Local Powers of Least-Squares-Based Test for Panel Fractional Ornstein-Uhlenbeck Process

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Abstract

In recent years, significant advancements have been made in the field of identifying financial bubbles, particularly through the development of time-series unit-root tests featuring fractionally integrated errors and panel unit-root tests. This study introduces an innovative approach for assessing the persistence parameter (α) sign within a panel fractional Ornstein-Uhlenbeck process, based on the least squares estimator of α. This method incorporates three distinct test statistics based on the Hurst parameter (H), which can take values in the range of $(1/2, 1)$, be equal to $1/2$, or fall within the interval of $(0, 1/2)$. The null hypothesis corresponds to $\alpha = 0$. Based on a panel of continuous record of observations, the null asymptotic distributions are obtained when the time span (T) is fixed and the number of cross sections (N) goes to infinity. The power function of the tests is obtained under the local alternative where α is close to zero in the order of $1/(T\sqrt{N})$. This alternative covers the departure from the unit root hypothesis from the explosive side, enabling the calculation of lower power in bubble tests. The hypothesis testing problem and the local power function are also considered when a panel of discrete-sampled observations is available under a sequential limit.

Keywords: Panel fractional Ornstein-Uhlenbeck process, Least squares, Asymptotic distribution, Local alternative, Local power

JEL Classification: C22, C23

1 Introduction

Inspired by the influential research of Phillips and Magdalinos (2007), a proliferation of techniques aimed at detecting explosiveness in single time series has emerged. The

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majority of these methods center around right-tailed unit root tests within the framework of autoregressive (AR) models, such as AR(1) or AR(p). Noteworthy instances encompass the contributions of Phillips et al. (2011), Phillips and Yu (2011), Phillips et al. (2014), Phillips et al. (2015a, 2015b), and Pedersen and Schütte (2020). These investigations consistently build upon the least squares estimator (LSE) derived from individual time series data, coupled with an AR model exhibiting weakly dependent errors. The local power function of the tests can be obtained using the local-to-unity approach suggested by Phillips (1987).

In recent times, the academic landscape has witnessed the emergence of novel bubble detection techniques that expand the horizons of traditional methods along two distinct trajectories. The initial trajectory continues to center on utilizing individual time series data, while now accommodating fractionally integrated errors. Notable contributions in this vein include the works of Magdalinos (2012), Lui et al. (2020, 2023), Wang et al. (2023), and Wang and Yu (2023). However, the precise characterization of the local power function for these tests remains elusive.

Moreover, an important stride has been taken by extending the realm of assumption from weakly dependent errors to encompass fractionally dependent errors. This extension holds empirical significance, as many economic and financial time series inherently exhibit the property of fractional dependence. This development contributes to a more comprehensive and accurate analysis, aligning with the intricacies often observed in real-world economic and financial data.

Another trajectory involves augmenting the potency of time-series tests through the application of the least squares estimator (LSE) to panel data. This approach has been embraced by Liu et al. (2023) within the framework of an autoregressive (AR) model featuring weakly dependent errors. The determination of the local power function for this test can be achieved by employing methodologies existing within the realm of panel unit root literature, as exemplified by the work of Breitung (2001). This strategic approach holds the potential to harness the collective information from panel data, potentially yielding enhanced detection capabilities.

In this paper, we combine these two extensions by considering unit root tests in a context of a panel of fractional Ornstein-Uhlenbeck (fO-U) processes. A univariate fO-U process with Hurst parameter H corresponds to an autoregressive fractionally integrated moving average (ARFIMA) model, or to be more precise, ARFIMA$(1, H - 0.5, 0)$; see Shi and Yu (2023). Hence, our approach not only allows for fractionally integrated errors but also a panel of time series, each following the fO-U process. Our proposed panel unit root tests are based on LSE. Moreover, we obtain the local power function of the panel unit root tests. The alternative hypothesis covers the unit root departure from the explosive side.

The fO-U process, which extends the specification of standard Ornstein-Uhlenbeck process, has found a wide range of applications in many fields, including but not limited to economics, finance, biology, physics, chemistry, medicine, and environmental studies. The fO-U process is described by the following stochastic differential equation:

$$dY(t) = \alpha Y(t)dt + dB^H(t), \tag{1.1}$$

where $\alpha \in \mathbb{R}$ is the persistence parameter and $B^H(t)$, a fractional Brownian motion (fBm) with the Hurst parameter $H \in (0,1)$, is a zero-mean Gaussian process with the following covariance

$$\mathbb{E}\left[B^H(t)B(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) := R_H(s,t). \tag{1.2}$$

Several methods have been proposed to estimate α in Model (1.1) when a continuous record of observations is available. For example, the maximum likelihood estimation method has been investigated in the ergodic case (i.e. $\alpha < 0$) by Kleptsyna and Le Breton (2002) and in the non-ergodic case (i.e. $\alpha > 0$) by Belfadli et al. (2011) and Tanaka (2015). The LSE has also been studied in the ergodic case by Hu and Nualart (2010) and Hu et al. (2019) and in the non-ergodic case by El Machkouri et al. (2016). Moreover, in the ergodic case, the minimum contrast estimator (see, e.g., Tanaka (2013)) and the method of moments estimator (see, e.g., Hu and Nualart (2010); Hu et al. (2019)) have been also investigated.

Unlike estimation, hypothesis testing in the fO-U process is less extensively studied. To the best of our knowledge, only a few studies are available in the literature. When a continuous record of observations is available, Moers (2012) proposed a test statistic, based on the function of $Y^2(T)$ and $\int_0^T Y^2(t)\,dt$, to test three types of hypothesis: (1) $\mathcal{H}_0$: $\alpha \geq 0$ against $\mathcal{H}_1$: $\alpha < 0$; (2) $\mathcal{H}_0$: $\alpha \leq 0$ against $\mathcal{H}_1$: $\alpha > 0$; (3) $\mathcal{H}_0$: $\alpha = 0$ against $\mathcal{H}_1$: $\alpha \neq 0$. For $H \geq 1/2$, Tanaka (2013) studied the testing problem $\mathcal{H}_0$: $\alpha = 0$ against $\mathcal{H}_1$: $\alpha < 0$ based on the maximum likelihood estimator (MLE) and minimum contrast estimator while Tanaka (2015) considered the testing problem $\mathcal{H}_0$: $\alpha = 0$ against $\mathcal{H}_1$: $\alpha > 0$ based on the MLE. For $H \in (0,1)$, Kukush et al. (2017) proposed a test statistic for the sign of α based on a logarithmic function of $Y(t)$. It is worth emphasizing that the test statistic proposed by Kukush et al. (2017) is based on the observation of Y at one point t. Kukush et al. (2017) presented some algorithms for testing $\mathcal{H}_0$: $\alpha \leq 0$ against $\mathcal{H}_1$: $\alpha > 0$ for all $H \in (0,1)$.

The discussion above focuses on the hypothesis testing problem of an fO-U process when a single time series is observed (i.e. $N = 1$). In practice, multiple time series may be observed (i.e. $N > 1$). Consequently, hypothesis testing in a panel framework is of interest and may significantly raise the power of time-series based unit root tests; see Liu et al (2023). Assuming that H is known and a panel of continuous records of observations is available, Tanaka (2019) investigates the hypothesis testing problem using the MLE of the panel fO-U process. The limiting distributions and the local power function are obtained in Tanaka (2019).

In practice, it is rare that a panel of continuous records of observations is available. When discrete-sampled data is available, it is not clear how to construct the likelihood function of a fractional continuous-time model. As a result, it is not known how to do maximum likelihood. This is why we consider the LSE of α and propose the test statistics based on the LSE in this paper.

To facilitate the construction of the LSE from discrete-sampled data, we first assume that a panel of continuous records of observations is available. We construct the LSE of α in the panel fO-U process based on the idea of the LSE for the discrete-time fractional local-to-unit root model. We then propose three test statistics, depending on $1/2 < H < 1$, $0 < H < 1/2$, or $H = 1/2$. The proposed statistics are used to test the null hypothesis that $\alpha = 0$. The null asymptotic distributions are obtained when N is assumed to go to infinity, where N is the number of cross sections. The limiting power function of the tests is obtained under the local alternative where α is close to zero in the order of $1/\left(T\sqrt{N}\right)$. The alternative covers the departure from the unit root from the explosive side. The limiting power function of the LSE-based tests is compared with that of the MLE-based test of Tanaka (2019).

When a panel of discrete-sampled observations is available, we introduce three versions of the LSE of α and three corresponding test statistics. The null asymptotic distributions are obtained when h is assumed to go to zero and then N is assumed to go to infinity, where h is the sampling interval between any two consecutive observations. The limiting power function of the tests is obtained. The limiting distributions and the power function are shown to be the same as those based on a panel of continuous records of observations.

The rest of this paper is organized as follows. Section 2 introduces the panel model, the LSE of α, the null and alternative hypotheses, and the test statistics when a panel of continuous records of observations is available. The asymptotic properties of the proposed test statistics and their limiting power function are also obtained and compared with those of the MLE-based test proposed. Section 3 constructs the LSE of α, the asymptotic properties of the proposed test statistics, and their limiting power functions when a panel of discrete-sampled observations is available. Section 4 contains some concluding remarks and directions of further works. All the proofs are collected in the Appendix.

We use the following notations throughout the paper: $\xrightarrow{p}$, $\xrightarrow{\mathcal{L}}$ and $\sim$ denote convergence in probability, convergence in distribution and asymptotic equivalence, respectively. Moreover, we will use the notation C for generic constants depending on H, which may change from line to line.

2 Model, Estimator, Test Statistics, and Power

2.1 Model and LSEs

Let $B^H(t)$ be defined on the complete filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,\infty)}\right)$. The panel fO-U model is

$$dY_i(t) = \alpha Y_i(t)dt + dB_i^H(t), Y_i(0) = 0, i = 1, \ldots, N,\ t \in [0, T]\,, \tag{2.1}$$

where $\alpha \in \mathbb{R}$ is an unknown persistence parameter and N is the cross-section dimension. Assume a continuous record of observations is available for $Y_i(t)$ for $0 \leq t \leq T$ and for all $i = 1, \ldots, N$. Following Tanaka (2019), we assume that $B_i^H(t)$ is independent of $B_j(t)$ for all $i \neq j$. When $\alpha < 0$, $Y_i(t)$ is ergodic for all i. When $\alpha > 0$, $Y_i(t)$ is explosive and hence non-ergodic for all i. When $\alpha = 0$, $Y_i(t)$ is null-recurrent for all i.

One may test the following one-sided hypothesis

$$\mathcal{H}_0 : \alpha = 0 \quad \text{against} \quad \mathcal{H}^L : \alpha < 0\,, \tag{2.2}$$

or

$$\mathcal{H}_0 : \alpha = 0 \quad \text{against} \quad \mathcal{H}^R : \alpha > 0. \tag{2.3}$$

Following Tanaka (2019), we consider the hypothesis testing problems with a local alternative, that is,

$$\mathcal{H}_0 : \alpha = 0 \quad \text{against} \quad \mathcal{H}_L^L : \alpha = \delta/\left(T\sqrt{N}\right) \quad \text{with } \delta < 0\,, \tag{2.4}$$

or

$$\mathcal{H}_0 : \alpha = 0 \quad \text{against} \quad \mathcal{H}_L^R : \alpha = \delta/\left(T\sqrt{N}\right) \quad \text{with } \delta > 0\,. \tag{2.5}$$

Note that $Y_i(t)$ reduces to $B_i^H(t)$ under $\mathcal{H}_0$.

Let $\Gamma(\cdot)$ denote the gamma function. Tanaka (2019) considered the MLE of α from a panel of continuous records of observations:

$$\breve{\alpha}\,(N, T) = \frac{\sum_{i=1}^N \int_0^T Q_i\,(t)\,dZ_i\,(t)}{\sum_{i=1}^N \int_0^T Q_i^2\,(t)\,d\omega\,(t)}\,, \tag{2.6}$$

where, for any $i \in \{1, ..., N\}$,

$$\begin{aligned}
Q_i(t) &= \frac{d}{d\omega(t)} \int_0^t k(t,s) Y_i(s) \, ds \,, \\
Z_i(t) &= \int_0^t k(t,s) \, dY_i(s) = \alpha_i \int_0^t Q_i(s) \, d\omega(s) + M(t) \,, \\
\omega(t) &= \lambda^{-1} t^{2-2H} \,, \\
k(t,s) &= \kappa^{-1} (s(t-s))^{\frac{1}{2}-H} \,, \\
\kappa &= 2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(H+\frac{1}{2}\right) \,, \\
\lambda &= \frac{2H\Gamma(3-2H)\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)} \,, \\
M(t) &= \int_0^t k(t,s) \, dB_i(s) \,.
\end{aligned}$$

Tanaka (2019) introduces the test statistic $\sqrt{N}T\breve{\alpha}(N,T)$ and shows that, under the null hypothesis $\mathcal{H}_0$, as $N \to \infty$,

$$\sqrt{N}T\breve{\alpha}(N,T) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{\beta_H}\right) \,, \tag{2.7}$$

where

$$\beta_H = \frac{1}{4} + \frac{1}{16H(1-H)} .$$

Under the alternative, regardless of $\mathcal{H}_L^L$ or $\mathcal{H}_L^R$, Tanaka (2019) shows that, as $N \to \infty$,

$$\sqrt{N}T\breve{\alpha}(N,T) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\delta, \frac{1}{\beta_H}\right) . \tag{2.8}$$

Therefore, the limiting (local) power function of the MLE-based test is

$$\mathbb{P}\left(\sqrt{N\beta_H}T\breve{\alpha}(N,T) < z_\gamma\right) \to \Phi\left(z_\gamma - \delta\sqrt{\beta_H}\right), \text{ as } N \to \infty \,, \tag{2.9}$$

where $\Phi(\cdot)$ is the distribution function of $\mathcal{N}(0,1)$ and z_γ is the $100 \times \gamma\%$ point of $\mathcal{N}(0,1)$. Tanaka (2019) also gives the exact power function of the test when T and N are finite.

The estimator considered in this paper is based on the least squares. To motivate the LSE from a panel of continuous records of observations, let us first review the idea of the LSE in the discrete-time model defined by

$$y_j = \rho y_{j-1} + \nu_j, \ (1-L)^{H-\frac{1}{2}} \nu_j = \varepsilon_j, \ y_0 = 0, \ \rho = 1 + \frac{\alpha}{n}, \varepsilon_j \overset{i.i.d.}{\sim} \left(0, \sigma^2\right) , \tag{2.10}$$

where $H \in (0,1)$. By definition, ν_j is a stationary and fractionally integrated process defined by

$$\nu_j = (1-L)^{-(H-1/2)} \varepsilon_j := \sum_{k=0}^{\infty} \frac{\Gamma(k+H-1/2)}{\Gamma(H-1/2)\Gamma(k+1)} \varepsilon_{j-k} .$$

Model (2.10) is a local-to-unit root model with a fractionally integrated error term. The LSE of ρ takes the form of

$$\hat{\rho} = \frac{\sum_{j=1}^{n} y_{j-1} y_j}{\sum_{j=1}^{n} y_{j-1}^2} = 1 + \frac{\sum_{j=1}^{n} (y_j - y_{j-1}) y_{j-1}}{\sum_{j=1}^{n} y_{j-1}^2} = 1 + \frac{\frac{1}{2}\left(y_n^2 - \sum_{j=1}^{n} (y_j - y_{j-1})^2\right)}{\sum_{j=1}^{n} y_{j-1}^2}. \tag{2.11}$$

From Tanaka (2017, page 605), for any $0 < H < 1$, as $n \to \infty$, we have

$$y_n^2 = \mathcal{O}_p\left(n^{2H}\right), \ \sum_{j=1}^{n} (y_j - y_{j-1})^2 = \mathcal{O}_p(n), \ \sum_{j=1}^{n} y_{j-1}^2 = \mathcal{O}_p\left(n^{2H+1}\right). \tag{2.12}$$

Denote as $Y(t)$ the solution of (1.1). Combining (2.11) with (2.12), we get

$$n(\hat{\rho} - 1) \xrightarrow{\mathcal{L}} \frac{\frac{1}{2} Y^2(1)}{\int_0^1 Y^2(t)\,dt}, \text{ when } 1/2 < H < 1, \tag{2.13}$$

$$n^{2H}(\hat{\rho} - 1) \xrightarrow{\mathcal{L}} \frac{-\frac{1}{2\sigma^2}\operatorname{Var}(y_i - y_{i-1})}{\int_0^1 Y^2(t)\,dt} = \frac{-\frac{1}{2} A_H}{\int_0^1 Y^2(t)\,dt}, \text{ when } 0 < H < 1/2, \tag{2.14}$$

$$n(\hat{\rho} - 1) \xrightarrow{\mathcal{L}} \frac{\int_0^1 Y(t)\,dY(t)}{\int_0^1 Y^2(t)\,dt}, \text{ when } H = 1/2, \tag{2.15}$$

where $A_H = \frac{\Gamma(2-2H)}{\Gamma^2\left(\frac{3}{2}-H\right)}$.

Borrowing the idea in (2.13), (2.14) and (2.15), we propose the following three LSEs of α in the panel fO-U model, depending on the true value of H,

$$\hat{\alpha}(N,T) = \frac{\frac{1}{2}\sum_{i=1}^{N} Y_i^2(T)}{\sum_{i=1}^{N} \int_0^T Y_i^2(t)\,dt}, \text{ when } 1/2 < H < 1, \tag{2.16}$$

$$\bar{\alpha}(N,T) = \frac{-\frac{1}{2}\sum_{i=1}^{N} A_H}{\sum_{i=1}^{N} \int_0^T Y_i^2(t)\,dt}, \text{ when } 0 < H < 1/2, \tag{2.17}$$

$$\tilde{\alpha}(N,T) = \frac{\sum_{i=1}^{N} \int_0^T Y_i(t)\,dY_i(t)}{\sum_{i=1}^{N} \int_0^T Y_i^2(t)\,dt}, \text{ when } H = 1/2. \tag{2.18}$$

To test the hypotheses specified in (2.4) and (2.5), we propose the following three test statistics, $\sqrt{N}\left(T\hat{\alpha}(N,T) - \left(H + \frac{1}{2}\right)\right)$, $\sqrt{N}\left(T^{2H+1}\bar{\alpha}(N,T) + \left(H + \frac{1}{2}\right)A_H\right)$, $\sqrt{N}T\tilde{\alpha}(N,T)$, depending on the true value of H. The reason why $T\hat{\alpha}(N,T)$ and $T^{2H+1}\bar{\alpha}(N,T)$ need to be re-centered will become clear soon.

2.2 Asymptotic properties and local power of the tests

Since the expressions of the LSE and the test statistic depend on the true value of H, we consider the hypothesis testing problem for $1/2 < H < 1$, $0 < H < 1/2$, and $H = 1/2$ separately. In all cases, $Y_i(t) = B_i(t)$ under the null hypothesis.

2.2.1 Case of $1/2 < H < 1$

We are now consider the asymptotic distribution of a properly centered $\sqrt{N}T\hat{\alpha}(N,T)$, which is presented in the following theorem.

Theorem 2.1 *For $1/2 < H < 1$ and under $\mathcal{H}_0$, as $N \to \infty$, the asymptotic distribution of a properly centered $\sqrt{N}T\hat{\alpha}(N,T)$ is*

$$\sqrt{N}\left(T\hat{\alpha}(N,T) - \left(H + \frac{1}{2}\right)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma_H^2\right), \tag{2.19}$$

where $\sigma_H^2 = (2H+1)^2\left[\frac{1}{2} + \frac{(2H+1)\Gamma^2(2H+1)}{4\Gamma(4H+2)} - \frac{(2H+1)(4H+3)}{8(4H+1)}\right]$. For $1/2 < H < 1$ and under $\mathcal{H}_L^L$ or $\mathcal{H}_L^R$, as $N \to \infty$, the asymptotic distribution of a properly centered $\sqrt{N}T\hat{\alpha}(N,T)$ is

$$\sqrt{N}\left(T\hat{\alpha}(N,T) - \left(H + \frac{1}{2}\right)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\frac{\delta}{4(H+1)}, (2H+1)^2 F_H\right), \tag{2.20}$$

where $F_H = \frac{1}{2} + \frac{2H+1}{4}B(2H+1, 2H+1) - \frac{(2H+1)(4H+3)}{8(4H+1)}$ and $B(\cdot,\cdot)$ is the beta function.

Remark 2.1 *When $1/2 < H < 1$, as $N \to \infty$, the limiting (local) power function of the LSE-based test is*

$$\mathbb{P}\left(\frac{\sqrt{N}\left(T\hat{\alpha}(N,T) - \left(H + \frac{1}{2}\right)\right)}{(2H+1)\sqrt{F_H}} \le z_\gamma\right) \to \Phi\left(z_\gamma - \frac{\delta}{4(H+1)(2H+1)\sqrt{F_H}}\right). \tag{2.21}$$

This limiting (local) power function compares to that of the MLE-based test in (2.9). Figure 1 plots the two sets of limiting (local) power functions for the ergodic alternatives. The limiting (local) power functions for the non-ergodic alternatives should be the mirror image to those for the ergodic alternatives for both tests. Clearly, the MLE-based test is always more powerful than the LSE-based test when $1/2 < H < 1$.

Remark 2.2 *Under $\mathcal{H}_0$, $T\hat{\alpha}(N,T)$ converges in probability to $H + \frac{1}{2}$. Hence, $\hat{\alpha}(N,T)$ is not a consistent estimator of α. The inconsistency in the LSE is also found in the case of a single time series; see Xiao and Yu (2019a).*

2.2.2 Case of $0 < H < 1/2$

The asymptotic distribution of a properly centered $\sqrt{N}T^{2H+1}\bar{\alpha}(N,T)$ is presented in the following theorem.

Theorem 2.2 *For $0 < H < 1/2$ and under $\mathcal{H}_0$, as $N \to \infty$, the asymptotic distribution of a properly centered $\sqrt{N}T^{2H+1}\bar{\alpha}(N,T)$ is*

$$\sqrt{N}\left(T^{2H+1}\bar{\alpha}(N,T) + \left(H + \frac{1}{2}\right)A_H\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \lambda_H^2\right), \tag{2.22}$$

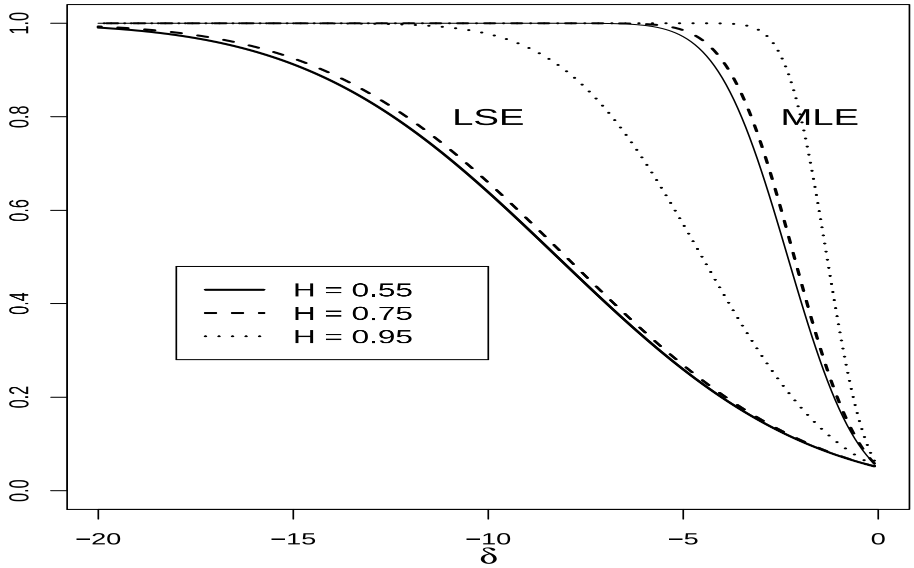

Figure 1: Limiting local powers of the LSE-based test and the MLE-based test against the ergodic alternative $\mathcal{H}_L^L : \alpha = \delta/(\sqrt{N}T)$ $(\delta < 0)$ when $1/2 < H < 1$.

where $A_H = \frac{\Gamma(2-2H)}{\Gamma^2\left(\frac{3}{2}-H\right)}$ *and* $\lambda_H^2 = \frac{A_H^2(2H+1)^4}{4}\left(\frac{4H+3}{(4H+1)(4H+2)} - \frac{2\Gamma^2(2H+1)}{\Gamma(4H+3)}\right) = \frac{A_H^2(2H+1)^4}{4} I_H$. *For* $0 < H < 1/2$ *and under* $\mathcal{H}_L^L$ *or* $\mathcal{H}_L^R$*, as* $N \to \infty$*, the asymptotic distribution of a properly centered* $T^{2H+1}\bar{\alpha}(N,T)$ *is*

$$\sqrt{N}\left(T^{2H+1}\bar{\alpha}(N,T) + \left(H + \frac{1}{2}\right)A_H\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\frac{(2H+1)A_H\delta}{4(H+1)}, \lambda_H^2\right). \tag{2.23}$$

Remark 2.3 *When* $0 < H < 1/2$*, as* $N \to \infty$*, the limiting (local) power function of the LSE-based test is*

$$\mathbb{P}\left(\frac{\sqrt{N}\left(T^{2H+1}\bar{\alpha}(N,T) + \left(H+\frac{1}{2}\right)A_H\right)}{\lambda_H} \le z_\gamma\right) \to \Phi\left(z_\gamma - \frac{\delta}{2(H+1)(2H+1)\sqrt{I_H}}\right). \tag{2.24}$$

This limiting (local) power function compares to that of the MLE-based test in (2.9). Figure 2 plots the two sets of limiting (local) power functions for the ergodic alternatives. The limiting (local) power functions for the non-ergodic alternatives should be the mirror image to those for the ergodic alternatives for both tests. Clearly, the MLE-based test is always more powerful than the LSE-based test when $0 < H < 1/2$.

Remark 2.4 *Under* $\mathcal{H}_0$, $T\bar{\alpha}(N,T)$ *converges in probability to* $-\left(H+\frac{1}{2}\right)$. *Hence,* $\bar{\alpha}(N,T)$ *is not a consistent estimator of* α. *The inconsistency in the LSE is also found in the case of a single time series; see Xiao and Yu (2019b).*

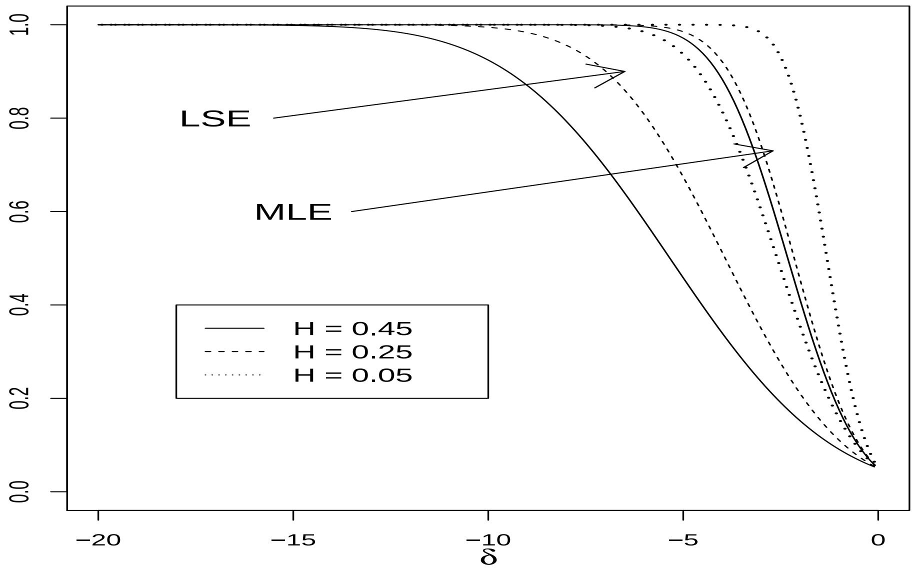

Figure 2: Limiting local powers of the LSE-based test and the MLE-based test against the ergodic alternative $\mathcal{H}_L^L : \alpha = \delta/(\sqrt{N}T)$ $(\delta < 0)$ when $0 < H < 1/2$.

2.2.3 Case of $H = 1/2$

When $H = 1/2$, $B_i(t) = W_i(t)$, where $W_i(t)$ denotes a standard Brownian motion. Under $\mathcal{H}_0$, we have

$$\tilde{\alpha}(N,T) = \frac{\sum_{i=1}^{N} \int_0^T W_i(t)\, dW_i(t)}{\sum_{i=1}^{N} \int_0^T W_i^2(t)\, dt} = \frac{\frac{1}{2}\sum_{i=1}^{N} \left(W_i^2(T) - T\right)}{T^2 \sum_{i=1}^{N} \int_0^1 W_i^2(t)\, dt}. \tag{2.25}$$

Consequently, we obtain

$$T\tilde{\alpha}(N,T) = \frac{\frac{1}{2}\sum_{i=1}^{N} \left(W_i^2(1) - 1\right)}{\sum_{i=1}^{N} \int_0^1 W_i^2(t)\, dt}. \tag{2.26}$$

The asymptotic distributions of $\sqrt{N}T\tilde{\alpha}(N,T)$ is presented in the following theorem.

Theorem 2.3 *For $H = 1/2$ and under $\mathcal{H}_0$, as $N \to \infty$, we have*

$$\sqrt{N}T\tilde{\alpha}(N,T) \xrightarrow{\mathcal{L}} \mathcal{N}(0,2). \tag{2.27}$$

For $H = 1/2$ and under $\mathcal{H}_L^L$ or $\mathcal{H}_L^R$, as $N \to \infty$, we have

$$\sqrt{N}T\tilde{\alpha}(N,T) \xrightarrow{\mathcal{L}} \mathcal{N}(\delta,2). \tag{2.28}$$

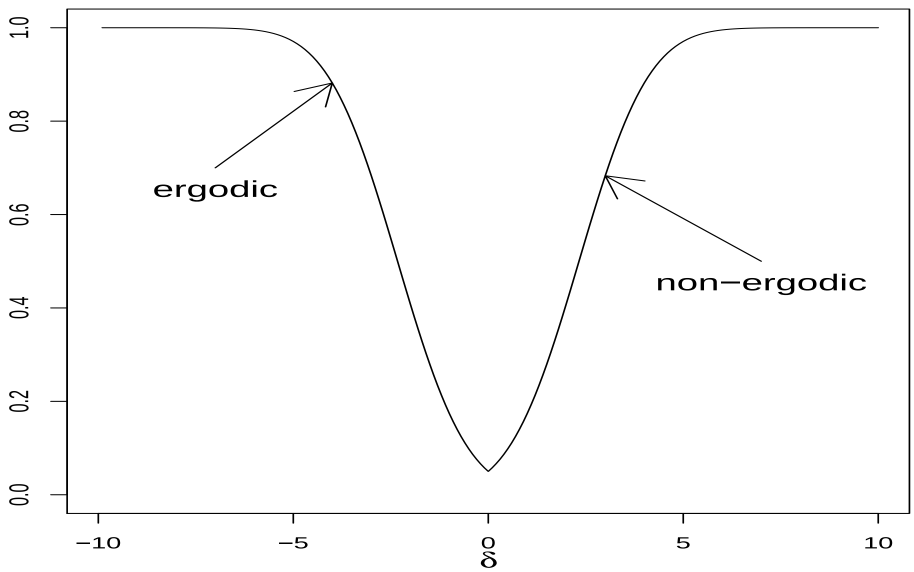

Figure 3: Limiting local powers of the LSE-based test against the ergodic alternative $\mathcal{H}_L^L : \alpha = \delta/(\sqrt{N}T)$ $(\delta < 0)$ and non-ergodic alternative $\mathcal{H}_L^R : \alpha = \delta/(\sqrt{N}T)$ $(\delta > 0)$ when $H = 1/2$.

Remark 2.5 *When $H = 1/2$, the limiting (local) power function of the LSE-based test is*

$$\mathbb{P}\left(\frac{\sqrt{N}T\tilde{\alpha}(N,T)}{\sqrt{2}} \le z_\gamma\right) \to \Phi\left(z_\gamma - \frac{\delta}{\sqrt{2}}\right). \tag{2.29}$$

Remark 2.6 *When $H = 1/2$, the LSE of α is the same as the MLE of α. It can be verified that $\beta_H = 1/2$ in this case. Hence, the limiting distributions in (2.27) and (2.28) are the same as those in (2.7) and (2.8). Not surprisingly, the limiting power function in (2.29) is the same as that in (2.9). Figure 3 plots the limiting local power function against the ergodic and the non-ergodic alternative when $H = 1/2$.*

3 Discrete-sampled Data

In Section 2 it is assumed that a panel of continuous-record observations is available. In practice, data are almost always available in discrete-time. Therefore, hypothesis testing for discrete-sampled fO-U processes is of great interest for practitioners. In this section we assume that a panel of discrete-sampled data $Y_i(jh)$, where $i = 1, \ldots, N$, $j = 0, 1, \ldots, M\left(:= \frac{T}{h}\right)$, is generated from model (2.1) and observed by econometricians.

Here, N is the number of cross-sectional units and $T := Mh$ is the time span of each time series with h being the sampling interval between any two consecutive observations. Based on these discrete-sampled data, we propose the following three LSEs of α:

$$\hat{\alpha}(N,T,h) = \frac{\frac{1}{2}\sum_{i=1}^{N} Y_i^2(Mh)}{\sum_{i=1}^{N}\sum_{j=0}^{M-1} hY_i^2(jh)}, \text{ when } 1/2 < H < 1, \tag{3.1}$$

$$\bar{\alpha}(N,T,h) = \frac{-\frac{1}{2}\sum_{i=1}^{N} A_H}{\sum_{i=1}^{N}\sum_{j=0}^{M-1} hY_i^2(jh)}, \text{ when } 0 < H < 1/2, \tag{3.2}$$

$$\tilde{\alpha}(N,T,h) = \frac{\frac{1}{2}\sum_{i=1}^{N}\left(Y_i^2(Mh) - Mh\right)}{\sum_{i=1}^{N}\sum_{j=0}^{M-1} hY_i^2(jh)}, \text{ when } H = 1/2. \tag{3.3}$$

Before deriving the power of our test, we first establish a few lemmas.

Lemma 3.1 *For all $H \in (0,1)$ and a fixed T, we have*

$$\mathbb{E}\left[\frac{1}{M}\sum_{j=0}^{M-1} B_i^2(jh) - \frac{1}{Mh}\int_0^{Mh} B_i^2(t)\,dt\right] \leq CT^{\frac{p}{2}(\epsilon+H)}h^{H-\epsilon}, \tag{3.4}$$

with $0 < \epsilon < H$ and $p \geq 1$, for any $i = 1, \ldots, N$.

Lemma 3.2 *For $\alpha < 0$, $H \in (0,1)$ and a fixed T, we have*

$$\mathbb{E}\left[\frac{1}{M}\sum_{j=0}^{M-1} Y_i^2(jh) - \frac{1}{Mh}\int_0^{Mh} Y_i^2(t)\,dt\right] \leq Ch^H, \tag{3.5}$$

for any $i = 1, \ldots, N$.

Lemma 3.3 *Let $\mathfrak{Z}$ be the class of nonnegative random variables ζ with the following property: there exists $C > 0$ independent of M such that $\mathbb{E}\exp\{x\zeta^2\} < \infty$ for any $0 < x < C$. For $\alpha > 0$, $H \in (0,1)$ and a fixed T, we have*

$$\left|\int_0^T Y_i^2(s)\,ds - h\sum_{k=0}^{M-1} Y_i^2(kh)\right| \leq Ce^{2\alpha T}\zeta^2 h, \tag{3.6}$$

where $\zeta \in \mathfrak{Z}$ and for any $i = 1, \ldots, N$.

We are now in the position to report the following three theorems under a sequential limit, that is, $h \to 0$ followed by $N \to \infty$.

Theorem 3.1 *For* $1/2 < H < 1$ *and under* $\mathcal{H}_0$*, as* $h \to 0$ *followed by* $N \to \infty$ *and* $Nh^{H-\epsilon} \to 0$ *with* $0 < \epsilon < H$*, the asymptotic distribution of a properly centered* $\sqrt{N}T\hat{\alpha}(N,T,h)$ *is*

$$\sqrt{N}\left(T\hat{\alpha}(N,T,h) - \left(H + \frac{1}{2}\right)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma_H^2\right), \tag{3.7}$$

where σ_H^2 *is defined in Theorem 2.1.*

For $1/2 < H < 1$ *and under* $\mathcal{H}_L^L$ *or* $\mathcal{H}_L^R$*, as* $h \to 0$ *followed by* $N \to \infty$ *and* $Nh^{H-\epsilon} \to 0$ *with* $0 < \epsilon < H$*, the asymptotic distribution of a properly centered* $\sqrt{N}T\hat{\alpha}(N,T)$ *is*

$$\sqrt{N}\left(T\hat{\alpha}(N,T,h) - \left(H + \frac{1}{2}\right)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\frac{\delta}{4(H+1)}, (2H+1)^2 F_H\right), \tag{3.8}$$

where F_H *is defined by Theorem 2.1.*

When $1/2 < H < 1$*, as* $h \to 0$ *followed by* $N \to \infty$ *and* $Nh^{H-\epsilon} \to 0$ *with* $0 < \epsilon < H$*, the limiting (local) power function of the LSE-based test is*

$$\mathbb{P}\left(\frac{\sqrt{N}\left(T\hat{\alpha}(N,T,h) - \left(H + \frac{1}{2}\right)\right)}{(2H+1)\sqrt{F_H}} \leq z_\gamma\right) \to \Phi\left(z_\gamma - \frac{\delta}{4(H+1)(2H+1)\sqrt{F_H}}\right). \tag{3.9}$$

Theorem 3.2 *For* $0 < H < 1/2$ *and under* $\mathcal{H}_0$*, as* $h \to 0$ *followed by* $N \to \infty$ *and* $Nh^{H-\epsilon} \to 0$ *with* $0 < \epsilon < H$*, the asymptotic distribution of a properly centered* $\sqrt{N}T^{2H+1}\bar{\alpha}(N,T,h)$ *is*

$$\sqrt{N}\left(T^{2H+1}\bar{\alpha}(N,T,h) + \left(H + \frac{1}{2}\right)A_H\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \lambda_H^2\right), \tag{3.10}$$

where A_H *and* λ_H *are defined in Theorem 2.2. For* $0 < H < 1/2$ *and under* $\mathcal{H}_L^L$ *or* $\mathcal{H}_L^R$*, as* $h \to 0$ *followed by* $N \to \infty$ *and* $Nh^{H-\epsilon} \to 0$ *with* $0 < \epsilon < H$*, the asymptotic distribution of a properly centered* $T^{2H+1}\bar{\alpha}(N,T,h)$ *is*

$$\sqrt{N}\left(T^{2H+1}\bar{\alpha}(N,T,h) + \left(H + \frac{1}{2}\right)A_H\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\frac{(2H+1)A_H\delta}{4(H+1)}, \lambda_H^2\right). \tag{3.11}$$

When $0 < H < 1/2$*, as* $h \to 0$ *followed by* $N \to \infty$ *and* $Nh^{H-\epsilon} \to 0$ *with* $0 < \epsilon < H$*, the limiting (local) power function of the LSE-based test is*

$$\mathbb{P}\left(\frac{\sqrt{N}\left(T^{2H+1}\bar{\alpha}(N,T,h) + \left(H + \frac{1}{2}\right)A_H\right)}{\lambda_H} \leq z_\gamma\right) \to \Phi\left(z_\gamma - \frac{\delta}{2(H+1)(2H+1)\sqrt{I_H}}\right), \tag{3.12}$$

where I_H *is defined in Theorem 2.2.*

Theorem 3.3 *For $H = 1/2$ and under $\mathcal{H}_0$, as $h \to 0$ followed by $N \to \infty$ and $Nh^{1/2-\epsilon} \to 0$ with $0 < \epsilon < 1/2$, we have*

$$\sqrt{N}T\tilde{\alpha}\left(N, T, h\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2\right) . \tag{3.13}$$

For $H = 1/2$ and under $\mathcal{H}_L^L$ or $\mathcal{H}_L^R$, as $h \to 0$ followed by $N \to \infty$ and $Nh^{1/2-\epsilon} \to 0$ with $0 < \epsilon < 1/2$, we have

$$\sqrt{N}T\tilde{\alpha}\left(N, T, h\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\delta, 2\right) . \tag{3.14}$$

When $H = 1/2$, as $h \to 0$ followed by $N \to \infty$ and $Nh^{1/2-\epsilon} \to 0$ with $0 < \epsilon < 1/2$,, the limiting (local) power function of the LSE-based test is

$$\mathbb{P}\left(\frac{\sqrt{N}T\tilde{\alpha}(N, T, h)}{\sqrt{2}} \leq z_\gamma\right) \to \Phi\left(z_\gamma - \frac{\delta}{\sqrt{2}}\right) . \tag{3.15}$$

4 Conclusion

This paper considers the estimation problem and the hypothesis testing problem for the persistence parameter, α, in the panel fO-U process based on the LSE, with a known Hurst index $H \in (0, 1)$. The proposed LSE takes different expressions depending on the true value of H, namely, whether $1/2 < H < 1$, or $0 < H < 1/2$, or $H = 1/2$. Similarly, the test statistics, which test the null of $\alpha = 0$, take different expressions under these three cases. When a panel of continuous record observations is available, we derive the local power functions of the test statistics in the three cases, facilitating the comparison of the efficiency of the proposed tests based on the LSE with those based on the MLE. It is shown that when $1/2 < H < 1$ and $0 < H < 1/2$, the proposed tests based on the LSE are less powerful than those based on the MLE. However, when $H = 1/2$, the proposed test based on the LSE is as powerful as that based on the MLE.

When a panel of discrete-sample data is available, it is not known yet how to apply the MLE. Hence, it is unclear how to construct the test statistic based on the MLE. We then propose the LSE of α based on a panel of discrete-sample data and construct the test statistic under each case of the true value of H, whether $1/2 < H < 1$, or $0 < H < 1/2$, or $H = 1/2$. We also derive the local power function of the test statistics in the three cases under the sequential limit.

In a recent paper, Wang and Yu (2023) develop asymptotic theory for the LSE of α based on time series data in the context of a local-to-unity model with fractionally integrated errors. Their asymptotic distributions also depend on the true values of the memory parameter. The asymptotic distributions can be used to construct unit root tests against both stationary and explosive alternatives. However, since such tests are constructed from the time-series estimate, we expect the power of the tests to be lower than our panel-data-based tests.

This study also suggests several important directions for future research. First, in the present paper the cross-sectional independence was assumed, that is, the fBm $B_1(t), \dots, B_N(t)$ which generate the panel fO-U processes are independent of each other. An extension to the cross-sectional dependence is an important topic to be pursued. Second, this paper assumes that a continuous record of the fO-U process is available for the development of asymptotic theory. In practice, it is usually only possible to observe these processes in discrete-time samples (e.g., stock prices collected once a day or, at best, at every tick). Therefore, the hypothesis testing of the sign of the mean-reversion parameter in the panel fO-U process based on discrete observations has been an active research area and at the same time it posed a challenging problem. Third, this paper considers the hypothesis testing problem of the panel fO-U process for all $H \in (0, 1)$. However, the fractional version of the Heston process, which is called the fractional Heston process, is extensively used for capturing the volatility of an asset price. Actually, under some mild conditions, this process is strictly positive and never hits zero. Due to zero probability of hitting zero, the fractional Heston process is suitable for modeling asset volatility and interest rates. Hence, statistical inference for the fractional Heston process has attracted much attention recently. The main difficulty lies in the fact that it is not clear whether the solution exists for the case $H < 1/2$. It would be interesting to estimate the unknown parameters or consider the hypothesis testing of the fractional Heston process, which is an ongoing project and will be reported in later work.

5 Appendix

5.1 Proof of Theorem 2.1

Under $\mathcal{H}_0$, using (2.16) and the scaling property of the fBm, we obtain

$$T\hat{\alpha}(N,T) = \frac{\frac{T}{2}\sum_{i=1}^{N} B_i^2(T)}{\sum_{i=1}^{N}\int_0^T B_i^2(t)\,dt} = \frac{\frac{T}{2}T^{2H}\sum_{i=1}^{N} B_i^2(1)}{T^{1+2H}\sum_{i=1}^{N}\int_0^1 B_i^2(t)\,dt} = \frac{\sum_{i=1}^{N} U_i(1)}{\sum_{i=1}^{N} V_i(1)}, \tag{5.1}$$

where $U_i(1) = \frac{1}{2}B_i^2(1)$ and $V_i(1) = \int_0^1 B_i^2(t)\,dt$.

Elementary calculations yield

$$\mathbb{E}(U_i(1)) = \frac{1}{2}\mathbb{E}\left(B_i^2(1)\right) = \frac{1}{2}, \tag{5.2}$$

$$Var(U_i(1)) = \frac{1}{4}Var\left(B_i^2(1)\right) = \frac{1}{2}, \tag{5.3}$$

$$\mathbb{E}(V_i(1)) = \int_0^1 \mathbb{E}\left(B_i^2(t)\right)dt = \int_0^1 t^{2H}dt = \frac{1}{2H+1}. \tag{5.4}$$

Using (5.1)-(5.4), we obtain, as $N \to \infty$,

$$T\hat{\alpha}(N,T) \xrightarrow{p} \frac{\frac{1}{2}}{\frac{1}{2H+1}} = H + \frac{1}{2}.$$

Consequently, we consider the following statistic

$$\sqrt{N}\left(T\hat{\alpha}(N,T) - \left(H+\frac{1}{2}\right)\right) = \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(U_i(1) - \left(H+\frac{1}{2}\right)V_i(1)\right)}{\frac{1}{N}\sum_{i=1}^{N}V_i(1)}. \tag{5.5}$$

Applying *Isserlis*' Theorem (or Wick's theorem) from Isserlis (1918) and (1.2), we obtain

$$\begin{aligned}\mathbb{E}\left[B_i^2(s)B_i^2(t)\right] &= \mathbb{E}\left[B_i(s)B_i(s)\right]\mathbb{E}\left[B_i(t)B_i(t)\right] + 2\mathbb{E}\left[B_i(s)B_i(t)\right]\mathbb{E}\left[B_i(s)B_i(t)\right] \\ &= \frac{1}{2}\left(s^{2H}+t^{2H}-|s-t|^{2H}\right)^2 + s^{2H}t^{2H}. \end{aligned} \tag{5.6}$$

Using (5.6), we have

$$\begin{aligned}\int_0^1 \mathbb{E}\left[B_i^2(1)B_i^2(t)\right]dt &= \frac{4H+3}{4H+1}\times\frac{1}{2} + \frac{1}{2H+1} - B(2H+1,2H+1), \\ \int_0^1\int_0^1 \mathbb{E}\left[B_i^2(s)B_i^2(t)\right]dsdt &= \frac{4H+3}{(4H+1)(4H+2)} - \frac{2\Gamma^2(2H+1)}{\Gamma(4H+3)} + \frac{1}{(2H+1)^2},\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the beta function. Using the above results, we have

$$\begin{aligned}&\mathbb{E}\left[\left(U_i(1) - \left(H+\frac{1}{2}\right)V_i(1)\right)^2\right] \\ =\ & \mathbb{E}\left[U_i^2(1) - (2H+1)U_i(1)V_i(1) + \left(H+\frac{1}{2}\right)^2 V_i^2(1)\right] \\ =\ & \frac{1}{4}\mathbb{E}\left[B_i^4(1)\right] - \frac{2H+1}{2}\mathbb{E}\left[B_i^2(1)\int_0^1 B_i^2(t)\,dt\right] + \left(H+\frac{1}{2}\right)^2\mathbb{E}\left[\left(\int_0^1 B_i^2(t)\,dt\right)^2\right] \\ =\ & \frac{3}{4} - \frac{2H+1}{2}\int_0^1 \mathbb{E}\left[B_i^2(1)B_i^2(t)\right]dt + \left(H+\frac{1}{2}\right)^2\int_0^1\int_0^1 \mathbb{E}\left[B_i^2(s)B_i^2(t)\right]dsdt \\ =\ & \frac{3}{4} - \left(H+\frac{1}{2}\right)\left[\frac{4H+3}{4H+1}\times\frac{1}{2} + \frac{1}{2H+1} - \frac{\Gamma^2(2H+1)}{\Gamma(4H+2)}\right] \\ & + \left(H+\frac{1}{2}\right)^2\left[\frac{4H+3}{(4H+1)(4H+2)} - \frac{2\Gamma^2(2H+1)}{\Gamma(4H+3)} + \frac{1}{(2H+1)^2}\right] \\ =\ & \frac{1}{2} + \frac{(2H+1)\Gamma^2(2H+1)}{4\Gamma(4H+2)} - \frac{(2H+1)(4H+3)}{8(4H+1)}. \end{aligned} \tag{5.7}$$

Combining (5.2), (5.4), (5.5) with (5.7), we obtain (2.19).

Now, we consider (2.20). Put $\delta_N = \delta/\sqrt{N}$ for notational simplicity. Using the scaling property of fBm and using $\alpha_i = \delta_N/T$, we can write the following result immediately

$$
\begin{aligned}
T\hat{\alpha}(N,T) &= \frac{\frac{1}{2}\sum_{i=1}^{N} e^{2\delta_N}\left(\int_0^1 e^{-u\delta_N} dB_i(u)\right)^2}{\sum_{i=1}^{N}\int_0^1 e^{2t\delta_N}\left(\int_0^t e^{-u\delta_N} dB_i(u)\right)^2 dt} \\
&= \frac{\frac{1}{2}\sum_{i=1}^{N} X_i^2(1)}{\sum_{i=1}^{N}\int_0^1 X_i^2(t)\,dt}. \qquad (5.8)
\end{aligned}
$$

Using the fact $e^x = 1 + x + \mathcal{O}(x^2)$ as $x \to 0$, we have, as $\delta_N \to 0$,

$$
\begin{aligned}
\mathbb{E}\left[X_i^2(t)\right] &= H(2H-1)e^{2\delta_N t}\int_0^t\int_0^t e^{-\delta_N(u+v)}|u-v|^{2H-2}dudv \\
&= H(2H-1)e^{2\delta_N t}\Big[\int_0^t\int_0^t |u-v|^{2H-2}dudv \\
&\quad -\delta_N\int_0^t\int_0^t (u+v)|u-v|^{2H-2}dudv + \mathcal{O}(\delta_N^2)\Big] \\
&= e^{2\delta_N t}\left[t^{2H} - \delta_N H(2H-1)\int_0^t\int_0^t (u+v)|u-v|^{2H-2}dudv + \mathcal{O}\left(\delta_N^2\right)\right] \\
&= e^{2\delta_N t}\left[t^{2H} - \delta_N t^{2H+1} + \mathcal{O}\left(\delta_N^2\right)\right] \\
&= t^{2H} + \delta_N t^{2H+1} + \mathcal{O}\left(\delta_N^2\right). \qquad (5.9)
\end{aligned}
$$

Combining (5.8) with (5.9), we get

$$
\lim_{N\to\infty}\mathbb{E}\left[T\hat{\alpha}(N,T)\right] = H + \frac{1}{2}. \qquad (5.10)
$$

Applying (5.8) and (5.10), we consider the following result

$$
\sqrt{N}\left(T\hat{\alpha}(N,T) - \left(H+\frac{1}{2}\right)\right) = \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{1}{2}X_i^2(1) - \left(H+\frac{1}{2}\right)\int_0^1 X_i^2(t)\,dt\right)}{\frac{1}{N}\sum_{i=1}^{N}\int_0^1 X_i^2(t)\,dt}. \qquad (5.11)
$$

Using (5.9), we have

$$
\begin{aligned}
\mathbb{E}\left[\frac{1}{2}X_i^2(1) - \left(H+\frac{1}{2}\right)\int_0^1 X_i^2(t)\,dt\right] &= \frac{1}{2}\left(1+\delta_N+\mathcal{O}\left(\delta_N^2\right)\right) \\
&\quad -\left(H+\frac{1}{2}\right)\left(\frac{1}{2H+1} + \frac{1}{2H+2}\delta_N + \mathcal{O}\left(\delta_N^2\right)\right) \\
&= \delta_N\left(\frac{1}{2} - \frac{H+\frac{1}{2}}{2H+2}\right) + \mathcal{O}\left(\delta_N^2\right) \\
&= \frac{1}{4(H+1)}\delta_N + \mathcal{O}\left(\delta_N^2\right). \qquad (5.12)
\end{aligned}
$$

Let $\omega_{11} = \mathbb{E}\left[X_i(s)\, X_i(s)\right]$, $\omega_{12} = \mathbb{E}\left[X_i(s)\, X_i(t)\right]$ and $\omega_{22} = \mathbb{E}\left[X_i(t)\, X_i(t)\right]$. Then, using (5.9), we have

$$\omega_{11} = s^{2H} + \delta_N s^{2H+1} + \mathcal{O}\left(\delta_N^2\right), \tag{5.13}$$

$$\omega_{22} = t^{2H} + \delta_N t^{2H+1} + \mathcal{O}\left(\delta_N^2\right). \tag{5.14}$$

Moreover, an elementary but tedious calculation yields

$$\begin{aligned}
\omega_{12} &= \mathbb{E}\left[e^{\delta_N s}\int_0^s e^{-\delta_N u} dB_i(u)\, e^{\delta_N t}\int_0^t e^{-\delta_N v} dB_i(v)\right] \\
&= e^{\delta_N(s+t)} H(2H-1)\int_0^s\int_0^t e^{-\delta_N(u+v)}|u-v|^{2H-2} du dv \\
&= e^{\delta_N(s+t)} H(2H+1)\int_0^s\left[\int_0^v \left(1-\delta_N(u+v)+\mathcal{O}\left(\delta_N^2\right)\right)(v-u)^{2H-2}\right. \\
&\quad \left. + \int_v^t \left(1-\delta_N(u+v)+\mathcal{O}\left(\delta_N^2\right)\right)(u-v)^{2H-2} du\right] dv \\
&= e^{\delta_N(s+t)}\left[\frac{1}{2}\left(s^{2H}+t^{2H}-|s-t|^{2H}\right)\right. \\
&\quad \left. + \frac{\delta_N}{2}\left(-s^{2H+1}-t^{2H+1}+(s+t)|s-t|^{2H}\right)+\mathcal{O}\left(\delta_N^2\right)\right] \\
&= \frac{1}{2}\left(s^{2H}+t^{2H}-|s-t|^{2H}\right)+\frac{\delta_N}{2}\left(st^{2H}+ts^{2H}\right)+\mathcal{O}\left(\delta_N^2\right).
\end{aligned} \tag{5.15}$$

Applying (5.13)-(5.15) and Isserlis' Theorem (or Wick's theorem) by Isserlis (1918), we can see that

$$\begin{aligned}
\mathbb{E}\left[X_i^2(s)\, X_i^2(s)\right] &= 2\omega_{12}^2 + \omega_{11}\omega_{22} \\
&= \frac{1}{2}\left(s^{2H}+t^{2H}-|s-t|^{2H}\right)^2 + s^{2H}t^{2H} \\
&\quad + \delta_N\Bigg(\left(s^{2H}+t^{2H}-|s-t|^{2H}\right)\left(st^{2H}+s^{2H}t\right) \\
&\quad + s^{2H+1}t^{2H} + s^{2H}t^{2H+1}\Bigg) + \mathcal{O}\left(\delta_N^2\right) \\
&= g(s,t) + \delta_N h(s,t) + \mathcal{O}\left(\delta_N^2\right),
\end{aligned} \tag{5.16}$$

where $h(s,t) = \left(s^{2H}+t^{2H}-|s-t|^{2H}\right)\left(st^{2H}+s^{2H}t\right) + s^{2H+1}t^{2H} + s^{2H}t^{2H+1}$ and $g(s,t) = \frac{1}{2}\left(s^{2H}+t^{2H}-|s-t|^{2H}\right)^2 + s^{2H}t^{2H}$.

Using (5.16), we can obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^1\int_0^1 X_i^2(s)\,X_i^2(t)\,dsdt\right] &= \int_0^1\int_0^1 g(s,t)\,dsdt+\delta_N\int_0^1\int_0^1 h(s,t)\,dsdt+\mathcal{O}\left(\delta_N^2\right)\\ &= \frac{4H+3}{(4H+1)(4H+2)}-\frac{2\Gamma^2(2H+1)}{\Gamma(4H+3)}+\frac{1}{(2H+1)^2}\\ &\quad+\delta_N\left(\frac{2}{(H+1)(2H+1)}+\frac{1}{4H+1}\right.\\ &\quad-\frac{1}{(H+1)(2H+1)(4H+3)}\\ &\quad\left.-\frac{4}{4H+3}B(2H+1,2H+2)\right)+\mathcal{O}\left(\delta_N^2\right). \qquad (5.17)\end{aligned}$$

Using (5.16) again and following similar arguments as above, we can derive that

$$\begin{aligned}\mathbb{E}\left[X_i^2(1)\int_0^1 X_i^2(t)\,dt\right] &= \int_0^1\left(g(1,t)+\delta_N h(1,t)+\mathcal{O}\left(\delta_N^2\right)\right)dt\\ &= \frac{4H+3}{2(4H+1)}+\frac{1}{2H+1}-B(2H+1,2H+1)\\ &\quad+\delta_N\left(\frac{1}{H+\frac{1}{2}}+\frac{1}{H+1}+\frac{1}{2}+\frac{1}{4H+1}\right.\\ &\quad\left.-B(2H+1,2H+1)-B(2,2H+1)\right)+\mathcal{O}\left(\delta_N^2\right)(5.18)\end{aligned}$$

Combining (5.17) with (5.18), we have

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{X_i^2(1)}{2}-\left(H+\frac{1}{2}\right)\int_0^1 X_i^2(t)\,dt\right)^2\right]\\ =\ &\frac{1}{4}\mathbb{E}\left(X_i^4(1)\right)-\left(H+\frac{1}{2}\right)\mathbb{E}\left(X_i^2(1)\int_0^1 X_i^2(t)\,dt\right)+\left(H+\frac{1}{2}\right)^2\mathbb{E}\left[\left(\int_0^1 X_i^2(t)\,dt\right)^2\right]\\ =\ &F_H+\delta_N G_H++\mathcal{O}\left(\delta_N^2\right), \qquad (5.19)\end{aligned}$$

where $F_H=\frac{1}{2}+\frac{2H+1}{4}B(2H+1,2H+1)-\frac{(2H+1)(4H+3)}{8(4H+1)}$, $G_H=-\frac{H\left(H-\frac{1}{2}\right)}{4H+1}-\frac{H+\frac{1}{2}}{2(H+1)(4H+3)}-\left(H+\frac{1}{2}\right)B(2,2H+1)+\frac{(H+1)(4H+3)}{4H+3}B(2H+1,2H+1)$.

Finally, applying (5.11), (5.12) and (5.19), we obtain (2.20). This concludes the proof of the theorem.

5.2 Proof of Theorem 2.2

We first focus on (2.22). Under $\mathcal{H}_0$ and using (2.17), we have

$$T^{2H+1}\bar{\alpha}\left(N,T\right)=\frac{-\frac{1}{2}T^{2H+1}NA_H}{\sum_{i=1}^{N}\int_0^T B_i^2\left(t\right)dt}=\frac{-\frac{1}{2}NA_H}{\sum_{i=1}^{N}\int_0^1 B_i^2\left(t\right)dt}. \tag{5.20}$$

A standard calculation yields

$$\int_0^1 \mathbb{E}\left[B_i^2\left(t\right)\right]dt=\frac{1}{2H+1}. \tag{5.21}$$

By combining (5.20) with (5.21), we can obtain

$$T^{2H+1}\bar{\alpha}\left(N,T\right)\xrightarrow{p}-\left(H+\frac{1}{2}\right)A_H. \tag{5.22}$$

Consequently, under $\mathcal{H}_0$, we have

$$\sqrt{N}\left(T^{2H+1}\bar{\alpha}\left(N,T\right)+\left(H+\frac{1}{2}\right)A_H\right)=-\frac{\frac{1}{2\sqrt{N}}A_H\sum_{i=1}^{N}\left(1-\left(2H+1\right)\int_0^1 B_i^2\left(t\right)dt\right)}{\frac{1}{N}\sum_{i=1}^{N}\int_0^1 B_i^2\left(t\right)dt},$$

which implies (2.22) by similar arguments as (5.7).

Under $\mathcal{H}_1$, we can write

$$T^{2H+1}\bar{\alpha}\left(N,T\right)=\frac{-\frac{1}{2}NA_H}{\sum_{i=1}^{N}\int_0^1 X_i^2\left(t\right)dt}, \tag{5.23}$$

where $X_i\left(t\right)=e^{\delta_N t}\int_0^t e^{-\delta_N s}dB_i\left(s\right)=\delta_N e^{\delta_N t}\int_0^t e^{-\delta_N s}B_i\left(s\right)ds+B_i\left(t\right)$.

Using (5.22) and (5.23), we consider the following statistic

$$\sqrt{N}\left(T^{2H+1}\bar{\alpha}\left(N,T\right)+\left(H+\frac{1}{2}\right)A_H\right)=\sqrt{N}\left(\frac{-\frac{1}{2}NA_H+\left(H+\frac{1}{2}\right)A_H\sum_{i=1}^{N}\int_0^1 X_i^2\left(t\right)dt}{\sum_{i=1}^{N}\int_0^1 X_i^2\left(t\right)dt}\right).$$

Using the definition of $X_i\left(t\right)$ and Corollary 1.44 in Kukush et al. (2018), we have

$$\begin{aligned}\mathbb{E}\left[X_i^2\left(t\right)\right] &= H\int_0^t u^{2H-1}e^{\delta_N u}du+H\int_0^t u^{2H-1}e^{\delta_N(2t-u)}du\\ &= H\int_0^t u^{2H-1}\left(1+\delta_N u+\mathcal{O}\left(\delta_N^2\right)\right)du\\ &\quad +H\int_0^t u^{2H-1}\left(1+\delta_N\left(2t-u\right)+\mathcal{O}\left(\delta_N^2\right)\right)du\\ &= t^{2H}+\delta_N t^{2H+1}+\mathcal{O}\left(\delta_N^2\right).\end{aligned} \tag{5.24}$$

Using (5.24), we can easily obtain

$$\mathbb{E}\left[1-\left(2H+1\right)\int_0^1 X_i^2\left(t\right)dt\right]=-\frac{2H+1}{2H+2}\delta_N+\mathcal{O}\left(\delta_N^2\right). \tag{5.25}$$

Combining (5.23) and (5.25) with (5.19), we obtain (2.23).

5.3 Proof of Theorem 2.3

Under $\mathcal{H}_0$, using (2.26), we can easily obtain

$$\sqrt{N}T\tilde{\alpha}(N,T) = \frac{\frac{1}{\sqrt{N}}\frac{1}{2}\sum_{i=1}^{N}\left(W_i^2(1)-1\right)}{\frac{1}{N}\sum_{i=1}^{N}\int_0^1 W_i^2(t)\,dt}. \quad (5.26)$$

Using the properties of the standard Brownian motion, we have

$$\mathbb{E}\left[\frac{1}{\sqrt{N}}\frac{1}{2}\sum_{i=1}^{N}\left(W_i^2(1)-1\right)\right] = 0, \quad (5.27)$$

$$\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\frac{1}{2}\sum_{i=1}^{N}\left(W_i^2(1)-1\right)\right)^2\right] = \frac{1}{2}, \quad (5.28)$$

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\int_0^1 W_i^2(t)\,dt\right] = \frac{1}{2}. \quad (5.29)$$

Combining (5.26), (5.27), (5.28) with (5.29), we can easily obtain (2.27).

On the other hand, under $\mathcal{H}_1 : \alpha = \frac{\delta_N}{T}$ with $\delta_N = \frac{\delta}{\sqrt{N}}$, we can obtain

$$T\tilde{\alpha}(N,T) = \frac{\frac{1}{2}\sum_{i=1}^{N}\left(X_i^2(1)-1\right)}{\sum_{i=1}^{N}\int_0^1 X_i^2(t)\,dt}, \quad (5.30)$$

where $X_i(t) = e^{\delta_N t}\int_0^t e^{-\delta_N s}dW_i(s)$.

Elementary calculations yield

$$\mathbb{E}\left[X_i^2(t)\right] = t + \delta_N t^2 + \mathcal{O}\left(\delta_N^2\right). \quad (5.31)$$

Combining (5.30) with (5.31), we have

$$\begin{aligned}\sqrt{N}T\tilde{\alpha}(N,T) &= \frac{\frac{1}{2\sqrt{N}}\sum_{i=1}^{N}\left(X_i^2(1)-1\right)}{\frac{1}{N}\sum_{i=1}^{N}\int_0^1 X_i^2(t)\,dt} \\ &\xrightarrow{\mathcal{L}} \mathcal{N}(\delta, 2), \quad (5.32)\end{aligned}$$

which is (2.28) and hence, completes the proof of the theorem.

5.4 Proof of Lemma 3.1

Let $\eta_1 := \sup_{t\neq s\in[0,T]} \frac{\left|B_i^H(t)-B_i(s)\right|}{|t-s|^{H-\epsilon}}$ with $0<\epsilon<H$ and $\eta_2 := 2\sup_{u\in[0,T]}|B_i(u)|$. Using the self-similarity property of the fBm, for any $p\geq 1$, we have $\mathbb{E}\left[\eta_1^p\right] = CT^{p\epsilon}$ and $\mathbb{E}\left[\eta_2^p\right] = CT^{pH}$.

Then, using the Cauchy–Schwarz inequality, we can calculate the difference between the integral and the corresponding integral sum as

$$\begin{aligned}
& \mathbb{E}\left[\frac{1}{M}\sum_{j=0}^{M-1} B_i^2\left(jh\right) - \frac{1}{Mh}\int_0^{Mh} B_i^2\left(t\right)dt\right] \\
= \;& \mathbb{E}\left[\frac{1}{Mh}\sum_{j=0}^{M-1} hB_i^2\left(jh\right) - \frac{1}{Mh}\int_0^{Mh} B_i^2\left(t\right)dt\right] \\
= \;& \mathbb{E}\left[\frac{1}{Mh}\sum_{j=0}^{M-1}\int_{jh}^{(j+1)h} B_i^2\left(jh\right)dt - \frac{1}{Mh}\sum_{j=0}^{M-1}\int_{jh}^{(j+1)h} B_i^2\left(t\right)dt\right] \\
= \;& \frac{1}{Mh}\sum_{j=0}^{M-1}\int_{jh}^{(j+1)h} \mathbb{E}\left[\left(B_i\left(jh\right) - B_i\left(t\right)\right)\left(B_i\left(jh\right) + B_i\left(t\right)\right)\right]dt \\
\leq \;& \frac{2}{T}\sum_{j=0}^{M-1}\int_{jh}^{(j+1)h} \left|jh - t\right|^{H-\epsilon}\mathbb{E}\left[\eta_1\eta_2\right]dt \\
\leq \;& \frac{2}{T}\sum_{j=0}^{M-1}\int_{jh}^{(j+1)h} \left|jh - t\right|^{H-\epsilon}\left(\mathbb{E}\left[\eta_1^2\right]\right)^{1/2}\left(\mathbb{E}\left[\eta_2^2\right]\right)^{1/2}dt \\
\leq \;& \frac{2}{T}T^{\frac{p}{2}(\epsilon+H)}h^{H-\epsilon}(Mh)\,, \qquad (5.33)
\end{aligned}$$

which is (3.4) and the proof of this lemma is completed.

5.5 Proof of Lemma 3.2

When $\alpha < 0$, (2.1) is stationary. For any $i = 1, \ldots, N$, we can see that (2.1) has a unique solution, which can be presented as

$$Y_i\left(t\right) = B_i\left(t\right) + \alpha e^{\alpha t}\int_0^t B_i\left(s\right)e^{-\alpha s}ds\,. \qquad (5.34)$$

From (5.34), for any $i = 1, \ldots, N$, we can easily obtain $\mathbb{E}\left[Y_i\left(t\right)\right] = 0$ and, as $t \to \infty$,

$$Var\left[Y_i\left(t\right)\right] = H\int_0^t z^{2H-1}\left(e^{\alpha z} + e^{\alpha(2t-z)}\right)dz \to \frac{H\Gamma(2H)}{\left|\alpha\right|^{2H}}.$$

Since $Y_i\left(t\right)$ is normally distributed for all $H \in (0,1)$ and $p \geq 1$, there exists a positive constant C such that

$$\mathbb{E}\left|Y_i\left(t\right)\right|^p \leq C\,. \qquad (5.35)$$

On the other hand, for any $i = 1, \ldots, N$, we can write the unique solution of (2.1) as

$$Y_i\left(t\right) = \alpha\int_0^t Y_i\left(s\right)ds + B_i\left(t\right)\,. \qquad (5.36)$$

Using (5.36), for any $i = 1, \ldots, N$, we get

$$|Y_i(t) - Y_i(s)| \le \alpha \int_s^t |Y_i(u)|\, du + |B_i(t) - B_i(s)| \,. \tag{5.37}$$

Applying the Cauchy-Schwarz inequality, (5.35) and (5.37), we obtain

$$\begin{aligned}
\mathbb{E}\left[Y_i(t) - Y_i(s)\right]^2 &\le 2\alpha^2 \mathbb{E}\left(\int_s^t |Y_i(u)|\, du\right)^2 + 2\mathbb{E}\left(B_i(t) - B_i(s)\right)^2 \\
&\le 2\alpha^2 (t-s) \int_s^t \mathbb{E}\,|Y_i(u)|^2\, du + 2\,(t-s)^{2H} \\
&\le C\,|t-s|^2 \,.
\end{aligned} \tag{5.38}$$

Moreover, since $Y_i(t) - Y_i(s)$ has a normal distribution, using (5.38), for all $H \in (0,1)$ and $p \ge 1$, there exists a positive constant C such that

$$\mathbb{E}\,|Y_i(t) - Y_i(s)|^p \le C|t-s|^{pH} \,. \tag{5.39}$$

Using the Cauchy-Schwarz inequality, (5.35) and (5.39), we have

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{M}\sum_{j=0}^{M-1} Y_i^2(jh) - \frac{1}{Mh}\int_0^{Mh} Y_i^2(t)\, dt\right] \\
= \;& \mathbb{E}\left[\frac{1}{Mh}\sum_{j=0}^{M-1} hY_i^2(jh) - \frac{1}{Mh}\int_0^{Mh} Y_i^2(t)\, dt\right] \\
= \;& \mathbb{E}\left[\frac{1}{Mh}\sum_{j=0}^{M-1} \int_{jh}^{(j+1)h} Y_i^2(jh)\, dt - \frac{1}{Mh}\sum_{j=0}^{M-1} \int_{jh}^{(j+1)h} Y_i^2(t)\, dt\right] \\
= \;& \frac{1}{Mh}\sum_{j=0}^{M-1} \int_{jh}^{(j+1)h} \mathbb{E}\left[\left(Y_i(jh) - Y_i(t)\right)\left(Y_i(jh) + Y_i(t)\right)\right] dt \\
\le \;& \frac{1}{T}\sum_{j=0}^{M-1} \int_{jh}^{(j+1)h} \sqrt{\mathbb{E}\left[\left(Y_i(jh) - Y_i(t)\right)^2\right]}\sqrt{\mathbb{E}\left[\left(Y_i(jh) + Y_i(t)\right)^2\right]} dt \\
\le \;& \frac{C}{T}\sum_{j=0}^{M-1} \int_{jh}^{(j+1)h} |jh - t|^H dt \\
\le \;& \frac{C}{T} h^H (Mh) \,,
\end{aligned} \tag{5.40}$$

which implies the desired result of (3.5).

5.6 Proof of Lemma 3.3

Using (5.34) and the well known result of $\sup_{0\le s\le t}|B_i(s)| \le \left(t^H \log^2 t + 1\right)\zeta$ (where $\zeta \in \mathfrak{Z}$ with $\mathfrak{Z}$ defined in Lemma 3.3), for any t and any $i = 1, \ldots, N$, we have

$$\begin{aligned}
\sup_{0\le u\le s}|Y_i(u)| &\le \alpha e^{\alpha s}\int_0^s e^{-\alpha u}\sup_{0\le t\le u}|B_i(t)|\,du + \sup_{0\le u\le s}|B_i(u)| \\
&\le \alpha e^{\alpha s}\zeta\int_0^s e^{-\alpha u}\left(u^H\log^2 u + 1\right)du + \left(s^H\log^2 s + 1\right)\zeta \\
&\le C\alpha e^{\alpha s}\zeta + \left(s^H\log^2 s + 1\right)\zeta \\
&\le \left(Ce^{\alpha s} + s^H\log^2 s\right)\zeta\,. \qquad (5.41)
\end{aligned}$$

Let us mention that, for any $0 < s < t < \infty$, we can obtain the important result of increments of the fBm, that is, $|B_i(t) - B_i(s)| \le (t-s)^H\left(|\log(t-s)|^{1/2} + 1\right)\zeta\log(t+2)$. See Remark 3 of Kukush et al. (2015) for details. Consequently, for any $kh < s \le (k+1)h$ and any $0 < r < H$, by a similar argument as Remark 3 of Kukush et al. (2015), we can obtain

$$\begin{aligned}
|B_i(s) - B_i(kh)| &\le \zeta(s-kh)^H\left(|\log(s-kh)|^{1/2} + 1\right)\log(s+2) \\
&= \zeta\left[(s-kh)^H|\log(s-kh)|^{1/2} + (s-kh)^H\right]\log(s+2) \\
&\le \zeta(s-kh)^{H-r}\log(s+2)\,. \qquad (5.42)
\end{aligned}$$

Using (5.36), (5.41) and (5.42), we have, for $s \in [kh, (k+1)h]$,

$$\begin{aligned}
\sup_{kh\le u\le s}|Y_i(u) - Y_i(kh)| &\le \alpha\int_{kh}^s |Y_i(u)|\,du + \sup_{kh\le u\le s}|B_i(u) - B_i(kh)| \\
&\le \zeta\left(hs^H\log^2 s + he^{\alpha s} + h^{H-r}\log(s+2)\right)\,. \qquad (5.43)
\end{aligned}$$

Let $\mathbf{1}_{x\in[a,b)}$ be an indicator function, which takes the value 1 if $x \in [a,b)$ and 0 otherwise. Then, using (5.41) and (5.43), we have

$$\begin{aligned}
&\left|\int_0^T Y_i^2(s)\,ds - h\sum_{k=0}^{M-1}Y_i^2(kh)\right| \\
\le\ & \int_0^T \left|\left(Y_i^2(s) - Y_i^2(kh)\right)\mathbf{1}_{s\in[kh,(k+1)h)}\right| ds \\
\le\ & \int_0^T |Y_i(s) - Y_i(kh)|\,|Y_i(s) + Y_i(kh)|\,\mathbf{1}_{s\in[kh,(k+1)h)}ds \\
\le\ & 2\int_0^T |Y_i(s) - Y_i(kh)|\sup_{0\le u\le s}|Y_i(u)|\,\mathbf{1}_{s\in[kh,(k+1)h)}ds \\
\le\ & 2\zeta^2\int_0^T \left[\left(hs^H\log^2 s + he^{\alpha s} + h^{H-r}\log(s+2)\right)\left(Ce^{\alpha s} + s^H\log^2 s\right)\right]ds \\
\le\ & Ce^{2\alpha T}\zeta^2 h\,, \qquad (5.44)
\end{aligned}$$

which implies (3.6).

5.7 Proof of Theorem 3.1

Under $\mathcal{H}_0$, using (3.1), (3.4), (2.19) and Slutsky's theorem, as $h \to 0$ followed by $N \to \infty$ and $Nh^{H-\epsilon} \to 0$ with $0 < \epsilon < H$, we obtain

$$\begin{aligned}
&\sqrt{N}\left(T\hat{\alpha}\left(N,T,h\right)-\left(H+\frac{1}{2}\right)\right)\\
=&\quad \sqrt{N}\left(\frac{\frac{T}{2}\sum_{i=1}^{N}B_i^2\left(T\right)}{\sum_{i=1}^{N}\sum_{j=0}^{M-1}hB_i^2(jh)}-\left(H+\frac{1}{2}\right)\right)\\
=&\quad \sqrt{N}\left(\frac{\frac{1}{2}\sum_{i=1}^{N}B_i^2\left(T\right)}{\sum_{i=1}^{N}\left[\frac{1}{M}\sum_{j=0}^{M-1}B_i^2(jh)-\frac{1}{Mh}\int_0^{Mh}B_i^2\left(t\right)dt+\frac{1}{T}\int_0^{T}B_i^2\left(t\right)dt\right]}-\left(H+\frac{1}{2}\right)\right)\\
\overset{\mathcal{L}}{\to}&\quad \mathcal{N}\left(0,\sigma_H^2\right)\ ,
\end{aligned}$$

which implies (3.7).

Similarly, under $\mathcal{H}_L^L$, using (3.1), (3.5), (2.20) and Slutsky's theorem, as $h \to 0$ followed by $N \to \infty$ and $Nh^H \to 0$, we can see that

$$\begin{aligned}
&\sqrt{N}\left(T\hat{\alpha}\left(N,T,h\right)-\left(H+\frac{1}{2}\right)\right)\\
=&\quad \sqrt{N}\left(\frac{\frac{T}{2}\sum_{i=1}^{N}Y_i^2\left(T\right)}{\sum_{i=1}^{N}\sum_{j=0}^{M-1}hY_i^2(jh)}-\left(H+\frac{1}{2}\right)\right)\\
=&\quad \sqrt{N}\left(\frac{\frac{1}{2}\sum_{i=1}^{N}Y_i^2\left(T\right)}{\sum_{i=1}^{N}\left[\frac{1}{M}\sum_{j=0}^{M-1}Y_i^2(jh)-\frac{1}{Mh}\int_0^{Mh}Y_i^2\left(t\right)dt+\frac{1}{T}\int_0^{T}Y_i^2\left(t\right)dt\right]}-\left(H+\frac{1}{2}\right)\right)\\
\overset{\mathcal{L}}{\to}&\quad \mathcal{N}\left(\frac{\delta}{4\left(H+1\right)},\left(2H+1\right)^2F_H\right)\ ,
\end{aligned}$$

which is (3.8) under $\mathcal{H}_L^L$.

Using (3.1), (3.6), (2.20), Slutsky's theorem and similar arguments as above, we can obtain (3.8) under $\mathcal{H}_L^R$. Moreover, (3.9) is a direct application of (3.8) and hence, completes the proof of the theorem.

5.8 Proof of Theorem 3.2

Under $\mathcal{H}_0$, using (3.2), (3.4), (2.22) and Slutsky's theorem, as $h \to 0$ followed by $N \to \infty$ and $Nh^{H-\epsilon} \to 0$ with $0 < \epsilon < H$, we can see that

$$\begin{aligned}
& \sqrt{N}\left(T^{2H+1}\bar{\alpha}(N,T,h) + \left(H+\frac{1}{2}\right)A_H\right) \\
= & \sqrt{N}\left(\frac{-\frac{1}{2}T^{2H+1}\sum_{i=1}^{N}A_H}{\sum_{i=1}^{N}\sum_{j=0}^{M-1}hB_i^2(jh)} + \left(H+\frac{1}{2}\right)A_H\right) \\
= & \sqrt{N}\left(\frac{-\frac{1}{2}\sum_{i=1}^{N}A_H}{\sum_{i=1}^{N}\frac{1}{T^{2H}}\left[\frac{1}{M}\sum_{j=0}^{M-1}B_i^2(jh) - \frac{1}{Mh}\int_0^{Mh}B_i^2(t)\,dt + \frac{1}{T}\int_0^T B_i^2(t)\,dt\right]} + \left(H+\frac{1}{2}\right)A_H\right) \\
\stackrel{\mathcal{L}}{\to} & \mathcal{N}\left(0,\lambda_H^2\right),
\end{aligned}$$

which implies (3.10).

Under $\mathcal{H}_L^L$, using (3.2), (3.5), (2.23) and Slutsky's theorem, as $h \to 0$ followed by $N \to \infty$ and $Nh^H \to 0$, we can see that

$$\begin{aligned}
& \sqrt{N}\left(T^{2H+1}\bar{\alpha}(N,T,h) + \left(H+\frac{1}{2}\right)A_H\right) \\
= & \sqrt{N}\left(\frac{-\frac{1}{2}T^{2H+1}\sum_{i=1}^{N}A_H}{\sum_{i=1}^{N}\sum_{j=0}^{M-1}hY_i^2(jh)} + \left(H+\frac{1}{2}\right)A_H\right) \\
= & \sqrt{N}\left(\frac{-\frac{1}{2}\sum_{i=1}^{N}A_H}{\sum_{i=1}^{N}\frac{1}{T^{2H}}\left[\frac{1}{M}\sum_{j=0}^{M-1}Y_i^2(jh) - \frac{1}{Mh}\int_0^{Mh}Y_i^2(t)\,dt + \frac{1}{T}\int_0^T Y_i^2(t)\,dt\right]} + \left(H+\frac{1}{2}\right)A_H\right) \\
\stackrel{\mathcal{L}}{\to} & \mathcal{N}\left(\frac{(2H+1)A_H\delta}{4(H+1)},\lambda_H^2\right),
\end{aligned}$$

which is (3.11) under $\mathcal{H}_L^L$.

Using (3.2), (3.6), (2.23), Slutsky's theorem and similar arguments as above, we can obtain (3.11) under $\mathcal{H}_L^R$. Moreover, (3.12) is a direct application of (3.11) and we complete the proof.

5.9 Proof of Theorem 3.3

Under $\mathcal{H}_0$, using (3.3), (3.4), (2.27) and Slutsky's theorem, as $h \to 0$ followed by $N \to \infty$ and $Nh^{1/2-\epsilon} \to 0$ with $0 < \epsilon < 1/2$, we can obtain

$$\begin{aligned}
\sqrt{N}T\tilde{\alpha}(N,T,h) &= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{2}\left(W_i^2(Mh)-Mh\right)}{\frac{1}{N}\frac{1}{T}\sum_{i=1}^{N}\sum_{j=0}^{M-1}hW_i^2(jh)} \\
&= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{2}\left(W_i^2(Mh)-Mh\right)\right]}{\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{M}\sum_{j=0}^{M-1}W_i^2(jh)-\frac{1}{Mh}\int_0^{Mh}W_i^2(t)\,dt+\frac{1}{T}\int_0^{T}W_i^2(t)\,dt\right]} \\
&\xrightarrow{\mathcal{L}} \mathcal{N}(0,2)\ ,
\end{aligned}$$

which implies (3.13).

Similarly, under $\mathcal{H}_L^L$ and $\mathcal{H}_L^R$, using (3.3), (3.5), (3.6), (2.28) and the Slutsky's theorem, as $h \to 0$ followed by $N \to \infty$ and $Nh^{1/2} \to 0$, we have

$$\begin{aligned}
\sqrt{N}T\tilde{\alpha}(N,T,h) &= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{2}\left(Y_i^2(Mh)-Mh\right)}{\frac{1}{N}\frac{1}{T}\sum_{i=1}^{N}\sum_{j=0}^{M-1}hY_i^2(jh)} \\
&= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{2}\left(Y_i^2(Mh)-Mh\right)\right]}{\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{M}\sum_{j=0}^{M-1}Y_i^2(jh)-\frac{1}{Mh}\int_0^{Mh}Y_i^2(t)\,dt+\frac{1}{T}\int_0^{T}Y_i^2(t)\,dt\right]} \\
&\xrightarrow{\mathcal{L}} \mathcal{N}(\delta,2)\ ,
\end{aligned}$$

which is (3.14). Moreover, a straightforward application of (3.14) yields (3.15) and we finish the proof.

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