Online Supplement on "Random Coefficient Continuous Systems: Testing for Extreme Sample Path Behaviour"

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This supplement provides additional simulation results and additional empirical results that are not shown in the paper.

1 Additional Simulation Results

Tables 1-6 report bias and standard errors under stationary, unstable, and locally explosive cases based on 10,000 replications. The results suggest that the proposed two-stage method is effective in estimating parameters in the continuous time model.

Table 1: Bias and standard errors of the two-stage estimates for different Δ and M and a fixed $T(=5)$. The parameter values are $\widetilde{\mu} = -1$, $\widetilde{\sigma}^2 = 1$, $\sigma^2 = 1$. $y_0 = 10$.

Params	$\Delta = 1/252$						$\Delta = 1/19656$					
	$M = 21$		$M = 63$		$M = 252$		$M = 21$		$M = 63$		$M = 252$	
	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.
Panel A: $\gamma = 0$												
$\widetilde{\mu}$	-0.2020	0.7148	-0.2021	0.7150	-0.2001	0.7174	-0.1961	0.7189	-0.1961	0.7189	-0.1961	0.7189
$\widetilde{\sigma}^2$	0.0133	0.0856	0.0097	0.0879	0.1100	0.0693	0.0017	0.0092	7.9e-04	0.0090	2.6e-04	0.0089
σ^2	0.0019	0.0926	0.0030	0.1140	0.0079	0.3065	0.0016	0.0124	7.6e-04	0.0122	2.3e-04	0.0119
κ	-0.1953	0.7115	-0.1972	0.7123	-0.1941	0.7153	-0.1952	0.7189	-0.1957	0.7189	-0.1959	0.7189
ρ	-7.9e-04	0.0028	-7.9e-04	0.0028	-7.9e-04	0.0028	-3.9e-05	2.9e-05	-1.0e-05	3.7e-05	-1.0e-05	3.7e-05
ϕ	-0.2086	0.7206	-0.2069	0.7204	-0.2060	0.7237	-0.1969	0.7188	-0.1965	0.7188	-0.1962	0.7188
Panel B: $\gamma = 0.8$												
$\widetilde{\mu}$	-0.2018	0.7469	-0.2033	0.7482	-0.2332	1.2467	-0.1913	0.7425	-0.1913	0.7425	-0.1913	0.7425
$\widetilde{\sigma}^2$	0.0233	0.1160	0.0212	0.1246	0.0438	0.1935	0.0018	0.0123	8.6e-04	0.0120	3.3e-04	0.0120
σ^2	0.0342	0.5280	0.0395	0.8242	0.4420	21.7866	0.0027	0.0399	0.0018	0.0394	0.0012	0.0387
γ	-0.0146	0.1177	-0.0160	0.1313	-0.0495	0.2450	-3.1e-04	0.0217	-2.5e-04	0.0212	-2.2e-04	0.0211
κ	-0.1901	0.7462	-0.1927	0.7493	-0.2113	1.2508	-0.1904	0.7426	-0.1909	0.7426	-0.1911	0.7426
ρ	-7.9e-04	0.0029	-8.0e-04	0.0030	-9.1e-04	0.0048	4.8e-05	1.5e-05	3.9e-05	2.9e-05	3.9e-05	2.9e-05
ϕ	-0.2134	0.7521	-0.2139	0.7522	-0.2551	1.2502	-0.1922	0.7424	-0.1917	0.7424	-0.1914	0.7424

Table 2: Bias and standard errors of the two-stage estimates for different Δ and M and a fixed $T(=5)$. The parameter values are $\widetilde{\mu} = -0.5$, $\widetilde{\sigma}^2 = 1$, $\sigma^2 = 1$. $y_0 = 10$.

Params	$\Delta = 1/252$						$\Delta = 1/19656$					
	$M = 21$		$M = 63$		$M = 252$		$M = 21$		$M = 63$		$M = 252$	
	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.
Panel A: $\gamma = 0$												
$\widetilde{\mu}$	-0.1916	0.6547	-0.1918	0.6546	-0.1913	0.6547	-0.1884	0.6646	-0.1884	0.6646	-0.1884	0.6646
$\widetilde{\sigma}^2$	0.0058	0.0728	0.0028	0.0746	0.0036	0.0915	0.0016	0.0080	7.0e-04	0.0079	1.7e-04	0.0078
σ^2	0.0545	0.7225	0.0738	1.0822	0.1807	2.9377	0.0016	0.1027	9.7e-04	0.1000	3.0e-04	0.1056
κ	-0.1887	0.6526	-0.1904	0.6526	-0.1896	0.6534	-0.1876	0.6647	-0.1880	0.6647	-0.1883	0.6647
ρ	-7.6e-04	0.0026	-7.6e-04	0.0026	-7.5e-04	0.0026	1.9e-05	2.0e-05	-9.6e-06	3.4e-05	-9.6e-06	3.4e-05
ϕ	-0.1945	0.6588	-0.1932	0.6586	-0.1931	0.6593	-0.1892	0.6645	-0.1887	0.6645	-0.1885	0.6645
Panel B: $\gamma = 0.8$												
$\widetilde{\mu}$	-0.1764	0.6908	-0.1778	0.6921	-0.2401	1.8586	-0.1646	0.6779	-0.1646	0.6779	-0.1646	0.6779
$\widetilde{\sigma}^2$	0.0212	0.1091	0.0208	0.1179	0.0491	0.1940	0.0020	0.0118	9.8e-04	0.0116	4.4e-04	0.0115
σ^2	0.4005	4.5552	0.5651	6.9179	1.4647	14.0834	0.0322	0.4455	0.0301	0.4253	0.0305	0.4646
γ	-0.0418	0.2102	-0.0464	0.2301	-0.1047	0.3632	-0.0034	0.3632	-0.0030	0.3632	-0.0028	0.3632
κ	-0.1658	0.6952	-0.1674	0.6987	-0.2155	1.8623	-0.1636	0.6783	-0.1641	0.6783	-0.1644	0.6783
ρ	-6.9e-04	0.0027	-7.0e-04	0.0027	-9.2e-04	0.0068	2.4e-05	1.0e-05	1.9e-05	2.0e-05	1.9e-05	2.0e-05
ϕ	-0.1869	0.6907	-0.1882	0.6904	-0.2646	1.8598	-0.1656	0.6776	-0.1651	0.6776	-0.1648	0.6776

Table 3: Bias and standard errors of the two-stage estimates for different Δ and M and a fixed $T(=5)$. The parameter values are $\widetilde{\mu} = 0.5$, $\widetilde{\sigma}^2 = 1$, $\sigma^2 = 1$. $y_0 = 10$.

| Params | $\Delta = 1/252$ | | | | | | $\Delta = 1/19656$ | | | | | |
| | $M = 21$ | | $M = 63$ | | $M = 252$ | | $M = 21$ | | $M = 63$ | | $M = 252$ | |
	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.
Panel A: $\gamma = 0$												
$\widetilde{\mu}$	-0.0847	0.5591	-0.0861	0.5586	-0.0960	0.5927	-0.0798	0.5786	-0.0817	0.5713	-0.0817	0.5713
$\widetilde{\sigma}^2$	-0.0064	0.0539	-0.0101	0.0561	-0.0161	0.0713	0.0015	0.0062	4.8e-04	0.0061	-6.1e-05	0.0061
σ^2	1.9927	7.9678	2.6250	11.3631	11.5247	109.1634	0.1054	0.8897	0.1025	0.8561	0.1045	0.8704
κ	-0.0880	0.5575	-0.0911	0.5570	-0.1041	0.5919	-0.0791	0.5785	-0.0815	0.5713	-0.0818	0.5713
ρ	-3.3e-04	0.0022	-3.4e-04	0.0022	-3.8e-04	0.0023	-1.4e-05	2.4e-05	-4.2e-06	2.9e-05	-4.2e-05	2.9e-05
ϕ	-0.0815	0.5620	-0.0810	0.5616	-0.0879	0.5956	-0.0806	0.5787	-0.0820	0.5714	-0.0817	0.5714
Panel B: $\gamma = 0.8$												
$\widetilde{\mu}$	-0.0864	0.5695	-0.0887	0.5694	-0.1088	1.5743	-0.0747	0.5720	-0.0747	0.5720	-0.0747	0.5720
$\widetilde{\sigma}^2$	0.0399	0.1135	0.0434	0.1294	0.1069	0.3211	0.0037	0.0120	0.0028	0.0119	0.0022	0.0118
σ^2	9.9413	32.2082	14.1995	59.1407	66.6901	364.0267	0.7251	2.5128	0.7103	2.4788	0.7149	2.5225
γ	-0.2871	0.5485	-0.3113	0.5699	-0.4528	0.6906	-0.0531	0.2409	-0.0535	0.2394	-0.0519	0.2396
κ	-0.0667	0.5769	-0.0670	0.5781	-0.1088	1.5758	-0.0728	0.5726	-0.0733	0.5727	-0.0736	0.5726
ρ	-3.4e-04	0.0023	-3.5e-04	0.0023	-4.1e-04	0.0061	-2.5e-05	7.8e-06	-2.1e-05	1.6e-05	-2.1e-05	1.6e-05
ϕ	-0.1063	0.5678	-0.1104	0.5680	-0.1622	1.5890	-0.0765	0.5714	-0.0761	0.5714	-0.0758	0.5714

Table 4: Bias and standard errors of the two-stage estimates for different T and M and a fixed $\Delta(= 1/252)$. The parameter values are $\widetilde{\mu} = -1$, $\widetilde{\sigma}^2 = 1$, $\sigma^2 = 1$. $y_0 = 10$.

| Params | $T = 30$ | | | | | | $T = 60$ | | | | | |
| | $M = 21$ | | $M = 63$ | | $M = 252$ | | $M = 21$ | | $M = 63$ | | $M = 252$ | |
	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.
Panel A: $\gamma = 0$												
$\widetilde{\mu}$	-0.0567	0.3556	-0.0529	0.3556	-0.0570	0.3557	-0.0294	0.2555	-0.0295	0.2555	-0.0296	0.2555
$\widetilde{\sigma}^2$	0.0077	0.0482	0.0035	0.0495	-2.5e-05	0.0574	0.0070	0.0362	0.0026	0.0371	-0.0019	0.0425
σ^2	-4.0e-07	0.0243	-3.5e-04	0.0250	4.0e-04	0.0297	1.7e-04	0.0173	-1.0e-04	0.0177	9.8e-04	0.0208
κ	-0.0529	0.3549	-0.0551	0.3552	-0.0570	0.3561	-0.0259	0.2554	-0.0282	0.2556	-0.0305	0.2568
ρ	-2.2e-04	0.0014	-2.2e-04	0.0014	-2.2e-04	0.0014	-1.2e-04	0.0010	-1.2e-04	0.0010	-1.2e-04	0.0010
ϕ	-0.0606	0.3580	-0.0586	0.3578	-0.0570	0.3576	-0.0330	0.2570	-0.0308	0.2568	-0.0187	0.2564
Panel B: $\gamma = 0.8$												
$\widetilde{\mu}$	-0.0495	0.2795	-0.0497	0.2796	-0.0499	0.2798	-0.0256	0.1928	-0.0256	0.1928	-0.0257	0.1928
$\widetilde{\sigma}^2$	0.0095	0.0517	0.0048	0.0538	0.0022	0.0668	0.0077	0.0374	0.0028	0.0387	-0.0011	0.0470
σ^2	-3.9e-04	0.0225	-0.0024	0.0232	-0.0031	0.0211	-3.7e-04	0.0156	-0.0024	0.0160	-0.0028	0.0191
γ	-0.0020	0.0129	-0.0023	0.0141	-0.0023	0.0286	-0.0018	0.0086	-0.0021	0.0091	-0.0020	0.0124
κ	-0.0448	0.2793	-0.0473	0.2799	-0.0488	0.2815	-0.0217	0.1929	-0.0242	0.1933	-0.0263	0.1942
ρ	-2.0e-04	0.0011	-2.0e-04	0.0011	-2.0e-04	0.0011	-1.0e-04	7.6e-04	-1.0e-04	7.6e-04	-1.0e-04	7.6e-04
ϕ	-0.0543	0.2821	-0.0520	0.2818	-0.0510	0.2821	-0.0294	0.1945	-0.0270	0.1942	-0.0252	0.1943

Table 5: Bias and standard errors of the two-stage estimates for different T and M and a fixed $\Delta(=1/252)$. The parameter values are $\widetilde{\mu} = -0.5$, $\widetilde{\sigma}^2 = 1$, $\sigma^2 = 1$. $y_0 = 10$.

Params	T = 30						T = 60					
	M = 21		M = 63		M = 252		M = 21		M = 63		M = 252	
	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.
Panel A: $\gamma = 0$												
$\widetilde{\mu}$	-0.0572	0.3125	-0.0573	0.3125	-0.0575	0.3125	-0.0300	0.2233	-0.0301	0.2233	-0.0302	0.2237
$\widetilde{\sigma}^2$	0.0024	0.0382	-0.0014	0.0391	-0.0035	0.0442	0.0024	0.0285	-0.0016	0.0291	-0.0044	0.0324
σ^2	2.1e-04	0.0266	-1.2e-04	0.0272	2.7e-04	0.0322	1.2e-04	0.0186	-2.1e-04	0.0190	5.0e-04	0.0221
κ	-0.0560	0.3122	-0.0581	0.3124	-0.0592	0.3129	-0.0288	0.2232	-0.0308	0.2234	-0.0324	0.2241
ρ	-2.3e-04	0.0012	-2.3e-04	0.0012	-2.3e-04	0.0012	-1.2e-04	8.8e-04	-1.2e-04	8.8e-04	-1.2e-04	8.8e-04
ϕ	-0.0584	0.3139	-0.0566	0.3138	-0.0557	0.3137	-0.0312	0.2243	-0.0293	0.2242	-0.0279	0.2237
Panel B: $\gamma = 0.8$												
$\widetilde{\mu}$	-0.0440	0.2309	-0.0441	0.2310	-0.0442	0.2311	-0.0222	0.1553	-0.0222	0.1553	-0.0223	0.1553
$\widetilde{\sigma}^2$	0.0036	0.0405	-2.8e-04	0.0413	-0.0021	0.0487	0.0028	0.0292	-0.0012	0.0298	-0.0038	0.0342
σ^2	-1.0e-04	0.0243	-0.0022	0.0248	-0.0029	0.0293	-2.7e-04	0.0168	-0.0024	0.0171	-0.0031	0.0193
γ	-6.8e-04	0.0102	-0.0010	0.0108	-0.0011	0.0140	-5.6e-04	0.0067	-9.6e-04	0.0069	-0.0011	0.0082
κ	-0.0422	0.2312	-0.0442	0.2315	0.0453	0.2323	-0.0208	0.1556	-0.0229	0.1558	-0.0242	0.1563
ρ	-1.7e-04	9.1e-04	-1.7e-05	9.1e-04	-1.7e-04	9.1e-04	5.0e-05	6.1e-04	-8.8e-05	6.1e-04	-8.8e-05	6.1e-04
ϕ	-0.0458	0.2325	-0.0439	0.2323	-0.0432	0.2325	-0.0236	0.1563	-0.0216	0.1561	-0.0204	0.1561

Table 6: Bias and standard errors of the two-stage estimates for different T and M and a fixed $\Delta(=1/252)$. The parameter values are $\widetilde{\mu} = 0.5$, $\widetilde{\sigma}^2 = 1$, $\sigma^2 = 1$. $y_0 = 10$.

Params	T = 30						T = 60					
	M = 21		M = 63		M = 252		M = 21		M = 63		M = 252	
	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.	Bias	S.E.
Panel A: $\gamma = 0$												
$\widetilde{\mu}$	-0.0329	0.2198	-0.0330	0.2198	-0.0336	0.2194	-0.0166	0.1501	-0.0166	0.1501	-0.0168	0.1500
$\widetilde{\sigma}^2$	-0.0021	0.0201	-0.0053	0.0206	-0.0059	0.0236	-0.0019	0.0139	-0.0051	0.0143	-0.0057	0.0163
σ^2	0.6625	4.8403	0.8266	6.3235	2.7896	23.1726	0.4609	3.8037	0.5752	5.4461	2.0818	23.0396
κ	-0.0340	0.2197	-0.0357	0.2196	-0.0366	0.2193	-0.0175	0.1502	-0.0191	0.1502	-0.0196	0.1501
ρ	-1.3e-04	8.7e-04	-1.3e-04	8.7e-04	-1.3e-04	8.7e-04	-6.6e-05	6.0e-04	-6.6e-05	6.0e-04	-6.7e-05	6.0e-04
ϕ	-0.0319	0.2204	-0.0304	0.2204	-0.0307	0.2202	-0.0156	0.1504	-0.0141	0.1504	-0.0139	0.1503
Panel B: $\gamma = 0.8$												
$\widetilde{\mu}$	0.0150	0.1701	0.0149	0.1701	0.0142	0.1938	0.0122	0.1195	0.0122	0.1195	0.0142	0.1644
$\widetilde{\sigma}^2$	-1.8e-04	0.0214	-0.0034	0.0219	-0.0032	0.0255	-0.0012	0.0139	-0.0044	0.0143	-0.0048	0.0164
σ^2	2.3272	12.1910	2.8084	15.2721	8.3717	83.5771	1.4188	8.9666	1.8492	12.4210	5.2935	58.9490
γ	-0.0741	0.2808	-0.0760	0.2880	-0.1056	0.3503	-0.0478	0.2277	-0.0477	0.2277	-0.0614	0.2700
κ	0.0149	0.1710	0.0132	0.1710	0.0125	0.1949	0.0116	0.1198	0.0100	0.1198	0.0118	0.1647
ρ	6.0e-05	6.8e-04	6.0e-05	6.8e-04	5.7e-05	7.7e-04	4.9e-05	4.8e-04	4.9e-05	4.8e-04	5.7e-05	6.6e-04
ϕ	0.0151	0.1700	0.0165	0.1700	0.0158	0.1934	0.0129	0.1196	0.0144	0.1196	0.0166	0.1645

2 Additional Empirical Results

For daily S&P 500 real prices over December 31, 1927 to June 29, 2018, time horizons of unstable and explosive episodes detected by random coefficient autoregressive models without endogeneity are reported in Table 7. This Table corresponds to Figure 5 in the main paper.

Table 7: Time Horizons of Unstable and Explosive Episodes Detected by Random Coefficient Autoregressive Models assuming No Endogeneity.

Unstable	Locally Explosive	Explosive
Feb. 1964 – May. 1966	Jan. 1965 – May. 1965	—
	Sep. 1965 – Apr. 1966	—
Jan. 1967 – Jan. 1968	—	—
Apr. 1968 – May. 1969	Sep. 1968 – Jan. 1969	—
May. 1991 – Jun. 2018	Jul. 1995 – Jun. 2018	Jul. 1997 – Jun. 2002
		May. 2003 – Aug. 2008
		Oct. 2010 – Jun. 2018

For daily S&P 500 real prices over December 31, 1927 to June 29, 2018, time horizons for explosive behavior detected by PWY and PSY are reported in Table 8. This Table corresponds to Figure 6 in the main paper.

Table 8: Horizons of Explosive Behaviour Detected by PWY and PSY Test.

PWY Test	PSY Test	
Apr. 1965 – May. 1965	Feb. 1942 – Dec. 1942	May. 1968 – Dec. 1968
Sep.1965 – Feb. 1966	Nov. 1945 – Jun. 1946	Jan. 1987 – Sep. 1987
Apr. 1968 – Jan. 1969	Jul. 1954 – Aug. 1956	Sep. 1995 – Oct. 2000
Jan. 1996 – Jan. 2001	Feb. 1961 – Mar. 1962	Nov. 2013 – Jul. 2015
Dec. 2016 – Jun. 2018	Mar. 1964 – Apr. 1966	Oct. 2017 – Feb. 2018

For daily S&P 500 real prices over December 31, 1927 to June 29, 2018, horizons of unstable and explosive behaviour detected by random coefficient autoregressive models with endogeneity are reported in Table 9. This table corresponds to Figure 7 in the main paper.

Table 9: Horizons of Unstable and Explosive Behaviour Detected by Random Coefficient Autoregressive Models with Endogeneity.

Unstable	Locally Explosive	Explosive
Mar. 1961 – Aug. 1962	Aug. 1961 – Apr. 1962	Dec. 1961 – Mar. 1962
Nov. 1962 – Mar. 1970	Jan. 1963 – Jul. 1966	Aug. 1963 – Jul. 1966
	Oct. 1966 – Dec. 1969	Oct. 1966 – Oct. 1969
Dec. 1970 – Oct. 1973	Mar. 1971 – May. 1971	—
	Dec. 1971 – Mar. 1973	Feb. 1972 – May. 1972
		Aug. 1972 – Feb. 1973
Jan. 1987 – Sep. 1987	Jun. 1987 – Sep. 1987	—
Feb. 1988 – Jun. 2018	Apr. 1989 – Jul. 1990	—
	Dec. 1990 – Aug. 2002	Dec. 1991 – Jun. 2001
	May. 2003 – Sep. 2008	—
	Dec. 2010 – Jul. 2011	—
	Jan. 2012 – Jun. 2018	Sep. 2017 – Jun. 2018