Online Supplement to "Modeling and Forecasting Realized Volatility with the Fractional Ornstein-Uhlenbeck Process"*

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This online appendix contains three sets of empirical results. The purpose of these additional empirical results is to show that all the empirical conclusions drawn in the main paper are qualitatively unchanged, when the log RV, log BV, and log RK for the three equities are predicted.

1 Forecasting Log RV for Equities

The main paper forecasts RV for three U.S. stock indices. Some papers in the literature compare alternative models in terms of ability to forecast log RV; see for example, Andersen et al. (2003). In this section, we compare the performance of six alternative models in forecasting log daily RV of the three U.S. equities.

Tables 1-2 report the ratio of RMSE of each candidate model and that of HAR and the MZ-R^2 of six competing models for h-day-ahead-forecast of RV with $h = 1, 2, \ldots, 10$ with the best result highlighted in boldface for each h. Most importantly, regardless of the RV series and forecasting horizon, fO-U always performs the best, followed by fBm and then by ARFIMA. ARFIMA model always outperforms the RW, AR(1), HAR models in both criteria and by a wide margin. This result confirms the finding of Andersen et al. (2003) that the ARFIMA model can generate accurate volatility forecasts.

Table 3 reports the DM test based on the squared forecast errors and the p-value (in parenthesis) with the benchmark being fO-U (boldface means statistically significant at the 10% level). We only report the results for HAR, ARFIMA and fBm in Table 3. In

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Table 1: The ratio of RMSE of different models and that of HAR for h-day-ahead-forecast of $\log\left(100\sqrt{RV \times 252}\right)$.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	1.0019	1.0876	1.0886	1.0982	1.0968	1.1174	1.0914	1.0652	1.0496	1.1188
AR(1)	0.9503	1.0152	1.0272	1.0517	1.0695	1.1011	1.1065	1.0985	1.0796	1.1090
HAR	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ARFIMA	0.8829	0.9325	0.9355	0.9398	0.9359	0.9470	0.9407	0.9289	0.9106	0.9349
fBm	0.8819	0.9281	0.9303	0.9328	0.9273	0.9362	0.9296	0.9166	0.8964	0.9164
fO-U	**0.8794**	**0.9266**	**0.9279**	**0.9313**	**0.9266**	**0.9318**	**0.9247**	**0.9123**	**0.8934**	**0.9148**
Panel B: DJIA										
RW	1.0681	1.0852	1.1141	1.1352	1.0799	1.0826	1.1342	1.0547	1.0676	1.1131
AR(1)	1.0066	1.0283	1.0802	1.1314	1.1276	1.1341	1.1773	1.1504	1.1354	1.0909
HAR	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ARFIMA	0.9193	0.9321	0.9490	0.9633	0.9426	0.9360	0.9584	0.9310	0.9241	0.9308
fBm	0.9163	0.9293	0.9454	0.9587	0.9393	0.9311	0.9510	0.9247	0.9157	0.9193
fO-U	**0.9130**	**0.9284**	**0.9443**	**0.9579**	**0.9377**	**0.9304**	**0.9486**	**0.9219**	**0.9137**	**0.9186**
Panel C: NASDAQ 100										
RW	0.9961	1.1251	1.1523	1.1523	1.1082	1.1173	1.1299	1.1000	1.0774	1.1296
AR(1)	0.9464	1.0366	0.9623	1.0706	1.0542	1.0704	1.0934	1.0946	1.0805	1.1101
HAR	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ARFIMA	0.8801	0.9483	0.9623	0.9623	0.9408	0.9444	0.9542	0.9449	0.9298	0.9528
fBm	0.8754	0.9359	0.9542	0.9542	0.9351	0.9614	0.9465	0.9385	0.9206	0.9376
fO-U	**0.8735**	**0.9343**	**0.9473**	**0.9473**	**0.9271**	**0.9296**	**0.9388**	**0.9301**	**0.9141**	**0.9338**

terms of DM, fO-U is statistically different from HAR always, from ARFIMA almost always, and from fBm in three cases at the 10% level.

Table 4 reports the p-value of the semi-quadratic statistic obtained from $2,000$ bootstrap iterations with a block length of 12. Values in boldface denote that the model belongs to the confidence set of the best models. HAR, AR(1), and RW are always rejected regardless of the RV series and forecasting horizon. In all but one case, ARFIMA is also rejected. Most importantly, in no case, fO-U and fBm are rejected.

To compare the forecasting performance across nested models, we calculate MSFE-adjusted statistics and p-values of the CW test of Clark and West (2007) that account for estimation errors. Table 5 reports the CW test and the p-value (in parenthesis) with the encompassing model being fO-U (boldface means statistically significant at the 10% level). According to the CW test, the performance of fO-U is statistically significantly better than RW and AR(1) always, and better than fBm in most cases.

Table 2: MZ-R^2 for h-day-ahead-forecast of $\log\left(100\sqrt{RV \times 252}\right)$ in different models.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	0.4274	0.2977	0.2450	0.1909	0.1483	0.1087	0.1070	0.0993	0.0770	0.0321
AR(1)	0.4276	0.2981	0.2457	0.1919	0.1495	0.1104	0.1092	0.1020	0.0804	0.0355
HAR	0.3521	0.2918	0.2329	0.1840	0.1421	0.1240	0.1025	0.0735	0.0420	0.0444
ARFIMA	0.4684	0.3467	0.2738	0.2140	0.1667	0.1294	0.1081	0.0845	0.0628	0.0422
fBm	0.4694	0.3475	0.2763	0.2112	0.1693	0.1314	0.1110	0.0875	0.0649	0.0422
fO-U	**0.4719**	**0.3511**	**0.2804**	**0.2180**	**0.1731**	**0.1414**	**0.1201**	**0.0958**	**0.0728**	**0.0500**
Panel B: DJIA										
RW	0.3637	0.2870	0.2219	0.1868	0.1924	0.1456	0.0988	0.1199	0.0767	0.0334
AR(1)	0.3641	0.2878	0.2234	0.1891	0.1952	0.1493	0.1039	0.1266	0.0847	0.0414
HAR	0.3386	0.2681	0.2274	0.2102	0.1606	0.1146	0.1136	0.0725	0.0435	0.0298
ARFIMA	0.4211	0.3313	0.2632	0.2239	0.1919	0.1453	0.1174	0.0970	0.0654	0.0392
fBm	0.4225	0.3319	0.2633	0.2242	0.1940	0.1470	0.1177	0.0989	0.0663	0.0397
fO-U	**0.4273**	**0.3341**	**0.2666**	**0.2277**	**0.1967**	**0.1499**	**0.1244**	**0.1047**	**0.0715**	**0.0443**
Panel C: NASDAQ 100										
RW	0.3673	0.2068	0.1699	0.1578	0.1587	0.1295	0.1097	0.1108	0.0858	0.0412
AR(1)	0.3674	0.2069	0.1703	0.1583	0.1588	0.1296	0.1099	0.1111	0.0863	0.0417
HAR	0.2681	0.2236	0.2073	0.1989	0.1588	0.1347	0.1209	0.0966	0.0558	0.0512
ARFIMA	0.4028	0.2690	0.2346	0.2137	0.1898	0.1563	0.1282	0.1114	0.0806	0.0534
fBm	0.4031	0.2703	0.2350	0.2164	0.1954	0.1617	0.1347	0.1186	0.0863	0.0552
fO-U	**0.4075**	**0.2775**	**0.2453**	**0.2267**	**0.2045**	**0.1712**	**0.1434**	**0.1271**	**0.0942**	**0.0636**

Table 3: DM statistic with fO-U being the benchmark model.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
HAR	-2.3620	-2.1489	-2.7014	-2.1975	-2.3658	-2.8187	-3.3169	-3.0484	-3.3693	-3.2553
	(0.0090)	**(0.0158)**	**(0.0034)**	**(0.0139)**	**(0.0089)**	**(0.0024)**	**(0.0004)**	**(0.0011)**	**(0.0003)**	**(0.0005)**
ARFIMA	-1.5004	-0.8978	-0.8978	-0.8386	-1.0459	-1.3378	-1.5789	-1.4807	-1.2558	-1.1517
	(0.0667)	(0.1846)	**(0.0205)**	(0.2008)	(0.1477)	**(0.0904)**	**(0.0571)**	**(0.0693)**	(0.1045)	(0.1247)
fBm	-0.7414	-0.7632	-0.7304	-0.5851	-0.4704	-1.2335	-1.5027	-1.2299	-0.5123	-0.4911
	(0.2292)	(0.2226)	(0.2325)	(0.2792)	(0.3190)	(0.1086)	**(0.0664)**	(0.1093)	(0.3041)	(0.3116)
Panel B: DJIA										
HAR	-2.5843	-1.8006	-1.6805	-2.1154	-2.9111	-3.4040	-2.7031	-3.2468	-4.0239	-4.1864
	(0.0048)	**(0.0358)**	**(0.0464)**	**(0.0171)**	**(0.0018)**	**(0.0003)**	**(0.0034)**	**(0.0005)**	**(0.0000)**	**(0.0000)**
ARFIMA	-2.5564	-1.1599	-1.6006	-1.7432	-1.9960	-1.9563	-2.3542	-2.3561	-2.0169	-3.7860
	(0.0052)	**(0.1230)**	**(0.0547)**	**(0.0406)**	**(0.0229)**	**(0.0252)**	**(0.0092)**	**(0.0092)**	**(0.0218)**	**(0.0000)**
fBm	-1.2297	-0.0604	-0.0328	-0.1004	-0.2044	-0.3914	-1.0826	-1.4731	-0.5659	-0.4287
	(0.1093)	(0.4758)	(0.4869)	(0.4599)	(0.4190)	(0.3477)	(0.1394)	**(0.0703)**	(0.2857)	(0.3340)
Panel C: NASDAQ 100										
HAR	-3.2115	-2.6699	-2.2984	-2.0592	-2.9021	-2.9575	-2.9859	-2.5789	-3.2425	-3.5785
	(0.0006)	**(0.0037)**	**(0.0107)**	**(0.0197)**	**(0.0018)**	**(0.0015)**	**(0.0014)**	**(0.0049)**	**(0.0005)**	**(0.0001)**
ARFIMA	-1.7765	-2.3980	-2.1917	-1.8985	-1.3069	-1.3184	-1.4442	-1.6582	-1.9338	-2.5427
	(0.0378)	**(0.0082)**	**(0.0141)**	**(0.0288)**	**(0.0956)**	**(0.0936)**	**(0.0743)**	**(0.0486)**	**(0.0265)**	**(0.0054)**
fBm	-0.6895	-1.4044	-0.9425	-0.1768	-0.6105	-0.1399	-0.2375	-0.3298	-0.0115	-0.7770
	(0.2452)	**(0.0800)**	(0.1729)	(0.4298)	(0.2707)	(0.4443)	(0.4061)	(0.3707)	(0.4953)	(0.2185)

Table 4: p-values of MSC to compare forecasts of $\log\left(100\sqrt{RV \times 252}\right)$ from six competing models.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	0.0020	0.0000	0.0000	0.0020	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
AR(1)	0.0030	0.0070	0.0025	0.0075	0.0030	0.0080	0.0040	0.0030	0.0025	0.0060
HAR	0.0220	0.0070	0.0025	0.0075	0.0030	0.0080	0.0040	0.0030	0.0025	0.0060
ARFIMA	**0.5690**	0.0315	0.0670	0.0075	0.0480	0.0080	**0.1375**	0.0065	0.0035	0.0060
fBm	**0.9340**	**0.2735**	**0.2880**	**0.5545**	**0.7980**	**0.1060**	**0.1375**	**0.2085**	**0.3130**	**0.6315**
fO-U	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
Panel B: DJIA										
RW	0.0000	0.0015	0.0010	0.0035	0.0025	0.0010	0.0000	0.0025	0.0020	0.0010
AR(1)	0.0025	0.0060	0.0050	0.0220	0.0025	0.0055	0.0075	0.0025	0.0070	0.0095
HAR	0.0075	0.0060	0.0050	0.0285	0.0055	0.0055	0.0085	0.0025	0.0070	0.0095
ARFIMA	0.0540	0.0125	0.0330	0.0325	0.0825	0.0490	0.0085	0.0045	0.0070	0.0095
fBm	**0.1670**	**0.5125**	**0.3805**	**0.5535**	**0.3840**	**0.6985**	**0.1980**	**0.1355**	**0.3065**	**0.7580**
fO-U	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
Panel C: NASDAQ 100										
RW	0.0000	0.0000	0.0010	0.0120	0.0030	0.0000	0.0000	0.0015	0.0030	0.0025
AR(1)	0.0000	0.0010	0.0020	0.0290	0.0090	0.0095	0.0275	0.0100	0.0180	0.0400
HAR	0.0000	0.0025	0.0030	0.0290	0.0090	0.0095	0.0275	0.0100	0.0180	0.0400
ARFIMA	0.0080	0.0230	0.0490	0.0555	0.0535	0.0710	0.0875	0.0930	0.0670	0.0735
fBm	**0.1890**	**0.5475**	**0.2860**	**0.1045**	**0.0865**	**0.1695**	**0.1935**	**0.1845**	**0.2665**	**0.5840**
fO-U	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**

Table 5: CW test with fO-U being the encompassing model.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	6.9937	7.2265	6.7051	7.0243	7.2835	7.4529	6.8242	6.9995	7.1601	7.6170
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
AR(1)	5.5013	5.2824	6.1855	6.5217	7.0099	7.6487	7.9651	8.2363	8.3440	8.43498
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
fBm	1.9200	1.1638	1.3328	0.9373	0.6028	2.3994	2.4491	2.1519	1.7594	0.9636
	(0.0274)	(0.1222)	**(0.0912)**	(0.1742)	(0.2732)	**(0.0082)**	**(0.0071)**	**(0.0157)**	**(0.0392)**	(0.1676)
Panel B: DJIA										
RW	7.6266	7.8135	6.9433	7.4795	6.6086	7.1967	7.1549	6.3739	6.5538	7.1341
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
AR(1)	6.3652	5.7428	6.8662	7.7331	8.4303	8.8022	9.1822	9.3105	9.2045	9.0997
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
fBm	1.8060	0.8261	0.9939	0.7660	1.1969	0.6292	1.9968	2.3137	1.6426	0.6502
	(0.0354)	(0.2043)	(0.1601)	(0.2218)	(0.1156)	(0.2645)	(0.0229)	**(0.0103)**	**(0.0502)**	(0.2577)
Panel C: NASDAQ 100										
RW	7.1515	7.4273	7.2146	6.5235	6.9304	7.8905	6.7338	6.4316	6.9278	7.4242
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
AR(1)	5.6051	5.2020	5.564	5.7187	6.1668	6.8636	6.9236	7.3456	7.4123	7.5117
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
fBm	1.2183	0.8429	1.4551	2.0687	2.5445	2.28309	2.3552	2.3832	2.0732	1.3373
	(0.1115)	(0.1996)	**(0.0728)**	**(0.0192)**	**(0.0054)**	**(0.0112)**	**(0.0092)**	**(0.0085)**	**(0.0190)**	**(0.0905)**

2 Forecast Log BV for Equities

The proposed fO-U model is a Gaussian process that does not allow for jumps in prices. However, RV may contain a component that is due to jumps in prices. It is known that the BV series are robust against jumps in prices (Barndorff-Nielsen and Shephard, 2004). Motivated by this property of BV, we apply the fO-U model, the proposed estimation method, and the developed asymptotic theory to the daily BV for the S&P 500, DJIA, and NASDAQ 100. The three BV series are obtained from the Ox-Mann realized library and based on 5-minute returns. The sample period is from January 3, 2000 to December 31, 2019. We fit the fO-U model to $\log\left(100\sqrt{BV \times 252}\right)$ which is the logarithmic annualized BV.

Table 6 reports the point estimates and the 95% confidence intervals implied by the developed asymptotic distributions for all four parameters in fO-U. In all cases, the estimated H is much less than $1/2$ and very similar to that obtained from the three RV series. These results give strong empirical support to fO-U once again, suggesting that the evidence of $H < 1/2$ is not driven by jumps.

We then compare the forecasting performance of the fO-U, fBm, ARFIMA, HAR, AR(1), RW models. We split the sample period in the same way as before. Tables 7-8 report the ratio of RMSE of each candidate model and that of HAR and the MZ-R^2 of six competing models for h-day-ahead-forecast of $\log\left(100\sqrt{BV \times 252}\right)$ with $h = 1, \ldots, 10$ with the best result highlighted in boldface for each h. In all cases, fO-U continues to provide the most accurate forecasts with the smallest RMSEs and the higher R^2 values, regardless of the BV series and forecasting horizon.

Table 9 reports the DM test based on the squared forecast errors and the p-value (in parenthesis) with the benchmark being fO-U (boldface means statistically significant at

Table 6: Empirical results for $\log(100\sqrt{BV \times 252})$ of the S&P 500, DJIA, and NASDAQ 100.

Name	H	σ	μ	κ
S&P 500	.2185	1.0479	2.3121	2.7196
	(.1774, .2595)	(1.0401, 1.0557)	(2.2392, 2.3849)	(2.6446, 2.7945)
DJIA	.1847	.8704	2.3037	1.4397
	(.1431, .2262)	(.8638, .8769)	(2.2003, 2.4070)	(1.3832, 1.4960)
NASDAQ 100	.2190	1.0468	2.4787	2.7887
	(.1778, .2600)	(1.0390, 1.0545)	(2.4075, 2.5498)	(2.7127, 2.8645)

Table 7: The ratio of RMSE of different models and that of HAR for h-day-ahead-forecast of $\log\left(100\sqrt{BV \times 252}\right)$.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	0.9912	1.0919	1.0641	1.1170	1.0912	1.1108	1.0946	1.0903	1.0565	1.1009
AR(1)	0.9397	1.0053	0.9747	1.0154	1.0101	1.0325	1.0342	1.0384	1.0188	1.0391
HAR	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ARFIMA	0.8771	0.9374	0.9209	0.9474	0.9347	0.9451	0.9392	0.9382	0.9172	0.9343
fBm	0.8726	0.9310	0.9125	0.9353	0.9230	0.9316	0.9266	0.9197	0.8979	0.9099
fO-U	**0.8707**	**0.9267**	**0.9077**	**0.9318**	**0.9194**	**0.9283**	**0.9225**	**0.9182**	**0.8967**	**0.9017**
Panel B: DJIA										
RW	1.0314	1.0984	1.0733	1.1391	1.1000	1.0765	1.1241	1.0772	1.0666	1.0967
AR(1)	0.9745	1.0140	0.9952	1.0515	1.0419	1.0319	1.0735	1.0580	1.0456	1.0456
HAR	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ARFIMA	0.9009	0.9404	0.9264	0.9591	0.9445	0.9292	0.9508	0.9346	0.9229	0.9269
fBm	0.8985	0.9352	0.9218	0.9514	0.9391	0.9226	0.9407	0.9245	0.9104	0.9086
fO-U	**0.8961**	**0.9312**	**0.9165**	**0.9469**	**0.9332**	**0.9172**	**0.9368**	**0.9207**	**0.9077**	**0.9009**
Panel C: NASDAQ 100										
RW	0.9901	1.1322	1.1391	1.1557	1.0982	1.1155	1.1295	1.1209	1.0882	1.1276
AR(1)	0.9403	1.0437	1.0491	1.0815	1.0600	1.0809	1.1023	1.1193	1.1085	1.1316
HAR	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ARFIMA	0.8771	0.9534	0.9524	0.9645	0.9370	0.9424	0.9499	0.9497	0.9341	0.9501
fBm	0.8729	0.9412	0.9418	0.9563	0.9318	0.9337	0.9402	0.9408	0.9242	0.9360
fO-U	**0.8707**	**0.9396**	**0.9375**	**0.9501**	**0.9242**	**0.9275**	**0.9341**	**0.9345**	**0.9185**	**0.9322**

the 10% level). To save space, we only report the results for HAR, ARFIMA and fBm in Table 9. According to the DM test, fO-U is always statistically different from HAR at the 10% level, statistically different from ARFIMA at the 10% level in most cases and statistically different from fBm at the 10% level in one case.

Table 10 reports the p-value of the MCS. Values in boldface denote that the model belongs to the confidence set of the best models. AR(1), RW and HAR are consistently tossed out of these MCSs regardless of the BV series and forecasting horizon. ARFIMA is rejected in all but four cases. fBm is rejected in a few cases. Most importantly, in no case, fO-U can be rejected.

Table 11 reports the CW test and the p-value (in parenthesis) with the benchmark being fO-U (boldface means statistically significant at the 10% level). According to the CW test, the performance of fO-U is statistically significantly better than RW and AR(1) in all cases and better than fBm in all but four cases.

In conclusion, all the empirical results obtained for the RV series qualitatively remain unchanged for the BV series, suggesting that empirical results, including the significance

Table 8: MZ-R^2 for h-day-ahead-forecast of $\log\left(100\sqrt{BV \times 252}\right)$ in different models.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	0.4254	0.2863	0.2434	0.1643	0.1430	0.1024	0.0940	0.0808	0.0762	0.0419
AR(1)	0.4255	0.2865	0.2437	0.1648	0.1432	0.1028	0.0944	0.0813	0.0771	0.0432
HAR	0.3416	0.2880	0.2107	0.1772	0.1363	0.1161	0.0960	0.0792	0.0501	0.0474
ARFIMA	0.4671	0.3365	0.2696	0.1979	0.1643	0.1247	0.1036	0.0886	0.0738	0.0534
fBm	0.4679	0.3402	0.2744	0.2030	0.1701	0.1308	0.1117	0.0908	0.0759	0.0540
fO-U	**0.4716**	**0.3458**	**0.2814**	**0.2099**	**0.1765**	**0.1375**	**0.1174**	**0.0992**	**0.0840**	**0.0706**
Panel B: DJIA										
RW	0.3942	0.2866	0.2483	0.1783	0.1819	0.1554	0.1107	0.1123	0.0859	0.0428
AR(1)	0.3943	0.2869	0.2488	0.1792	0.1827	0.1563	0.1124	0.1141	0.0886	0.0456
HAR	0.3429	0.2905	0.2251	0.2143	0.1791	0.1271	0.1253	0.0892	0.0569	0.0375
ARFIMA	0.4430	0.3407	0.2831	0.2287	0.2040	0.1632	0.1378	0.1127	0.0860	0.0561
fBm	0.4445	0.3425	0.2864	0.2312	0.2091	0.1691	0.1401	0.1158	0.0879	0.0570
fO-U	**0.4474**	**0.3481**	**0.2930**	**0.2387**	**0.2158**	**0.1759**	**0.1483**	**0.1233**	**0.0955**	**0.0717**
Panel C: NASDAQ 100										
RW	0.3729	0.2041	0.1752	0.1533	0.1607	0.1242	0.0994	0.0950	0.0796	0.0428
AR(1)	0.3730	0.2043	0.1757	0.1540	0.1613	0.1249	0.1004	0.0964	0.0815	0.0448
HAR	0.2671	0.2270	0.2006	0.1959	0.1506	0.1265	0.1085	0.0952	0.0595	0.0524
ARFIMA	0.4061	0.2676	0.2361	0.2099	0.1876	0.1538	0.1238	0.1116	0.0865	0.0615
fBm	0.4065	0.2691	0.2362	0.2118	0.1925	0.1566	0.1271	0.1145	0.0888	0.0616
fO-U	**0.4109**	**0.2762**	**0.2463**	**0.2219**	**0.2010**	**0.1666**	**0.1365**	**0.1240**	**0.0981**	**0.0710**

of $H < 1/2$ and the superiority of fO-U over other competing models, are not driven by jumps.

Table 9: DM statistic with fO-U being the benchmark model.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
HAR	-3.4424	-2.4115	-3.3245	-2.5220	-2.9072	-2.4221	-3.6272	-3.0321	-4.0020	-5.4340
	(0.0002)	**(0.0079)**	**(0.0004)**	**(0.0058)**	**(0.0018)**	**(0.0077)**	**(0.0001)**	**(0.0012)**	**(0.0000)**	**(0.0000)**
ARFIMA	-2.1306	-2.2981	-2.1441	-2.3773	-2.4214	-2.3551	-3.2702	-2.6404	-2.9038	-4.2023
	(0.0165)	**(0.0107)**	**(0.0160)**	**(0.0087)**	**(0.0077)**	**(0.0092)**	**(0.0005)**	**(0.0041)**	**(0.0018)**	**(0.0000)**
fBm	-0.7422	-1.1117	-0.8773	-0.7843	-0.2584	-0.3828	-0.0561	-0.5652	-0.9217	-0.4295
	(0.2289)	(0.1331)	(0.1901)	(0.2164)	(0.3980)	(0.3509)	(0.4776)	(0.2859)	(0.1783)	(0.3337)
Panel B: DJIA										
HAR	-3.1722	-2.4639	-3.3385	-3.4332	-3.3258	-3.4806	-2.6958	-3.2280	-3.8733	-4.2085
	(0.0007)	**(0.0068)**	**(0.0004)**	**(0.0002)**	**(0.0004)**	**(0.0002)**	**(0.0035)**	**(0.0006)**	**(0.0001)**	**(0.0000)**
ARFIMA	-2.7186	-2.0128	-2.8765	-2.3206	-2.8048	-3.3786	-2.0833	-1.1477	-2.4224	-0.9813
	(0.0032)	**(0.0220)**	**(0.0020)**	**(0.0101)**	**(0.0025)**	**(0.0003)**	**(0.0186)**	(0.1255)	**(0.0077)**	(0.1632)
fBm	-0.2302	-0.4819	-0.0145	-0.0455	-0.2569	-0.1804	-0.0208	-0.0137	-0.4349	-0.7442
	(0.4089)	(0.3149)	(0.4941)	(0.4818)	(0.3986)	(0.4283)	(0.4916)	(0.4945)	(0.3317)	(0.2283)
Panel C: NASDAQ 100										
HAR	-3.2291	-2.3267	-2.0896	-2.2264	-1.9468	-2.1591	-1.7822	-2.1452	-2.2070	-2.6798
	(0.0006)	**(0.0099)**	**(0.0183)**	**(0.0129)**	**(0.0257)**	**(0.0154)**	**(0.0373)**	**(0.0159)**	**(0.0136)**	**(0.0036)**
ARFIMA	-1.8042	-1.9497	-1.8453	-1.5773	-1.3537	-1.4726	-1.6720	-2.0740	-2.1733	-2.1880
	(0.0355)	**(0.0256)**	**(0.0324)**	**(0.0573)**	**(0.0879)**	**(0.0704)**	**(0.0472)**	**(0.0190)**	**(0.0148)**	**(0.0143)**
fBm	-0.6977	-1.4900	-0.9611	-0.4087	0.2618	-0.4091	-0.3155	-0.0921	-0.1147	-0.7080
	(0.2426)	**(0.0681)**	(0.3413)	(0.1682)	(0.6032)	(0.3412)	(0.3761)	(0.4632)	(0.4543)	(0.2394)

Table 10: p-values of MSC to compare forecasts of $\log\left(100\sqrt{BV \times 252}\right)$ from 6 competing models.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	0.0000	0.0000	0.0000	0.0010	0.0005	0.0000	0.0000	0.0010	0.0005	0.0010
AR(1)	0.0000	0.0005	0.0010	0.0085	0.0065	0.0175	0.0325	0.0105	0.0125	0.0060
HAR	0.0000	0.0020	0.0435	0.0450	0.0535	0.0175	0.0325	0.0790	0.0625	0.0060
ARFIMA	0.0070	0.0170	0.0595	0.0565	0.0535	0.0635	0.0550	0.0790	0.0625	0.0060
fBm	**0.1475**	0.0945	**0.1930**	**0.4040**	**0.4045**	**0.5140**	**0.4630**	**0.8000**	**0.8095**	**0.1740**
fO-U	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
Panel B: DJIA										
RW	0.0000	0.0000	0.0005	0.0035	0.0040	0.0000	0.0010	0.0050	0.0060	0.0025
AR(1)	0.0005	0.0055	0.0035	0.0365	0.0185	0.0140	0.0540	0.0250	0.0265	0.0125
HAR	0.0005	0.0055	0.0270	0.0365	0.0185	0.0865	0.0540	0.0250	0.0265	0.0125
ARFIMA	0.0120	0.0200	0.0610	0.0815	0.0775	0.0865	0.0845	0.1130	0.0820	0.0125
fBm	0.0890	0.0965	0.0905	**0.2495**	**0.1600**	**0.2205**	**0.4405**	**0.4895**	**0.6295**	**0.1660**
fO-U	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
Panel C: NASDAQ 100										
RW	0.0000	0.0000	0.0005	0.0040	0.0025	0.0000	0.0025	0.0025	0.0010	0.0035
AR(1)	0.0000	0.0010	0.0010	0.0210	0.0135	0.0095	0.0280	0.0225	0.0215	0.0415
HAR	0.0000	0.0070	0.0010	0.0225	0.0135	0.0095	0.0280	0.0225	0.0215	0.0415
ARFIMA	0.0080	0.0205	0.0635	0.0585	0.0810	0.0835	0.0930	**0.1010**	0.0905	0.0855
fBm	**0.1305**	**0.5230**	**0.2495**	**0.1395**	**0.1225**	**0.2535**	**0.3050**	**0.3220**	**0.3650**	**0.5885**
fO-U	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**

Table 11: CW test with fO-U being the benchmark model.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	6.7718	7.6366	6.6570	7.0669	7.1830	8.1270	6.9916	7.1664	7.2294	7.1787
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
AR(1)	5.2867	5.5222	5.2366	5.3667	5.5703	5.9586	6.1059	6.4797	6.6318	6.9218
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
fBm	1.2907	1.7097	1.7817	1.4728	1.6031	1.3819	1.5634	0.8931	0.7989	2.4765
	(0.0983)	**(0.0436)**	**(0.0373)**	**(0.0703)**	**(0.0544)**	**(0.0834)**	**(0.0589)**	(0.1858)	(0.2121)	**(0.0066)**
Panel B: DJIA										
RW	7.0302	7.9005	7.0723	7.4112	7.0138	8.0551	7.0859	7.0081	7.0809	7.1271
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
AR(1)	5.6480	5.8583	5.5844	5.9178	6.3138	6.4097	6.9182	6.8691	7.0586	7.0434
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
fBm	1.8193	1.7776	2.0416	1.8157	2.2083	2.0266	1.5404	1.5127	1.1838	2.6542
	(0.0344)	**(0.0377)**	**(0.0205)**	**(0.0347)**	**(0.0136)**	**(0.0213)**	**(0.0617)**	**(0.0651)**	(0.1182)	**(0.0039)**
Panel C: NASDAQ 100										
RW	6.7758	7.6341	7.1164	6.8321	7.3451	8.0035	7.3128	7.7108	7.8725	7.9120
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
AR(1)	5.2776	5.4941	5.8622	6.1028	6.6423	7.2167	6.4906	6.7656	7.3082	7.6132
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
fBm	1.3592	0.8519	1.4716	1.9600	2.4524	2.0251	1.9822	2.0672	1.9028	1.3747
	(0.0870)	(0.1971)	**(0.0705)**	**(0.0249)**	**(0.0070)**	**(0.0214)**	**(0.0237)**	**(0.0193)**	**(0.0285)**	**(0.0846)**

3 Forecasting Log RK for Equities

We have used the daily RV or daily BV constructed from 5-minute returns of equities. The use of 5-minute returns, not of returns at much shorter intervals, to calculate RV/BV is to deal with the effect of microstructure noise because at much higher frequencies, say second-by-second, returns are subject to microstructure noise. Unfortunately, this sparse sampling technique throws away a lot of data. To make use of more information, one can use the daily realized kernel (RK) of Barndorff-Nielsen et al. (2008). Compared with RV, RK should incur much smaller estimation errors as many more observations are used. Motivated by this property of RK, we apply the fO-U model and the proposed method to the daily RK for the S&P 500, DJIA, and NASDAQ 100. The three RK series are obtained from the Ox-Mann realized library and based on 5-minute returns. The sample period is from January 3, 2000 to December 31, 2019. We fit the fO-U model to $\log\left(100\sqrt{RK \times 252}\right)$ which is the logarithmic annualized RK.

The point estimates and the 95% asymptotic confidence intervals for all four parameters in fO-U are reported in Table 12. In all cases, the estimated H is much less than 1/2 and very similar to that obtained from the three RV series and the three BV series. These results give strong empirical support to fO-U once again, suggesting that the evidence of $H < 1/2$ is not driven RV.

Table 12: Empirical results for $\log(100\sqrt{RK \times 252})$ of the S&P 500, DJIA, and NASDAQ 100.

Name	H	σ	μ	κ
S&P 500	.1973	.8417	2.3657	1.0817
	(.1559, .2386)	(.8353, .8479)	(2.2276, 2.5037)	(1.0334, 1.1299)
DJIA	.1550	.6402	2.3483	.3624
	(.1130, .1969)	(.6353, .6450)	(2.0721, 2.6244)	(.3331, .3916)
NASDAQ 100	.2501	1.0576	2.3972	2.0598
	(.2094, .2907)	(1.0498, 1.0653)	(2.2904, 2.5038)	(1.9964, 2.1232)

Tables 13-14 report the ratio of RMSE of each candidate model and that of HAR and the MZ-R^2 of six competing models for h-day-ahead-forecast of $\log\left(100\sqrt{RK \times 252}\right)$ with $h = 1, \ldots, 10$ with the best result highlighted in boldface for each h. In all cases, fBm continues to provide the most accurate forecasts with the smallest RMSEs and the higher R^2 values, regardless of RK series and forecasting horizon.

Table 15 reports the DM statistic and the p-value with the benchmark being fO-U.

Table 13: The ratio of RMSE of different models and that of HAR for h-day-ahead-forecast of $\log\left(100\sqrt{RK \times 252}\right)$.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	0.9576	1.0521	1.0700	1.0607	1.0746	1.0639	1.0599	1.0507	1.0277	1.0670
AR(1)	0.9194	0.9906	1.0076	1.0123	1.0384	1.0468	1.0540	1.0569	1.0434	1.0634
HAR	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ARFIMA	0.8638	0.9217	0.9350	0.9328	0.9415	0.9387	0.9369	0.9336	0.9177	0.9309
fBm	0.8617	0.9201	0.9329	0.9323	0.9383	0.9330	0.9325	0.9254	0.9062	0.9154
fO-U	**0.8591**	**0.9151**	**0.9269**	**0.9248**	**0.9320**	**0.9276**	**0.9266**	**0.9212**	**0.9034**	**0.9143**
Panel B: DJIA										
RW	1.0153	1.0418	1.0549	1.0910	1.0818	1.0411	1.1123	1.0371	1.0400	1.0904
AR(1)	0.9660	0.9746	0.9870	1.0200	1.0263	1.0059	1.0616	1.0055	1.0204	1.0348
HAR	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ARFIMA	0.8981	0.9206	0.9308	0.9499	0.9502	0.9278	0.9623	0.9356	0.9257	0.9405
fBm	0.8974	0.9203	0.9293	0.9476	0.9474	0.9235	0.9563	0.9291	0.9155	0.9253
fO-U	**0.8953**	**0.9161**	**0.9242**	**0.9423**	**0.9419**	**0.9189**	**0.9525**	**0.9257**	**0.9140**	**0.9199**
Panel C: NASDAQ 100										
RW	0.9652	1.0822	1.0873	1.1148	1.1016	1.0630	1.0861	1.0793	1.0362	1.0835
AR(1)	0.9340	1.0287	1.0340	1.0715	1.0800	1.0669	1.0963	1.1138	1.0958	1.1297
HAR	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ARFIMA	0.8653	0.9344	0.9349	0.9525	0.9522	0.9344	0.9458	0.9486	0.9269	0.9466
fBm	0.8647	0.9298	0.9304	0.9500	0.9516	0.9341	0.9422	0.9443	0.9215	0.9333
fO-U	**0.8600**	**0.9233**	**0.9217**	**0.9393**	**0.9395**	**0.9214**	**0.9309**	**0.9336**	**0.9118**	**0.9275**

According to the DM test, fO-U is almost always statistically different from HAR at the 10% level. It is statistically different from ARFIMA (or fBm) in thirteen (or one) cases.

Table 14: MZ-R^2 for h-day-ahead-forecast of $\log\left(100\sqrt{RK \times 252}\right)$ in different models.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	0.5127	0.3685	0.2994	0.2522	0.1988	0.1702	0.1418	0.1206	0.1037	0.0621
AR(1)	0.5129	0.3687	0.2997	0.2526	0.1991	0.1706	0.1421	0.1212	0.1044	0.0630
HAR	0.4191	0.3481	0.2902	0.2329	0.1954	0.1599	0.1283	0.1007	0.0668	0.0583
ARFIMA	0.5431	0.4071	0.3259	0.2602	0.2081	0.1707	0.1302	0.1033	0.0815	0.0576
fBm	0.5437	0.4108	0.3322	0.2715	0.2176	0.1782	0.1422	0.1217	0.1172	0.0637
fO-U	**0.5471**	**0.4146**	**0.3365**	**0.2742**	**0.2213**	**0.1832**	**0.1445**	**0.1252**	**0.1199**	**0.0658**
Panel B: DJIA										
RW	0.4653	0.3719	0.3000	0.2382	0.2182	0.1966	0.1315	0.1189	0.0831	0.0527
AR(1)	0.4654	0.3720	0.3001	0.2386	0.2183	0.1966	0.1321	0.1195	0.0842	0.0540
HAR	0.4150	0.3357	0.2740	0.2435	0.2112	0.1490	0.1512	0.1201	0.0865	0.0533
ARFIMA	0.5091	0.4025	0.3226	0.2629	0.2261	0.1973	0.1546	0.1234	0.0871	0.0565
fBm	0.5098	0.4061	0.3268	0.2688	0.2327	0.1989	0.1608	0.1297	0.0917	0.0584
fO-U	**0.5119**	**0.4091**	**0.3311**	**0.2730**	**0.2370**	**0.1997**	**0.1650**	**0.1337**	**0.0961**	**0.0687**
Panel C: NASDAQ 100										
RW	0.4640	0.3083	0.2618	0.2153	0.2068	0.2034	0.1582	0.1413	0.0745	0.0699
AR(1)	0.4641	0.3084	0.2619	0.2157	0.2069	0.2034	0.1582	0.1415	0.0747	0.0702
HAR	0.3631	0.3081	0.2626	0.2464	0.2247	0.2043	0.1536	0.1307	0.0806	0.0677
ARFIMA	0.4964	0.3610	0.3110	0.2663	0.2386	0.2120	0.1689	0.1421	0.1080	0.0744
fBm	0.4982	0.3647	0.3148	0.2726	0.2473	0.2215	0.1773	0.1499	0.1166	0.0773
fO-U	**0.5019**	**0.3713**	**0.3242**	**0.2819**	**0.2560**	**0.2311**	**0.1870**	**0.1594**	**0.1249**	**0.0867**

Table 15: DM statistic with fO-U being the benchmark model.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
HAR	-2.8007	-1.7858	-2.6354	-2.7049	-2.5671	-2.8321	-3.0921	-3.1460	-2.5391	-3.8731
	(0.0025)	**(0.0370)**	**(0.0042)**	**(0.0042)**	**(0.0051)**	**(0.0023)**	**(0.0009)**	**(0.0008)**	**(0.0005)**	**(0.0001)**
ARFIMA	-1.4157	-0.9647	-0.6994	-0.3975	-0.8362	-0.3829	-0.6940	-0.7058	-0.7257	-1.2948
	(0.0784)	(0.1673)	(0.2421)	(0.3454)	(0.2015)	(0.3508)	(0.24381)	(0.2401)	(0.2339)	**(0.0977)**
fBm	-0.3575	-0.1566	-0.0060	-0.7474	-0.5721	-0.7018	-0.6831	-0.0968	-0.6926	-0.6333
	(0.3603)	(0.4377)	(0.4975)	(0.2273)	(0.2836)	(0.2414)	(0.2472)	(0.4614)	(0.2442)	(0.2632)
Panel B: DJIA										
HAR	-3.0100	-2.5222	-2.5801	-2.8720	-2.5571	-2.5571	-2.3944	-3.0176	-2.2271	-3.8490
	(0.0013)	**(0.0058)**	**(0.0049)**	**(0.0020)**	**(0.0052)**	**(0.0052)**	**(0.0083)**	**(0.0012)**	**(0.0129)**	**(0.0001)**
ARFIMA	-2.5350	-1.2496	-1.1366	-1.0520	-1.1469	-1.1469	-1.1146	-0.9818	-0.8378	-0.9494
	(0.0056)	(0.1057)	(0.1278)	(0.1465)	(0.1257)	(0.1257)	(0.1325)	(0.1631)	(0.2010)	(0.1711)
fBm	-0.9570	-0.8174	-0.9558	-1.0330	-0.9394	-0.9394	-0.6002	-0.6292	-0.1690	-0.7389
	(0.1692)	(0.2068)	(0.1695)	(0.1508)	(0.1737)	(0.1737)	(0.2741)	(0.2646)	(0.4329)	(0.2299)
Panel C: NASDAQ 100										
HAR	-3.6812	-2.5282	-2.7085	-2.4055	-2.753	-3.0917	-3.1700	-2.8422	-4.0364	-3.6021
	(0.0001)	**(0.0057)**	**(0.0033)**	**(0.0080)**	**(0.0029)**	**(0.0009)**	**(0.0007)**	**(0.0022)**	**(0.0000)**	**(0.0001)**
ARFIMA	-2.6184	-1.9246	-1.7349	-2.1378	-1.7080	-1.6673	-1.7630	-2.0019	-1.7886	-2.5792
	(0.0044)	**(0.0271)**	**(0.0413)**	**(0.0162)**	**(0.0438)**	**(0.0477)**	**(0.0389)**	**(0.0226)**	**(0.0368)**	**(0.0049)**
fBm	-0.3777	-0.4521	-0.0538	-0.7936	-1.3630	-0.9573	-0.8430	-0.9171	-0.6015	-0.5492
	(0.3528)	(0.3255)	(0.4785)	(0.2137)	**(0.0865)**	(0.1692)	(0.1996)	(0.1795)	(0.2737)	(0.2914)

Table 16 reports the p-value of the MCS of Hansen et al. (2011). Values in boldface

denote that the model belongs to the confidence set of the best performing models. RW and AR(1) are consistently tossed out of these MCSs regardless of the RV series and forecasting horizon. HAR, ARFIMA, fBm is rejected in twenty-six, eighteen, thirteen cases, respective. Most importantly, in no case, fO-U is rejected.

Table 16: p-values of MSC to compare forecasts of $\log\left(100\sqrt{RK \times 252}\right)$ from 6 competing models.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	0.0040	0.0000	0.0015	0.0100	0.0050	0.0015	0.0040	0.0065	0.0045	0.0055
AR(1)	0.0055	0.0055	0.0055	0.0145	0.0225	0.0145	0.0360	0.0155	0.0230	0.0305
HAR	0.0215	0.0100	0.0300	0.0750	0.0225	0.0145	0.0360	0.0155	0.0230	0.0305
ARFIMA	0.0675	0.0385	0.0965	0.0990	**0.1020**	0.0820	**0.1480**	**0.1140**	0.0990	0.0805
fBm	0.0705	0.0385	0.0965	0.0990	**0.1020**	**0.1815**	**0.2010**	**0.3705**	**0.5920**	**0.8360**
fO-U	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
Panel B: DJIA										
RW	0.0025	0.0005	0.0005	0.0110	0.0060	0.0030	0.0040	0.0125	0.0185	0.0040
AR(1)	0.0050	0.0110	0.0030	0.0565	0.0525	0.0190	**0.1055**	0.0355	0.0600	0.0480
HAR	0.0060	0.0220	0.0395	0.0565	0.0525	**0.1325**	**0.1055**	**0.1425**	**0.1385**	0.0480
ARFIMA	0.0475	0.0490	0.0450	0.0565	0.0975	**0.1325**	**0.1280**	**0.1425**	**0.1385**	0.0480
fBm	0.0475	0.0490	0.0450	0.0635	**0.1140**	**0.2010**	**0.3640**	**0.3845**	**0.6835**	**0.2170**
fO-U	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
Panel C: NASDAQ 100										
RW	0.0000	0.0000	0.0000	0.0085	0.0025	0.0005	0.0010	0.0070	0.0045	0.0085
AR(1)	0.0000	0.0010	0.0000	0.0145	0.0155	0.0060	0.0275	0.0245	0.0300	0.0460
HAR	0.0005	0.0050	0.0010	0.0145	0.0265	0.0060	0.0300	0.0255	0.0300	0.0640
ARFIMA	0.0170	0.0220	0.0480	0.0785	0.0820	**0.1115**	**0.1120**	**0.1475**	**0.1705**	**0.1630**
fBm	0.0170	0.0355	0.0480	0.0785	0.0820	**0.1115**	**0.1120**	**0.1495**	**0.2120**	**0.4950**
fO-U	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**

Table 17 reports the CW test and the p-value (in parenthesis) with the benchmark being fO-U (boldface means statistically significant at the 10% level). According to the CW test, the performance of fO-U is statistically significantly better than RW and AR(1) in all cases and better than fBm in all but one cases.

Table 17: CW test with fO-U being the benchmark model.

h	1	2	3	4	5	6	7	8	9	10
Panel A: S&P 500										
RW	7.3512	7.7297	6.9357	6.6764	7.2390	7.4588	6.9354	7.4066	7.2426	7.5789
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
AR(1)	5.5211	5.2217	5.3018	5.3061	5.9083	6.3375	6.6065	7.0040	7.2410	7.4959
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
fBm	2.9036	2.7206	2.7555	2.7177	3.2774	3.5974	3.4979	3.9181	4.2888	4.4688
	(0.0018)	**(0.0032)**	**(0.0029)**	**(0.0032)**	**(0.0005)**	**(0.0001)**	**(0.0002)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
Panel B: DJIA										
RW	7.5719	7.6392	6.7952	7.1504	6.9643	7.2609	7.1015	6.7910	6.7112	7.5932
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
AR(1)	5.8816	4.6827	4.9033	5.1514	5.4539	5.7568	6.0860	6.0828	6.0341	6.1490
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
fBm	2.6690	2.4730	2.8007	2.9660	3.1318	3.3041	3.8002	1.6572	1.0080	2.4264
	(0.0038)	**(0.0066)**	**(0.0025)**	**(0.0015)**	**(0.0008)**	**(0.0004)**	**(0.0000)**	**(0.0487)**	(0.1567)	**(0.0076)**
Panel C: NASDAQ 100										
RW	7.5751	7.8562	7.3093	6.9636	7.2085	7.4081	6.8766	6.9174	6.8780	7.2646
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
AR(1)	6.0832	5.7414	5.9513	6.0283	6.3099	7.0914	6.1772	7.8194	8.1051	8.5498
	(0.0000)	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**	**(0.0000)**
fBm	2.5740	2.2633	2.4934	2.8649	3.3013	3.2343	3.0717	2.9049	2.6423	1.7923
	(0.0050)	**(0.0118)**	**(0.0063)**	**(0.0021)**	**(0.0048)**	**(0.0006)**	**(0.0011)**	**(0.0018)**	**(0.0041)**	**(0.0365)**

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