Limit Theory for Dating the Origination and Collapse of Mildly Explosive Periods in Time Series Data*

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Abstract

Some limit theory is developed for estimators suggested in Phillips, Wu and Yu (2009) for dating bubble pheonoma in time series data. The models involve mildly explosive autoregressions and the tests rely on right sided recursive unit root tests. The estimates locate the origination and collapse dates of bubbles involving mildly explosive episodes set within longer periods where the data evolve as a stochastic trend. The dating estimators are shown to be consistent under mild regularity conditions on the process. Some simulation evidence on the performance of the estimators is reported. The proposed method works well in finite samples and conforms with the limit theory.

Keywords: Bubble, Date stamping, Explosive behavior, Mildly explosive process, Right sided unit root tests, Structural breaks.

JEL classification: C15, G12

1 Introduction

The global financial turmoil over 2008-2009 and its effects on real economic activity have led to renewed interest among economists in financial bubbles. One important econometric aspect of this phenomenon is date stamping. Dating the origination and collapse of financial asset bubbles is obviously of interest in its own right but it is also important from an economic perspective.

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Date stamping provides precision in the assessment of bubble phenomena, providing information on the temporal extent, the magnitude, and the course of the bubble. Dating also facilitates comparisons between alternative economic hypotheses about bubbles. For example, in a recent study, Caballero, Farhi and Gourinchas (2009) put forward a sequential hypothesis about bubble creations and collapses during the course of the current financial turmoil in the U.S. economy. According to this hypothesis, asset bubbles emerged and collapsed creating in their wake a sequence of bubble effects in other asset markets. Liquidity shortages crashed the real estate bubble but then created bubbles in commodities and oil markets as investors transferred financial assets. The deepening financial crisis then sharply slowed down economic growth, which in turn destroyed the commodity bubbles. This story makes strong predictions concerning the timing of the origination and the collapse of various bubble phenomena in different markets. To evaluate the evidence in support of such interpretations of the events, consistent date stamping of those events is critical.

The present paper seeks to provide a rigorous econometric approach to dating bubble phenomena. In particular, this paper derives a limit theory for estimators of the origination and collapse dates of mildly explosive bubble episodes that were proposed in recent work by Phillips, Wu and Yu (2009, PWY hereafter). PWY used forward recursive right sided unit root tests and mildly explosive regression asymptotics to assess empirical evidence for explosive behavior. These methods were shown capable of detecting the existence of mildly explosive episodes in time series data and were used to date stamp the origination and collapse of the Nasdaq bubble in the 1990s, corresponding to Greenspan's famous remark about irrational exuberance in financial market which was made in December 1996.

One of the contributions of the current paper is to show that the PWY approach provides consistent estimates of the timing of bubble episodes. The paper also examines the finite sample performance of the estimators in simulations. The asymptotic and finite sample results both indicate that the method is reliable for data stamping the origination and collapse dates of bubble episodes in time series data.

The remainder of the paper is organized as follows. Section 2 develops an econometric model of financial bubbles based on the successive conjunction of unit root and mildly explosive processes, so that regime shifts in the model are contained within the same model family and involve only moderate (local) changes in the autoregressive coefficient. This section also proposes date stamping estimators using recursive regressions and right sided unit root tests of the type considered in PWY. Section 3 derives the limit theory for this dating procedure showing that the estimates of both origination and termination of the bubble are consistent. Section 4 checks the finite sample performance of these estimates. Section 5 concludes. Appendices collect together the proof of the main results in the paper.

2 Econometric models of bubble behavior

PWY propose the use of recursive regression techniques for the detection of bubble behavior. The key idea is to use right sided unit root tests to assess evidence for mildly explosive behavior in the data. In particular, for time series $\{X_t\}_{t=1}^n$, we apply unit root tests (such as Dickey-Fuller or Augmented Dickey-Fuller tests) based on either the estimated coefficient or the t statistic and with standard null asymptotics against the alternative of an explosive or mildly explosive root, so that the test is a right sided test, as distinct from the standard left sided tests for stationarity. The recursions use subsets of the sample data incremented by one observation at each pass. That is, we estimate the following autoregressive specification (or suitably augmented versions) by least squares

$$X_t = \mu + \delta X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{ iid } (0, \sigma^2). \tag{1}$$

The null hypothesis is $H_0 : \delta = 1$ and the right-tailed alternative hypothesis is $H_1 : \delta > 1$. If the first regression involves $\tau_0 = [nr_0]$ observations, for some fraction r_0 of the total sample where $[\,]$ denotes the integer part of its argument, subsequent regressions employ this originating data set supplemented by successive observations giving a sample of size $\tau = [nr]$ for $r_0 \leq r \leq 1$. Denote the corresponding Dickey-Fuller test statistics by DF_r^δ and DF_r^t, namely,

$$DF_r^\delta := \tau\left(\hat{\delta}_\tau(\tau) - 1\right), \quad DF_r^t := \left(\frac{\sum_{j=1}^{\tau} \tilde{X}_{j-1}^2}{\hat{\sigma}_\tau^2}\right)^{1/2} \left(\hat{\delta}_\tau(\tau) - 1\right), \tag{2}$$

where $\hat{\delta}_\tau$ is the least squares estimate of δ based on the first $\tau = [nr]$ observations, $\hat{\sigma}_\tau^2$ is the corresponding estimate of σ^2, and $\tilde{X}_{j-1} = X_{j-1} - \tau^{-1}\sum_{j=1}^{\tau} X_{j-1}$. Obviously, DF_1^δ and DF_1^t correspond to the full sample test statistics. Under the null hypothesis of the pure unit root dynamics and standard weak convergence methods, we have the following limit theory as $\tau = [nr] \to \infty$ for all $r \in [r_0, 1]$

$$DF_r^\delta \Rightarrow \frac{\int_0^1 \widetilde{W} dW}{\int_0^1 \widetilde{W}^2}, \quad DF_r^t \Rightarrow \frac{\int_0^1 \widetilde{W} dW}{\left(\int_0^1 \widetilde{W}^2\right)^{1/2}},$$

where W is standard Brownian motion and $\widetilde{W}(r) = W(r) - \int_0^1 W$ is demeaned Brownian motion.

There are two major advantages in the forward recursive regression approach. First, it significantly improves the power of the full sample tests that have been considered earlier in the literature (e.g. Diba and Grossman, 1988) when the bubbles are subject to collapse. The improvements are especially large when the probability of a bubble originating and collapsing is small but positive, which is an empirically realistic situation leading to a single bubble in models such as that of Evans (1994). Second, it allows one to estimate the origination and collapse of a bubble through sequential analysis of the recursion.

The idea is as follows: We date the origination of an explosive episode as $\hat{\tau}_e = [n\hat{r}_e]$ where

$$\hat{r}_e = \inf_{s \geq r_0} \left\{ s : DF_s^{\delta} > cv_{\beta_n}^{\delta} \right\}, \text{ or } \hat{r}_e = \inf_{s \geq r_0} \left\{ s : DF_s^{t} > cv_{\beta_n}^{df} \right\}, \tag{3}$$

and $cv_{\beta_n}^{\delta}$ ($cv_{\beta_n}^{df}$) is the right side $100\beta_n\%$ critical value of the limit distribution of the DF_r^{δ} (DF_r^{t}) statistic based on $\tau_s = [ns]$ observations, and β_n is the size of the one-sided test. Conditional on finding some originating date $\hat{r}_e$ for explosive behavior, we date the collapse of the explosive episode by $\hat{\tau}_f = [n\hat{r}_f]$ where

$$\hat{r}_f = \inf_{s \geq \hat{r}_e + \frac{\log(n)}{n}} \left\{ s : DF_s^{\delta} < cv_{\beta_n}^{\delta} \right\}, \text{ or } \hat{r}_f = \inf_{s \geq \hat{r}_e + \frac{\log(n)}{n}} \left\{ s : DF_s^{t} < cv_{\beta_n}^{df} \right\}. \tag{4}$$

The dating rule (4) for determining the collapse date $\hat{r}_f$ involves a search over $\tau_s \geq [n\hat{r}_e + \log(n)]$ in the recursive comparisons of DF_s^{δ} and DF_s^{t}. This rule ensures that the duration of the bubble is significant – at least a small infinity, i.e., episodes of smaller order than $O(\log n)$ are not considered significant in the dating algorithm for τ_f.

This recursive method can be used in the same way in connection with other unit root tests, such as the Augmented Dickey-Fuller (ADF) test and Phillips-Perron tests developed in Phillips (1987) and Phillips and Perron (1988). Since there is no material change in the use of these more general procedures and no change in the asymptotic theory, the analysis that follows will deal with the simple model (1). Extensions to the general semiparametric case will be examined later.

There many ways to model financial exuberance, bubble formation and collapsing behavior. For example, one mechanism that can capture exuberance is to allow for a period of mildly explosive behavior, as in the following model

$$\begin{aligned} X_t &= X_{t-1} 1\{t < \tau_e \text{ or } t > \tau_f\} + \delta_n X_{t-1} 1\{\tau_e \leq t \leq \tau_f\} + \varepsilon_t, \ t = 1, ..., \tau \\ \delta_n &= 1 + \frac{c}{n^{\alpha}}, c > 0, \alpha \in (0,1), \end{aligned} \tag{5}$$

where ε_t is $\text{iid}(0, \sigma^2)$ and initialization of the process is assumed to occur at $t = 0$ from some $X_0 = O_p(1)$. As indicated, it is possible to extend the theory to allow for weakly dependent ε_t, and the main results given below will continue to hold for such extensions under some regularity conditions analogous to those in Phillips (1987).

The autoregressive parameter $\delta_n = 1 + \frac{c}{n^\alpha} > 1$ for all n when $c > 0$ and δ_n is, in the terminology of Phillips and Magdalinos (2007a, hereafter PM_a) a mildly integrated root (as distinct from a root local to unity) on the explosive side of unity and this root correspondingly leads to mildly explosive behavior in the data over the subperiod $t \in [\tau_e, \tau_f]$. Model (5) allows for two regimes – a unit root regime and an explosive regime. The system switches regimes from unit root behavior to mildly explosive behavior at τ_e and from the explosive root back to a unit root at τ_f. In this model, the process X_t does not "collapse" at τ_f but rather resumes unit root wandering behavior from the new level X_{τ_f}. The mechanism of mildly explosive growth in this case effectively changes the level of the martingale to a new plateau of origination. If there is no mildly explosive episode, then $c = 0$ and $\delta_n = 1$.

An alternative mechanism that can capture both exuberance and collapse, involves re-initialization of the process under collapse, possibly with some transitional dynamics. For instance, the following model specifies a new initial value, or re-initialization of the process, when the explosive period comes to the end, so that the initial value of the new unit root period differs from the end value of the explosive period

$$\begin{aligned} X_t &= X_{t-1} 1\{t < \tau_e\} + \delta_n X_{t-1} 1\{\tau_e \le t \le \tau_f\} \\ &\quad + \left(\sum_{k=\tau_f+1}^{t} \varepsilon_k + X_{\tau_f}^* \right) 1\{t > \tau_f\} + \varepsilon_t \, 1\{t \le \tau_f\} \\ \delta_n &= 1 + \frac{c}{n^\alpha}, \quad c > 0, \quad \alpha \in (0,1). \end{aligned} \tag{6}$$

With the re-initialization at τ_f, the process jumps to a different level $X_{\tau_f}^*$. The new initial value $X_{\tau_f}^*$ may be related to the earlier period of martingale behavior in the process, perhaps with some random deviation, in which case we would have $X_{\tau_f}^* = X_{\tau_e} + X^*$ for some $O_p(1)$ random quantity X^*. The model may be further adapted to allow for a short period transitional dynamic, which could be mean reverting to the level $X_{\tau_f}^*$. As in (5), the model (6) is assumed to initiate at $t = 0$ from some $O_p(1)$ random variable X_0.

An intercept may be added to the model formulations (5) and (6), as in the fitted autoregression (1), but this has consequences on the properties of X_t. In particular, a non zero intercept in (5) and (6) introduces drift to the process X_t for $t < \tau_e$, which can be realistic, at least for small μ, in empirical applications during periods where there is a unit root and stochastic trend in the data. However, during periods of exuberance for $\tau_e \le t \le \tau_f$, where there is a mildly explosive root in the autoregression, a non zero intercept leads to a dominating deterministic component in X_t with the following explosive form

$$\mu \frac{\delta_n^{t-\tau_e} - 1}{(\delta_n - 1)} = \frac{\mu n^\alpha}{c} e^{cn^{1-\alpha}(r-r_e)} \{1 + o(1)\},$$

which is empirically unrealistic for most economic and financial time series. Although an intercept is included in the fitted regression (1), we therefore anticipate that the generating mechanism (6) is unlikely to involve a non zero intercept, at least during mildly explosive episodes. The theory below is therefore given for the generating mechanism (6). In fact, the limit theory given here continues to hold in the case where there is a non-zero intercept in the model, including the empirically relevant case where the intercept is non zero only during the episodes of unit root behavior. This extension requires some modification in the proofs, and these will be indicated later.

Both models (5) and (6) may be analyzed using the methods in PM_a. In particular, PM_a show that a central limit theory can be developed for mildly explosive time series like those in (5) and (6), so that robust econometric inference is possible in this environment. The limit theory methods in PM_a and Phillips and Magdalinos (2007b, hereafter PM_b) are used extensively in the development of the asymptotic theory that follows. Our attention will focus on Model (6) because this model allows for both the initiation of mildly explosive behavior, representing an episode of market exuberance, and the subsequent collapse of that exuberance to a realized earlier level of the process. Such a model has the inherent capability of capturing financial market behavior of the type witnessed during the recent financial crisis in several different asset markets.

3 Dating the Origination and Collapse of an Explosive Episode with Re-initialization

This section develops limit theory for dating the origination and collapse of a mildly explosive period under Model (6). We are particularly interested in establishing the consistency of the dating estimator using recursive regressions. In what follows it is assumed that recursive autoregressions are run with data $\{X_t : t = 1, 2, ..., \tau = [nr]\}$ originating from $\tau_0 = [nr_0]$, so that the minimum amount of data used for the regressions is τ_0. We date the origination of the explosive episode as $\hat{\tau}_e = [n\hat{r}_e]$ where

$$\hat{r}_e = \inf_{s \geq r_0} \left\{ s : DF_s^\delta > cv_{\beta_n}^\delta (s) \right\} \text{ or } \hat{r}_e = \inf_{s \geq r_0} \left\{ s : DF_s^t > cv_{\beta_n}^{DF} (s) \right\}, \tag{7}$$

and $cv_{\beta_n}^{\delta}(s)$ $(cv_{\beta_n}^{df}(s))$ is the right side $100\beta_n\%$ critical value of the two DF_s statistics based on $\tau_s = [ns]$ observations, and β_n is the size of the one-sided test. We will allow $\beta_n \to 0$ as $n \to \infty$ because in this event $cv_{\beta_n}^{\delta} \to \infty$ and $cv_{\beta_n}^{df} \to \infty$. Conditional on finding some originating date $\hat{r}_e$ for explosive behavior, we date the collapse of the explosive episode by $\hat{\tau}_f = [n\hat{r}_f]$ where

$$\hat{r}_f = \inf_{s \geq \hat{r}_e + \frac{\log(n)}{n}} \left\{ s : DF_s^{\delta} < cv_{\beta_n}^{\delta}(s) \right\} \quad \text{or} \quad \hat{r}_f = \inf_{s \geq \hat{r}_e + \frac{\log(n)}{n}} \left\{ s : DF_s^{t} < cv_{\beta_n}^{df}(s) \right\}. \tag{8}$$

It is assumed that $\tau_e = [nr_e]$ and $\tau_f = [nr_f]$ with $r_e < r_f$, so that asymptotically there is $O(n)$ of data (a large infinity of data) separating origination and collapse. In the recursive procedure for determining the collapse date $\hat{r}_f$ in (8), we construct the dating rule so that all data $\tau_s \geq [n\hat{r}_e + \log(n)]$ is considered in the evaluation of the recursive comparisons of DF_s^{δ} and DF_s^{t} against the critical values $cv_{\beta_n}^{\delta}(s)$ and $cv_{\beta_n}^{df}(s)$. As mentioned earlier, this dating rule ensures that small separations between τ_e and τ_f, viz. those of order $o(\log n)$, are not considered in the dating algorithm for τ_f.

3.1 Dating the Origination of an Explosive Episode

We first develop the limit theory under the null hypothesis and then consider detection of the origination and termination dates under the alternative.

THEOREM 3.1 *Under the null hypothesis of no episode of explosive behavior ($c = 0$ and $\delta_n = 1$ in Model (6)) and provided $cv_{\beta_n}^{\delta} \to \infty$ and $cv_{\beta_n}^{df} \to \infty$, the probability of detecting the origination of a bubble using DF^{δ} or DF^{t} is zero as $n \to \infty$, so that $P(\hat{r}_e \in [r_0, 1]) \to 0$.*

We now determine the limit behavior of $\hat{r}_e$ under the alternative hypothesis of the initiation of a period of mildly explosive behavior at $\tau_e = [nr_e]$ for some $r_e > r_0$. Model (6) implies that the process X_t follows the generating mechanism

$$X_t = X_{t-1} 1\{t < \tau_e\} + \delta_n X_{t-1} 1\{t \geq \tau_e\} + \varepsilon_t, \quad \delta_n = 1 + \frac{c}{n^{\alpha}}, \quad c > 0, \quad \alpha \in (0,1), \tag{9}$$

over $t \in [\tau_0, \tau_f]$, where ε_t is $\text{iid}(0, \sigma^2)$.

THEOREM 3.2

(i) *If*

$$\frac{1}{cv_{\beta_n}^{\delta}} + \frac{cv_{\beta_n}^{\delta}}{n^{1-\alpha}} \to 0, \tag{10}$$

then under the alternative hypothesis of mildly explosive behavior in Model (6), $\hat{r}_e \xrightarrow{p} r_e$ as $n \to \infty$, where $\hat{r}_e$ is obtained from the DF^{δ} test.

(ii) *If*

$$\frac{1}{cv_{\beta_n}^{df}} + \frac{cv_{\beta_n}^{df}}{n^{1-\alpha/2}} \to 0, \tag{11}$$

then under the alternative hypothesis of explosive behavior in Model (6), $\hat{r}_e \xrightarrow{p} r_e$ as $n \to \infty$,*where* $\hat{r}_e$ *is obtained from the* DF^t *test.*

Remarks

(a) As shown in the proof of the theorem, when data from the explosive period are included in estimating the autoregressive coefficient, these observations govern the asymptotics of the estimate since the signal from the explosive period dominates that from the unit root model. This difference in signal between the two periods provides identifying information and explains why the two test procedures consistently estimate the origination date.

(b) The DF test based on the coefficient diverges at the rate $O\left(n^{1-\alpha}\right)$, while the DF test based on the t statistic diverges at the rate $A_n = O\left(n^{1-\alpha/2}\right)$, as shown in (31) and (33) in the Appendix. We therefore expect these statistics to perform in a very similar way under the alternative hypothesis.

(c) For practical implementation, we might set the critical value sequences$\left\{cv_{\beta_n}^{\delta}, cv_{\beta_n}^{df}\right\}$ according to some (arbitrary) expansion rule such as $cv_{\beta_n}^{\delta} = \log\log^2 n$ and $cv_{\beta_n}^{df} = \frac{2}{3}\log\log^2 n$. Both critical values diverge at a slowly varying rate with $cv_{\beta_n}^{df} < cv_{\beta_n}^{\delta}$. This setting is justified by the fact that the critical values for the DF^t test are well known to be smaller (in magnitude) than those for DF^{δ}. (For example, the 1% asymptotic critical value for DF^{δ} is 1.04 whereas the 1% asymptotic critical value for DF^t is 0.60.) Noting that fixed (right side) critical values like $cv_{0.05}^{\delta}$ (and critical values for higher significance levels) are negative rather than positive for right side testing against a unit root, it is clear that such a rule is conservative and places a higher (and increasing) bar on the assessment for explosive behavior as $n \to \infty$. For DF^{δ}, when $n = 50$, the 10% critical value is -0.81, the 1% critical value is 1.22, and $\log\log^2 50 = 1.36$. In such cases using a rule like $cv_{\beta_n}^{\delta} = \log\log^2 n$ will produce a significance level close to the 1% level for samples of this magnitude, while the level slowly goes to 0 as $n \to \infty$. For DF^t, when $n = 50$, the 10% critical value is -0.40, the 1% critical value is 0.66, and $\frac{2}{3}\log\log^2 50 = 0.90$. In such cases using a rule like $cv_{\beta_n}^{\delta} = \frac{2}{3}\log\log^2 n$ will produce a significance level close to the 1% level for samples of this magnitude, while again the level slowly goes to 0 as $n \to \infty$. A conservative 1% rule seems acceptable because there is only a small risk of choosing the explosive alternative when it is not true and thereby only a small degree of possible underestimation of the origination date. On the other hand, choosing a large critical value expansion rate will result in overestimation bias in the origination date estimate.

3.2 Dating the Collapse of an Explosive Episode

This section develops the limit theory for dating the collapse of a mildly explosive period under model (6). Again, it is convenient for the development to assume that ε_t is $iid\left(0,\sigma^2\right)$, although this can be relaxed to allow for weak dependence in the residuals in view of the (equivalent) limit theory for the mildly explosive case given in PM_b under weakly dependent errors and the modified semiparametric unit root limit theory (e.g., Phillips, 1987) for the unit root case. The parameter $\tau_e = [nr_e]$ is the origination date and $\tau_f = [nr_f]$ is the collapse date of the explosive episode. If there is no mildly explosive episode, then $c = 0$ and $\delta_n = 1$ throughout the sample. It is further assumed that there is only a single explosive episode in the data, but the methods are applicable when there are repeated episodes and the asymptotic theory given here continues to apply in such cases, although we do not prove this here. It is also assumed that $\tau_f - \tau_e > \log(n)$ in order to achieve date separation that exceeds a small infinity of data asymptotically, as discussed earlier

Model (6) implies that data $t = \tau_e, ..., \tau > \tau_f$ are generated according to

$$\begin{aligned} X_t &= X_{t-1}1\{t < \tau_e\} + \delta_n X_{t-1}1\{\tau_e \leq t \leq \tau_f\} \\ &\quad + \left(\sum_{k=\tau_f+1}^{t} \varepsilon_k + X^*_{\tau_f}\right) 1\{t > \tau_f\} + \varepsilon_t 1\{t \leq \tau_f\} \qquad (12) \\ \delta_n &= 1 + \frac{c}{n^\alpha}, \quad c > 0, \quad \alpha \in (0,1), \end{aligned}$$

where $X^*_{\tau_f}$ is a (random) re-initialization following the explosive period. This generating mechanism ensures that the explosive episode switches off at τ_f and the data subsequently follows a unit root process evolving from the initialization $X^*_{\tau_f} = X_{\tau_e} + X^*$ for some random quantity $X^* = O_p(1)$. Thus, for $t > \tau_f$, X_t follows a unit root evolution originating at $X^*_{\tau_f}$, which is determined to be within some $O_p(1)$ interval of the initiation of the explosive episode X_{τ_e}. Initializing the post explosive episode at $X^*_{\tau_f}$ ensures that the bubble collapses completely and reinitializes to a level within $O_p(1)$ of its origination.

Note that if $c = 0$ and $\delta_n = 1$ in (12) there is again no mildly explosive episode in the process and $X_t = O_p(\sqrt{n})$ for all $t \geq \tau$. Under model (12) and the rate conditions (10) and (11), Theorem 3.2 holds as before, so that again $\hat{r}_e \to_p r_e$. The consistency of $\hat{r}_f$ is given in the following theorem.

THEOREM 3.3

(i) *Under the null hypothesis of no episode of explosive behavior* $P(\hat{r}_e \in [r_0, 1]) \to 0$ *as in theorem 3.1 and correspondingly* $P(\hat{r}_f \in [r_0, 1]) \to 0$ as $n \to \infty$.

(ii) *Suppose conditions (10) and (11) hold and the alternative hypothesis of Model (6) with a mildly explosive episode applies. Conditional on some* $\hat{r}_e > r_0$, *as* $n \to \infty$, *we then have* $\hat{r}_f \xrightarrow{p} r_f$ *where* $\hat{r}_f$ *is obtained from the right sided DF test based on either the coefficient or the* t *statistic.*

Remarks

(d) When the system switches back to the unit root model with the re-initialization taking place within an $O_p(1)$ neighborhood of X_{τ_e}, the signal from the explosive period dominates that from the unit root model and so governs the asymptotics of the estimate. This domination by initial conditions is analogous to the domination by infinitely distant initializations that arises in unit root limit theory (see Phillips and Magdalinos, 2009).

(e) It is shown in the proof of Theorem 3.3 that $\hat{\delta}_n(\tau) \to_p 1$ when $\tau > \tau_f$ but its limit distribution in this case involves a (second order) downward bias. This bias is explained by the fact that $\hat{\delta}_n(\tau)$ is computed with data that involves the explosive episode ($\tau_e \leq t \leq \tau_f$) as well as post explosive data ($\tau > \tau_f$), which makes the post-collapse data look mean reverting, producing a second order downward bias below unity in $\hat{\delta}_n(\tau)$.

(f) More specifically, $n^{\alpha}\left(\hat{\delta}_n(\tau) - 1\right)$ converges to $-c$ as $n \to \infty$, and so the statistic $DF_{\tau}^{\delta} = \tau\left(\hat{\delta}_n(\tau) - 1\right)$ diverges to $-\infty$ at the rate $n^{1-\alpha}$. The DF_{τ}^{t} also diverges to $-\infty$ and at the rate $n^{(1+\alpha)/2}$, which exceeds $n^{1-\alpha}$ when $\alpha > 1/3$. The difference is explained by the fact that the equation standard error is also sensitive to the collapse in the mildly explosive period. These differences in asymptotic behavior suggest that in some cases the DF t test may be more powerful in identifying the termination of an explosive episode than the DF coefficient test due to the sensitivity of both the fitted coefficient and the equation standard error to the post-collapse data.

4 Monte Carlo Simulations

This section reports some brief simulations examining the finite sample performance of the above dating estimation procedure and the accuracy of the asymptotic theory. Two experiments are used, one based on the DF_{τ}^{δ} test and the other on the DF_{τ}^{t} test. Both experiments use 1,000 sample path replications with data simulated from Model (6). From each sample path, we obtain $\hat{r}_e$ and $\hat{r}_f$ based on the DF_r^δ or DF_r^t statistic for $r \in [0.2, 1]$, thereby setting $r_0 = 0.2$. We use the critical values $\log\log^2 n$ for DF_r^δ and $\frac{2}{3}\log\log^2 n$ for DF_r^t. In all cases, we set the true values of the origination and collapse dates as $r_e = 0.4$ and $r_f = 0.6$ and impose the (small infinity) duration condition $\hat{r}_f - \hat{r}_e \geq [\log(n)]/n$. Three different sample sizes are considered in both experiments: $n = 100, 400, 800$. To assess the sensitivity of the dating estimators to the parameters determining the nature and extent of mildly explosive bubble activity, we fix $\alpha = 0.5$ but allow c to take four different values so that the implied autoregressive coefficient takes the following four values, 1.035, 1.040, 1.045, 1.050. These mildly explosive roots are empirically realistic given recent experience in the financial markets (see PWY for discussion).

Tables 1-3 report results for the DF_τ^δ test, giving means, standard errors, and root mean square errors (RMSE) and the percentage of replications that determine the true date stamp, for $\hat{r}_e$ and $\hat{r}_f$, when $n = 100, 400, 800$, respectively. Several results emerge from Tables 1-3. First, in all cases, the true values of r_e and r_f can be estimated with high accuracy, reflected by a small bias and a small standard error (and hence a small RMSE) in each case. In fact, in all cases, the true values of r_e and r_f are always within the 2 (simulation) standard deviations of the estimated values. When $n = 100$, $\hat{r}_f = r_f$ for 99% of the time. For 1% of the time, there is a downward bias in $\hat{r}_f$, which arises from the Type I error when estimating r_e. This sharp resolution of r_f is explained by the substantial effect of the bubble collapse on the limit theory. Second, when the explosive behavior is stronger, it is easier to estimate r_e and in this case both the bias and the standard error become smaller. It is interesting to note that the estimate of r_f remains unchanged when the explosive behavior is stronger - essentially the collapse magnitude is sufficient for determination of r_f even when the bubble has smaller overall magnitude. Third, when the sample size increases, it is easier to estimate r_e and r_f, both the bias and the standard error becoming smaller, corroborating the consistency result. In particular, when $n = 800$, $\hat{r}_f = r_f$ 100% of the time.

Tables 4-6 report findings for the DF_τ^t test, reporting the means, standard errors, and root mean square errors (RMSE) and the percentage of replications that hit the true date stamp for $\hat{r}_e$ and $\hat{r}_f$, when $n = 100, 400, 800$, respectively. The results are very similar to those for the DF_τ^δ test. In particular, r_e and r_f are again estimated with high accuracy and stronger explosive behavior facilitates estimation of r_e. Noticeably, the estimation of r_f is improved using the DF_τ^t test. In particular, $\hat{r}_f = r_f$ 100% of the time for sample size $n = 400$ as well as 800, thereby indicating the increased power in this test which corroborates the asymptotic theory.

Comparison of Tables 1-3 and Table 4-6 reveals several results. First, DF^δ and DF^t perform similarly as far as $\hat{r}_e$ is concerned. Both the bias and the standard error are close to each others. This result is consistent with the finding of the similar divergence rates in DF^{δ} and DF^{t} (see Remark (b)). Second, DF^{t} performs better than DF^{δ} for estimating r_f. The estimate $\hat{r}_f$ from DF^{t} has smaller bias and standard error in all cases. This result is consistent with the finding of different divergence rates in DF^{δ} and DF^{t} (see Remark (f)).

5 Conclusions

This paper develops limit theory for dating the origination and collapse of explosive periods, based on a procedure originally proposed in PWY (2009). That procedure involves recursive calculations of right sided unit root tests for estimating the emergence and termination of mildly explosive episodes in time series data. It is shown here that, under general regularity conditions, the estimates of both the origination and collapse dates are consistent. Simulation evidence shows that the dating procedure works well in finite samples provided the explosive episode is sufficiently sustained, so that the duration of the period is at least of $O(\log n)$, where n is the overall sample size.

6 Appendix

Proof of Theorem 3.1: The result is straightforward to prove and is a direct consequence of controlling size in the test so that $\beta_n \to 0$ as $n \to \infty$. Since $DF_s^{\delta} \Rightarrow \int_0^1 \widetilde{W} dW / \int_0^1 \widetilde{W}^2$ and $DF_s^{t} \Rightarrow \int_0^1 \widetilde{W} dW / \left(\int_0^1 \widetilde{W}^2\right)^{1/2}$ under the null hypothesis of no episode of explosive behavior, we have under that null hypothesis

$$\lim_{n\to\infty} P\left(DF_s^{\delta} > cv_{\beta_n}^{\delta}\right) = P\left(\frac{\int_0^1 \widetilde{W} dW}{\int_0^1 \widetilde{W}^2} = \infty\right) = 0, \tag{13}$$

and

$$\lim_{n\to\infty} P\left(DF_s^{t} > cv_{\beta_n}^{df}\right) = P\left(\frac{\int_0^1 \widetilde{W} dW}{\left(\int_0^1 \widetilde{W}^2\right)^{1/2}} = \infty\right) = 0, \tag{14}$$

because $cv_{\beta_n}^{\delta} \to \infty$ and $cv_{\beta_n}^{df} \to \infty$. Hence, in the limit as $n \to \infty$ under the null, no origination point for an explosive period in the data will be detected. ■

Proof of Theorem 3.2: For time series data sampled from (9)

$$X_t = X_{t-1} 1\{t < \tau_e\} + \delta_n X_{t-1} 1\{t \geq \tau_e\} + \varepsilon_t, \quad \delta_n = 1 + \frac{c}{n^{\alpha}}, c > 0, \alpha \in (0,1), \tag{15}$$

over $t = 1, ..., \tau = [nr]$ prior to $\tau_e = [nr_e]$, conventional unit root asymptotics apply. So, for $r < r_e$ we have the functional convergence

$$\frac{1}{\sqrt{n}} X_{[nr]} \Longrightarrow B(r) = \sigma W(r) \equiv BM\left(\sigma^2\right), \tag{16}$$

and following limit theory for the tests

$$DF_r^{\delta} = \tau\left(\hat{\delta}_n(\tau) - 1\right) \Rightarrow \frac{\int_0^1 \widetilde{W} dW}{\int_0^1 \widetilde{W}^2}, \quad DF_r^t \Rightarrow \frac{\int_0^1 \widetilde{W} dW}{\left(\int_0^1 \widetilde{W}^2\right)^{1/2}},$$

where $\hat{\delta}_n(\tau) = \sum_{j=1}^{\tau} \tilde{X}_j \tilde{X}_{j-1} / \sum_{j=1}^{\tau} \tilde{X}_{j-1}^2$ is the least squares regression coefficient from the fitted equation (1) and $\tilde{X}_j$ denotes demeaned X_j. Clearly, (13) and (14) hold for all $r < r_e$. Hence, under this model and the alternative hypothesis (9), we have for $\tau < \tau_e$

$$P\{\hat{r}_e < r_e\} \to 0, \quad \text{as } n \to \infty. \tag{17}$$

Next suppose that data is sampled over $t = 1, ..., \tau = [nr]$ for $r_e < r \le r_f$. In this case, the data $\{X_t : t = \tau_e, ..., \tau\}$ satisfy

$$X_t = \delta_n X_{t-1} 1\{t \ge \tau_e\} + \varepsilon_t = \sum_{j=0}^{t-\tau_e} \delta_n^j \varepsilon_{t-j} + \delta_n^{t-\tau_e+1} X_{\tau_e-1}. \tag{18}$$

and the components of $\hat{\delta}_n$ are dominated asymptotically by this part of the time series when $r > r_e$. Note that $\delta_n^{-(t-\tau_e)} X_t = \sum_{j=0}^{t-\tau_e} \delta_n^{j-(t-\tau_e)} \varepsilon_{t-j} + \delta_n X_{\tau_e-1}$, and, from Phillips and Magdalinos (2007a, lemma 4.2), as $t - \tau_e \to \infty$ the following central limit theory holds

$$\frac{1}{n^{\alpha/2}} \sum_{j=0}^{t-\tau_e} \delta_n^{j-(t-\tau_e)} \varepsilon_{t-j} \Rightarrow X_c \equiv N\left(0, \sigma^2/2c\right), \tag{19}$$

whereas $n^{-1/2} X_{\tau_e-1} \Rightarrow B(r_e)$ from (16). Suppose $\tau = [np]$ for some $p > r_e$. Then, as $t - \tau_e \to \infty$ we have

$$\frac{\delta_n^{-(t-\tau_e)}}{n^{1/2}} X_t = \frac{1}{n^{(1-\alpha)/2}} \frac{1}{n^{\alpha/2}} \sum_{j=0}^{t-\tau_e} \delta_n^{-(t-\tau_e-j)} \varepsilon_{t-j} + \frac{\delta_n}{n^{1/2}} X_{\tau_e-1} \Rightarrow B(r_e), \tag{20}$$

so that

$$X_t = \delta_n^{(t-\tau_e)} X_{\tau_e} \{1 + o_p(1)\} \sim n^{1/2} \delta_n^{(t-\tau_e)} B(r_e), \tag{21}$$

for all $\tau - \tau_e \to \infty$.

Now consider the centred quantities $\tilde{X}_t = X_t - \tau^{-1} \sum_{j=1}^{\tau} X_j$. For $\tau = [nr]$ and $r_e < r \le r_f$, We have

$$\frac{1}{\tau\sqrt{n}} \sum_{j=1}^{\tau} X_j = \frac{1}{\tau\sqrt{n}} \sum_{j=\tau_e}^{\tau} X_j + \frac{\tau_e}{\tau} \tau_e^{-1} \sum_{j=1}^{\tau_e-1} \frac{X_j}{\sqrt{n}} \sim \frac{1}{\tau\sqrt{n}} \sum_{j=\tau_e}^{\tau} X_j + \frac{r_e}{r} \int_0^1 B(s)\, ds,$$

and

$$\begin{aligned}
\frac{1}{\tau}\sum_{j=\tau_e}^{\tau} X_j &= \frac{1}{\tau}\sum_{j=\tau_e}^{\tau} \delta_n^{(j-\tau_e)}\left(\delta_n^{-(j-\tau_e)} X_j\right) = \frac{X_{\tau_e}}{\tau}\sum_{k=0}^{\tau-\tau_e} \delta_n^{k}\left\{1+o_p(1)\right\} \\
&= X_{\tau_e}\frac{\delta_n^{\tau-\tau_e+1}-1}{\tau\left(\delta_n-1\right)}\left\{1+o_p(1)\right\} \\
&= X_{\tau_e}\frac{n^{\alpha}\delta_n^{\tau-\tau_e}}{\tau c}\left\{1+o_p(1)\right\}, \qquad (22)
\end{aligned}$$

so that

$$\tau^{-1}\sum_{j=1}^{\tau} X_j = X_{\tau_e}\frac{n^{\alpha}\delta_n^{\tau-\tau_e}}{\tau c}\left\{1+o_p(1)\right\}. \qquad (23)$$

It follows that

$$\tilde{X}_t = X_t - \tau^{-1}\sum_{j=1}^{\tau} X_j = \left[\delta_n^{(t-\tau_e)} - \frac{n^{\alpha}\delta_n^{\tau-\tau_e}}{\tau c}\right] X_{\tau_e}\left\{1+o_p(1)\right\} \qquad (24)$$

and so $\tilde{X}_t$ behaves asympotically like X_t for $t=[np]$ when $r_e < p \le r_f$. Using (22), we have for $j \le \tau_e$

$$\tilde{X}_j = X_j - \tau^{-1}\sum_{j=1}^{\tau} X_j = -X_{\tau_e}\frac{n^{\alpha}\delta_n^{\tau-\tau_e+1}}{\tau c}\left\{1+o_p(1)\right\},$$

so that the sample mean dominates over this time period and

$$\sum_{j=1}^{\tau_e} \tilde{X}_{j-1}^2 = \frac{n^{2\alpha}\tau_e\delta_n^{2(\tau-\tau_e)}}{\tau^2 c^2} X_{\tau_e}^2\left\{1+o_p(1)\right\}. \qquad (25)$$

Using these results in conjunction with standard unit root limit theory, we have

$$\begin{aligned}
\sum_{j=1}^{\tau} \tilde{X}_{j-1}^2 &\sim \sum_{j=\tau_e}^{\tau} \tilde{X}_{j-1}^2 + \sum_{j=1}^{\tau_e-1} \tilde{X}_{j-1}^2 = \sum_{j=\tau_e}^{\tau} \tilde{X}_{j-1}^2\left\{1+o_p(1)\right\}, \qquad (26) \\
\sum_{j=1}^{\tau} \tilde{X}_{j-1}\left(X_j - \delta_n X_{j-1}\right) &= \sum_{j=\tau_e}^{\tau} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=1}^{\tau_e-1} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{n^{\alpha}}X_{j-1}\right) \\
&= \sum_{j=\tau_e}^{\tau} \tilde{X}_{j-1}\varepsilon_j\left\{1+o_p(1)\right\}. \qquad (27)
\end{aligned}$$

Explicit limits may be obtained from (24) for the mildly explosive components of the sums in (26) and (27). In particular, for $\tau=[nr]$ with $r > r_e$, we obtain

$$\begin{aligned}
\sum_{j=\tau_e}^{\tau} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_e}^{\tau} \delta_n^{2(j-\tau_e)} X_{\tau_e}^2\left\{1+o_p(1)\right\} = \frac{\delta_n^{2(\tau-\tau_e+1)}-\delta_n^2}{\delta_n^2-1} X_{\tau_e}^2\left\{1+o_p(1)\right\} \\
&= \frac{n^{\alpha}\delta_n^{2(\tau-\tau_e)}}{2c} X_{\tau_e}^2\left\{1+o_p(1)\right\},
\end{aligned}$$

which dominates $\sum_{j=1}^{\tau_e} \tilde{X}_{j-1}^2$ as given in (25). Also,

$$\begin{aligned}\sum_{j=\tau_e}^{\tau} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_e}^{\tau} \delta_n^{j-1-\tau_e} X_{\tau_e}\varepsilon_j \{1+o_p(1)\} = \delta_n^{\tau-\tau_e} X_{\tau_e} \sum_{j=\tau_e}^{\tau} \delta_n^{-(\tau-j+1)}\varepsilon_j \{1+o_p(1)\} \\ &= \left(n^{\alpha/2}\delta_n^{\tau-\tau_e} X_{\tau_e}\right)\left\{\frac{1}{n^{\alpha/2}}\sum_{j=\tau_e}^{\tau} \delta_n^{-(\tau-j+1)}\varepsilon_j\right\}\{1+o_p(1)\} \\ &\sim n^{\alpha/2+1/2}\delta_n^{\tau-\tau_e} B(r_e) X_c,\end{aligned}$$

where X_c is $N\left(0,\sigma^2/2c\right)$ using (19). Thus, as $n\to\infty$

$$\frac{2c\delta_n^{-2(\tau-\tau_e)}}{n^{1+\alpha}} \sum_{j=\tau_e}^{\tau} \tilde{X}_{j-1}^2 \Rightarrow B(r_e)^2, \tag{28}$$

Further,

$$\frac{\delta_n^{-(\tau-\tau_e)}}{n^{\alpha/2+1/2}} \sum_{j=\tau_e}^{\tau} \tilde{X}_{j-1}\varepsilon_j \Rightarrow B(r_e) X_c,$$

where $B(r_e)$ and X_c are independent Gaussian variates. It follows that for $\tau=[nr]$ and $r>r_e$, we have

$$\begin{aligned}\frac{n^{(1+\alpha)/2}\delta_n^{\tau-\tau_e}}{2c}\left(\hat{\delta}_n(\tau)-\delta_n\right) &= \frac{\frac{\delta_n^{-(\tau-\tau_e)}}{n^{\alpha/2+1/2}}\sum_{j=\tau_e}^{\tau} X_{j-1}\varepsilon_j\{1+o_p(1)\}}{\frac{2c\delta_n^{-2(\tau-\tau_e)}}{n^{1+\alpha}}\sum_{j=\tau_e}^{\tau} X_{j-1}^2\{1+o_p(1)\}} \\ &\Rightarrow \frac{X_c}{B(r_e)},\end{aligned} \tag{29}$$

where the rate of convergence is $n^{(1+\alpha)/2}\delta_n^{\tau-\tau_e}$. Then, since $\alpha\in(0,1)$ and $c>0$ we have

$$\begin{aligned}\tau\left(\hat{\delta}_n(\tau)-1\right) &= \tau\left(\hat{\delta}_n(\tau)-\delta_n\right)+\tau(\delta_n-1) \\ &= \tau(\delta_n-1)+o_p\left(\frac{\tau}{n^{(1+\alpha)/2}\delta_n^{\tau-\tau_e}}\right) \\ &= n^{1-\alpha}rc+o_p(1)\to\infty.\end{aligned} \tag{30}$$

Correspondingly,

$$DF_r^{\delta} = \tau\left(\hat{\delta}_\tau(\tau)-1\right) = n^{1-\alpha}rc\{1+o_p(1)\}\to\infty. \tag{31}$$

The regression residual variance estimate is, using (29),

$$
\begin{aligned}
\hat{\sigma}_{\tau}^{2} &= \tau^{-1}\sum_{j=1}^{\tau}\left(\tilde{X}_{j}-\hat{\delta}_{n}\left(\tau\right)\tilde{X}_{j-1}\right)^{2} \\
&= \tau^{-1}\sum_{j=1}^{\tau}\left(\varepsilon_{j}-\left(\hat{\delta}_{n}\left(\tau\right)-\delta_{n}\right)\tilde{X}_{j-1}1\left\{j\geq\tau_{e}\right\}-\left(\hat{\delta}_{n}\left(\tau\right)-1\right)\tilde{X}_{j-1}1\left\{j<\tau_{e}\right\}\right)^{2} \\
&= \tau^{-1}\sum_{j=1}^{\tau}\varepsilon_{j}^{2}+o_{p}\left(\frac{\tau^{-1}\sum_{j=\tau_{e}}^{\tau}\tilde{X}_{j-1}^{2}}{n^{1+\alpha}\delta_{n}^{2(\tau-\tau_{e})}}\right)+o_{p}\left(\frac{\tau^{-1}\sum_{j=\tau_{e}}^{\tau}\tilde{X}_{j-1}\varepsilon_{j}}{n^{(1+\alpha)/2}\delta_{n}^{\tau-\tau_{e}}}\right) \\
&\quad +O_{p}\left(\frac{n^{-2\alpha}}{\tau}\sum_{j=1}^{\tau_{e}}\tilde{X}_{j-1}^{2}\right)+O_{p}\left(\frac{\sum_{j=1}^{\tau_{e}}\tilde{X}_{j-1}\varepsilon_{j}}{n^{\alpha}\tau}\right) \\
&= \frac{\delta_{n}^{2(\tau-\tau_{e})}\tau_{e}}{\tau^{3}c^{2}}X_{\tau_{e}}^{2}\left\{1+o_{p}\left(1\right)\right\}=\frac{\delta_{n}^{2(\tau-\tau_{e})}r_{e}}{\tau^{2}c^{2}r}X_{\tau_{e}}^{2}\left\{1+o_{p}\left(1\right)\right\}, \qquad (32)
\end{aligned}
$$

since $\sum_{j=1}^{\tau_e}\tilde{X}_{j-1}^2=\frac{n^{2\alpha}\tau_e\delta_n^{2(\tau-\tau_e)}}{\tau^2c^2}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\}$ from (25). Using these results , the DF^t statistic for $\tau=[nr]$ and $r>r_e$ is

$$
\begin{aligned}
DF_{r}^{t} &= \left(\frac{\sum_{j=1}^{\tau}\tilde{X}_{j-1}^{2}}{\hat{\sigma}_{\tau}^{2}}\right)^{1/2}\left(\hat{\delta}_{n}\left(\tau\right)-1\right) \\
&= \left(\frac{\tau^{-2}\sum_{j=1}^{\tau}\tilde{X}_{j-1}^{2}}{\hat{\sigma}_{\tau}^{2}}\right)^{1/2}\tau\left(\hat{\delta}_{n}\left(\tau\right)-1\right) \\
&= \left(\frac{\tau^{-2}\sum_{j=\tau_{e}}^{\tau}\tilde{X}_{j-1}^{2}}{\hat{\sigma}_{\tau}^{2}}\right)^{1/2}n^{1-\alpha}rc\left\{1+o_{p}\left(1\right)\right\} \\
&= \left(\frac{\frac{n^{\alpha}\delta_{n}^{2(\tau-\tau_{e})}}{\tau^{2}2c}X_{\tau_{e}}^{2}}{\frac{\delta_{n}^{2(\tau-\tau_{e})}r_{e}}{\tau^{2}c^{2}r}X_{\tau_{e}}^{2}}\right)^{1/2}n^{1-\alpha}rc\left\{1+o_{p}\left(1\right)\right\} \\
&= \left(\frac{n^{\alpha}cr}{2r_{e}}\right)^{1/2}n^{1-\alpha}rc\left\{1+o_{p}\left(1\right)\right\} \\
&= n^{1-\alpha/2}\frac{c^{3/2}r^{3/2}}{2^{1/2}r_{e}^{1/2}}\left\{1+o_{p}\left(1\right)\right\}, \qquad (33)
\end{aligned}
$$

which diverges at the rate $n^{1-\alpha/2}$ as $n\rightarrow\infty$.

Define the (asymptotic fractional r) critical values of the coefficient and DF tests as

$$
cv_{\beta_{n}}^{\delta}=cv_{\beta_{n}}\left\{\frac{\int_{0}^{1}\tilde{W}dW}{\int_{0}^{1}\tilde{W}^{2}}\right\}, \quad \text{and } cv_{\beta_{n}}^{df}=cv_{\beta_{n}}\left\{\frac{\int_{0}^{r}\tilde{W}dW}{\left(\int_{0}^{r}\tilde{W}^{2}\right)^{1/2}}\right\}, \qquad (34)
$$

and, as indicated earlier, we set $\beta_n \to 0$ so that $cv^{\delta}_{\beta_n}, cv^{df}_{\beta_n} \to \infty$ for all $r \in [r_0, 1]$.

We deduce from (30) that for all $\tau = [nr]$ and $r > r_e$

$$P\left(\tau\left(\hat{\delta}_n(\tau) - 1\right) > cv^{\delta}_{\beta_n}\right) \to 1, \tag{35}$$

provided $\frac{cv^{\delta}_{\beta_n}}{n^{1-\alpha}} \to 0$, and from (33) that

$$P\left(DF^t_r > cv^{df}_{\beta_n}\right) \to 1, \tag{36}$$

provided $\frac{cv^{df}_{\beta_n}}{n^{1-\alpha/2}} \to 0$. According to (7) we have $\hat{r}_e = \inf_s \left\{s : DF^t_s > cv^{df}_{\beta_n}\right\}$. It follows that for any $\eta > 0$

$$P\left\{\hat{r}_e > r_e + \eta\right\} \to 0,$$

since $P\left(DF^t_{r_e+a_\eta} > cv^{df}_{\beta_n}\right) \to 1$ for all $0 < a_\eta < \eta$. Since $\eta > 0$ is arbitrary and since $P\left\{\hat{r}_e < r_e\right\} \to 0$ from (17), we deduce that $P\left\{|\hat{r}_e - r_e| > \eta\right\} \to 0$ as $n \to \infty$, provided

$$\frac{1}{cv^{df}_{\beta_n}} + \frac{cv^{df}_{\beta_n}}{n^{1-\alpha/2}} \to 0, \tag{37}$$

for all $r \in [r_0, 1]$. Hence, as $n \to \infty$

$$\hat{r}_e = \inf_s \left\{s : DF_s > cv^{df}_{\beta_n}; s \in [r_0, 1]\right\} \to_p r_e. \tag{38}$$

In a similar way for the coefficient test, we have

$$\hat{r}_e = \inf_s \left\{s : [ns]\left(\hat{\delta}_n([ns]) - 1\right) > cv^{\delta}_{\beta_n}; s \in [r_0, 1]\right\} \to_p r_e,$$

provided

$$\frac{1}{cv^{\delta}_{\beta_n}} + \frac{cv^{\delta}_{\beta_n}}{n^{1-\alpha}} \to 0, \tag{39}$$

for all $r \in [r_0, 1]$. ■

Proof of Theorem 3.3: When $c = 0$ and $\delta_n = 1$ in (12) there is no explosive episode in the data and no origination date is detected as $n \to \infty$, so that (i) holds by Theorem 3.1.

To prove (ii) we consider the behavior the tests for various values of τ. First suppose that data is sampled over $t = 1, ..., \tau = [nr]$ with $r \in (r_e, r_f)$. The data $\{X_t : t = \tau_e, ..., \tau\}$ satisfy (18), and the test statistics DF^{δ} and DF^t have the same behavior as in (31) and (33), just as before. The case $r_e < r \leq r_f$ therefore follows from the earlier analysis. In particular, over this range of r, mildly explosive asymptotics apply and we have from (30)

$$\tau\left(\hat{\delta}_n(\tau) - 1\right) = n^{1-\alpha} rc + o_p(1) \to \infty,$$

so that

$$P\left(\tau\left(\hat{\delta}_n(\tau)-1\right)<cv_{\beta_n}^{\delta}\right)\to 0, \tag{40}$$

for all fixed critical values $cv_{\beta_n}^{\delta}$. Similarly, from (33) we have

$$P\left(DF_r^t<cv_{\beta_n}^{df}\right)\to 0, \tag{41}$$

for all fixed critical values $cv_{\beta_n}^{df}$. Since the two termination date estimation criteria are

$$\hat{r}_f=\inf_{s\geq\hat{r}_e+\frac{\log(n)}{n}}\left\{s:DF_{s\geq\hat{r}_e}^{a}<cv_{\beta_n}^{df}\right\},\quad\text{for } a=\delta,t$$

it follows from (40) - (41) that

$$\lim_{n\to\infty}P\left(\hat{r}_f<r_f\right)\to 0, \tag{42}$$

for both the coefficient test and t test. Hence, as $n\to\infty$, $\hat{r}_f$ never underestimates the collapse date r_f.

Next consider the case where $t>\tau_f$. In this event the data X_t satisfy

$$X_t=\sum_{k=\tau_f+1}^{t}\varepsilon_k+X_{\tau_f}^{*}=\sum_{k=\tau_f+1}^{t}\varepsilon_k+X_{\tau_e}+X=O_p\left(n^{1/2}\right),$$

while the demeaned data $\tilde{X}_t=X_t-\tau^{-1}\sum_{j=1}^{\tau}X_j=X_t-\bar{X}_\tau$ depend on the full sequence $\{X_j\}_1^{\tau}$ which satisfies

$$X_j=X_{j-1}1\{j<\tau_e\}+\delta_nX_{j-1}1\{\tau_e\leq j\leq\tau_f\}+\left(\sum_{k=\tau_f+1}^{j}\varepsilon_k+X_{\tau_f}^{*}\right)1\{j>\tau_f\}+\varepsilon_j1\{j\leq\tau_f\}.$$

For $\tau_e<j\leq\tau_f$ we have from (21)

$$X_j=\delta_n^{(j-\tau_e)}X_{\tau_e}\{1+o_p(1)\}\sim n^{1/2}\delta_n^{(j-\tau_e)}B(r_e). \tag{43}$$

Hence, the sample mean is

$$\begin{aligned}\tau^{-1}\sum_{j=1}^{\tau}X_j &= \frac{1}{\tau}\sum_{j=\tau_f+1}^{\tau}\left(\sum_{k=\tau_f+1}^{j}\varepsilon_k+X_{\tau_e}+X\right)+\frac{1}{\tau}\sum_{j=\tau_e}^{\tau_f}\left(\sum_{k=0}^{j-\tau_e}\delta_n^k\varepsilon_{t-k}+\delta_n^{j-\tau_e}X_{\tau_e}\right)\\ &\quad+\frac{1}{\tau}\sum_{j=1}^{\tau_e-1}\left(\sum_{k=1}^{j}\varepsilon_k+X_0\right)\\ &= \frac{1}{\tau}\left(\sum_{j=\tau_e}^{\tau_f}\delta_n^{(j-\tau_e)}\right)X_{\tau_e}+O_p\left(\frac{n^{3/2}}{\tau}\right)\\ &= \frac{1}{\tau}\frac{\delta_n^{\tau_f-\tau_e+1}-1}{\delta_n-1}X_{\tau_e}+O_p\left(\frac{n^{3/2}}{\tau}\right)\\ &= \frac{n^{\alpha}}{\tau c}\delta_n^{\tau_f-\tau_e+1}X_{\tau_e}\{1+o_p(1)\}\sim\frac{n^{\alpha+1/2}}{\tau c}\delta_n^{\tau_f-\tau_e+1}B(r_e).\end{aligned} \tag{44}$$

Then, for $\tau_f < t \leq \tau$

$$\begin{aligned}\tilde{X}_t &= X_t - \tau^{-1}\sum_{j=1}^{\tau} X_j = \left(\sum_{k=\tau_f+1}^{t} \varepsilon_k + X_{\tau_e} + X\right) - \frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e} X_{\tau_e}\left\{1+o_p(1)\right\} \\ &= -\frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e} X_{\tau_e}\left\{1+o_p(1)\right\},\end{aligned}$$

which is dominated by the sample mean $\bar{X}_\tau$.

It follows that

$$\sum_{j=\tau_f+1}^{\tau} \tilde{X}_{j-1}^2 = \frac{n^{2\alpha}(\tau-\tau_f)}{\tau^2 c^2}\delta_n^{2(\tau_f-\tau_e)} X_{\tau_e}^2\left\{1+o_p(1)\right\}, \tag{45}$$

and

$$\sum_{j=\tau_f+1}^{\tau} \tilde{X}_{j-1}\varepsilon_j = -\frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e} X_{\tau_e}\sum_{j=\tau_f+1}^{\tau}\varepsilon_j\left\{1+o_p(1)\right\}. \tag{46}$$

Next we need to consider $\tilde{X}_t = X_t - \tau^{-1}\sum_{j=1}^{\tau} X_j$ for $\tau_e \leq t \leq \tau_f$. Over this period the behavior of X_t is given in (43) and so, using (44),

$$\tilde{X}_t = X_t - \tau^{-1}\sum_{j=1}^{\tau} X_j = \delta_n^{t-\tau_e} X_{\tau_e}\left\{1+o_p(1)\right\} - \frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e} X_{\tau_e}\left\{1+o_p(1)\right\}. \tag{47}$$

Since

$$\sum_{j=\tau_e}^{\tau_f}\delta_n^{(j-\tau_e)} = \frac{\delta_n^{(\tau_f-\tau_e+1)}-\delta_n}{\delta_n-1} = \frac{n^\alpha\delta_n^{\tau_f-\tau_e}}{c}\left\{1+o_p(1)\right\},$$

we have

$$\begin{aligned}\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}^2 &= \left\{\sum_{j=\tau_e}^{\tau_f}\delta_n^{2(j-\tau_e)} - 2\left(\sum_{j=\tau_e}^{\tau_f}\delta_n^{(j-\tau_e)}\right)\left(\frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}\right) + (\tau_f-\tau_e)\left(\frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}\right)^2\right\} \\ &\quad\times\ X_{\tau_e}^2\left\{1+o_p(1)\right\} \\ &= \left\{\frac{\delta_n^{2(\tau_f-\tau_e+1)}-\delta_n^2}{\delta_n^2-1} - \frac{2}{\tau}\left(\frac{n^\alpha\delta_n^{\tau_f-\tau_e}}{c}\right)^2 + (\tau_f-\tau_e)\left(\frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}\right)^2\right\} \\ &\quad\times\ X_{\tau_e}^2\sum_{k=0}^{\tau-\tau_e}\delta_n^{2k}\left\{1+o_p(1)\right\} \\ &= \left\{\frac{n^\alpha\delta_n^{2(\tau_f-\tau_e)}}{2c} - \frac{2}{\tau}\left(\frac{n^\alpha\delta_n^{\tau_f-\tau_e}}{c}\right)^2 + (\tau_f-\tau_e)\left(\frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}\right)^2\right\} X_{\tau_e}^2\left\{1+o_p(1)\right\} \\ &= \frac{n^\alpha\delta_n^{2(\tau_f-\tau_e)}}{2c} X_{\tau_e}^2\left\{1+o_p(1)\right\},\end{aligned} \tag{48}$$

so that $\sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}^2 = \sum_{j=\tau_e}^{\tau_f} X_{j-1}^2 \{1 + o_p(1)\}$. Similarly, we find using (47) that

$$\begin{aligned}
\sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_e}^{\tau_f} \left[\delta_n^{j-1-\tau_e} - \frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}\right] X_{\tau_e}\varepsilon_j \{1 + o_p(1)\} \\
&= \left\{\sum_{j=\tau_e}^{\tau_f} \delta_n^{j-1-\tau_e}\varepsilon_j - \frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}\sum_{j=\tau_e}^{\tau_f}\varepsilon_j\right\} X_{\tau_e}\{1 + o_p(1)\} \\
&= \left\{\delta_n^{\tau_f-\tau_e}\sum_{j=\tau_e}^{\tau_f} \delta_n^{-(\tau_f-j+1)}\varepsilon_j - \frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}\sum_{j=\tau_e}^{\tau_f}\varepsilon_j\right\} X_{\tau_e}\{1 + o_p(1)\} \\
&= \left\{n^{\alpha/2}\delta_n^{\tau_f-\tau_e}\frac{1}{n^{\alpha/2}}\sum_{j=\tau_e}^{\tau_f} \delta_n^{-(\tau_f-j+1)}\varepsilon_j - \frac{n^{\alpha+1/2}}{\tau c}\delta_n^{\tau_f-\tau_e}\frac{\sum_{j=\tau_e}^{\tau_f}\varepsilon_j}{\sqrt{n}}\right\} X_{\tau_e}\{1 + o_p(1)\} \\
&= \left\{n^{\alpha/2}\delta_n^{\tau_f-\tau_e}\frac{1}{n^{\alpha/2}}\sum_{j=\tau_e}^{\tau_f} \delta_n^{-(\tau_f-j+1)}\varepsilon_j\right\} X_{\tau_e}\{1 + o_p(1)\} \\
&= \sum_{j=\tau_e}^{\tau} X_{j-1}\varepsilon_j \{1 + o_p(1)\} \sim n^{\alpha/2+1/2}\delta_n^{\tau_f-\tau_e} B(r_e) X_c, \qquad (49)
\end{aligned}$$

using (19), since $\alpha/2 - (\alpha - 1/2) = (1 - \alpha)/2 > 0$. Then, (48) and (49) give

$$\frac{2c\delta_n^{-2(\tau_f-\tau_e)}}{n^\alpha}\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}^2 \Rightarrow B(r_e)^2, \qquad (50)$$

$$\frac{\delta_n^{-(\tau_f-\tau_e)}}{n^{(1+\alpha)/2}}\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}\varepsilon_j \Rightarrow B(r_e) X_c. \qquad (51)$$

Using (45), (50) and (44) we deduce that

$$\begin{aligned}
\sum_{j=1}^{\tau}\tilde{X}_{j-1}^2 &= \sum_{j=\tau_f+1}^{\tau}\tilde{X}_{j-1}^2 + \sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}^2 + \sum_{j=1}^{\tau_e-1}\tilde{X}_{j-1}^2 \\
&= \sum_{j=\tau_f+1}^{\tau}\tilde{X}_{j-1}^2 + \sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}^2 + \sum_{j=1}^{\tau_e-1}X_{j-1}^2 + (\tau_e - 1)\bar{X}_\tau^2 - 2\bar{X}_\tau\sum_{j=1}^{\tau_e-1}X_{j-1} \\
&= \frac{n^{2\alpha}(\tau - \tau_f)}{\tau^2c^2}\delta_n^{2(\tau_f-\tau_e)}X_{\tau_e}^2\{1 + o_p(1)\} + \frac{n^\alpha\delta_n^{2(\tau_f-\tau_e)}}{2c}X_{\tau_e}^2\{1 + o_p(1)\} \\
&\quad + (\tau_e - 1)\left(\frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}X_{\tau_e}\right)^2\{1 + o_p(1)\} \\
&= \frac{n^\alpha\delta_n^{2(\tau_f-\tau_e)}}{2c}X_{\tau_e}^2\{1 + o_p(1)\}
\end{aligned}$$

and

$$\frac{2c\delta_n^{-2\left(\tau_f-\tau_e\right)}}{n^\alpha}\sum_{j=1}^{\tau}\tilde{X}_{j-1}^2 \Rightarrow B\left(r_e\right)^2.$$

Now suppose $\tau = [nr] > \tau_f$ and consider

$$\begin{aligned}\tilde{X}_j - \tilde{X}_{j-1} &= X_j - X_{j-1} - \frac{1}{\tau}\sum_{t=1}^{\tau}\left(X_t - X_{t-1}\right) = X_j - X_{j-1} - \frac{1}{\tau}\left(X_\tau - X_0\right) \\ &= \left\{\begin{array}{cc} \varepsilon_j - \frac{1}{\tau}\left(X_\tau - X_0\right) & j < \tau_e \\ \varepsilon_j + \frac{c}{n^\alpha}X_{j-1} - \frac{1}{\tau}\left(X_\tau - X_0\right) & \tau_e \leq j \leq \tau_f \\ \varepsilon_{\tau_f+1} + X_{\tau_e} + X - X_{\tau_f} - \frac{1}{\tau}\left(X_\tau - X_0\right) & j = \tau_{f+1} \\ \varepsilon_j - \frac{1}{\tau}\left(X_\tau - X_0\right) & j > \tau_f \end{array}\right. .\end{aligned}$$

Observe that $X_\tau = \sum_{k=\tau_f+1}^{\tau}\varepsilon_k + X_{\tau_f}^* = \sum_{k=\tau_f+1}^{\tau}\varepsilon_k + X_{\tau_e} + X = O_p\left(\sqrt{n}\right)$, so that $\tau^{-1}X_\tau = O_p\left(n^{-1/2}\right)$. Further, $X_{\tau_f} = \delta_n^{\left(\tau_f-\tau_e\right)}X_{\tau_e}\left\{1 + o_p(1)\right\}$ from (43). So when $j = \tau_f + 1$ we have

$$\begin{aligned}\tilde{X}_{\tau_f+1} - \tilde{X}_{\tau_f} &= X_{\tau_e} - X_{\tau_f} + \left(\varepsilon_{\tau_f+1} + X\right) + O_p\left(n^{-1/2}\right) \\ &= -\delta_n^{\left(\tau_f-\tau_e\right)}X_{\tau_e}\left\{1 + o_p(1)\right\}. \end{aligned} \tag{52}$$

Next, using (46), (49), (50), and (52) we have

$$\begin{aligned}&\sum_{j=1}^{\tau}\tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \\ &= \sum_{j=\tau_f+2}^{\tau}\tilde{X}_{j-1}\varepsilon_j + \tilde{X}_{\tau_f}\left(\tilde{X}_{\tau_f+1} - \tilde{X}_{\tau_f}\right) + \sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}\left(\varepsilon_j + \frac{c}{n^\alpha}X_{j-1}\right) + \sum_{j=1}^{\tau_e-1}\tilde{X}_{j-1}\varepsilon_j \\ &= -\frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}X_{\tau_e}\sum_{j=\tau_f+2}^{\tau}\varepsilon_j\left\{1 + o_p(1)\right\} - \delta_n^{2\left(\tau_f-\tau_e\right)}X_{\tau_e}^2\left\{1 + o_p(1)\right\} \\ &\quad + \sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}\left(\varepsilon_j + \frac{c}{n^\alpha}X_{j-1}\right) - \frac{n^\alpha}{\tau c}\delta_n^{\tau_f-\tau_e}X_{\tau_e}\sum_{j=1}^{\tau_e-1}\varepsilon_j\left\{1 + o_p(1)\right\} + O_p(n)\end{aligned}$$

$$
\begin{aligned}
&= -n\delta_n^{2(\tau_f-\tau_e)}\frac{X_{\tau_e}^2}{n}\{1+o_p(1)\} \\
&\quad -\frac{n^{1+\alpha}}{\tau c}\delta_n^{\tau_f-\tau_e}\frac{X_{\tau_e}}{\sqrt{n}}\frac{\sum_{j=\tau_f+2}^{\tau}\varepsilon_j+\sum_{j=1}^{\tau_e-1}\varepsilon_j}{\sqrt{n}}\{1+o_p(1)\} \\
&\quad +n^{\alpha/2+1/2}\delta_n^{\tau_f-\tau_e}\left(\frac{1}{n^{\alpha/2}}\sum_{j=\tau_e}^{\tau_f}\delta_n^{-(\tau_f-j+1)}\varepsilon_j\right)\frac{X_{\tau_e}}{\sqrt{n}}\{1+o_p(1)\} \\
&\quad +\frac{c}{n^\alpha}\frac{n^\alpha\delta_n^{2(\tau_f-\tau_e)}}{2c}X_{\tau_e}^2\{1+o_p(1)\}+O_p(n) \\
&= -n\delta_n^{2(\tau_f-\tau_e)}\frac{X_{\tau_e}^2}{n}\{1+o_p(1)\}+\frac{n\delta_n^{2(\tau_f-\tau_e)}}{2}\frac{X_{\tau_e}^2}{n}\{1+o_p(1)\} \\
&= -\frac{n\delta_n^{2(\tau_f-\tau_e)}}{2}\frac{X_{\tau_e}^2}{n}\{1+o_p(1)\}.
\end{aligned}
$$

Hence, for $\tau=[nr]$ and $r>r_f$, we have

$$
\hat{\delta}_n(\tau)-1=\frac{\sum_{j=1}^{\tau}\tilde{X}_{j-1}\left(\tilde{X}_j-\tilde{X}_{j-1}\right)}{\sum_{j=1}^{\tau}\tilde{X}_{j-1}^2}=\frac{-\frac{n\delta_n^{2(\tau_f-\tau_e)}}{2}\frac{X_{\tau_e}^2}{n}\{1+o_p(1)\}}{\frac{n^{1+\alpha}\delta_n^{2(\tau_f-\tau_e)}}{2c}\frac{X_{\tau_e}^2}{n}\{1+o_p(1)\}}=-\frac{c}{n^\alpha}\{1+o_p(1)\}.
$$

Then $n^\alpha\left(\hat{\delta}_n(\tau)-1\right)=-c+o_p(1)$ and

$$
\tau\left(\hat{\delta}_n(\tau)-1\right)=-c\frac{\tau}{n^\alpha}+o_p(1)=-crn^{1-\alpha}\to-\infty. \tag{53}
$$

for all $\tau-\tau_e\to\infty$.

We deduce from (53) that, for $\tau>\tau_f$, $\hat{\delta}_n(\tau)\to_p 1$. Observe that there is some downward bias (below unity) in $\hat{\delta}_n(\tau)$ even in the limit distribution. This is due to the fact that $\hat{\delta}_n(\tau)$ is computed with data that involves the explosive episode ($\tau_e\le t\le\tau_f$) which makes the post-collapse data ($\tau>\tau_f$) look mean reverting and (eventually) leads to a second order downward bias (that is in the limit distribution) in $\hat{\delta}_n(\tau)$ below unity.

Next consider the residual variance estimate that appears in the DF^t statistic. As in (32), we have

$$
\begin{aligned}
& \hat{\sigma}_\tau^2 \\
= \; & \frac{1}{\tau} \sum_{j=1}^{\tau} \left(\tilde{X}_j - \hat{\delta}_n(\tau) \tilde{X}_{j-1} \right)^2 \\
= \; & \frac{1}{\tau} \sum_{j=1, j \neq \tau_f + 1}^{\tau} \left(\varepsilon_j - \left(\hat{\delta}_n(\tau) - \delta_n \right) \tilde{X}_{j-1} 1\{\tau_e \leq j \leq \tau_f\} - \left(\hat{\delta}_n(\tau) - 1 \right) \tilde{X}_{j-1} 1\{j > \tau_f \text{ or } j < \tau_e\} \right)^2 \\
& + \frac{1}{\tau} \left\{ \left(\varepsilon_{\tau_f+1} + X_{\tau_e} + X - \bar{X}_\tau \right) - \hat{\delta}_n(\tau) \left(X_{\tau_f} - \bar{X}_\tau \right) \right\}^2 \\
= \; & \frac{1}{\tau} \sum_{j=1}^{\tau} \varepsilon_j^2 + O_p\left(\frac{\tau^{-1} \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}^2}{n^2 \delta_n^{2(\tau - \tau_e)}} \right) + O_p\left(\frac{\tau^{-1} \left[\sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1} \varepsilon_j + \sum_{j=\tau_f}^{\tau} \tilde{X}_{j-1} \varepsilon_j \right]}{n \delta_n^{\tau - \tau_e}} \right) \\
& + O_p\left(\frac{n^{-2\alpha}}{\tau} \left[\sum_{j=1}^{\tau_e} \tilde{X}_{j-1}^2 + \sum_{j=\tau_f+1}^{\tau} \tilde{X}_{j-1}^2 \right] \right) + O_p\left(\frac{\sum_{j=1}^{\tau_e} \tilde{X}_{j-1} \varepsilon_j + \sum_{j=\tau_f+1}^{\tau} \tilde{X}_{j-1} \varepsilon_j}{n^\alpha \tau} \right) \\
& + \tau^{-1} X_{\tau_f}^2 \{1 + o_p(1)\} \\
= \; & \tau^{-1} \delta_n^{2(\tau_f - \tau_e)} X_{\tau_e}^2 \{1 + o_p(1)\},
\end{aligned}
$$

since $X_{\tau_f} = \delta_n^{(\tau_f - \tau_e)} X_{\tau_e} \{1 + o_p(1)\}$. Hence, the DF^t statistic has the form

$$
\begin{aligned}
& \left(\frac{\sum_{j=1}^{\tau} \tilde{X}_{j-1}^2}{\hat{\sigma}_\tau^2} \right)^{1/2} \left(\hat{\delta}_n(\tau) - 1 \right) \\
= \; & \left(\frac{\frac{n^\alpha \delta_n^{2(\tau_f - \tau_e)}}{2c} X_{\tau_e}^2 \{1 + o_p(1)\}}{\tau^{-1} \delta_n^{2(\tau_f - \tau_e)} X_{\tau_e}^2 \{1 + o_p(1)\}} \right)^{1/2} \left(\hat{\delta}_n(\tau) - 1 \right) \\
= \; & \left(\frac{\tau n^\alpha}{2c} \right)^{1/2} \left(\hat{\delta}_n(\tau) - 1 \right) \{1 + o_p(1)\} \\
= \; & -\frac{n^{(1+\alpha)/2} r^{1/2} c^{1/2}}{\sqrt{2}} \{1 + o_p(1)\} \to -\infty
\end{aligned}
$$

and diverges to minus infinity at the rate $O\left(n^{(1+\alpha)/2}\right)$.

The limit theory for the terminal estimate $\hat{r}_f$ under the alternative when $r > r_f$ now follows. For all $\tau = [nr]$ with $r > r_f$ we have

$$
P\left(\tau \left(\hat{\delta}_n(\tau) - 1 \right) < cv_{\beta_n}^{\delta} \right) = P\left(-n^{1-\alpha} rc \{1 + o_p(1)\} < cv_{\beta_n}^{\delta} \right) \to 1, \tag{54}
$$

and similarly

$$
P\left(DF_r^t < cv_{\beta_n}^{df} \right) = P\left(-\frac{n^{(1+\alpha)/2} c^{1/2}}{\sqrt{2}} \{1 + o_p(1)\} < cv_{\beta_n}^{df} \right) \to 1. \tag{55}
$$

Note that both (54) and (55) hold even for fixed critical values $cv_{\beta_n}^{\delta}$ and $cv_{\beta_n}^{df}$. The remainder of the proof follows as before and we have

$$\hat{r}_f = \inf_{s \geq \hat{r}_e + \frac{\log(n)}{n}} \left\{ s : DF_s^t < cv_{\beta_n}^{df}; s \in (\hat{r}_e, 1) \right\} \rightarrow_p r_f.$$

and

$$\hat{r}_f = \inf_{s \geq \hat{r}_e + \frac{\log(n)}{n}} \left\{ s : n \left(\hat{\delta}_n \left([ns] \right) - 1 \right) < cv_{\beta_n}^{\delta}; s \in (\hat{r}_e, 1) \right\} \rightarrow_p r_f,$$

which hold for all $cv_{\beta_n}^{\delta}, cv_{\beta_n}^{df} \geq 0$. ■

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Table 1: Estimates of r_e and r_f based on the DF coefficient test. We simulate 1,000 sample paths, each with 100 observations, from the model defined by (6). The true values for r_e is 0.4 and the true value for r_f is 0.6. We impose the rule that $\hat{r}_f - \hat{r}_e > [\log(n)]/n$. Critical values are $\log\log^2(\tau)$.

	$1 + c/n^\alpha = 1.035$		$1 + c/n^\alpha = 1.040$		$1 + c/n^\alpha = 1.045$		$1 + c/n^\alpha = 1.050$	
	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$
Mean	.4496	.5968	.4425	.5968	.4369	.5968	.4326	.5968
Std	.0353	.0275	.0311	.0275	.0279	.0275	.0253	.0275
RMSE	.0609	.0277	.0526	.0277	.0462	.0277	.0413	.0277
Exact Est (%)	2.22	98.67	2.22	98.67	3.78	98.67	4.22	98.67

Table 2: Estimates of r_e and r_f based on the DF coefficient test. We simulate 1,000 sample paths, each with 400 observations, from the model defined by (6). The true values for r_e is 0.4 and the true value for r_f is 0.6. We impose the rule that $\hat{r}_f - \hat{r}_e > [\log(n)]/n$. Critical values are $\log\log^2(\tau)$.

	$1 + c/n^\alpha = 1.035$		$1 + c/n^\alpha = 1.040$		$1 + c/n^\alpha = 1.045$		$1 + c/n^\alpha = 1.050$	
	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$
Mean	.4246	.5993	.4213	.5993	.4188	.5993	.4169	.5993
Std	.0167	.0150	.0149	.0150	.0135	.0150	.0125	.0150
RMSE	.0298	.0150	.0260	.0150	.0231	.0150	.0210	.0150
Exact Est (%)	0.22	99.78	0.44	99.78	0.44	99.78	0.66	99.78

Table 3: Estimates of r_e and r_f based on the DF coefficient test. We simulate 1,000 sample paths, each with 800 observations, from the model defined by (6). The true values for r_e is 0.4 and the true value for r_f is 0.6. We impose the rule that $\hat{r}_f - \hat{r}_e > [\log(n)]/n$. Critical values are $\log\log^2(\tau)$.

	$1 + c/n^\alpha = 1.035$		$1 + c/n^\alpha = 1.040$		$1 + c/n^\alpha = 1.045$		$1 + c/n^\alpha = 1.050$	
	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$
Mean	.4182	.6000	.4157	.6000	.4139	.6000	.4125	.6000
Std	.0135	.0000	.0116	.0000	.0103	.0000	.0092	.0000
RMSE	.0226	.0000	.0196	.0000	.0173	.0000	.0155	.0000
Exact Est (%)	0.00	100	0.00	100	0.00	100	0.00	100

Table 4: Estimates of r_e and r_f based on the DF t test. We simulate 1,000 sample paths, each with 100 observations, from the model defined by (6). The true values for r_e is 0.4 and the true value for r_f is 0.6. We impose the rule that $\hat{r}_f - \hat{r}_e > [\log(n)]/n$. Critical values are $\frac{2}{3}\log\log^2(\tau)$.

	$1 + c/n^\alpha = 1.035$		$1 + c/n^\alpha = 1.040$		$1 + c/n^\alpha = 1.045$		$1 + c/n^\alpha = 1.050$	
	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$
Mean	.4579	.5978	.4492	.5978	.4435	.5978	.4387	.5978
Std	.0354	.0230	.0311	.0230	.0280	.0230	.0256	.0230
RMSE	.0679	.0231	.0582	.0231	.0517	.0231	.0464	.0231
Exact Est (%)	1.33	99.11	2.00	99.11	2.22	99.11	2.67	99.11

Table 5: Estimates of r_e and r_f based on the DF t test. We simulate 1,000 sample paths, each with 400 observations, from the model defined by (6). The true values for r_e is 0.4 and the true value for r_f is 0.6. We impose the rule that $\hat{r}_f - \hat{r}_e > [\log(n)]/n$. Critical values are $\frac{2}{3}\log\log^2(\tau)$.

	$1 + c/n^\alpha = 1.035$		$1 + c/n^\alpha = 1.040$		$1 + c/n^\alpha = 1.045$		$1 + c/n^\alpha = 1.050$	
	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$
Mean	.4284	.6000	.4248	.6000	.4220	.6000	.4198	.6000
Std	.0152	.0000	.0134	.0000	.0118	.0000	.0107	.0000
RMSE	.0323	.0000	.0282	.0000	.0250	.0000	.0225	.0000
Exact Est (%)	0.00	100	0.00	100	0.00	100	0.00	100

Table 6: Estimates of r_e and r_f based on the DF t test. We simulate 1,000 sample paths, each with 800 observations, from the model defined by (6). The true values for r_e is 0.4 and the true value for r_f is 0.6. We impose the rule that $\hat{r}_f - \hat{r}_e > [\log(n)]/n$. Critical values are $\frac{2}{3}\log\log^2(\tau)$.

	$1 + c/n^\alpha = 1.035$		$1 + c/n^\alpha = 1.040$		$1 + c/n^\alpha = 1.045$		$1 + c/n^\alpha = 1.050$	
	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$	$\hat{r}_e$	$\hat{r}_f$
Mean	.4203	.6000	.4177	.6000	.4157	.6000	.4141	.6000
Std	.0134	.0000	.0116	.0000	.0102	.0000	.0091	.0000
RMSE	.0243	.0000	.0211	.0000	.0187	.0000	.0168	.0000
Exact Est (%)	0.00	100	0.00	100	0.00	100	0.00	100