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Bubble testing under polynomial trends[1]

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Summary: This paper develops the asymptotic theory of the least squares estimator of the autoregressive (AR) coefficient in an AR(1) regression with intercept when data is generated from a polynomial trend model in different forms. It is shown that the commonly used right-tailed unit root tests tend to favour the explosive alternative. A new procedure, which implements the right-tailed unit root tests in an AR(2) regression, is proposed. It is shown that, when the data generating process has a polynomial trend, the test statistics based on the new procedure cannot find evidence of explosiveness. Whereas, when the data generating process is mildly explosive, the new procedure finds evidence of explosiveness. Hence, it enables robust bubble testing under polynomial trends. Empirical application of the proposed procedure using data from the US real estate market reveals some interesting findings. In particular, all the negative bubble episodes flagged by the traditional method are no longer regarded as bubbles by the new procedure.

Keywords: *Autoregressive regressions*, *right-tailed unit root test*, *mildly explosive processes*, *polynomial trends*, *coefficient-based statistic*, *t statistic*.

JEL codes: *C12*, *C22*, *C52*.

1. INTRODUCTION

Often, a financial bubble immediately precedes a financial crisis. It is widely acknowledged that a financial crisis can have catastrophic effects on financial markets and severe negative impacts on the real economy. Not surprisingly, many econometric methods have been developed to test for the presence of bubbles in financial markets, and to timestamp the origination date and the termination date of each bubble in real time. Well-known methods include the recursive right-tailed unit root testing procedures proposed in Phillips et al. (2011, PWY hereafter), Phillips and Yu (2009, 2011), and Phillips et al. (2015a, 2015b, PSY hereafter). Harvey et al. (2016) and Harvey et al. (2019, 2020) extended the procedures to deal with the case with heteroscedasticity. Excellent surveys and comparisons of alternative methods in bubble testing and time-stamping can be found in Homm and Breitung (2012) and Phillips and Shi (2021).

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The testing and date-stamping methods for bubbles rely on a technique of fitting to time series data (i.e., prices adjusted by fundamentals, denoted by p_t); the following AR(1) model, where the autoregressive root (denoted by β) takes a value greater than unity (i.e. an explosive model) during a bubble period, but take a unit value (i.e. a unit root model) during a normal period:

$$p_t = \alpha + \beta p_{t-1} + \varepsilon_t. \tag{1.1}$$

To test $H_0 : \beta = 1$ against $H_1 : \beta > 1$, both the Dickey−Fuller (DF) t test and the DF coefficient-based test (Dickey and Fuller, 1979), based on the least squares (LS) estimate of β, have been used.[1] When these tests are implemented recursively, they can detect when a time series switches from a unit root model to an explosive model followed by a crash, and vice versa, as shown in Phillips and Yu (2009) and PWY (2011) in the context of single bubble, and PSY (2015a, 2015b) in the context of multiple bubbles.

The trajectory generated by a unit root model is different from that generated by an explosive AR model. While the sample path of a unit root process may display a linear-trend-type behaviour, the sample path of an explosive AR process evolves as an explosive trend, and hence exhibits a nonlinear curvature. That is the reason why right-tailed unit root tests can distinguish a unit root process from the explosive AR process.

In a recent study, Phillips and Shi (2019) show how the DF t test can spuriously lead to the identification of a bubble when the data is generated from a unit root model followed by a linear trend model. Intuitively, the sample path from the structural break model has a curvature similar to that from an explosive AR model. Consequently, the DF t test has difficulties in distinguishing between the two models.

The trending behaviour in stock prices, at least in the short to medium run, is not new to the literature as there is a large volume of studies on momentum. See, for example, Hong and Stein (1999) and Daniel et al. (1998) for alternative theories that can generate momentum, and Jegadeesh and Titman (1993) for empirical evidence for the profitability of momentum strategies. However, momentum may correspond to a nonlinear trend in prices, as shown in Hong and Stein (1999). Moreover, existence of momentum does not necessarily require a structural break.

In this paper, we provide an alternative explanation for the false identification of bubble episodes, even in the absence of structural breaks. We show that if the data generating process (DGP) has a polynomial trend, the right-tailed unit root tests (DF t and coefficient-based tests) tend to reject the unit root null hypothesis in favour of the explosive alternative. The intuition why the right-tailed unit root tests have difficulties distinguishing between the polynomial trend model and the explosive AR model is that the sample path from these two models have a similar nonlinear curvature.

We then propose a robust testing approach that can successfully distinguish the explosive AR model from the polynomial trend model. The proposal is to apply the right-tailed unit root testing procedure based on the following AR(2) model

$$p_t = \alpha + \beta p_{t-1} + \psi \Delta p_{t-1} + \varepsilon_t. \tag{1.2}$$

It is shown that, when the DGP is a polynomial trend model or a model that switches from a unit root to a polynomial trend, the estimated β in (1.2) converges to unity at a rate faster than $O_p(1/n)$, as the sample size n goes to infinity. As a result, both the DF t and DF coefficient-based tests tend not to reject the unit root null hypothesis. Moreover, when the data is generated from

[1] Some recent studies have proposed methods based on the weighted LS estimate to deal with heteroscedasticity; see, for example, Harvey et al. (2019).

a mildly explosive AR process, both test statistics diverge to positive infinity as n increases. Thus, the two unit root tests tend to reject the unit root null hypothesis in favour of the explosive alternative.

In the empirical study, we apply the robust testing procedure based on (1.2) to the price–rent ratio of the US housing market. Two new empirical results emerge. First, the robust procedure detects only two bubble episodes. Compared to the testing results based on Model (1.1), the termination dates of both bubbles estimated by our robust method are much earlier and match better with the turning points in the data. Second, the robust testing procedure no longer flags the two collapsing periods as bubble episodes. This result suggests that the data in the two collapsing periods could be better fitted by the polynomial trend model.

The rest of the paper is organized as follows. Section 2 proves that the existing right-tailed unit root tests based on the AR(1) regression cannot distinguish the explosive AR model from the polynomial trend model. Section 3 advocates the usage of the AR(2) regression given by (1.2) and proves that the new testing approach can successfully distinguish the mildly explosive AR model from the polynomial trend model. Section 4 presents simulation evidence to support the theoretical results. The proposed procedure is used to analyse a real time series in Section 5. Section 6 concludes. The proofs of the main theoretical results are given in the Appendix. The Online Supplement contains the proof of Theorem 3.1.3, the extension of some theories to the general AR(k) model with $k > 2$, and additional simulation results.

2. RIGHT-TAILED UNIT ROOT TESTS UNDER POLYNOMIAL TRENDS

In this section, we study the asymptotic properties of the existing right-tailed unit root tests based on the AR(1) regression (1.1) when the data is actually generated from the following structural break model:

$$p_t = \begin{cases} p_{t-1} + \varepsilon_t & \text{if } t \in N \equiv [1, n_c] \\ p_{n_c} + \delta\,(t - n_c)^m + \varepsilon_t & \text{if } t \in T \equiv (n_c, n]\,, \end{cases} \tag{2.1}$$

where $\delta \neq 0$, $m \geq 2$, $\{\varepsilon_t\}$ an independent and identically distributed (iid) sequence with mean zero and finite variance (denoted σ^2), and $E\left(\varepsilon_t^4\right) < \infty$.[2] The first regime (called regime N) is the normal period where p_t evolves as a pure random walk. The second regime (called regime T) is the abnormal period where p_t evolves as a polynomial trend, as we assume $m \geq 2$. Let $\tilde{\beta}_{r_1}^{r_2}$ denote the LS estimate of β in the AR(1) regression (1.1) based on the sample from n_1 to n_2 where $n_i = \lfloor nr_i \rfloor$ represents the integer part of nr_i with $r_i \in [0, 1]$ for $i = 1, 2$, $se\left(\tilde{\beta}_{r_1}^{r_2}\right)$ be the standard error of $\tilde{\beta}_{r_1}^{r_2}$, and $t\left(\tilde{\beta}_{r_1}^{r_2}\right)$ be the corresponding t statistic.

The empirical usefulness of the right-tailed unit root tests has been made clear in PSY (2015a) for testing for the presence of bubbles and date-stamping each bubble when there are multiple bubbles. The PSY procedure relies on repeated calculations of the t statistic in autoregression in a recursive manner, where the ending point r_2 (fraction) of each sample takes a value between r_0 to 1, and the starting point r_1 (fraction) of the sample takes a value between 0 to $r_2 - r_0$ with r_0 (fraction) being the smallest sample window. Thus, $\lfloor nr_0 \rfloor$ is the minimum window size in the calculations. PSY (2015a) proposed the *GSADF* t statistic to be the largest t statistic in the double

[2] As shown in an earlier version of the present paper (Wang and Yu, 2017), many results reported in this paper continue to hold when ε_t is replaced with u_t where $u_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$ with $c_0 = 1$ and $\sum_{j=0}^{\infty} j|c_j| < \infty$.

Table 1. 99% CVs of DF coefficient-based, DF t, *GSADFc*, and *GSADFt* statistics based on Model (1.1) or Model (1.2).

n	20		50		40		100	
r_0	–	–	–	–	0.3		0.2	
Regression	AR(1)	AR(2)	AR(1)	AR(2)	AR(1)	AR(2)	AR(1)	AR(2)
DF c-stat	1.3727	2.2381	1.2320	1.4401	–	–	–	–
DF t-stat	0.8004	0.9153	0.7083	0.7641	–	–	–	–
GSADFc	–	–	–	–	5.8972	10.154	5.5287	8.5815
GSADFt	–	–	–	–	2.8485	2.9212	2.5077	2.7497

Notes: Calculations are based on 10,000 replications for the DF coefficient-based and t tests, and 2,000 for the *GSADFc* and *GSADFt* tests. For the DF coefficient-based and DF t statistics, we set $n = 20, 50$. For the *GSADFc* and *GSADFt* statistics, we set $n = 40$, $r_0 = 0.3$ and $n = 100$, $r_0 = 0.2$.

recursion over all possible combinations of r_1 and r_2, namely

$$GSADFt = \sup_{r_2 \in [r_0,1], r_1 \in [0, r_2 - r_0]} \left\{ t\left(\tilde{\beta}_{r_1}^{r_2}\right) \right\}. \tag{2.2}$$

The asymptotic distribution of *GSADFt* was also given in PSY (2015a) when the null hypothesis is a unit root process, from which the asymptotic right-tailed critical values (CVs) can be obtained.

After a bubble is detected, one can timestamp it. For example, the origination date and the termination date of the first bubble can be estimated by

$$\hat{r}_e = \inf\nolimits_{r_2 \in [r_0,1]} \left\{ r_2 : BSADFt\left(r_2\right) > cv \right\}, \tag{2.3}$$

$$\hat{r}_f = \inf\nolimits_{r_2 \in [\hat{r}_e,1]} \left\{ r_2 : BSADFt\left(r_2\right) < cv \right\}, \tag{2.4}$$

where

$$BSADFt\left(r_2\right) = \sup_{r_1 \in [0, r_2 - r_0]} \left\{ t\left(\tilde{\beta}_{r_1}^{r_2}\right) \right\}, \tag{2.5}$$

and cv is the critical value of the sup t statistic. PWY (2011) developed the asymptotic distribution of the sup t statistic as $\sup_{r \in [r_0,1]} \int_0^r \widetilde{W}_r dW / \left(\int_0^r \widetilde{W}_r^2\right)^{1/2}$, where $W(\cdot)$ denotes a standard Brownian motion and $\widetilde{W}_r(s) = W(s) - \frac{1}{r}\int_0^r W(t)\,dt$ is the demeaned Brownian motion. The 90%, 95%, 99% asymptotic and finite sample CVs of the sup t statistic were reported in Table 1 of PSY.

The t statistics in (2.2) and (2.5) can be replaced with the coefficient-based statistics, $\left(n_2 - n_1\right)\left(\tilde{\beta}_{r_1}^{r_2} - 1\right)$. The corresponding double supremum and backward supremum of the coefficient-based statistics are denoted as $GSADFc$ and $BSADFc\left(r_2\right)$, respectively. Phillips and Yu (2011) obtained the asymptotic distribution of the sup coefficient-based statistic as $\sup_{r \in [r_0,1]} r \int_0^r \widetilde{W}_r dW / \int_0^r \widetilde{W}_r^2$.

The main results of this section are in the following theorem when the AR(1) Model (1.1) is estimated by LS.[3]

[3] All the results in this theorem continue to hold when the DGP in the abnormal period is replaced with the alternative processes, such as $p_t = p_{t-1} + \delta\left(t - n_c\right)^{m-1} + \varepsilon_t$ with $\delta \neq 0$ and $m \geq 2$, and $p_t = \theta + \delta_1\left(t - n_c\right) + \cdots + \delta_m\left(t - n_c\right)^m + \epsilon_t$ with $\delta_m \neq 0$.

THEOREM 2.1. *Assume $\{p_t\}$ is generated from Model (2.1). Let $\tilde{\beta}_{r_1}^{r_2}$ be the LS estimate of β in the AR(1) regression (1.1) based on the sample from n_1 to n_2 where $n_i = \lfloor nr_i \rfloor$ with $r_i \in 0, 1$for $i = 1, 2$.*

(1) Let $n_1 = n_c$ and $n_c < n_2 \leq n$, such that $\{p_t\}_{t=n_1}^{n_2}$comes entirely from a polynomial trend model. Define $n_s := n_2 - n_1$. When $n \to \infty$, it has

$$n_s\left(\tilde{\beta}_{r_c}^{r_2} - 1\right) \xrightarrow{p} \frac{(2m+1)\left(m^2-1\right)}{2m^2} \quad \textit{and} \quad n_s^{-1/2} t\left(\tilde{\beta}_{r_c}^{r_2}\right) \xrightarrow{p} \sqrt{4m^2-1}.$$

(2) As $n \to \infty$,

$$GSADFt \xrightarrow{p} +\infty.$$

(3) For any r_2 such that $r_0 + r_c < r_2 \leq 1$, as $n \to \infty$,

$$BSADFt\,(r_2) \xrightarrow{p} +\infty.$$

REMARK 2.1. Theorem 2.1.1 shows that, when the data comes entirely from a polynomial trend model, the unit root tests based on the AR(1) regression (1.1) tend to reject the unit root null hypothesis in favour of explosiveness. To clearly explain this argument, we take the case of $m = 2$ as an example, in which a quadratic trend model generates the data $\{p_t\}$ in the abnormal period. When $m = 2$, Theorem 2.1.1 shows that $(n_2 - n_1)\left(\tilde{\beta}_{r_1}^{r_2} - 1\right) \xrightarrow{p} 15/8$ and $t\left(\tilde{\beta}_{r_1}^{r_2}\right) \xrightarrow{p} +\infty$ as n increases. Whereas for the AR(1) Model (1.1), the 99% asymptotic CV of the DF coefficient-based test and the DF t test is 1.04 and 0.6 (see tables B.5–B.6 in Hamilton (1994)). Therefore, both tests based on the AR(1) regression would reject the unit root null hypothesis in favour of explosiveness. In other words, the unit root tests based on the AR(1) regression could not distinguish the explosive AR model from the polynomial trend model. The reason behind this result is that the sample path of a quadratic trend model has a curvature similar to that generated by an explosive AR model. This result complements Phillips and Shi (2019), where a structural break is needed for the standard unit root test to identify explosiveness falsely.

REMARK 2.2. Theorem 2.1.2–2.1.3 follows immediately from Theorem 2.1.1, as the supremum of the t statistics in subsamples cannot be smaller than the t statistic when the data comes entirely from a polynomial trend model. According to Theorem 2.1.2, the *GSADFt* test spuriously leads to the identification of explosiveness when the data is generated from Model (2.1). According to Theorem 2.1.3, the *BSADFt* test provides an estimate of the origination date of the spurious bubble. Similar spurious results can also be found for the *GSADFc* and $BSADFc$ tests.

3. NEW RIGHT-TAILED UNIT ROOT TESTS

To obtain tests that are robust to polynomial trends, a simple solution is to construct right-tailed unit root tests based on the AR(2) model given by (1.2).[4] Let $\check{\beta}_{r_1}^{r_2}$ denote the LS estimate of β in the AR(2) regression (1.2) based on the sample from $\lfloor nr_1 \rfloor$ to $\lfloor nr_2 \rfloor$, $se\left(\check{\beta}_{r_1}^{r_2}\right)$ be the standard error of $\check{\beta}_{r_1}^{r_2}$, and $t\left(\check{\beta}_{r_1}^{r_2}\right)$ be the corresponding t statistic.

[4] We can extend the results for Model (1.2) to the AR(k) model with $k > 2$, as shown in the Online Supplement.

If $\{p_t\}$ is generated from $p_t = p_{t-1} + \varepsilon_t$ in the whole sample period, it is easy to get that, for any $r_1, r_2 \in [0, 1]$ as $n \to \infty$,

$$n_s\left(\check{\beta}_{r_1}^{r_2} - 1\right) \Rightarrow \frac{\int_{r_1}^{r_2} \widetilde{W}_{r_1,r_2}(s)\, dW(s)}{\int_{r_1}^{r_2} \left[\widetilde{W}_{r_1,r_2}(s)\right]^2 ds} \text{ and } t\left(\check{\beta}_{r_1}^{r_2}\right) \Rightarrow \frac{\int_{r_1}^{r_2} \widetilde{W}_{r_1,r_2}(s)\, dW(s)}{\left\{\int_{r_1}^{r_2} \left[\widetilde{W}_{r_1,r_2}(s)\right]^2 ds\right\}^{1/2}}, \tag{3.1}$$

where

$$\widetilde{W}_{r_1,r_2}(s) := W(s) - \frac{1}{r_2 - r_1}\int_{r_1}^{r_2} W(t)\, dt$$

$W(\cdot)$ is a standard Brownian motion and $\Rightarrow$ denotes weak convergence. The asymptotic distributions given in (3.1) are the same as those developed in PSY (2015a, 2015b) for the test statistics based on AR(1) regression. Hence, under the unit root null hypothesis, the supremum test statistics GSADF and BSADF still apply to the AR(2) regression with the same asymptotic CVs suggested in PSY (2015a, 2015b).

When the data is generated from the model of (2.1), Theorem 3.1 below shows that the probability of *GSADFc* or *GSADFt* rejecting the null hypothesis of unit root is very low. Hence, it is difficult for these two tests to lead to a spurious bubble detection. Also proved in Theorem 3.1 is that, when the data is generated from a process that switches from a unit root model to a mildly explosive AR model as

$$p_t = \begin{cases} p_{t-1} + \varepsilon_t & \text{if } t \in N \equiv [1, n_c] \\ \left(1 + \frac{\gamma}{n_e^{\theta}}\right) p_{t-1} + \varepsilon_t, \text{ with } \gamma > 0, \theta \in (0,1), & \text{if } t \in T \equiv (n_c, n], \end{cases} \tag{3.2}$$

where $n_e = n - n_c$, the *GSADFc* or *GSADFt* tests can detect explosiveness and identify the bubble period successfully. Hence, the *GSADFc* or *GSADFt* tests based on the AR(2) regression are robust to polynomial trends.

THEOREM 3.1. *Let $\check{\beta}_{r_1}^{r_2}$ be the LS estimate of β in the AR(2) regression (1.2) based on the sample from $n_1 = \lfloor nr_1 \rfloor$ to $n_2 = \lfloor nr_2 \rfloor$, $t\left(\check{\beta}_{r_1}^{r_2}\right)$ be the corresponding t statistic, $r_i \in [0, 1]$.*

(1) If $\{p_t\}$ is generated from Model (2.1), for any $n_1, n_2 \in T$, as $n \to \infty$, it has

$$(n_2 - n_1)\left(\check{\beta}_{r_1}^{r_2} - 1\right) \xrightarrow{p} 0 \text{ and } t\left(\check{\beta}_{r_1}^{r_2}\right) \xrightarrow{p} \begin{cases} 0, & \text{when } m = 2 \\ -\infty, & \text{when } m > 2, \end{cases}$$

(2) If $\{p_t\}$ is generated from Model (2.1), for any $n_1 \in N$ and $n_2 \in T$, as $n \to \infty$, it has

$$(n_2 - n_1)\left(\check{\beta}_{r_1}^{r_2} - 1\right) \xrightarrow{p} 0 \text{ and } t_{r_1}^{r_2} \xrightarrow{p} -\infty.$$

(3) If $\{p_t\}$ is generated from Model (3.2), then as $n \to \infty$,

$$GSADFc \xrightarrow{p} +\infty, \; GSADFt \xrightarrow{p} +\infty.$$

Moreover, for any $r_2 > r_c + r_0$, as $n \to \infty$,

$$BSADFc(r_2) \xrightarrow{p} +\infty, \; BSADFt(r_2) \xrightarrow{p} +\infty.$$

REMARK 3.1. If $p_t = p_{n_c} + \delta(t - n_c)^m + \varepsilon_t$ with $m = 2$, the DGP can also be expressed as $p_t = 2\delta + p_{t-1} + \Delta p_{t-1} + \Delta^2\varepsilon_t$, where Δ and Δ^2 are operators for the first- and second-order difference, respectively. This alternative representation is covered by the suggested AR(2) regression (1.2) and explains why both DF statistics converge to zero. When $m > 2$, the DGP of p_t is not covered by the suggested regression model (1.2) anymore. For example, when $m = 3$, it has $p_t = -6\delta + p_{t-1} + \Delta p_{t-1} + 6\delta (t - n_c) + \Delta^2 \varepsilon_t$, which is not covered by the regression model (1.2). This difference is the reason why the asymptotics of the test statistics obtained in Theorem 3.1.1 when $m = 2$ are different from that when $m > 2$.

REMARK 3.2. As discussed earlier and well-known in the literature, when the data comes entirely from a unit root model, both the DF coefficient-based statistic and t statistic converge to well-defined distributions whose 99% CVs are positive. According to Theorem 3.1.1–3.1.2, if *GSADFc* or *GSADFt* rejects the null hypothesis, the supremum of the coefficient-based statistics or the supremum of the t statistics cannot correspond to a subsample whose ending point is in regime T. In other words, the probability that *GSADFc* or *GSADFt* rejects the null hypothesis is 1%. This is in sharp contrast with the probability of false detecting explosiveness approaching one in the case when *GSADFc* and *GSADFt* are obtained from the AR(1) regression of (1.1).

REMARK 3.3. According to Theorem 3.1.3, if the DGP is Model (3.2) that switches from a unit root model to a mildly explosive root AR model at the point n_c, the supremum tests GSADF and BSADF can successfully detect the explosiveness and consistently estimate the bubble origination date. Therefore, Theorem 3.1 proves that the recursive bubble tests based on the AR(2) regression is robust to polynomial trends.

4. SIMULATION STUDIES

We first obtain 99% finite sample CVs of DF coefficient-based, DF t, *GSADFc*, and *GSADFt* test statistics obtained from AR(1) and AR(2) via simulations. In particular, data are simulated from a random walk model and then fitted to either Model (1.1) or Model (1.2) to calculate the DF coefficient-based, DF t, *GSADFc*, and *GSADFt* test statistics. The number of replications is set to 10,000 for the DF coefficient-based and DF t statistics. The number of replications is set to 2,000 for the *GSADFc* and *GSADFt* statistics. Table 1 reports the 99% CVs of the four statistics. For the DF coefficient-based and DF t statistics, we set $n = 20, 50$. For the *GSADFc* and *GSADFt* statistics, we set $n = 40$, $r_0 = 0.3$ and $n = 100$, $r_0 = 0.2$. Comparing results in Table 1 with those reported in table 1 of PSY (2015a), the CVs for the *GSADFt* statistic are similar to each other. The CVs for the *GSADFc* statistic are new to the literature.

4.1. Quadratic trend model

We then simulate data from Model (2.1) with $m = 2$, $\delta = 10$ or 20, $r_c = 0.5$, $\varepsilon_t \overset{iid}{\sim} N(0, 1)$, that is, the first half of the sample comes from the random walk model and the second half from a quadratic trend model. We consider two sample sizes, $n = 40$ or 100. Hence, either 20 or 50 observations are obtained from a quadratic trend model, which are used to examine the finite sample property of the DF coefficient-based and DF t tests. The entire 40 or 100 observations are used to examine the finite sample property of the *GSADFc* and *GSADFt* tests.

Tables 2–3 report the simulation results based on 1,000 replications when Model (1.1) or Model (1.2) is fitted to the simulated data, respectively. Each table reports the average value (across all 1,000 replications) of estimated β, the average value of DF coefficient-based statistic, the percentage of replications, where the unit root null hypothesis is rejected by the DF coefficient-based statistic, the average value of DF t statistic, and the percentage of replications, where the unit root null hypothesis is rejected by the DF t statistic. Here, only the second half of the sample is used and the structural break is irrelevant. Moreover, each table also reports the average value

Table 2. Statistical results when Model (1.1) is fitted to the data simulated from Model (2.1) with $m = 2$ and $r_c = 0.5$.

δ	10	20	10	20
Effective sample size	20	20	50	50
Mean of estimated β	1.0944	1.0944	1.0376	1.0376
Mean of c-stat	1.7933	1.7933	1.8422	1.8422
% rejection of H_0 by c-stat	1.0000	1.0000	1.0000	1.0000
Mean of t-stat	17.8470	17.8653	27.6803	27.6853
% rejection of H_0 by t-stat	1.0000	1.0000	1.0000	1.0000
Effective sample size	40	40	100	100
Mean of *GSADFc*	39.9456	53.1392	59.6868	88.9255
% rejection of H_0 by *GSADFc*	1.0000	1.0000	1.0000	1.0000
Mean of *GSADFt*	30.2003	30.5064	67.8973	68.5116
% rejection of H_0 by *GSADFt*	1.0000	1.0000	1.0000	1.0000

Notes: Calculations are based on 1,000 replications when Model (1.1) is fitted to the simulated sample path. For the DF coefficient-based and DF t statistics, the simulated sample is from the polynomial trend model with $n = 20, 50$. For the *GSADFc* and *GSADFt* statistics, the simulated sample is from the model that switches from the random walk to the polynomial trend with $n = 40$, $r_0 = 0.3$ and $n = 100$, $r_0 = 0.2$.

Table 3. Statistical results when Model (1.2) is fitted to the data simulated from Model (2.1) with $m = 2$ and $r_c = 0.5$.

δ	10	20	10	20
Effective sample size	20	20	50	50
Mean of estimated β	1.0005	1.0001	1.0000	1.0000
Mean of c-stat	0.0093	0.0024	0.0018	0.0002
% rejection of H_0 by c-stat	0.0000	0.0000	0.0000	0.0000
Mean of t-stat	0.1963	0.1012	0.1008	0.0417
% rejection of H_0 by t-stat	0.0400	0.0240	0.0080	0.0060
Effective sample size	40	40	100	100
Mean of *GSADFc*	17.3568	23.1654	17.2030	22.9204
% rejection of H_0 by *GSADFc*	0.6050	0.6090	0.6600	0.6930
Mean of *GSADFt*	1.9319	1.7130	1.9910	1.8293
% rejection of H_0 by *GSADFt*	0.1090	0.0800	0.1440	0.1278

Notes: Calculations are based on 1,000 replications when Model (1.2) is fitted to the simulated sample path. For the DF coefficient-based and DF t statistics, the simulated sample is from the polynomial trend model with $n = 20, 50$. For the *GSADFc* and *GSADFt* statistics, the simulated sample is from the model that switches from the random walk to the polynomial trend with $n = 40$, $r_0 = 0.3$ and $n = 100$, $r_0 = 0.2$.

of *GSADFc*, the percentage of replications, where the unit root null hypothesis is rejected by the *GSADFc* statistic, the average value of *GSADFt*, and the percentage of replications, where the unit root null hypothesis is rejected by the *GSADFc* statistic. Here, the full sample that covers the break is used. In all cases, the 99% CVs from Table 1 are used to calculate the percentage of the rejection of the unit root null hypothesis.

Table 2 reports the simulation results when Model (1.1) is fitted to the simulated data. Some conclusions can be made from Table 2. First, the average value of $\tilde{\beta}$ is greater than 1 in all four cases, although it gets closer to 1 as n increases. Second, consistent with the asymptotic theory given in Theorem 2.1.1, the DF coefficient-based statistics take values around $15/8 \approx 1.8750$

and are always larger than the respective 99% finite sample CVs. Hence, the DF coefficient-based test always spuriously suggests explosiveness in the unit root test. Third, as suggested by the asymptotic theory, the DF t statistics take values around $\sqrt{15n}$ and are always larger than the respective 99% finite sample CVs. Hence, just as the DF coefficient-based test, the DF t test always spuriously suggests explosiveness in the unit root test. Fourth, both the *GSADFc* and the *GSADFt* statistics are always larger than the respective 99% finite sample CVs. They become larger when the sample size increases. Hence, both tests always spuriously suggest explosiveness, as predicted by Theorem 2.1.2.

The findings from Table 2 have important empirical implications, as the implementation of the right-tailed unit root testing in the literature has been often based on Model (1.1). When Model (1.1) is fitted, all four tests cannot distinguish between the quadratic trend model and the explosive AR model. They always spuriously suggest explosiveness when data come from either a quadratic trend model or a model that switches from unit root to quadratic trend.

Table 3 reports the simulation results when Model (1.2) is fitted to the simulated data. Several conclusions are made from Table 3. First, $\check{\beta}$ converges to 1 very quickly so that the DF coefficient-based statistics take values very close to 0, as suggested by the asymptotic theory given in Theorem 3.1.1. For all replications, they are smaller than the 99% finite sample CVs. Hence, the right-tailed DF coefficient-based test from the AR regression model (1.2) does not reject the null hypothesis of unit root. Second, the DF t statistic goes to zero when n increases, as suggested by the asymptotic theory given in Theorem 3.1.1. For almost all replications, the average value of the t statistic is smaller than the 99% critical value, indicating that the DF t test has a great chance not to reject the unit root hypothesis. Hence, we can conclude that both DF tests can distinguish the explosive AR model and the quadratic trend model if they are calculated from Model (1.2), although the DF coefficient-based test is slightly more powerful. Third, the *GSADFc* statistic from Model (1.2) takes values much smaller than that from Model (1.1). As a result, for a significant proportion of replications (ranging between 30% to 40%), we cannot reject the unit root null hypothesis. Fourth, just like the *GSADFc* statistic, the *GSADFt* statistic from Model (1.2) also takes values much smaller than that from Model (1.1). For a significant proportion of replications (ranging between 85% to 92%), we cannot reject the unit root null hypothesis. Hence, we can conclude that both the *GSADFc* and *GSADFt* tests have some power in distinguishing the explosive AR model, and the quadratic trend model when they are calculated from Model (1.2). Moreover, the *GSADFt* test is more powerful than the *GSADFc* test.

4.2. *Explosive AR model*

We now carry out a simulation study based on 10,000 replications for Model (1.2) when the true DGP is $p_t = \beta y_{t-1} + \varepsilon_t$ with $y_0 = 10$ and $\varepsilon_t \overset{iid}{\sim} N(0, 1)$. To find the power of the DF coefficient-based and DF t tests, we first find the 99% critical values for $n = 50, 100, 500$ when the true DGP is $p_t = p_{t-1} + u_t$ with $y_0 = 10$ and $\varepsilon_t \overset{iid}{\sim} N(0, 1)$.[5] These critical values are used to test the null hypothesis of unit root against the explosive alternative in the AR regression model (1.2). When the test statistics take values larger than the corresponding critical values, the evidence of explosiveness is found.

Table 4 reports the proportions of replications, out of 10,000 replications, where the DF coefficient-based test and the DF t test reject the unit root null hypothesis in favour of explosiveness, when the true DGP is explosive with $\beta = 1.03$ or $\beta = 1.05$. These two values of β for

[5] To save space, these critical values are not reported and may be requested from the authors.

Table 4. Proportion of replications, where the null hypothesis of unit root is rejected in favour of explosiveness for the DF coefficient-based and the DF t tests based on Model (1.2).

	quadratic trend		explosive ($\beta = 1.03$)		explosive ($\beta = 1.05$)	
n	c-stat	t-stat	c-stat	t-stat	c-stat	t-stat
50	0	0.0071	0.2933	0.6897	0.9875	0.9950
100	0	0.0020	0.9873	0.9873	1.000	1.000
250	0	0	1.000	1.000	1.000	1.000

Notes: Calculations are based on 10,000 replications when Model (1.2) is fitted to the simulated sample from either a quadratic trend model or an explosive AR model.

the explosive process are empirically reasonable. For the purpose comparison, we also report the proportions of replications when the true DGP is a quadratic trend model (i.e. no structural break in the DGP) in Table 4. The overall conclusion from this simulation study is that our proposed procedure can effectively distinguish the polynomial trend processes from the explosive process. In particular, when data is generated from the polynomial trend model, in no replication the DF coefficient-based test based on Model (1.2) finds evidence of explosiveness in all three sample sizes considered. In a small number of replications, the DF t test based on Model (1.2) finds the evidence of explosiveness in unit root testing. The proportion becomes smaller when the sample size increases. When data is generated from an explosive process with a stronger explosive behaviour ($\beta = 1.05$), the two tests almost always find evidence of explosiveness. When the explosive behaviour is not so strong ($\beta = 1.03$) and the sample size is small ($n = 50$), the two statistics, especially the coefficient-based statistic, has difficulty in rejecting the unit root hypothesis. When $n = 100$ or 200, the two unit root tests almost always find the evidence of explosiveness regardless of $\beta = 1.03$ or 1.05. All these results are consistent with what Theorem 3.1.3 predicts.

From the above simulations, it is clear that there is a trade-off between the DF coefficient-based test and the DF t test. While the DF coefficient-based statistic is more robust against polynomial trends in data, it is less powerful in identifying a mildly explosive behaviour in small samples. For conservative users whose primary concern is on the robustness property of the right-tailed unit root testing against the trend stationary behaviour, our recommendation is to use the DF coefficient-based test.

5. AN EMPIRICAL STUDY

In the empirical study, we analyse the data of the US real estate market, which contains the monthly S&P/Case-Shiller US National Home Price Index and the monthly rent of primary residence in US city average (Federal Reserve Bank of St. Louis, 2022), both over the period from January 1981 to June 2017 (438 monthly observations in the full sample).[6] The price–rent ratio is calculated for the sample period. Since the simulation study suggests that the DF coefficient-based statistic is more robust against polynomial trends in data than the DF t statistic in the context of the AR regression model (1.2), in the empirical study reported below, we only calculate the coefficient-based statistic.

[6] The data is downloaded from Federal Reserve Bank of St Louis. The series code for the home price index is CSUSHPISA, while the code for the rent is CUSR0000SEHA.

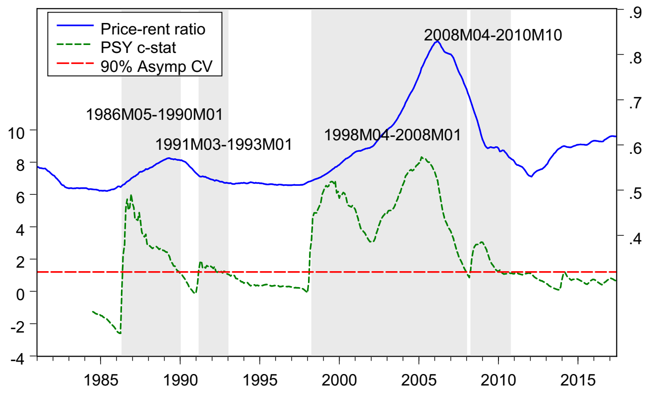

Figure 1. Date-stamping bubble periods in the US home price–rent ratio based on the PSY method.

For the purpose of comparison, we first apply the PSY method (based on the sup coefficient-based statistic obtained from AR regression model (1.1)) to the price–rent ratio. Figure 1 plots the price–rent ratio, the sequence of PSY coefficient-based statistics that are calculated in a recursive backward manner with the minimum window size chosen as $0.01 + 1.8/\sqrt{438} \approx 42$, and the sequence of the 95% asymptotic critical values of the sup coefficient-based statistic reported in table 1 of PSY (2015a). It can be seen that four bubble episodes are identified by the PSY method; namely, May 1986 to January 1990, March 1991 to January 1993, April 1998 to January 2008, and April 2008 to October 2010. The first and third episodes correspond to the well-known periods of US real estate market expansions. However, some post-peak periods are included as a part of bubble expansion. The price–rent ratio reached a peak in May 1989, whereas the estimated termination date of the first bubble by PSY method is April 1990. The price–rent ratio reached another peak in March 2006, whereas the PSY method estimates January 2008 to be the termination date of the third bubble. During the second and last detected bubble periods, the real estate market experienced market downturns. Hence, according to PSY, these two periods must have negative bubbles.

We then apply the proposed procedure (i.e., fit the AR regression model (1.2)) to the price–rent ratio of the US real estate market. Figure 2 plots the price–rent ratio, the sequence of the coefficient-based statistics from the proposed procedure calculated in a recursive backward manner with the same minimum window size as in PSY method, and the sequence of the 95% asymptotic critical values of the sup coefficient-based statistic reported in table 1 of PSY (2015a). Only two bubble periods have been identified by the proposed method; namely, May 1986 to May 1989 and July 1998 to March 2006. Although these two bubble periods correspond to the first and third bubble periods identified by the PSY method, they are shorter in the sense that both bubbles ended much earlier. The first bubble ended in May 1989 according to the proposed method, eleven months earlier than that identified by the PSY method. The second bubble ended in March 2006 according to the proposed method, 22 months earlier than that identified by the PSY method. As noted earlier, both May 1989 and March 2006 are two peaks of the time series. Clearly, the estimated termination dates synchronize the turning points by the proposed procedure. Moreover, the third and the last bubble periods identified by PSY method are not flagged as negative bubble episodes by the proposed method. This observation indicates that the

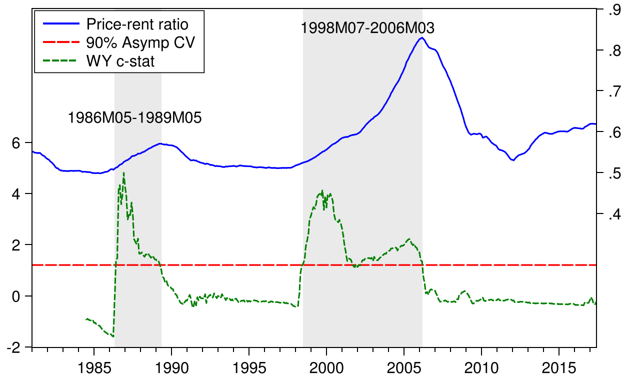

Figure 2. Date-stamping bubble periods in the US home price–rent ratio based on the proposed regression method.

market downturns in 1991–1993 and 2008–2011 may be better explained by deterministic trend models than pure AR models.

6. CONCLUSION

This paper concerns the robustness of the right-tailed unit root tests against the explosive alternative when the true DGP is not an explosive process. In a recent interesting study, Phillips and Shi (2019) introduced a model with a structural break. Before the break, data is generated from a unit root model. After the break, a linear trend model generates the data. While there is no explosiveness in this DGP, they show that the right-tailed DF t statistic, obtained from a sample covering both data before and after the break, tends to find explosive behaviour. As a result, the DF t and *GSADFt* tests cannot distinguish between the structural break model and the explosive AR model.

In this paper, we provide an alternative explanation for the spurious identification of bubbles without the help of a structural break. Our idea is to use polynomial trend models. We show that, when there is a polynomial trend in DGP, the conventional right-tailed unit root tests based on the AR(1) regression also tend to reject the null hypothesis of unit root in favour of the explosive alternative. As a result, the *GSADFc* and the *GSADFt* tests based on the AR(1) regression cannot distinguish between the explosive AR model and the polynomial trend model. The intuition is that the polynomial trend model can generate a curvature similar to the explosive AR model. Extensive simulation studies reinforce the asymptotic properties of the test statistics.

For the right-tailed unit root tests to be able to distinguish the polynomial trend model from the explosive AR model, we propose a new autoregressive procedure by including the lagged first difference as an additional regressor. Asymptotic properties for the test statistics under the null hypothesis of unit root and several different alternatives are studied. When data is generated from a model that switches from a random walk to a polynomial trend, we show that the *GSADFc* and *GSADFt* tests often cannot reject the unit root null hypothesis. Whereas, when data is generated from an explosive autoregressive process, we show that both the tests reject the null hypothesis of unit root. Hence, the two tests have power in distinguishing between the explosive AR model and the polynomial trend model. Interestingly, our proposed procedure is nearly identical to the augmented DF procedure discussed in PSY (2015a, equation (4)) with an important distinction. Our method requires k to be one, whereas, in equation (4) of PSY, k is set to zero (i.e., no augmented term) in the empirical implementation.

We have applied our proposed method to real data. For the price–rent ratio of the US real estate market from January 1981 to June 2017, our testing results are different from those obtained by using the PSY method with $k = 0$. Two negative bubbles identified by the PSY method are not flagged as bubbles anymore. In addition, the termination dates of two identified positive bubble periods are estimated to be much earlier than those estimated by the PSY method.

The asymptotic results obtained in this paper continue to hold when the DGP in the abnormal period of Model (2.1) is replaced with the following nonexplosive AR process,[7]

$$p_t = p_{t-1} + \delta (t - n_c)^{m-1} + \varepsilon_t, \quad t = n_c + 1, \ldots, n, \tag{6.1}$$

where $\delta \neq 0$ and $m \geq 2$. This alternative DGP (6.1) has a deterministic trend of a lower order $m - 1$ as well as p_{t-1}. It is easy to rewrite the DGP in (6.1) as

$$p_t = p_{n_c} + \delta \sum_{s=n_c+1}^{t} (s - n_c)^{m-1} + \sum_{s=n_c+1}^{t} \varepsilon_s = p_{n_c} + \frac{\delta}{m} (t - n_c)^m + O((t - n_c)^{m-1}) + \sum_{s=n_c+1}^{t} \varepsilon_s.$$

The leading term is $\delta (t - n_c)^m / m$, and has the same order as that of p_t under the DGP (2.1). If we start from the DGP in (2.1), the first difference gives

$$p_t = p_{t-1} + \delta m (t - 1 - n_c)^{m-1} + O((t - 1 - n_c)^{m-2}) + \varepsilon_t - \varepsilon_{t-1}.$$

This is similar to the alternative process in (6.1). From the proofs, we note that the asymptotic results are fully determined by the leading terms of p_t and $\Delta p_t = p_t - p_{t-1}$. By the same argument, it is easy to see that the asymptotic results continue to hold when the DGP of the abnormal period in Model (2.1) is replaced with the alternative process

$$p_t = p_{n_c} + \delta_1 (t - n_c) + \cdots + \delta_m (t - n_c)^m + \varepsilon_t, \text{ with } \delta_m \neq 0.$$

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[7] We thank a referee for pointing out this equivalence to us.

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SUPPORTING INFORMATION

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Online Appendix
Replication Package

Co-editor Dennis Kristensen handled this manuscript.

APPENDIX: PROOF OF THEOREMS

Proof of Theorem 2.1: We here prove the asymptotics of the statistics with the sample from $n_c + 2$ to n_2, which is the same as that with the sample from n_c to n_2. Define $\sum = \sum_{n_c+2}^{n_2}$, $\Delta\varepsilon_t = \varepsilon_t - \varepsilon_{t-1}$, $g(t) = \delta(t - n_c)^m$, and $\Delta g(t) = \delta(t - n_c)^m - \delta(t - n_c - 1)^m$. For any $t > n_c + 1$, it has

$$p_t = p_{n_c} + \delta(t - n_c)^m + \varepsilon_t = p_{t-1} + \Delta g(t) + \Delta\varepsilon_t.$$

Thus, the centred LS estimators of the parameters in the AR(1) model (1.1) can be expressed as

$$\begin{pmatrix} \tilde{\alpha}_{r_c}^{r_2} - 0 \\ \tilde{\beta}_{r_c}^{r_2} - 1 \end{pmatrix} = \begin{pmatrix} \sum 1 & \sum p_{t-1} \\ \sum p_{t-1} & \sum p_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum [\Delta g(t) + \Delta\varepsilon_t] \\ \sum p_{t-1}[\Delta g(t) + \Delta\varepsilon_t] \end{pmatrix}.$$

Considering that $m \geq 2$ and $\Delta g(t) = \delta m(t - n_c - 1)^{m-1} + O\left((t - n_c - 1)^{m-2}\right)$, we then have, as $n \to \infty$,

$$\begin{pmatrix} n_s^{-m} \sum [\Delta g(t) + \Delta\varepsilon_t] \\ n_s^{-2m} \sum p_{t-1}[\Delta g(t) + \Delta\varepsilon_t] \end{pmatrix} = \begin{pmatrix} n_s^{-m} \sum \delta m(t - n_c - 1)^{m-1} + o_p(1) \\ n_s^{-2m} \sum \delta^2 m(t - n_c - 1)^{2m-1} + o_p(1) \end{pmatrix} \xrightarrow{p} \begin{pmatrix} \delta \\ \delta^2/2 \end{pmatrix},$$

where the limits are from the fact of $n_s^{-(j+1)} \sum (t - n_c)^j \to 1/(j+1)$.

Note that, as $n \to \infty$, it has

$$n_s^{-(m+1)} \sum p_{t-1} = n_s^{-(m+1)} \sum \delta(t - n_c - 1)^m + o_p(1) \xrightarrow{p} \frac{\delta}{m+1},$$

$$n_s^{-(2m+1)} \sum p_{t-1}^2 = n_s^{-(2m+1)} \sum \delta^2(t - n_c - 1)^{2m} + o_p(1) \xrightarrow{p} \frac{\delta^2}{2m+1}.$$

Hence,

$$\begin{pmatrix} n_s^{-m+1} & 0 \\ 0 & n_s \end{pmatrix} \begin{pmatrix} \sum 1 & \sum p_{t-1} \\ \sum p_{t-1} & \sum p_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} n_s^m & 0 \\ 0 & n_s^{2m} \end{pmatrix} = \begin{pmatrix} \frac{\sum 1}{n_s} & \frac{\sum p_{t-1}}{n_s^{m+1}} \\ \frac{\sum p_{t-1}}{n_s^{m+1}} & \frac{\sum p_{t-1}^2}{n_s^{2m+1}} \end{pmatrix}^{-1} \xrightarrow{p} \begin{pmatrix} 1 & \frac{\delta}{m+1} \\ \frac{\delta}{m+1} & \frac{\delta^2}{2m+1} \end{pmatrix}^{-1}.$$

We then get that, as $n \to \infty$,

$$\begin{pmatrix} n_s^{-m+1} \tilde{\alpha}_{r_c}^{r_2} \\ n_s\left(\tilde{\beta}_{r_c}^{r_2} - 1\right) \end{pmatrix} \xrightarrow{p} \begin{pmatrix} 1 & \frac{\delta}{m+1} \\ \frac{\delta}{m+1} & \frac{\delta^2}{2m+1} \end{pmatrix}^{-1} \begin{pmatrix} \delta \\ \delta^2/2 \end{pmatrix} = \begin{pmatrix} \frac{(m+1)^2}{m^2} & -\frac{(2m+1)(m+1)}{\delta m^2} \\ -\frac{(2m+1)(m+1)}{\delta m^2} & \frac{(2m+1)(m+1)^2}{\delta^2 m^2} \end{pmatrix} \begin{pmatrix} \delta \\ \delta^2/2 \end{pmatrix},$$

which gives the limit of $n_s\left(\tilde{\beta}_{r_c}^{r_2} - 1\right)$ as presented in the first part of Theorem 2.1.1.

The standard error of $\tilde{\beta}_{r_c}^{r_2}$, denoted by $se\left(\tilde{\beta}_{r_c}^{r_2}\right)$, takes the form of

$$se\left(\tilde{\beta}_{r_c}^{r_2}\right) = \left[(0\ 1) \begin{pmatrix} \sum 1 & \sum p_{t-1} \\ \sum p_{t-1} & \sum p_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(\frac{\sum \left(p_t - \tilde{\alpha}_{r_c}^{r_2} - \tilde{\beta}_{r_c}^{r_2} p_{t-1}\right)^2}{n_s - 3} \right)\right]^{1/2}.$$

From the results in the above proof, it has, as $n \to \infty$,

$$\begin{aligned} &(0\ n_s) \begin{pmatrix} \sum 1 & \sum p_{t-1} \\ \sum p_{t-1} & \sum p_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ n_s^{2m} \end{pmatrix} \\ &= (0\ 1) \begin{pmatrix} n_s^{-m+1} & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} \sum 1 & \sum p_{t-1} \\ \sum p_{t-1} & \sum p_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} n_s^m & 0 \\ 0 & n_s^{2m} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\xrightarrow{p} (0\ 1) \begin{pmatrix} 1 & \frac{\delta}{m+1} \\ \frac{\delta}{m+1} & \frac{\delta^2}{2m+1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{(2m+1)(m+1)^2}{\delta^2 m^2}. \end{aligned}$$

Given that $p_t = p_{t-1} + \Delta g(t) + \Delta\varepsilon_t$, we have

$$\sum \left(p_t - \tilde{\alpha}_{r_c}^{r_2} - \tilde{\beta}_{r_c}^{r_2} p_{t-1}\right)^2 = \sum \left(\Delta g(t) + \Delta\varepsilon_t - \tilde{\alpha}_{r_c}^{r_2} - \left(\tilde{\beta}_{r_c}^{r_2} - 1\right) p_{t-1}\right)^2$$
$$= \sum \left(\Delta g(t) + \Delta\varepsilon_t - \tilde{\alpha}_{r_c}^{r_2} - \left(\tilde{\beta}_{r_c}^{r_2} - 1\right) p_{t-1}\right) \left(\Delta g(t) + \Delta\varepsilon_t\right),$$

where the second equation comes from the first-order condition of the LS regression. As $m \geq 2$ and $\Delta g(t) = \delta m (t - n_c - 1)^{m-1} + O\left((t - n_c - 1)^{m-2}\right)$, we further get

$$n_s^{-2m+1} \sum \left(\Delta g(t) + \Delta\varepsilon_t\right)^2 = n_s^{-2m+1} \sum \left(\Delta g(t)\right)^2 \xrightarrow{p} \delta^2 \frac{m^2}{2m-1}.$$

From the earlier proof, we have $n_s^{-m+1}\tilde{\alpha}_{r_c}^{r_2} \xrightarrow{p} \frac{\delta}{2}\frac{(m+1)}{m^2}$ and $n_s\left(\tilde{\beta}_{r_c}^{r_2} - 1\right) \xrightarrow{p} \frac{(2m+1)\left(m^2-1\right)}{2m^2}$. Thus, we can prove that

$$n_s^{-2m+1}\tilde{\alpha}_{r_c}^{r_2} \sum \left(\Delta g(t) + \Delta\varepsilon_t\right) = \left(n_s^{-m+1}\tilde{\alpha}_{r_c}^{r_2}\right)\left(n_s^{-m} \sum \Delta g(t)\right) \xrightarrow{p} \frac{\delta^2}{2}\frac{(m+1)}{m^2},$$

and

$$n_s^{-2m+1}\left(\tilde{\beta}_{r_c}^{r_2} - 1\right) \sum p_{t-1}\left(\Delta g(t) + \Delta\varepsilon_t\right)$$
$$= \delta^2 m n_s \left(\tilde{\beta}_{r_c}^{r_2} - 1\right) n_s^{-2m} \sum (t - n_c - 1)^{2m-1} + o_p(1) \xrightarrow{p} \delta^2 \frac{(2m+1)\left(m^2-1\right)}{4m^2}.$$

Consequently, it is obtained that

$$n_s^{-2m+1} \sum \left(p_t - \tilde{\alpha}_{r_c}^{r_2} - \tilde{\beta}_{r_c}^{r_2} p_{t-1}\right)^2 \xrightarrow{p} \delta^2 \frac{(m-1)^2}{4m^2(2m-1)}.$$

Therefore,

$$n_s^3 \left[se\left(\tilde{\beta}_{r_c}^{r_2}\right)\right]^2 \xrightarrow{p} \frac{(2m+1)\left(m^2-1\right)^2}{4m^4(2m-1)}.$$

Finally, we get

$$n_s^{-1/2} t\left(\tilde{\beta}_{r_c}^{r_2}\right) = \frac{n_s\left(\tilde{\beta}_{r_c}^{r_2} - 1\right)}{n_s^{3/2} se\left(\tilde{\beta}_{r_c}^{r_2}\right)} \xrightarrow{p} \frac{\frac{(2m+1)\left(m^2-1\right)}{2m^2}}{\left[\frac{(2m+1)\left(m^2-1\right)^2}{4m^4(2m-1)}\right]^{1/2}} = \left(4m^2 - 1\right)^{1/2}.$$

This proves the second part of Theorem 2.1.1.

The proof of Theorem 2.1.2–2.1.3 is straightforward. If we choose n_1 and n_2 such that $n_1 = \lfloor nr_c \rfloor = n_c$ and $n_2 = \lfloor nr_2 \rfloor \in T$, according to Theorem 2.1.1, we have $t\left(\tilde{\beta}_{r_1}^{r_2}\right) \to +\infty$ as $n \to \infty$. Consequently,

$$GSADFt(r_0) = \sup_{r_2 \in [r_0, 1], r_1 \in [0, r_2 - r_0]} t\left(\tilde{\beta}_{r_1}^{r_2}\right) \to +\infty,$$
$$BSADFt(r_2) = \sup_{r_1 \in [0, r_2 - r_0]} t\left(\tilde{\beta}_{r_1}^{r_2}\right) \to +\infty, \text{ if } r_0 + r_c < r_2 < 1.$$

Proof of Theorem 3.1.1: In any subsample period where $n_1 \in T$ and $n_2 \in T$, $\{p_t\}$ is generated by a polynomial trend model. Without loss of generality, we give the proof for the case where $n_1 = n_c + 3$. The same approach can be applied to the case where $n_1 > n_c + 3$. Let $n_s = n_2 - n_1$ and $\sum = \sum_{n_1+3}^{n_2}$.

When $t > n_c$, it has $p_t = p_{n_c} + \delta (t - n_c)^m + \varepsilon_t$. In the following, we prove the case where $m = 2$. The proof for the case $m > 2$ will be given in the Online Supplement. When $m = 2$, for any $t \geq n_c + 3$, the second-order difference of p_t has the representation of $\Delta^2 p_t = 2\delta + \Delta^2\varepsilon_t$, which leads to

$$p_t = 2\delta + p_{t-1} + \Delta p_{t-1} + \Delta^2\varepsilon_t.$$

Hence, for the regression model (1.2), it has

$$\begin{bmatrix} \breve{\alpha}_{r_1}^{r_2} - 2\delta \\ \breve{\beta}_{r_1}^{r_2} - 1 \\ \breve{\psi}_{r_1}^{r_2} - 1 \end{bmatrix} = \left(\sum \begin{bmatrix} 1 & p_{t-1} & \Delta p_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\Delta p_{t-1} \\ \Delta p_{t-1} & p_{t-1}\Delta p_{t-1} & (\Delta p_{t-1})^2 \end{bmatrix} \right)^{-1} \begin{bmatrix} \sum \Delta^2 \varepsilon_t \\ \sum p_{t-1}\Delta^2\varepsilon_t \\ \sum \Delta p_{t-1}\Delta^2\varepsilon_t \end{bmatrix}.$$

Let $j = t - n_c$. It is easy to get that as $n \to \infty$,

$$\begin{aligned} n_s^{-2} \sum p_{t-1}\Delta^2\varepsilon_t &= n_s^{-2} \sum \left[p_{n_c} + \delta (t-1-n_c)^2 + \varepsilon_{t-1} \right] \Delta^2 \varepsilon_t \\ &= n_s^{-2} \sum \left[\delta (t-1-n_c)^2 \right] \Delta^2 \varepsilon_t + O_p\left(n_s^{-1}\right) \\ &= \delta n_s^{-2} \left[(n_s - 1)^2 \Delta\varepsilon_{n_2} - 2 \sum_{j=2}^{n_s-1} j \Delta\varepsilon_{n_c+j} + \sum_{j=3}^{n_s-1} \Delta\varepsilon_{n_c+j} \right] + O_p\left(n_s^{-1}\right) \\ &= \delta \Delta\varepsilon_{n_2} + O_p\left(n_s^{-1}\right), \end{aligned}$$

and

$$\begin{aligned} n_s^{-1} \sum \Delta p_{t-1}\Delta^2\varepsilon_t &= n_s^{-1} \sum \left(2\delta(t-n_c) - 3\delta + \Delta\varepsilon_{t-1}\right) \Delta^2\varepsilon_t \\ &= 2\delta n_s^{-1} (n_2 - n_c) \Delta\varepsilon_{n_2} + n_s^{-1} \sum \Delta\varepsilon_{t-1}\Delta^2\varepsilon_t + O_p\left(n_s^{-1}\right) \\ &= 2\delta\Delta\varepsilon_{n_2} - 3\sigma^2 + o_p(1). \end{aligned}$$

We then have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & n_s^{-2} & 0 \\ 0 & 0 & n_s^{-1} \end{pmatrix} \begin{bmatrix} \sum \Delta^2 \varepsilon_t \\ \sum p_{t-1}\Delta^2\varepsilon_t \\ \sum \Delta p_{t-1}\Delta^2\varepsilon_t \end{bmatrix} = \begin{bmatrix} \Delta\varepsilon_{n_2} - \Delta\varepsilon_{n_c+2} \\ \delta\Delta\varepsilon_{n_2} \\ 2\delta\Delta\varepsilon_{n_2} - 3\sigma^2 \end{bmatrix} + o_p(1).$$

Next, note that, as $n \to \infty$,

$$\begin{aligned} n_s^{-2} \sum \Delta p_{t-1} &= n_s^{-2} \left(p_{n_2-1} - p_{n_1+1} \right) = \delta + O_p\left(n_s^{-1}\right), \\ n_s^{-4} \sum p_{t-1}\Delta p_{t-1} &= n_s^{-4} \sum \left[p_{n_c} + \delta(t-n_c-1)^2 + \varepsilon_{t-1} \right] \left[2\delta(t-n_c) - 3\delta + \Delta\varepsilon_{t-1} \right] \\ &= 2\delta^2 n_s^{-4} \sum (t-n_c)^3 + O_p\left(n_s^{-1}\right) = \delta^2/2 + O_p\left(n_s^{-1}\right), \\ n_s^{-3} \sum (\Delta p_{t-1})^2 &= n_s^{-3} \sum \left(2\delta(t-n_c) - 3\delta + \Delta\varepsilon_{t-1}\right)^2 \\ &= \frac{4\delta^2}{n_s^3} \sum (t-n_c)^2 + O_p\left(n_s^{-1}\right) = \frac{4\delta^2}{3} + O_p\left(n_s^{-1}\right). \end{aligned}$$

Together with the following limits

$$\begin{aligned} n_s^{-3} \sum p_{t-1} &= n_s^{-3} \sum \left(p_{n_c} + \delta(t-n_c-1)^2 + \varepsilon_{t-1} \right) = \delta/3 + O_p\left(n_s^{-1}\right), \\ n_s^{-5} \sum p_{t-1}^2 &= n_s^{-5} \sum \left(p_{n_c} + \delta(t-n_c-1)^2 + \varepsilon_{t-1} \right)^2 = \delta^2/5 + O_p\left(n_s^{-1}\right), \end{aligned}$$

we can get

$$\begin{aligned} &\begin{pmatrix} n_s & 0 & 0 \\ 0 & n_s^3 & 0 \\ 0 & 0 & n_s^2 \end{pmatrix} \left(\sum \begin{bmatrix} 1 & p_{t-1} & \Delta p_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\Delta p_{t-1} \\ \Delta p_{t-1} & p_{t-1}\Delta p_{t-1} & (\Delta p_{t-1})^2 \end{bmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & n_s^2 & 0 \\ 0 & 0 & n_s \end{pmatrix} \\ &= \begin{bmatrix} 1 & \delta/3 & \delta \\ \delta/3 & \delta^2/5 & \delta^2/2 \\ \delta & \delta^2/2 & 4\delta^2/3 \end{bmatrix}^{-1} + O_p\left(n_s^{-1}\right). \end{aligned}$$

As a result, we have

$$\begin{pmatrix} n_s & 0 & 0 \\ 0 & n_s^3 & 0 \\ 0 & 0 & n_s^2 \end{pmatrix} \begin{bmatrix} \breve{\alpha}_{r_1}^{r_2} - 2\delta \\ \breve{\beta}_{r_1}^{r_2} - 1 \\ \breve{\psi}_{r_1}^{r_2} - 1 \end{bmatrix} = \begin{bmatrix} 1 & \delta/3 & \delta \\ \delta/3 & \delta^2/5 & \delta^2/2 \\ \delta & \delta^2/2 & 4\delta^2/3 \end{bmatrix}^{-1} \begin{bmatrix} \Delta\varepsilon_{n_2} - \Delta\varepsilon_{n_c+2} \\ \delta\Delta\varepsilon_{n_2} \\ 2\delta\Delta\varepsilon_{n_2} - 3\sigma^2 \end{bmatrix} + o_p(1),$$

which leads to

$$n_s\left(\breve{\beta}_{r_1}^{r_2} - 1\right) = O_p\left(n_s^{-2}\right) \xrightarrow{p} 0, \text{ as } n \to \infty.$$

To get the limit of the DF t statistic, we first study the large sample theory of $se\left(\breve{\beta}_{r_1}^{r_2}\right)$ in Model (1.2). Note that $p_t = 2\delta + p_{t-1} + \Delta p_{t-1} + \Delta^2\varepsilon_t$. Together with the fact that $\breve{\alpha}_{r_1}^{r_2}$, $\breve{\beta}_{r_1}^{r_2}$, and $\breve{\psi}_{r_1}^{r_2}$ are all consistent as proved above, it is straightforward to get that

$$\frac{\sum \breve{e}_t^2}{n_s} = \frac{1}{n_s}\sum\left(p_t - \breve{\alpha}_{r_1}^{r_2} - \breve{\beta}_{r_1}^{r_2} p_{t-1} - \breve{\psi}_{r_1}^{r_2}\Delta p_{t-1}\right)^2 = \frac{\sum\left(\Delta^2\varepsilon_t\right)^2}{n_s} + o_p(1) = 6\sigma^2 + o_p(1).$$

Consequently, it has

$$\begin{aligned} n_s^5\left[se\left(\breve{\beta}_{r_1}^{r_2}\right)\right]^2 &= n_s^5 (0\ 1\ 0)\left(\sum\begin{bmatrix} 1 & p_{t-1} & \Delta p_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\Delta p_{t-1} \\ \Delta p_{t-1} & p_{t-1}\Delta p_{t-1} & (\Delta p_{t-1})^2 \end{bmatrix}\right)^{-1}\begin{pmatrix}0\\1\\0\end{pmatrix}\frac{\sum \breve{e}_t^2}{n_s - 5} \\ &= (0\ 1\ 0)\begin{bmatrix} 1 & \delta/3 & \delta \\ \delta/3 & \delta^2/5 & \delta^2/2 \\ \delta & \delta^2/2 & 4\delta^2/3 \end{bmatrix}^{-1}\begin{pmatrix}0\\1\\0\end{pmatrix} 6\sigma^2 + o_p(1). \end{aligned}$$

Finally, the limit of the DF t statistic is obtained as

$$t\left(\breve{\beta}_{r_1}^{r_2}\right) = n_s^{-1/2}\frac{n_s^3\left(\breve{\beta}_{r_1}^{r_2} - 1\right)}{n_s^{5/2} se\left(\breve{\beta}_{r_1}^{r_2}\right)} = O_p\left(n_s^{-1/2}\right).$$

Proof of Theorem 3.1.2: We give the proof for the case where $m = 2$. The same approach applies directly to the case where $m > 2$, but with tedious details. As $n_1 \in N$ and $n_2 \in T$, it has $n_1 < n_c < n_2$. With the subsample from n_1 to n_2, the AR(2) regression (1.2) leads to the following LS estimators:

$$\begin{bmatrix} \breve{\alpha}_{r_1}^{r_2} \\ \breve{\beta}_{r_1}^{r_2} \\ \breve{\psi}_{r_1}^{r_2} \end{bmatrix} = Q^{-1}\sum_{t=n_1}^{n_2}\begin{bmatrix} 1 \\ p_{t-1} \\ \Delta p_{t-1} \end{bmatrix} p_t \text{ with } Q = \sum_{t=n_1}^{n_2}\begin{bmatrix} 1 & p_{t-1} & \Delta p_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\Delta p_{t-1} \\ \Delta p_{t-1} & p_{t-1}\Delta p_{t-1} & (\Delta p_{t-1})^2 \end{bmatrix}.$$

When $n_1 \le t < n_c$, the DGP is $p_t = p_{t-1} + \varepsilon_t$. Whereas, when $n_c < t \le n_2$, p_t follows a quadratic trend model that is

$$p_t = p_c + \delta(t - n_c)^2 + \varepsilon_t = 2\delta + p_{t-1} + \Delta p_{t-1} + \Delta^2\varepsilon_t.$$

Hence, it has

$$Q = \sum_{t=n_1}^{n_c}\begin{bmatrix} 1 & p_{t-1} & \varepsilon_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\varepsilon_{t-1} \\ \varepsilon_{t-1} & p_{t-1}\varepsilon_{t-1} & \varepsilon_{t-1}^2 \end{bmatrix} + \sum_{t=n_c+1}^{n_2}\begin{bmatrix} 1 & p_{t-1} & \Delta p_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\Delta p_{t-1} \\ \Delta p_{t-1} & p_{t-1}\Delta p_{t-1} & (\Delta p_{t-1})^2 \end{bmatrix}$$

$$\sum_{t=n_1}^{n_c}\begin{bmatrix} 1 \\ p_{t-1} \\ \Delta p_{t-1} \end{bmatrix} p_t = \sum_{t=n_1}^{n_c}\begin{bmatrix} 1 & p_{t-1} & \varepsilon_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\varepsilon_{t-1} \\ \varepsilon_{t-1} & p_{t-1}\varepsilon_{t-1} & \varepsilon_{t-1}^2 \end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} + \sum_{t=n_1}^{n_c}\begin{bmatrix} 1 \\ p_{t-1} \\ \varepsilon_{t-1} \end{bmatrix}\varepsilon_t,$$

$$\sum_{t=n_c+1}^{n_2}\begin{bmatrix} 1 \\ p_{t-1} \\ \Delta p_{t-1} \end{bmatrix} p_t = \sum_{t=n_c+1}^{n_2}\begin{bmatrix} 1 & p_{t-1} & \Delta p_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\Delta p_{t-1} \\ \Delta p_{t-1} & p_{t-1}\Delta p_{t-1} & (\Delta p_{t-1})^2 \end{bmatrix}\begin{bmatrix}2\delta\\1\\1\end{bmatrix} + \sum_{t=n_c+1}^{n_2}\begin{bmatrix} 1 \\ p_{t-1} \\ \Delta p_{t-1} \end{bmatrix}\Delta^2\varepsilon_t.$$

Consequently, we have

$$\begin{bmatrix} \check{\alpha}_{r_1}^{r_2} - 2\delta \\ \check{\beta}_{r_1}^{r_2} - 1 \\ \check{\psi}_{r_1}^{r_2} - 1 \end{bmatrix} = Q^{-1} \left\{ \sum_{t=n_1}^{n_c} \begin{bmatrix} 1 & p_{t-1} & \varepsilon_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\varepsilon_{t-1} \\ \varepsilon_{t-1} & p_{t-1}\varepsilon_{t-1} & \varepsilon_{t-1}^2 \end{bmatrix} \begin{bmatrix} -2\delta \\ 0 \\ -1 \end{bmatrix} + \sum_{t=n_1}^{n_c} \begin{bmatrix} 1 \\ p_{t-1} \\ \varepsilon_{t-1} \end{bmatrix} \varepsilon_t + \sum_{t=n_c+1}^{n_2} \begin{bmatrix} 1 \\ p_{t-1} \\ \Delta p_{t-1} \end{bmatrix} \Delta^2 \varepsilon_t \right\}.$$

Note that, as $n \to \infty$, it has

$$\begin{pmatrix} n^{-1} & & \\ & n^{-3} & \\ & & n^{-2} \end{pmatrix} \sum_{t=n_1}^{n_c} \begin{bmatrix} 1 & p_{t-1} & \varepsilon_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\varepsilon_{t-1} \\ \varepsilon_{t-1} & p_{t-1}\varepsilon_{t-1} & \varepsilon_{t-1}^2 \end{bmatrix} \begin{pmatrix} 1 & & \\ & n^{-2} & \\ & & n^{-1} \end{pmatrix} \Rightarrow \begin{pmatrix} r_c - r_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} n^{-1} & & \\ & n^{-3} & \\ & & n^{-2} \end{pmatrix} \sum_{t=n_c+1}^{n_2} \begin{bmatrix} 1 & p_{t-1} & \Delta p_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\Delta p_{t-1} \\ \Delta p_{t-1} & p_{t-1}\Delta p_{t-1} & (\Delta p_{t-1})^2 \end{bmatrix} \begin{pmatrix} 1 & & \\ & n^{-2} & \\ & & n^{-1} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} r_2 - r_c & \delta (r_2 - r_c)^3/3 & \delta (r_2 - r_c)^2 \\ \delta (r_2 - r_c)^3/3 & \delta^2 (r_2 - r_c)^5/5 & \delta^2 (r_2 - r_c)^4/2 \\ \delta (r_2 - r_c)^2 & \delta^2 (r_2 - r_c)^4/2 & 4\delta^2 (r_2 - r_c)^3/3 \end{pmatrix},$$

where the first limit comes from the large sample properties of the unit root process, and the second limit is obtained directly from the large sample properties of the quadratic trend process that have been explored in the proof of Theorem 3.1.1. Therefore, we get

$$\begin{pmatrix} 1 & & \\ & n^2 & \\ & & n^1 \end{pmatrix} Q^{-1} \begin{pmatrix} n & & \\ & n^3 & \\ & & n^2 \end{pmatrix} \Rightarrow \begin{pmatrix} r_2 - r_1 & \delta (r_2 - r_c)^3/3 & \delta (r_2 - r_c)^2 \\ \delta (r_2 - r_c)^3/3 & \delta^2 (r_2 - r_c)^5/5 & \delta^2 (r_2 - r_c)^4/2 \\ \delta (r_2 - r_c)^2 & \delta^2 (r_2 - r_c)^4/2 & 4\delta^2 (r_2 - r_c)^3/3 \end{pmatrix}^{-1}.$$

Moreover, from the large sample properties of the unit root process, we can have

$$\begin{pmatrix} n^{-1} & & \\ & n^{-3} & \\ & & n^{-2} \end{pmatrix} \sum_{t=n_1}^{n_c} \begin{bmatrix} 1 & p_{t-1} & \varepsilon_{t-1} \\ p_{t-1} & p_{t-1}^2 & p_{t-1}\varepsilon_{t-1} \\ \varepsilon_{t-1} & p_{t-1}\varepsilon_{t-1} & \varepsilon_{t-1}^2 \end{bmatrix} \begin{bmatrix} -2\delta \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{pmatrix} n^{-1} & & \\ & n^{-3} & \\ & & n^{-2} \end{pmatrix} \sum_{t=n_1}^{n_c} \begin{bmatrix} -2\delta - \varepsilon_{t-1} \\ -2\delta p_{t-1} - p_{t-1}\varepsilon_{t-1} \\ -2\delta \varepsilon_{t-1} - \varepsilon_{t-1}^2 \end{bmatrix} \Rightarrow \begin{bmatrix} -2\delta (r_c - r_1) \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{pmatrix} n^{-1} & & \\ & n^{-3} & \\ & & n^{-2} \end{pmatrix} \sum_{t=n_1}^{n_c} \begin{bmatrix} 1 \\ p_{t-1} \\ \varepsilon_{t-1} \end{bmatrix} \varepsilon_t \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

As proved in Theorem 3.1.1, it has

$$\begin{pmatrix} n^{-1} & & \\ & n^{-3} & \\ & & n^{-2} \end{pmatrix} \sum_{t=n_c+1}^{n_2} \begin{bmatrix} 1 \\ p_{t-1} \\ \Delta p_{t-1} \end{bmatrix} \Delta^2 \varepsilon_t = n^{-1} \begin{bmatrix} O_p(1) \\ O_p(1) \\ O_p(1) \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, we have

$$\begin{pmatrix} 1 & & \\ & n^2 & \\ & & n \end{pmatrix} \begin{bmatrix} \check{\alpha}_{r_1}^{r_2} - 2\delta \\ \check{\beta}_{r_1}^{r_2} - 1 \\ \check{\psi}_{r_1}^{r_2} - 1 \end{bmatrix} \Rightarrow \begin{pmatrix} r_2 - r_1 & \delta (r_2 - r_c)^3/3 & \delta (r_2 - r_c)^2 \\ \delta (r_2 - r_c)^3/3 & \delta^2 (r_2 - r_c)^5/5 & \delta^2 (r_2 - r_c)^4/2 \\ \delta (r_2 - r_c)^2 & \delta^2 (r_2 - r_c)^4/2 & 4\delta^2 (r_2 - r_c)^3/3 \end{pmatrix}^{-1} \begin{bmatrix} -2\delta (r_c - r_1) \\ 0 \\ 0 \end{bmatrix},$$

which leads to

$$n^2 \left(\check{\beta}_{r_1}^{r_2} - 1 \right) \Rightarrow -\frac{60 (r_c - r_1)}{(r_2 - r_c)^2 [9 (r_2 - r_1) - 8 (r_2 - r_c)]},$$

and

$$(n_2 - n_1)^2 \left(\check{\beta}_{r_1}^{r_2} - 1 \right) \Rightarrow -\frac{60 (r_c - r_1)(r_2 - r_1)^2}{(r_2 - r_c)^2 [9 (r_2 - r_1) - 8 (r_2 - r_c)]}.$$

To derive the large sample theory of $t\left(\breve{\beta}_{r_1}^{r_2}\right)$, we first study the limit of $\sum_{t=n_1}^{n_2} \breve{e}_t^2 = \sum_{t=n_1}^{n_c} (\breve{e}_t)^2 + \sum_{t=n_c+1}^{n_2} (\breve{e}_t)^2$, where $\breve{e}_t = p_t - \breve{\alpha}_{r_1}^{r_2} - \breve{\beta}_{r_1}^{r_2} p_{t-1} - \breve{\psi}_{r_1}^{r_2} \Delta p_{t-1}$. From the asymptotic properties of the unit root process, it can be proved that, as $n \to \infty$,

$$n^{-1} \sum_{t=n_1}^{n_c} \breve{e}_t^2 = n^{-1} \left[\sum_{t=n_1}^{n_c} \varepsilon_t^2 + \sum_{t=n_1}^{n_c} \left(\breve{\alpha}_{r_1}^{r_2}\right)^2 + \breve{\psi}_{r_1}^{r_2} \sum_{t=n_1}^{n_c} \varepsilon_{t-1}^2 - 2\breve{\psi}_{r_1}^{r_2} \sum_{t=n_1}^{n_c} \varepsilon_t \varepsilon_{t-1} \right] + o_p(1) = O_p(1),$$

where the last two equalities come from the facts that $\breve{\alpha}_{r_1}^{r_2} - 2\delta = O_p(1)$, $n^2\left(\breve{\beta}_{r_1}^{r_2} - 1\right) = O_p(1)$, and $n\left(\breve{\psi}_{r_1}^{r_2} - 1\right) = O_p(1)$. Moreover, from the asymptotic properties of the quadratic trend model, we can get

$$\begin{aligned} n^{-1} \sum_{t=n_c+1}^{n_2} \breve{e}_t^2 = n^{-1} \sum_{t=n_c+1}^{n_2} & \Big\{ \left(\Delta^2 \varepsilon_t\right)^2 + \left(\breve{\alpha}_{r_1}^{r_2} - 2\delta\right)^2 + \left(\breve{\beta}_{r_1}^{r_2} - 1\right)^2 p_{t-1}^2 + \left(\breve{\psi}_{r_1}^{r_2} - 1\right)^2 (\Delta p_{t-1})^2 \\ & -2\left(\breve{\psi}_{r_1}^{r_2} - 1\right) \Delta p_{t-1} \Delta^2 \varepsilon_t + 2\left(\breve{\alpha}_{r_1}^{r_2} - 2\delta\right)\left(\breve{\beta}_{r_1}^{r_2} - 1\right) p_{t-1} \\ & +2\left(\breve{\alpha}_{r_1}^{r_2} - 2\delta\right)\left(\breve{\psi}_{r_1}^{r_2} - 1\right) \Delta p_{t-1} + 2\left(\breve{\beta}_{r_1}^{r_2} - 1\right)\left(\breve{\psi}_{r_1}^{r_2} - 1\right) p_{t-1} \Delta p_{t-1} \Big\} + o_p(1) \\ = {} & O_p(1). \end{aligned}$$

Hence, $n^{-1} \sum_{t=n_1}^{n_2} (\breve{e}_t)^2 = O_p(1)$. We then have

$$n^5 \left[se\left(\breve{\beta}_{r_1}^{r_2}\right)\right]^2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & n^2 & \\ & & n \end{pmatrix} Q^{-1} \begin{pmatrix} n & & \\ & n^3 & \\ & & n^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \frac{\sum_{t=n_1}^{n_2} \breve{e}_t^2}{n_2 - n_1} = O_p(1).$$

Therefore, it has

$$t\left(\breve{\beta}_{r_1}^{r_2}\right) = \frac{\breve{\beta}_{r_1}^{r_2} - 1}{se\left(\breve{\beta}_{r_1}^{r_2}\right)} = n^{1/2} \frac{n^2\left(\breve{\beta}_{r_1}^{r_2} - 1\right)}{n^{5/2} se\left(\breve{\beta}_{r_1}^{r_2}\right)} \to -\infty,$$

where the last limit comes from the fact that $\lim_{n\to\infty} n^2\left(\breve{\beta}_{r_1}^{r_2} - 1\right) < 0$, as proved above.