive unit root statistic under the alternative hypothesis are as follows:

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \frac{r^{3/2}}{\sqrt{2(r_{ie}-r_1)}} & \text{if } r \in B_1 \\ -T^{(1-\alpha)/2}\left(\frac{1}{2}cr\right)^{1/2} & \text{if } r \in N_1 \cup N_2 \\ -T^{(1-\alpha)/2}\left(\frac{1}{2}cr\right)^{1/2} & \text{if } r \in B_2 \text{ and } \tau_{1f} - \tau_{1e} > \tau - \tau_{2e} \\ T^{1-\alpha/2}\left[\frac{cr^3}{2\left(r_{1e}-r_1+r_{2e}-r_{1f}\right)}\right]^{1/2} & \text{if } r \in B_2 \text{ and } \tau_{1f} - \tau_{1e} \leq \tau - \tau_{2e} \end{cases}. \tag{13}$$

It is obvious that if $r \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{DF_{r_2} > cv^{\beta_T}\right\} = \Pr\left\{F_r(W) = \infty\right\} = 0.$$

If $r \in B_1$, $\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_1$, $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\left\{\hat{r}_{1e} > r_{1e} + \eta\right\} \to 0 \text{ and } \Pr\left\{\hat{r}_{1f} < r_{1f} - \gamma\right\} \to 0,$$

due to the fact that $\Pr\left\{BDF_{r_{1e}+a_\eta} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{DF_{r_{1f}-a_\gamma} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\left\{\hat{r}_{1e} < r_{1e}\right\} \to 0$ and $\Pr\left\{\hat{r}_{1f} > r_{1f}\right\} \to 0$, we deduce that $\Pr\left\{\left|\hat{r}_{1e} - r_{1e}\right| > \eta\right\} \to 0$ and $\Pr\left\{\left|\hat{r}_{1f} - r_{1f}\right| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.$$

The strategy can therefore consistently estimate r_{1e} and r_{1f}.

If $r \in B_2$ and $\tau_{1f} - \tau_{1e} > \tau - \tau_{2e}$, $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$ since $cv^{\beta_T} \to \infty$. If $r \in B_2$ and $\tau_{1f} - \tau_{1e} \leq \tau - \tau_{2e}$, $\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$ in view of the final panel entry of (13). If $r \in N_1$, $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$. This implies that the strategy cannot identify the second bubble when $\tau_{1f} - \tau_{1e} > \tau_2 - \tau_{2e}$. However, when $\tau_{1f} - \tau_{1e} \leq \tau_2 - \tau_{2e}$ it can identify the second bubble provided that

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.$$

This suggests that estimated second bubble origination date $\hat{r}_{2e}$ will be biased, taking values of $r_{2e} + r_{1f} - r_{1e}$ (in view of the condition $\tau_{1f} - \tau_{1e} \leq \tau - \tau_{2e}$ under which the final panel entry of (13) holds). So, there will be delayed detection of the second bubble in general. The termination point of the second bubble r_{2f} is consistently estimated. This proves Theorem 7.

2.2.2 The PSY algorithm

The origination dates of bubble expansion (r_{1e}, r_{2e}) and the dates of bubble collapse (r_{1f}, r_{2f}) that are based on the backward sup DF test are identified by crossing time events as follows:

$$\begin{aligned}
\hat{r}_{1e} &= \inf_{r\in[r_0,1]} \left\{r : BSDF_r\left(r_0\right) > scv^{\beta_T}\right\} \text{ and } \hat{r}_{1f} = \inf_{r\in[\hat{r}_{1e}+r_T,1]} \left\{r : BSDF_r\left(r_0\right) < scv^{\beta_T}\right\}, \\
\hat{r}_{2e} &= \inf_{r\in\left(\hat{r}_{1f},1\right]} \left\{r : BSDF_r\left(r_0\right) > scv^{\beta_T}\right\} \text{ and } \hat{r}_{2f} = \inf_{r\in[\hat{r}_{2e}+r_T,1]} \left\{r : BSDF_r\left(r_0\right) < scv^{\beta_T}\right\},
\end{aligned}$$

where $r_T = \frac{L_T}{T}$, and $L_T \to \infty$ is a slowly varying function at infinity. We know that when $\beta_T \to 0$, $scv^{\beta_T} \to \infty$.

Suppose the minimum window size is smaller than the distance separating the termination dates of two bubbles (i.e. $r_0 < r_{2f} - r_{1f}$). The asymptotic distributions of the backward sup DF statistic under the alternative hypothesis are

$$BSDF_r\left(r_0\right) \sim_a \begin{cases} F_r\left(W, r_0\right) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \sup\limits_{r_1\in[0,r-r_0]} \left\{\frac{(r-r_1)^{3/2}}{\sqrt{2(r_{ie}-r_1)}}\right\} & \text{if } r \in B_i \\ -T^{(1-\alpha)/2} \sup\limits_{r_1\in[0,r-r_0]} \left(\frac{1}{2}cr_w\right)^{1/2} & \text{if } r \in N_1 \cup N_2 \end{cases}. \tag{14}$$

It is obvious that if $r \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{BSDF_r\left(r_0\right) > scv^{\beta_T}\right\} = \Pr\left\{F_r\left(W, r_0\right) = \infty\right\} = 0.$$

If $r \in B_i$ with $i = 1, 2$, $\lim_{T\to\infty} \Pr\left\{BSDF_r(r_0) > scv^{\beta_T}\right\} = 1$ provided that $\frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_i$ with $i = 1, 2$, $\lim_{T\to\infty} \Pr\left\{BSDF_r(r_0) < scv^{\beta_T}\right\} = 1$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\{\hat{r}_{ie} > r_{ie} + \eta\} \to 0 \text{ and } \Pr\{\hat{r}_{if} < r_{if} - \gamma\} \to 0,$$

since $\Pr\left\{BSDF_{r_{ie}+a_\eta}(r_0) > scv^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{BSDF_{r_{if}-a_\gamma}(r_0) > scv^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\{\hat{r}_{ie} < r_{ie}\} \to 0$ and $\Pr\{\hat{r}_{if} > r_{if}\} \to 0$, we deduce that $\Pr\{|\hat{r}_{ie} - r_{ie}| > \eta\} \to 0$ and $\Pr\{|\hat{r}_{if} - r_{if}| > \gamma\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{scv^{\beta_T}} + \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0 \ .$$

Therefore, the date-stamping strategy based on the recursive sup ADF test procedure of PSY consistently estimates each of the dates r_{1e}, r_{1f}, r_{2e} and r_{2f}. This proves Theorem 8.

2.2.3 The sequential PWY procedure

The origination dates of bubble expansion (r_{1e}, r_{2e}) and bubble collapse dates (r_{1f}, r_{2f}) that are based on the sequential PWY procedure using recursive DF tests are identified by the following crossing times:

$$\hat{r}_{1e} = \inf_{r\in[r_0,1]} \left\{r : DF_r > cv^{\beta_T}\right\} \text{ and } \hat{r}_{1f} = \inf_{r\in[\hat{r}_{1e}+r_T,1]} \left\{r : DF_r < cv^{\beta_T}\right\},$$
$$\hat{r}_{2e} = \inf_{r\in\left(\hat{r}_{1f}+r_0,1\right]} \left\{r :_{\hat{r}_{1f}} DF_r > cv^{\beta_T}\right\} \text{ and } \hat{r}_{2f} = \inf_{r\in[\hat{r}_{2e}+r_T,1]} \left\{r :_{\hat{r}_{1f}} DF_r < cv^{\beta_T}\right\},$$

where ${}_{\hat{r}_{1f}}DF_r$ is the DF statistic calculate over $(\hat{r}_{1f}, r]$, $r_T = \frac{L_T}{T}$, and $L_T \to \infty$ is a slowly varying function at infinity. Importantly, the search for the second bubble origination date $\hat{r}_{2e}$ commences after $\hat{r}_{1f}+r_0$, that is after a minimum time has elapsed (r_0) following the termination of the first bubble $(\hat{r}_{1f})$. We know that when $\beta_T \to 0$, $cv^{\beta_T} \to \infty$.

The asymptotic distributions of the recursive unit root statistic under the alternative hypothesis are

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2}\frac{r^{3/2}}{\sqrt{2(r_{ie}-r_1)}} & \text{if } r \in B_1 \\ -T^{(1-\alpha)/2}\left(\frac{1}{2}cr\right)^{1/2} & \text{if } r \in N_1 \end{cases}$$

and

$$
{}_{\hat{r}_{1f}}DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_1 \\ T^{1-\alpha/2}\frac{(r-r_{1f})^{3/2}}{\sqrt{2(r_{ie}-r_1)}} & \text{if } r \in B_2 \\ -T^{(1-\alpha)/2}\left[\frac{1}{2}c\left(r-r_{1f}\right)\right]^{1/2} & \text{if } r \in N_2 \end{cases}.
$$

It is obvious that if $r \in N_0$,

$$
\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = \Pr\left\{F_{r_2}(W) = \infty\right\} = 0.
$$

If $r \in B_1$, $\lim_{T\to\infty}\Pr\left\{DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_1$, $\lim_{T\to\infty}\Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$ and $\lim_{T\to\infty}\Pr\left\{{}_{\hat{r}_{1f}}DF_r > cv^{\beta_T}\right\} = \Pr\left\{F_r(W) = \infty\right\} = 0$. If $r \in B_2$, $\lim_{T\to\infty}\Pr\left\{{}_{\hat{r}_{1f}}DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1/2}\delta_T^{\tau-\tau_{2e}}} \to 0$. This implies that provided that $\frac{cv^{\beta_T}}{T^{1/2}} \to 0$, $\lim_{T\to\infty}\Pr\left\{{}_{\hat{r}_{1f}}DF_r > cv^{\beta_T}\right\} = 1$ for any $r \in B_2$. If $r \in N_2$, $\lim_{T\to\infty}\Pr\left\{{}_{\hat{r}_{1f}}DF_r < cv^{\beta_T}\right\} = 1$.

It follows that for any $\eta, \gamma > 0$,

$$
\Pr\left\{\hat{r}_{1e} > r_{1e} + \eta\right\} \to 0 \text{ and } \Pr\left\{\hat{r}_{1f} < r_{1f} - \gamma\right\} \to 0,
$$

since $\Pr\left\{DF_{r_{1e}+a_\eta} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{DF_{r_{1f}-a_\gamma} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\left\{\hat{r}_{1e} < r_{1e}\right\} \to 0$ and $\Pr\left\{\hat{r}_{1f} > r_{1f}\right\} \to 0$, we deduce that $\Pr\left\{|\hat{r}_{1e} - r_{1e}| > \eta\right\} \to 0$ and $\Pr\left\{|\hat{r}_{1f} - r_{1f}| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$
\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.
$$

Thus, this date-stamping strategy consistently estimates r_{1e} and r_{1f}.

For any $\phi, \kappa > 0$,

$$
\Pr\left\{\hat{r}_{2e} > r_{2e} + \phi\right\} \to 0 \text{ and } \Pr\left\{\hat{r}_{2f} < r_{2f} - \kappa\right\} \to 0,
$$

since $\Pr\left\{{}_{\hat{r}_{1f}}DF_{r_{2e}+a_\phi} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\phi < \phi$ and $\Pr\left\{{}_{\hat{r}_{1f}}DF_{r_{2f}-a_\kappa} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\kappa < \kappa$. Since $\phi, \kappa > 0$ is arbitrary and $\Pr\left\{r_{1f} < \hat{r}_{2e} < r_{2e}\right\} \to 0$ and $\Pr\left\{\hat{r}_{2f} > r_{2f}\right\} \to 0$, we deduce that $\Pr\left\{|\hat{r}_{2e} - r_{2e}| > \eta\right\} \to 0$ and $\Pr\left\{|\hat{r}_{2f} - r_{2f}| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$
\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.
$$

Therefore, the alternative sequential implementation of the PWY procedure consistently estimates r_{2e} and r_{2f} provided $r_{2e} \geq r_{1f} + r_0$. This proves Theorem 9.

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