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Volatility Puzzle: Long Memory or Antipersistency

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Received: January 6, 2022	Abstract. The log realized volatility (RV) is often modeled as an autoregressive fractionally
Accepted: March 3, 2022	integrated moving average model $ARFIMA(1, d, 0)$. Two conflicting empirical results have
Published Online in Articles in Advance:	been found in the literature. One stream shows that log RV has a long memory (i.e., the frac-
November 7, 2022	tional parameter $d > 0$). The other stream suggests that the autoregressive coefficient α is near
	unity with antipersistent errors (i.e., $d < 0$). This paper explains how these conflicting empiri-
https://doi.org/10.1287/mnsc.2022.4552	cal findings can coexist in the context of $\overrightarrow{ARFIMA}(1, d, 0)$ model by examining the finite sam-
	ple properties of popular estimation methods, including semiparametric methods and
Copyright: © 2022 INFORMS	parametric maximum likelihood methods. The finite sample results suggest that it is challeng-
	ing to distinguish Model 1 (ARFIMA(1, d, 0) with α close to 0 and d close to 0.5) from Model 2
	(ARFIMA(1, d , 0) with α close to unity and d close to -0.5). An intuitive explanation is given.
	For the 10 financial assets considered, despite that no definitive conclusions can be drawn
	regarding the data-generating process, we find that the frequency domain maximum likeli-
	hood (or Whittle) method can generate the most accurate out-of-sample forecasts.
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	Funding: S. Shi acknowledges research support from the Australian Research Council [Project DE190100840]. J. Yu acknowledges financial support from the Ministry of Education–Singapore Tier 2 Academic Re- search Fund [Project MOE-T2EP402A20-0002] and the Lee Foundation.
	Supplemental Material: The data files are available at https://doi.org/10.1287/mnsc.2022.4552.

Keywords: long memory • fractional integration • roughness • short-run dynamics • realized volatility

1. Introduction

The availability of intraday prices of financial assets fosters the development of high-frequency financial econometrics, making an accurate measurement of daily "realized" volatility (RV) possible. The estimated daily RV has been shown to be helpful for various purposes, including forecasting macroeconomic fundamentals (Andersen et al. 2005), making investment decisions (Fleming et al. 2003), pricing options (Christoffersen et al. 2014), managing financial risk (Christoffersen and Diebold 2000), and estimating model parameters (Phillips and Yu 2009, Tao et al. 2019).

A class of autoregressive fractionally integrated moving average (ARFIMA) models, particularly ARFIMA(p, d, q) with p = 1 and q = 0, has gained much prominence in modeling daily log RV. For notational convenience, in the rest of this paper, we re-label ARFIMA(1,d,0) by AR1FI(α , d), representing an autoregressive order one and fractional integrated model with coefficients α and d. When d > 0, the autocorrelation function (ACF) of AR1FI(α , d) decays hyperbolically and is not absolutely summable. This feature matches well with the empirical ACF observed in real data. The value of the fractional parameter d has important implications for both the theoretical and empirical analyses of RV. As such, the main focus of the literature has been on the estimation of *d*.

Several estimation techniques for d have been proposed, including the local Whittle estimation (LWE) method (Künsch 1987, Robinson 1995a) and the log periodogram estimation method (Geweke and Porter-Hudak 1983, Robinson 1995b). These two methods rely on the asymptotic behavior of the spectral density at frequencies near zero (ignoring short-run dynamics) and hence, are often referred to as semiparametric methods. When the two semiparametric methods are applied to log RV, it is often found that the point estimate of *d* is around 0.5. See, for example, Andersen and Bollerslev (1997), Andersen et al. (2001a, b, 2003), and Baillie et al. (2019). Such an estimate implies that the log RV has a long memory,¹ which has been widely regarded as a stylized fact. Furthermore, one could estimate the short-run parameter from the prefiltered data (based on the estimated d). The estimated short-run parameter typically suggests weak short-run behavior or strong mean reversion. When an autoregressive order one model is fitted to the filtered data, the estimated autoregressive parameter α is often close to zero. For convenience, we label the AR1FI(α , d) process with α close to 0 and d near 0.5 by Model 1.

One advantage of the semiparametric methods is their asymptotic robustness to short-run dynamics, as short-run behavior does not change the asymptotic spectral density at near-zero frequencies. This insensitive relationship, however, does not necessarily hold in finite samples. In particular, AR1FI(α , d) with α close to unity is similar to AR1FI(0, d + 1), and the spectral density that ignores the near-unity behavior is expected to approximate the actual spectral density poorly, even with a large sample size, at frequencies near zero.² This concern might have important empirical implications.

It is known that the ARFIMA(0, d, 0) model with $d \in$ (0, 1/2) is asymptotically equivalent to the fractional Gaussian noise (fGn) with the Hurst parameter H with H = d + 0.5. The fGn is the increment of the fractional Brownian motion (fBm), denoted by $B^{H}(t)$, whose sample path is (locally) Hölder continuous up to order H. Based on fBm, Wang et al. (2021) consider a fractional Ornstein-Uhlenbeck (fOU) process for log RV. Under the in-fill asymptotic scheme, the discrete-time representation of the fOU process is a local to unity (Phillips 1987) process with fGn, which is asymptotically equivalent to AR1FI(α , d) with α close to unity and d = H - 0.5.³ A change of frequency method is proposed to estimate H in fOU in Wang et al. (2021). The estimated H from several log RV series suggests H < 0.5 (i.e., d < 0). Similar empirical estimates of *H* are found in Bolko et al. (2022) when the fOU model is used to capture the movement of log spot volatility and the generalized method of moments is used to estimate H. The (pre-imposed) local to unity dynamic of the fOU model generates strong persistency that is attenuated by the antipersistent errors. We label the AR1FI(α , d) process with α close to unity and d < 0 by Model 2.

Clearly, the empirical evidence for log RV by semiparametric methods is at odds with that obtained from the fOU model. Although the semiparametric methods suggest a process with a weak short-run dynamic and long memory errors (i.e., Model 1) for log RV, the empirical evidence obtained from the fOU model reveals near-unity behavior and antipersistent errors (i.e., Model 2). We investigate the causes of this volatility puzzle in the present paper.

The first goal of this paper is to understand how these conflicting empirical findings coexist in the literature. To achieve this goal, we examine the finite sample properties of several popular estimation methods for AR1FI(α , d) under a wide range of parameter settings. The methods include two semiparametric methods and two parametric maximum likelihood (ML) methods. The semiparametric methods are LWE applied to log RV and LWE applied to the first difference of log RV. We refer to the latter as LWE(diff) hereafter. The two ML approaches are the modified profile time-domain likelihood (MPL) method and the frequency domain maximum likelihood (Whittle) method. Both classes of estimation approaches have some finite sample problems under one or both of Model 1 and Model 2.

For the semiparametric methods, it is found that when the true data-generating process (DGP) is Model 2, LWE points to Model 1. When the true DGP is Model 1, LWE(diff) points to AR1FI(1, d) with d being negative. These findings hold true even when the sample size is very large in an empirically realistic situation. Moreover, the LWE (LWE(diff)) estimator is substantially biased when the autoregressive coefficient deviates far from zero (unity).⁴ In contrast, the two parametric ML methods generally perform well. However, it is possible for both MPL and Whittle to mix up Model 1 and Model 2 in finite samples. Specifically, when the DGP is Model 1, with a small and nonnegligible probability, both methods lead to Model 2. On the other hand, when the true DGP is Model 2, the parametric ML methods could point to Model 1. These problems arise because there are two modes in their likelihood functions, and the mode around the true parameter values may be lower than the other mode in finite samples. As a result, the finite sample distribution can be bimodal. The simulation findings call for cautious interpretation of empirical estimation results when using those techniques.

We consider the log RV time series of 10 financial assets spanning over a decade from 2010 to 2021. The estimates from LWE lead to Model 1 ($\alpha \in [-0.162, 0.004]$) and $d \in [0.54, 0.70]$), whereas the two ML methods point to Model 2 ($\alpha \in [0.995, 0.999]$ and $d \in [-0.47, -0.38]$). Results of LWE(diff) are close to those from the ML methods but with $\alpha = 1$ by assumption. Both LWE and LWE(diff) suggest nonstationarity, whereas by assumption, the estimated processes from two ML methods are stationary. Because the ML methods are relatively reliable, Model 2 is more likely to be the true model than Model 1. Nevertheless, there is still a small chance that Model 1 is the true DGP. Despite the inconclusive estimation results, we show that the Whittle method can provide the best out-of-sample forecast out of the four estimation techniques (especially at long forecasting horizons) followed by MPL.

Our paper contributes to the literature in two aspects. First, our simulation findings explain how Model 1 and Model 2 coexist in the RV literature. Although the simulation studies in the existing literature⁵ have found a substantial upward bias in d with the semiparametric methods when α takes a large positive value and examined the performance of the ML methods under various parameter settings, the simulation designs adopted in these studies prevent them from discovering the difficulty of the two classes of estimation methods in distinguishing Model 1 from Model 2. In particular, the selected values for α are too far away from unity so that the bias generated by semiparametric methods is not substantial enough and ML methods can well distinguish Model 1 from Model 2. We find that under Model 1 and Model 2, the finite sample distributions of the two ML methods are bimodal. One mode is in the parameter ranges of Model 1, and the other one corresponds to Model 2. Consequently, the traditional summary statistics of the estimates, such as means (or the biases) and standard errors (or root mean squared errors), are poor choices. Alternative measures are proposed. Second, although we cannot draw definitive conclusions regarding the DGP with the estimation methods in empirical applications because of their finite sample issues, we find that the Whittle method provides the best out-of-sample forecasts out of the four.

The paper is organized as follows. Section 2 introduces the RV estimator. Section 3 presents the model specification and reviews some statistical properties of the model. Section 4 introduces the four popular estimation methods. Section 5 presents the simulation designs and reports the finite sample properties of the estimation approaches. Section 6 reports empirical estimation and forecasting results. Section 7 concludes. The appendix reviews a technique, known as tapering, for the Whittle method. We also examine the robustness of our empirical results using alternative volatility measures in the appendix.

2. Realized Volatility

Assume data are observed at a regular frequency. Let t = 1, ..., T and $n = T/\delta$ be the total number of intraday observations available within the sample period, where δ is the distance between two consecutive observations. Let $X_{t,i}$ be the observed *i*th log prices at period *t*. The traditional realized volatility is constructed as

$$RV_t = \sum_{i=2}^{1/\delta} (\Delta X_{t,i})^2, \text{ with } \Delta X_{t,i} = X_{t,i} - X_{t,i-1}.$$
(1)

Under a standard Itō semimartingale process, the realized volatility is shown to be a consistent estimator of the quadratic variation of the process.

One of the most recent contributions in the volatility estimation literature was made by Da and Xiu (2021), who developed a quasi-maximum likelihood (QML) approach providing uniform valid inference on volatility under an extremely general model setting with both moving average infinity, MA (∞), market microstructure noises and jumps.⁶ The model specification considered by Da and Xiu (2021) is as follows. The observed log asset prices consist of two components:

$$X_t^o = X_t + U_t,$$

where X_t is the underlying log efficient price and U_t is the noise component. The noise process U_t is assumed to have flexible serial correlations, modeled as a MA(∞) process. The underlying price is assumed to be an Itō semimartingale process defined on some filtered probability space ($\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}$) and satisfies

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \left(\delta \mathbf{1}_{\{|\delta| \le 1\}} * (\eta - \upsilon)\right)_t \\ &+ \left(\delta \mathbf{1}_{\{|\delta| > 1\}}\right) \mu_t, \end{aligned} \tag{2}$$

where μ_t and σ_t are adapted and locally bounded, *W* is a standard Brownian motion, η is a Poisson random measure on $\mathbb{R}_+ \times E$ with a nonrandom intensity measure $v(dt, ds) = dt \otimes \lambda(ds)$, and λ is a σ -finite measure on (E, ξ) , which is a Polish space. The last two components of (2) capture the dynamics of jumps. See, for example, Jacod et al. (2017) or Da and Xiu (2021) for more details of the assumptions.

The likelihood function of QML is taken from a much-simplified process, assuming the efficient price follows a Brownian motion with constant volatility and a Gaussian MA(q) noise component. The QML estimator of the volatility, denoted by $\hat{\sigma}^2(\hat{q})$ with \hat{q} obtained from the Akaike information criterion, is shown to converge to the following quadratic variation,

$$\delta \left[\int_{t-1}^t \sigma_s^2 ds + \sum_{i=2}^{1/\delta} \left(\Delta X_{t,i} \right)^2 \right],$$

which comprises both continuous (integrated variance) and discontinuous (jump) components.

This paper investigates the dynamics of the QML volatility estimates of various financial assets. The QML realized volatility data are conveniently provided by the Risk Lab⁷ and computed using transaction prices sampled at the highest available frequency. For convenience, we refer to the QML volatility estimator as log RV (QML) subsequently, and with a slight abuse of notation, we denote the log RV (QML) volatility estimator log $\hat{\sigma}^2(\hat{q})$ by y_t .

3. Model Specification

Consider the AR1FI(α , d) model

$$(1 - \alpha L)(y_t - \mu) = \sigma_u u_t, \tag{3}$$

where *L* is the lag operator and u_t is the error term. The error term is a fractionally integrated process (Granger and Joyeux 1980) such that

$$u_t = (1 - L)^{-d} \varepsilon_t \text{ with } \varepsilon_t \sim_{iid} N(0, 1), \tag{4}$$

where *d* is the memory parameter. The AR1FI(α , *d*) model is one of the most popular specifications for modeling log RV in the literature. See, for example, Andersen et al. (2003) and Wang et al. (2021).⁸ For any real number *d*, the fractional integrated error process can be rewritten as

$$u_t = \sum_{k=0}^{\infty} \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} \varepsilon_{t-k},$$
(5)

where $\Gamma(\cdot)$ is the Gamma function. See Beran (1994, p. 60). The long-run variance of u_t is one when d = 0, ∞ when d > 1/2, and zero when d < 1/2. Assuming $|\alpha| < 1$, we say that y_t is a long memory process whenever d > 0 as in Phillips and Shimotsu (2004) and a rough process when d < 0. The AR1FI model reduces to a standard autoregressive process when d = 0.

When $d \in (-1/2, 1/2)$, u_t is stationary and invertible (Bloomfield 1985).⁹ Let $\gamma_u(k) := Cov(u_t, u_{t-k})$ be the *k*th order autocovariance of u_t . Under the specification of (4), according to Hosking (1981), the autocovariance function of u_t is

$$\gamma_u(k) = \frac{(-1)^k (-2d)!}{(k-d)!(-k-d)!} = \frac{(-1)^k \Gamma(1-2d)}{\Gamma(k-d+1)\Gamma(1-k-d)},$$
 (6)

where $(\cdot)!$ is the factorial of the argument. The *k*th order ACF of u_t is

$$\rho_u(k) = \frac{(-d)!(k+d-1)!}{(d-1)!(k-d)!} \sim \frac{(-d)!}{(d-1)!} k^{2d-1} \text{ as } k \to \infty.$$

The correlation coefficient $\rho_u(k)$ decays at a hyperbolic rate as k goes to infinity. This is in contrast to the exponential decaying rate of an autoregressive moving average model.

If $|\alpha| < 1$ and $d \in (-1/2, 1/2)$, y_t is covariance stationary, and hence, we can write $\gamma_y(k) := Cov(y_t, y_{t-k})$. Let $-\pi \le \lambda \le \pi$ be the Fourier frequency. The spectral density of y_t is

$$f_{y}(\lambda) = \frac{\sigma^{2}}{2\pi} \frac{(2 - 2\cos(\lambda))^{-d}}{1 - 2\alpha\cos(\lambda) + \alpha^{2}} \sim C\lambda^{-2d}$$

when λ is near zero. (7)

This is also the "pseudo spectral" density (Velasco and Robinson 2000) of AR1FI(α , d) when $d \in (1/2, 1)$.

4. Estimation Methods

In this section, we review four alternative estimation methods, namely LWE, LWE(diff), the time-domain ML method, and the Whittle ML method.

4.1. LWE and LWE(diff)

Künsch (1987) and Robinson (1995a) investigate a class of models whose spectral densities satisfy the following property:

$$f_y(\lambda) \sim C\lambda^{-2d} \text{ as } \lambda \to 0^+,$$
 (8)

with *C* being a positive constant. The property concerns only frequencies approaching zero.

The LWE method of Künsch (1987) and Robinson (1995a) is defined as

$$(\hat{C}, \hat{d}) = \arg \max_{C, d} \frac{1}{m} \sum_{j=1}^{m} \left[-\log f_y(\lambda_j) - \frac{I(\lambda_j)}{f_y(\lambda_j)} \right]$$
(9)

$$= \arg \max_{C,d} \frac{1}{m} \sum_{j=1}^{m} \left[-\log C + 2d \log \lambda_j - \frac{1}{C} \lambda^{2d} I(\lambda_j) \right],$$
(10)

where $I(\lambda_j)$ denotes the periodogram at the *j*th Fourier frequency $\lambda_j = 2\pi j/T$ with j = 1, 2, ..., m and *T* being the

sample size. Specifically,

$$I(\lambda_j) = \frac{1}{2\pi T} \left| \sum_{t=0}^T y_t \exp\left(-it\lambda_j\right) \right|^2,$$
(11)

which is a nonparametric estimate of the density. The parameter *m* satisfies the condition $m \le (T-1)/2$ and diverges to infinity at a rate that is slower than *T* as $T \rightarrow \infty$. The analytical solution of LWE is

$$\hat{d} = \arg \max_{d} \left[-\log \hat{C}(d) + 2d \frac{1}{m} \sum_{j=1}^{m} \log \lambda_j \right] \text{and}$$
$$\hat{C}(d) = \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d} I(\lambda_j).$$
(12)

Robinson (1995a) shows that the local Whittle estimator is consistent at the \sqrt{m} rate and asymptotically normal with variance 1/(4m): that is,

$$\sqrt{m}(\hat{d}-d) \to_d N(0,1/4),$$

when $d \in (-1/2, 1/2)$. Velasco (1999) investigates the possibility of using LWE for some nonstationary situations (i.e., $1/2 \le d < 3/2$), showing that the consistency of LWE holds for $d \in (-1/2, 1)$ and the asymptotic normality holds for d < 3/4 with the same variance as in the stationary situation.¹⁰

There are significant advantages to using LWE. First, it works for a wider range of *d*, which goes beyond the stationary region $d \in (-1/2, 1/2)$. Second and perhaps most importantly, it is asymptotically robust against the short-run dynamic, which is determined by α in the AR1FI(α , *d*) model. However, the robustness comes with the cost of a reduced rate of convergence (\sqrt{m} instead of \sqrt{T}). Moreover, a more significant and potentially empirically relevant problem is that LWE may have poor finite sample properties when the short-run dynamic is near unity. In this case, we expect that $f_y(\lambda)$ is poorly approximated by $C\lambda^{-2d}$. At an intuitive level, the AR1FI(0, *d*) model is observationally equivalent to the AR1FI(1, *d* – 1) model because

$$(1-L)(y_t-\mu) = \sigma_u(1-L)^{-d+1}\varepsilon_t$$

can be rewritten as

$$y_t - \mu = \sigma_u (1 - L)^{-d} \varepsilon_t.$$

As a result, it is expected that the spectral density of the AR1FI(α , d) model, whose α is strictly less than but very close to unity, is better approximated by $C\lambda^{-2d-2}$ when λ is close to zero. A detailed comparison between $\log (f_y(\lambda))$ and $\log (C\lambda^{-2d})$ will be made later (see Figure 2).

When α is very close to unity, a sensible method to estimate *d* is to apply LWE to Δy_t , resulting in LWE(diff). If the estimated memory parameter by LWE(diff) is

 \hat{d} , within the class of AR1FI(α , d), it implies that the estimated model for y_t is either AR1FI($0, \hat{d} + 1$) or AR1FI($1, \hat{d}$). Although LWE(diff) does not yield a consistent estimator of d when α is not exactly unity as discussed in Section 5.4, it may have good finite sample performances when α is very close to unity.

4.2. Time-Domain ML Estimation

To implement the ML methods, we assume y_t is stationary: that is, $|\alpha| < 1$ and $d \in (-1/2, 1/2)$. The stationary assumption is imposed for two reasons. First, stationarity ensures that the likelihood function is relatively easier to calculate as elements in the variance-covariance matrix are finite and time invariant. Second, most asset pricing models have been developed based on the condition that volatility is stationary. Examples include bond pricing (Duffie and Kan 1996) and option pricing (Hull and White 1987, Heston 1993). See also the remark made by Robert Engle in Diebold (2003) against nonstationary volatility models.

Let $y = (y_1, y_2, \dots, y_T)'$ and $\theta = (\alpha, d)$. Under the model specification of (3) with u_t specified as (4), $y_t - \mu$ follows a normal distribution with mean zero and variance-covariance matrix, denoted by Σ_y . The objective function of the ML estimator is given by

$$(\hat{\boldsymbol{\theta}}, \hat{\sigma}_u) = \arg \max_{\boldsymbol{\theta}, \sigma_u} \log L_N(\mu, \sigma_u, \boldsymbol{\theta}),$$

where

$$\log L_N(\mu, \sigma_u, \boldsymbol{\theta}) \propto \frac{1}{2T} \log |\boldsymbol{\Sigma}_y| + \frac{1}{2T} (\boldsymbol{y} - \mu \boldsymbol{l})' \boldsymbol{\Sigma}_y^{-1} (\boldsymbol{y} - \mu \boldsymbol{l})$$
(13)

and l = (1, ..., 1)'.

For the case of known mean value μ , the limiting properties of $\hat{\theta}$ were derived by Hannan (1973) for short memory processes and Yajima (1985) for long memory processes. That is, under some mild regularity conditions,

$$\sqrt{T}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \rightarrow_d N(0, \boldsymbol{\Xi}_{\boldsymbol{\theta}_0}^{-1}),$$

where θ_0 is the true parameter vector and Ξ_{θ_0} is the Fisher information matrix.

4.2.1. Modified Profile Likelihood. Dahlhaus (1989) extends the results of Yajima (1985) to the case with unknown mean. In case of unknown μ , a plug-in method is required. The plug-in method substitutes μ by a consistent estimator of the mean (e.g., the sample mean). Although the method provides a \sqrt{T} consistent and asymptotically normal estimator, it is contaminated by an additional second-order negative bias (Lieberman 2005) because of the need of estimating μ .

An alternative solution is the MPL estimator proposed by Cox and Reid (1987). The idea of the MPL estimator is to use a linear transformation of parameters of interest to make them orthogonal to nuisance parameters (μ and σ_u). The modified profile likelihood is given by

$$\log L_{M}(\boldsymbol{y}, \hat{\boldsymbol{\mu}}, \boldsymbol{\theta}) \propto \left(\frac{1}{T} - \frac{1}{2}\right) \log |\boldsymbol{R}| - \frac{1}{2} \log (\boldsymbol{l}' \boldsymbol{R}^{-1} \boldsymbol{l}) + \frac{3 - T}{2} \log \left[T^{-1} (\boldsymbol{y} - \hat{\boldsymbol{\mu}} \boldsymbol{l})' \boldsymbol{R}^{-1} (\boldsymbol{y} - \hat{\boldsymbol{\mu}} \boldsymbol{l})\right],$$
(14)

where $\mathbf{R} = \mathbf{\Sigma}_y / \sigma_u^2$ and $\hat{\mu} = (\mathbf{l}' \mathbf{R}^{-1} \mathbf{l})^{-1} \mathbf{l}' \mathbf{R}^{-1} \mathbf{y}$. The asymptotic distribution of the MPL estimator is unchanged compared with the exact ML but eliminates some degree of bias in the exact ML (An and Bloomfield 1993, Hauser 1999).

4.2.2. Variance-Covariance Matrix Σ_y . Let the $(t,s)^{th}$ element of Σ_y be $\gamma_y(k)$, where $t, s = 1, \dots, T$ and k = |t - s|. The covariance function of the ARFIMA(p, d, q) process was derived by Hosking (1981, lemma 1(c)) and Sowell (1992, equations 8 and 9) and approximated to improve computational speed by Chung (1994). In the special case of p = 1 and q = 0, the covariance function of Hosking (1981) is

$$\gamma_y(k) = \frac{\sigma_u^2}{1 - \alpha^2} \gamma_u(k) A(k, \alpha), \tag{15}$$

where $A(k, \alpha) = C(k, \alpha) + C(-k, \alpha) - 1$, $C(k, \alpha) = F(d + k, 1; 1 - d + k; \alpha)$, and $F(\cdot)$ is the hypergeometric function.

The hypergeometric function is computational costly and extremely large when *k* is large and α is far from unity,¹¹ leading to extreme behavior of the covariance function. As an alternative, one may compute the covariance function as

$$\gamma_{y}(k) = \sigma_{u}^{2} \sum_{i=k}^{\infty} \sum_{j=0}^{\infty} \alpha^{i+j-k} \gamma_{u}(|i-j|), \qquad (16)$$

which resembles that of a standard AR(1) process. Unreported simulations show that the computed covariance values from (16) with a truncation of 20,000 for both summands are identical to those obtained from (15) when the hypergeometric behaves normally (e.g., α = 0.996 and k < 1,000). This method is, however, quite computationally intensive.

Another method, which is proposed by Bertelli and Caporin (2002), is referred to as the splitting approach. It is based on the following property of the covariance function for stationary processes (Brockwell and Davis 1987):

$$\gamma_{y}(k) = \sum_{s=-\infty}^{\infty} \tilde{\gamma}(s) \gamma_{u}(k-s), \qquad (17)$$

where $\tilde{\gamma}(s)$ is the autocovariance of the pure AR

component. For practical implementation, the summand is truncated at *K*.

To provide some practical guidance for the choice of *K*, in Figure 1, we show the differences between the log determinants of Σ_{ν} computed from (16) and (17) for each value of K, ranging from 1,000 to 10,000 (with an increment of 100). It is expected that one would need a larger K to ensure the estimation accuracy when the data series is highly persistent and has a long memory (i.e., when α is close to one and *d* is close to 0.5), as both γ_s and γ_u decay slower. We consider the autoregressive coefficient $\alpha = \{0.5, 0.8, 0.9, 0.95, 0.99, 0.995, 0.999\}, n =$ 3,000 for the dimension of the covariance matrix, and d = 0.45¹² Evidently, the splitting method can provide very accurate estimation for the variance-covariance matrix with K = 200 when $\alpha \le 0.9$, which is consistent with the finding of Bertelli and Caporin (2002). However, for processes with autoregressive root close to unity, one would need a substantially larger value of K to ensure accuracy. Based on the simulation results presented in Figure 1, we recommend using K = 300 for $0.9 < \alpha \le 0.95$, K = 1,700 for $0.95 < \infty$ $\alpha \le 0.99$, K = 3,000 for $0.99 < \alpha \le 0.995$, and K = 7,000for $0.995 < \alpha < 1$ so that the differences between the log determinants are at a maximum order of 10^{-3} when d = 0.45.

4.3. Whittle ML Estimation

To avoid inverting Σ_y that is required in calculating the time-domain likelihood function, following Whittle (1953, 1954), one can approximate Σ_y^{-1} by $(2\pi)^{-2} \int_{-\pi}^{\pi} f_y(\lambda)^{-1} \cos((i-j)\lambda) d\lambda$ and $\log |\Sigma_y|$ by $T(2\pi)^{-1} \int_{-\pi}^{\pi} \log f_y(\lambda) d\lambda$ for a stationary process. The discrete-time version of the Whittle likelihood function (up to a scale multiplication) is

$$\log L_W(\boldsymbol{\theta}, \sigma_u^2) \propto -\sum_{j=1}^m \log f_y(\lambda_j | \boldsymbol{\theta}, \sigma_u^2) - \sum_{j=1}^m \frac{I(\lambda_j)}{f_y(\lambda_j | \boldsymbol{\theta}, \sigma_u^2)}.$$
(18)

The Whittle likelihood function was presented in Künsch (1987) and Dahlhaus (1988). Fox and Taqqu (1986) show that the asymptotic properties of the estimators remain the same if we simplify the objective function to the following:

$$\log L_W(\boldsymbol{\theta}, \, \sigma_u^2) \propto -\sum_{j=1}^m \frac{I(\lambda_j)}{f_y(\lambda_j | \, \boldsymbol{\theta}, \sigma_u^2)}$$
(19)

where the distance between the spectrum density $f_y(\lambda_j | \theta, \sigma_u^2)$ and $I(\lambda_j)$ is minimized. We employ the simplified objective function for the estimation.¹³ Like MPL, the parameter μ does not enter the objective function of



Figure 1. (Color online) The Differences Between the Log Determinants of Σ_y Computed from (16) and (17)

Notes. The insets within each subplot are zoomed-in versions of the graphs within a particular range. (a) $\alpha = \{0.5, 0.8, 0.9, 0.95\}$. (b) $\alpha = 0.99$. (c) $\alpha = 0.995$. (d) $\alpha = 0.999$.

Paper	Relevant tables	Relevant estimation methods	Sample size
1			1
Smith et al. (1997)	Tables I and VI	ML and LPE ($m = T^{0.5}, T^{0.6}, T^{0.7}$)	256
Nielsen and Frederiksen (2005)	Tables 8 and 9	Exact ML, MPL, Whittle, conditional ML LWE and LPE ($m = T^{0.5}$, $T^{0.65}$)	128,256,512
Nadarajah et al. (2021)	Tables 6 and 7	ML and LPE $(m = T^{0.65})$	96, 576

Table 1. Existing Monte Carlo Studies

the Whittle method as the zero frequency is not included. The spectral density of AR1FI(α , d) is given in (7). The Whittle ML method yields \sqrt{T} -consistent, asymptotically normal, and efficient parameter estimates (Hannan 1973, Fox and Taqqu 1986, Giraitis and Surgailis 1990) when $d \in (0, 1/2)$.

5. Monte Carlo Simulations

We now examine the finite sample properties of various estimation techniques. The DGP is AR1FI(α , d),¹⁴ covering both Model 1 and Model 2. We assume α takes a value in {-0.2, 0, 0.3, 0.5, 0.7, 0.9, 0.99, 0.996} and d takes a value in {-0.4, 0, 0.4}. We set $\sigma_u = 1$ and μ to zero but assume them unknown. The initial value of each simulated sample path is set to the long-run mean (i.e., $\mu/(1 - \alpha)$), which is zero under this setting. The first 5,000 observations are discarded from each simulated

sample path to minimize the impact of the initial value. The number of replications is 1,000.

We investigate the estimation accuracy of the semiparametric methods and the ML methods for both the memory parameter d and the short-run dynamic parameter α . Table 1 provides a brief summary of the existing literature on the simulation and their Monte Carlo designs. Our Monte Carlo design extends those in the existing studies by considering more empirically relevant parameter values. In particular, we (1) allow the maximum value of α to be much closer to the unity (i.e., 0.996 versus 0.8 in Smith et al. 1997 and Nielsen and Frederiksen 2005 and 0.9 in Nadarajah et al. 2021) and (2) consider larger sample sizes (i.e., T = 1,024 as well as *T* = 2,048, 4,096 in the case $\alpha \ge 0.9$ for LWE). The choices of near unit α and sample size are guided by the empirical results that will be reported later.

Figure 2. (Color online) The Difference Between the Theoretical Spectral Density and the Approximate Spectral Density for AR1FI(α , d): log ($f(\lambda)$) – log ($C\lambda^{-2d}$)



Notes. (a) d = -0.4. (b) d = 0. (c) d = 0.4. (d) $\alpha = 0.$ (e) $\alpha = 0.3$. (f) $\alpha = 0.9$.

Implementing the time-domain ML method under these parameter settings is not straightforward, as existing methods for computing the variance-covariance matrix do not work. As discussed in Section 4.2.2, the hypergeometric function in the formulas of Hosking (1981), Sowell (1992), and Chung (1994) behaves abnormally when the dimension of the matrix (which is $T \times T$) is large. The suggested truncation of K = 200 in (17) of the splitting method cannot provide accurate results when α is close to unity. We propose an alternative truncation scheme as detailed in Section 4.2.2. For the semiparametric methods, in addition to the popular LWE and LPE methods, we also investigate the finite sample performance of LWE(diff). The LWE(diff) method has already been used in empirical applications (e.g., Phillips and Shimotsu 2004), but its finite sample performance has yet been studied.

Several interesting findings emerge from the simulations. In particular, we document the poor finite sample performance of LWE (LWE(diff)) when the short-run dynamic is strong (weak) and explain why. We show that the ML estimators have a bimodal distribution under certain parameter settings, leading to a possible misidentification between Model 1 and Model 2. Under this circumstance, the traditional performance measures, such as means and standard deviations, are not appropriate. Alternative measures are used to present the estimation results. A summary of the simulation findings is provided in Section 5.3.

5.1. Semiparametric Methods

We first investigate the performance of LWE and LWE(diff). The parameter support for *d* is (-1,3/2), and the bandwidth $m = \{\lfloor T^{0.55} \rfloor, \lfloor T^{0.65} \rfloor, \lfloor T^{0.75} \rfloor, \lfloor T^{0.85} \rfloor\}$, where $\lfloor . \rfloor$ denotes the integer part of the argument. The objective functions are optimized with the command fminbnd in MATLAB, as there is only one model parameter. Table 2 reports the means and standard errors (in brackets) of the LWE and LWE(diff) estimates of \hat{d} , obtained from 1,000 replications. The sample size is set at T = 1,024.¹⁵ There are several interesting observations from Table 2.

Table 2. Means and Standard Errors (in Parentheses) of the LWE and LWE(diff) Estimates for *d* with $m = \{T^{0.55}, T^{0.65}, T^{0.75}, T^{0.85}\}$ and T = 1,024

	LWE			LWE(diff)				
	$m = T^{0.55}$	$m = T^{0.65}$	$m = T^{0.75}$	$m = T^{0.85}$	$m = T^{0.55}$	$m = T^{0.65}$	$m = T^{0.75}$	$m = T^{0.85}$
$\alpha = -0.2$								
d = -0.4	-0.39 (0.10)	-0.40 (0.06)	-0.41(0.04)	-0.45(0.03)	-0.49 (0.26)	-0.68(0.21)	-0.82 (0.15)	-0.92 (0.09)
d = 0	-0.01(0.09)	-0.01 (0.06)	-0.03(0.04)	-0.07 (0.03)	-0.64(0.21)	-0.74 (0.16)	-0.82 (0.12)	-0.88(0.09)
d = 0.4	0.40 (0.09)	0.39 (0.06)	0.38 (0.04)	0.31 (0.03)	-0.55 (0.11)	-0.57 (0.07)	-0.59 (0.05)	-0.63 (0.03)
$\alpha = 0$								
d = -0.4	-0.39 (0.10)	-0.40 (0.06)	-0.39 (0.04)	-0.38 (0.03)	-0.56 (0.25)	-0.73 (0.19)	-0.85 (0.13)	-0.92 (0.09)
d = 0	-0.01(0.09)	-0.01 (0.06)	-0.00(0.04)	-0.00 (0.03)	-0.69 (0.20)	-0.77 (0.15)	-0.83 (0.11)	-0.84(0.08)
d = 0.4	0.40 (0.09)	0.40 (0.06)	0.40 (0.04)	0.38 (0.03)	-0.56 (0.10)	-0.57(0.07)	-0.57(0.05)	-0.56(0.03)
$\alpha = 0.3$								
d = -0.4	-0.39 (0.09)	-0.37 (0.06)	-0.32(0.04)	-0.21 (0.03)	-0.66 (0.23)	-0.80 (0.16)	-0.88(0.12)	-0.90 (0.10)
d = 0	-0.00 (0.09)	0.02 (0.06)	0.07 (0.04)	0.17 (0.03)	-0.74(0.17)	-0.80 (0.13)	-0.81 (0.09)	-0.73 (0.05)
d = 0.4	0.41 (0.09)	0.42 (0.06)	0.47 (0.04)	0.55 (0.03)	-0.56(0.10)	-0.56(0.07)	-0.51(0.04)	-0.39 (0.03)
$\alpha = 0.5$								
d = -0.4	-0.38 (0.09)	-0.33 (0.06)	-0.22 (0.04)	-0.04(0.03)	-0.73 (0.21)	-0.84(0.14)	-0.88(0.11)	-0.85 (0.09)
d = 0	0.01 (0.09)	0.06 (0.06)	0.17 (0.04)	0.33 (0.03)	-0.77 (0.15)	-0.80 (0.11)	-0.74(0.07)	-0.58(0.04)
d = 0.4	0.42 (0.09)	0.46 (0.06)	0.57 (0.04)	0.71 (0.03)	-0.55 (0.10)	-0.52 (0.06)	-0.41(0.04)	-0.23 (0.03)
$\alpha = 0.7$								
d = -0.4	-0.33 (0.09)	-0.21 (0.06)	-0.02 (0.04)	0.18 (0.03)	-0.80 (0.18)	-0.86 (0.13)	-0.84(0.11)	-0.71 (0.05)
d = 0	0.06 (0.09)	0.18 (0.06)	0.37 (0.04)	0.55 (0.03)	-0.78 (0.13)	-0.73 (0.09)	-0.58 (0.05)	-0.38 (0.03)
d = 0.4	0.47 (0.09)	0.58 (0.06)	0.77 (0.04)	0.92 (0.03)	-0.51 (0.10)	-0.41 (0.06)	-0.21 (0.04)	-0.01 (0.03)
$\alpha = 0.9$								
d = -0.4	-0.04(0.10)	0.17 (0.07)	0.34 (0.05)	0.43 (0.03)	-0.80(0.15)	-0.72 (0.09)	-0.60 (0.05)	-0.50 (0.03)
d = 0	0.36 (0.09)	0.57 (0.07)	0.74 (0.05)	0.80 (0.03)	-0.58 (0.10)	-0.40(0.07)	-0.24(0.05)	-0.13 (0.03)
d = 0.4	0.76 (0.10)	0.96 (0.07)	1.12 (0.05)	1.16 (0.04)	-0.23 (0.10)	-0.02 (0.07)	0.15 (0.05)	0.24 (0.03)
$\alpha = 0.99$								
d = -0.4	0.52 (0.09)	0.55 (0.06)	0.56 (0.04)	0.56 (0.03)	-0.45(0.10)	-0.43 (0.06)	-0.41(0.04)	-0.39 (0.03)
d = 0	0.91 (0.09)	0.94 (0.06)	0.96 (0.04)	0.93 (0.03)	-0.08(0.09)	-0.05 (0.06)	-0.03(0.04)	-0.01(0.03)
d = 0.4	1.21 (0.11)	1.21 (0.11)	1.20 (0.11)	1.14 (0.11)	0.32 (0.09)	0.35 (0.06)	0.37 (0.04)	0.36 (0.03)
$\alpha = 0.996$								
d = -0.4	0.57 (0.09)	0.58 (0.06)	0.58 (0.04)	0.57 (0.03)	-0.41 (0.09)	-0.41 (0.06)	-0.40 (0.04)	-0.38 (0.03)
d = 0	0.97 (0.09)	0.97 (0.06)	0.97 (0.04)	0.94 (0.03)	-0.03 (0.09)	-0.02 (0.06)	-0.01(0.04)	-0.01(0.03)
d = 0.4	1.19 (0.13)	1.17 (0.12)	1.14 (0.11)	1.08 (0.11)	0.37 (0.09)	0.38 (0.06)	0.38 (0.04)	0.38 (0.03)

First, LWE works very well in estimating *d* when α is near zero (say $\alpha \leq 0.3$), with negligible biases and small standard errors. This is especially true when $m = \lfloor T^{0.65} \rfloor$. Together with its asymptotic robustness property against short-run dynamics, the good finite sample property may be the reason why it has been popular in estimating d for log RVs. However, it leads to a substantial upward bias (spurious long memory) when the process becomes more persistent. The substantial upward bias in d by the semiparametric methods when $\alpha = 0.8$ or 0.9 and $T = \{96, 256, 512, 576\}$ has been documented in Smith et al. (1997), Nielsen and Frederiksen (2005), and Nadarajah et al. (2021). Our results indicate that this upward bias problem continues to hold when $\alpha = \{0.99, 0.996\}$ and T = 1,028. Interestingly, the bias increases toward one as α gets closer to unity. For example, when α is 0.996 and d is -0.4 (i.e., Model 2 is the DGP), with a small standard error of 0.06, the estimated *d* (with $m = \lfloor T^{0.65} \rfloor$) is located around 0.58, always suggesting spurious long memory. This is expected because when α is very close to unity, the spectral density is better approximated by $C\lambda^{-2d-2}$, and hence, LWE essentially estimates d + 1.

To better understand this point, we show the gaps between the theoretical spectral density of y_t and the approximate spectral density $C\lambda^{-2d}$ used by LWE under various parameter settings. The larger the distance between $f(\lambda)$ and $C\lambda^{-2d}$ is, the less accurate estimated results are expected from LWE. Figure 2 plots the quantity $\log(f(\lambda)) - \log(C\lambda^{-2d})$ against the frequency λ . We choose the value of *C* such that the quantity takes value zero at frequency zero. It is obvious that the distances at frequencies close to zero are affected substantially by α but not so much by *d*. This is consistent with our findings in Table 2 that LWE leads to a substantial bias when α is close to unity, whereas the bias is similar across various values of *d* given a value of α .

Second, there is a trade-off between bias and standard error with the different choice of *m* for LWE. When the bandwidth parameter *m* reduces from $\lfloor T^{0.85} \rfloor$ to $\lfloor T^{0.55} \rfloor$, the bias of the *d* estimate decreases. However, the use of smaller tuning parameter *m* does not alleviate the problem of severe bias in LWE of *d* when α is very close to unity (i.e., $\alpha = 0.996$).

The finite sample problem in LWE for the case when α is close to unity naturally suggests that one may use LWE(diff) to estimate *d*. The right side of Table 2 reports the means and standard errors of LWE(diff) under the same parameter settings. There are several interesting observations. First, as expected, the performance of LWE(diff) is good when $\alpha = 0.996$. The mean and standard error are nearly a mirror image of those of LWE when $\alpha = 0$. In general, LWE(diff) works very well in estimating *d* when α is near unity (say $\alpha \ge 0.99$), with negligible biases and small standard errors. The results are relatively stable across different settings of *m*, with $m = \lfloor T^{0.85} \rfloor$ providing estimates with the smallest

variations. Second, LWE(diff) leads to a substantial downward bias in *d* when α is not so close to zero, including the case $\alpha = 0.9$. The further α is away from unity, the larger the downward bias is. For example, when the true value of *d* is 0.4 and the true value of α is 0.3 (i.e., the true DGP is Model 1), with a small standard error of 0.03, the estimated *d* (with $m = T^{0.85}$) is located around -0.39, always suggesting spurious antipersistent errors.

To understand if larger sample sizes can help address the finite sample problems in LWE, in Table 3 we report the means and standard errors of LWE with $m = \lfloor T^{0.65} \rfloor$ when $T = \{2,048,4,096\}$ and $\alpha = \{0.9.0.99,0.996\}$, obtained from 1,000 replications. The sample sizes $T = \{2,$ $048,4,096\}$ are large but remain empirically reasonable for RV. For the ease of comparison, we also report the results when T = 1,024. It is clear that although the standard error reduces as *T* increases, the bias remains substantial. For example, when α is 0.996, *d* is -0.4 (i.e., Model 2 is the DGP), and T = 4,096, with a small standard error of 0.04, the estimated *d* is located around 0.57, always suggesting spurious long memory. Similar finite sample problems apply to LWE(diff) when $T = \{2048, 4096\}$ and α is far away from unity.

To obtain an estimate of α using LWE, we fit an AR(1) model to a prefiltered data series using \hat{d} obtained from LWE.¹⁶ This two-stage approach has been used in the literature; see, for example, Andersen et al. (2003). The last column of Table 4 reports the means and standard errors of $\hat{\alpha}$, based on LWE ($m = \lfloor T^{0.65} \rfloor$), for the same parameter setting as before and T = 1,024. From Table 4, the estimated α from LWE is fairly close to its true value when $\alpha \leq 0.3$. However, when $\alpha > 0.3$, the upward biases in \hat{d} lead to equally significant downward biases in $\hat{\alpha}_1$. When d = -0.4 and $\alpha = \{0.99, 0.996\}$ (i.e., the true DGP is Model 2), with a small standard error, LWE tends to conclude that α is located around zero. Together with the simulation results on \hat{d} reported earlier, we conclude that LWE always suggests that the

Table 3. Means and Standard Errors (in Parentheses) of the LWE Estimates for *d* with $m = \lfloor T^{0.65} \rfloor$ and T = 1,024, 2,048, 4,096

	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 0.996$
T = 1,024			
d = -0.4	0.17 (0.07)	0.55 (0.06)	0.58 (0.06)
d = 0	0.57 (0.07)	0.94 (0.06)	0.97 (0.06)
d = 0.4	0.96 (0.07)	1.21 (0.11)	1.17 (0.12)
T = 2,048			
d = -0.4	0.09 (0.05)	0.54 (0.05)	0.57 (0.05)
d = 0	0.49 (0.05)	0.93 (0.05)	0.97 (0.05)
d = 0.4	0.89 (0.05)	1.24 (0.09)	1.21 (0.11)
T = 4,096			
d = -0.4	0.00 (0.04)	0.52 (0.04)	0.57 (0.04)
d = 0	0.40 (0.04)	0.92 (0.04)	0.96 (0.04)
d = 0.4	0.80 (0.04)	1.26 (0.07)	1.24 (0.10)

	MPL	Whittle	Whittle (taper)	LWE	LWE(diff)
$\alpha = -0.2$					
d = -0.4	-0.40(0.04)	-0.40(0.04)	-0.41(0.04)	-0.40(0.06)	-0.97 (0.17)
d = 0	-0.00(0.04)	-0.01(0.04)	-0.01 (0.04)	-0.01(0.06)	-0.88(0.10)
d = 0.4	0.40 (0.04)	0.39 (0.04)	0.40 (0.04)	0.39 (0.06)	-0.63 (0.03)
$\alpha = 0$					
d = -0.4	-0.40 (0.05)	-0.41(0.04)	-0.42 (0.05)	-0.40 (0.06)	-0.97 (0.15)
d = 0	-0.01(0.04)	-0.01(0.04)	-0.02 (0.05)	-0.01 (0.06)	-0.84(0.08)
d = 0.4	0.29 (0.28)	0.29 (0.28)	0.31 (0.26)	0.40 (0.06)	-0.56 (0.03)
$\alpha = 0.3$					
d = -0.4	-0.41 (0.06)	-0.41 (0.06)	-0.42 (0.06)	-0.37 (0.06)	-0.92 (0.12)
d = 0	-0.01(0.08)	-0.03 (0.08)	-0.04 (0.11)	0.02 (0.06)	-0.73 (0.05)
d = 0.4	0.35 (0.17)	0.33 (0.17)	0.34 (0.18)	0.42 (0.06)	-0.39 (0.03)
$\alpha = 0.5$					
d = -0.4	-0.41 (0.07)	-0.42 (0.07)	-0.43 (0.07)	-0.33 (0.06)	-0.85 (0.09)
d = 0	-0.03 (0.10)	-0.06 (0.11)	-0.07 (0.13)	0.06 (0.06)	-0.58 (0.04)
d = 0.4	0.38 (0.10)	0.35 (0.11)	0.36 (0.12)	0.46 (0.06)	-0.23 (0.03)
$\alpha = 0.7$					
d = -0.4	-0.39 (0.08)	-0.42 (0.08)	-0.42 (0.08)	-0.21 (0.06)	-0.71 (0.05)
d = 0	-0.00 (0.09)	-0.04(0.09)	-0.03 (0.10)	0.18 (0.06)	-0.38 (0.03)
d = 0.4	0.39 (0.08)	0.36 (0.08)	0.38 (0.09)	0.58 (0.06)	-0.01 (0.03)
$\alpha = 0.9$					
d = -0.4	-0.39 (0.06)	-0.40 (0.06)	-0.40 (0.07)	0.17 (0.07)	-0.50 (0.03)
d = 0	0.01 (0.06)	-0.00 (0.05)	0.00 (0.07)	0.57 (0.07)	-0.13 (0.03)
d = 0.4	0.41 (0.05)	0.36 (0.06)	0.40 (0.06)	0.96 (0.07)	0.24 (0.03)
$\alpha = 0.99$					
d = -0.4	-0.38 (0.13)	-0.38 (0.13)	-0.37 (0.15)	0.55 (0.06)	-0.39 (0.03)
d = 0	0.00 (0.03)	0.00 (0.03)	0.00 (0.03)	0.94 (0.06)	-0.01 (0.03)
d = 0.4	0.43 (0.05)	0.19 (0.12)	0.41 (0.03)	1.21 (0.11)	0.36 (0.03)
$\alpha = 0.996$					
d = -0.4	-0.38 (0.12)	-0.38 (0.12)	-0.36 (0.18)	0.58 (0.06)	-0.38 (0.03)
d = 0	0.00 (0.03)	0.00 (0.03)	0.01 (0.03)	0.97 (0.06)	-0.01 (0.03)
d = 0.4	0.44 (0.05)	0.13 (0.11)	0.42 (0.03)	1.17 (0.12)	0.38 (0.03)

Table 4. Means and Standard Errors (in Parentheses) of \hat{d} when T = 1,024

Notes. The bandwidth parameter $m = \lfloor T^{0.65} \rfloor$ for LWE and $m = \lfloor T^{0.85} \rfloor$ for LWE(diff). Bold corresponds to cases where the DGP is Model 1 or Model 2.

estimated model is Model 1 when the DGP is Model 2. Once again, this finding is not surprising as AR1FI(α , d) with α = 0.99, 0.996 is very similar to AR1FI(0, d + 1).

Our simulation studies suggest that one should be cautious against using LWE and LWE(diff). LWE tends to point to Model 1 when the DGP is Model 2; LWE(diff) tends to point to Model 2 when the DGP is Model 1. Because we do not know the value of α ex ante in practice, we generally do not know if we should use LWE or LWE(diff).

5.2. Parametric Methods

For the Whittle method, we use a grid searching method to choose the "optimal" initial values of *d* and *a*. The grids range from –0.499 to 0.499 for *d* and from –0.999 to 0.999 for α , with an increment of 0.005. We evaluate the Whittle log likelihood for all possible combinations of *d* and α . The pair that produces the highest log-likelihood value is taken as our initial values for the Whittle method. For MPL, we set the initial values of the two parameters to be the estimates of the Whittle method. For both MPL and Whittle, the parameter supports for α and *d* are (–1,1) and (–0.5,0.5), respectively. The log

likelihoods of the two parametric ML methods are optimized using the fmincon function in MATLAB with the sequential quadratic programming algorithm. The two ML methods estimate both d and α simultaneously.

Following the common practice in the literature, in Tables 4 and 5, we report the means and standard errors of \hat{d} and $\hat{\alpha}$ for MPL, Whittle, and Whittle (taper) under the same parameter settings as before with T = 1,024, obtained from all 1,000 replications.¹⁷ For the ease of comparison, we repeat results of LWE and LWE(diff) in the last two columns of Table 4 and results of LWE in the last column of Table 5.

As evidenced in the two tables, although the two ML methods generally work well across all parameter settings for both *d* and α , there are two important exceptions that are empirically relevant to RV. When the true DGP is Model 1 or Model 2 (highlighted in bold in the two tables), the standard errors are unusually large. For example, from Table 5, if $\alpha = 0$, the standard error of \hat{d} for both MPL and Whittle is 0.28 when d = 0.4, which is about six times larger than those when d = 0; if $\alpha = 0.99$, the standard errors of \hat{d} for MPL and Whittle (taper) are 0.13 and 0.15, respectively, when d = -0.4, which are

Table 5. Means and Standard Errors (in Parentheses) of $\hat{\alpha}$ when T = 1,024

	MPL	Whittle	Whittle (taper)	LWE
$\alpha = -0.2$				
d = -0.4	-0.20 (0.05)	-0.20 (0.04)	-0.19 (0.05)	-0.20 (0.06)
d = 0	-0.20 (0.04)	-0.19 (0.04)	-0.19 (0.05)	-0.19 (0.06)
d = 0.4	-0.20 (0.04)	-0.20 (0.04)	-0.20 (0.05)	-0.19 (0.06)
$\alpha = 0$				
d = -0.4	0.00 (0.06)	0.01 (0.05)	0.01 (0.06)	-0.00 (0.07)
d = 0	0.00 (0.05)	0.01 (0.05)	0.01 (0.06)	0.01 (0.07)
d = 0.4	0.11 (0.31)	0.11 (0.31)	0.09 (0.28)	0.00 (0.07)
$\alpha = 0.3$				
d = -0.4	0.30 (0.07)	0.31 (0.07)	0.32 (0.07)	0.27 (0.07)
d = 0	0.31 (0.09)	0.33 (0.09)	0.34 (0.12)	0.28 (0.07)
d = 0.4	0.35 (0.17)	0.36 (0.17)	0.36 (0.17)	0.28 (0.07)
$\alpha = 0.5$				
d = -0.4	0.50 (0.09)	0.52 (0.07)	0.52 (0.08)	0.43 (0.07)
d = 0	0.52 (0.10)	0.55 (0.11)	0.56 (0.12)	0.44 (0.06)
d = 0.4	0.52 (0.10)	0.55 (0.11)	0.53 (0.11)	0.43 (0.06)
$\alpha = 0.7$				
d = -0.4	0.69 (0.08)	0.71 (0.07)	0.71 (0.08)	0.52 (0.06)
d = 0	0.69 (0.08)	0.72 (0.07)	0.72 (0.09)	0.53 (0.06)
d = 0.4	0.70 (0.07)	0.72 (0.07)	0.71 (0.08)	0.53 (0.06)
$\alpha = 0.9$				
d = -0.4	0.89 (0.05)	0.89 (0.04)	0.89 (0.05)	0.37 (0.08)
d = 0	0.89 (0.04)	0.89 (0.04)	0.89 (0.05)	0.37 (0.08)
d = 0.4	0.89 (0.03)	0.91 (0.03)	0.90 (0.03)	0.39 (0.08)
$\alpha = 0.99$				
d = -0.4	0.97 (0.12)	0.97 (0.12)	0.96 (0.14)	0.05 (0.07)
d = 0	0.99 (0.01)	0.98 (0.01)	0.98 (0.01)	0.06 (0.08)
d = 0.4	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	-0.03 (0.46)
$\alpha = 0.996$				
d = -0.4	0.98 (0.11)	0.97 (0.11)	0.95 (0.17)	0.02 (0.07)
d = 0	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	0.03 (0.09)
d = 0.4	0.99 (0.01)	0.99 (0.00)	0.99 (0.00)	-0.01 (0.56)

Notes. The bandwidth parameter $m = \lfloor T^{0.65} \rfloor$ for LWE. Bold corresponds to cases where the DGP is Model 1 or Model 2.

about four times larger than those when d = 0. Similar features are observed for $\hat{\alpha}$ from Table 4.

These unusually large standard errors motivate us to examine the finite sample property in different ways. In Figure 3, we report the scatterplots of the estimated *d* and α by MPL and Whittle from the 1,000 replications when the true parameter values are $\alpha = 0$ and d = 0.4 (Model 1) and $\alpha = 0.996$ and d = -0.4 (Model 2). The scatterplots indicate that when the true DGP is Model 1 or Model 2, two disjoint clusters are obtained: one located around the true parameter values and the other one far away. Interestingly, one of the clusters corresponds to Model 1, and the other corresponds to Model 2. That is, the finite sample distribution is a mixture of two disjoint distributions. In the figures, we also report the percentage of replications (of 1,000 replications) that fall in each cluster. When the true parameter values are $\alpha = 0$ and d = 0.4, with probability 89% (or for 890 replications), the two ML methods yield estimates around the true values; with probability 11%, the two ML methods yield an estimate of α near unity and an estimate of d near -0.5. When the true parameter values are $\alpha = 0.996$ and d = -0.4, with

probability 98.2%, the two ML methods yield estimates around the true values; with probability 1.8% (or for 18 replications), the two ML methods yield an estimate of α near zero and an estimate of *d* near 0.5.

The scatterplots indicate that it is very plausible that there are two modes in the likelihood functions for MPL and Whittle. Figure 4 displays the contour plots of the log-likelihood surfaces of MPL (Figure 4, (a) and (c)) and Whittle (Figure 4, (b) and (d)) for one simulated sample path under the setting of $\alpha = 0$ and d = 0.4 (Figure 4, (a) and (b)) or $\alpha = 0.996$ and d = -0.4 (Figure 4, (c) and (d)). We remove log-likelihood values that are smaller than certain thresholds to obtain better visualization of the surface at the peak. The two modes can be seen in all cases.

When the distribution is bimodal, mean and standard error are not ideal performance measures. In Table 6, for each of the two parametric ML methods, we report the means and standard errors of \hat{d} and $\hat{\alpha}$ for each cluster (instead of the whole distribution) and the probability of false identification (PFI) when the DGP is from Model 1 (i.e., $(d, \alpha) = (0.4, 0)$ or (0.4, 0.3)) and from Model 2 (i.e., $(d, \alpha) = (-0.4, 0.99)$ or (-0.4, 0.996)). We also report the means and standard errors for the whole distribution (all 1,000 replications) for comparison. It can be seen that when α is close to zero, the means and standard errors of the correct cluster for both ML methods compare well with those of LWE. When α is close to unity, the means and standard errors of the correct cluster for both ML methods compare well with those of LWE(diff). Moreover, when d = 0.4 and $\alpha = 0$, the standard error of d for the two ML methods, obtained from all 1,000 replications, is larger than that in other cases. This is because PFI is the largest (11%) in this case.

In Figure 5, we plot the kernel densities of d and $\hat{\alpha}$ (solid lines) for MPL and Whittle when d = 0.4 and $\alpha = 0$ from 1,000 replications. The bimodality in the finite sample distribution can be seen clearly. In the same figure, we also show the 95% highest density intervals (the shortest confidence intervals), identified by the two segments of the dotted line around each mode. Not surprisingly, in all cases, the highest density interval contains two disjoint intervals.

5.3. Summary of Monte Carlo Studies

Our simulation results suggest that the LWE and LWE(diff) methods show significant bias when the autoregressive coefficient of the process is not close to zero and unity, respectively. Most importantly, the semiparametric methods cannot separate Model 1 and Model 2 Although LWE can provide accurate estimation results under Model 1, it always falsely points to Model 1 when the true DGP is Model 2, even with large sample sizes. This finding explains why research papers employing semiparametric methods (LWE or LPE) tend to find evidence of long memory with *d* around 0.5



Figure 3. (Color online) Scatterplots of the Estimated *d* and α by MPL and Whittle from the 1,000 Simulated Paths When $\alpha = 0$ and d = 0.4

Notes. The numbers on the graphs are the percentages of replications where the estimates fall in the wrong parameter region. (a) MPL: $\alpha = 0$ and d = 0.4 (Model 1). (b) Whittle: $\alpha = 0$ and d = 0.4 (Model 1). (c) MPL: $\alpha = 0.996$ and d = -0.4 (Model 2). (d) Whittle: $\alpha = 0.996$ and d = -0.4 (Model 2).

Figure 4. Contour Plots of the Log-Likelihood Surfaces of MPL and Whittle for a Simulated Sample Path Under the Settings of $\alpha = 0$ and d = 0.4 ((a) and (b)) and $\alpha = 0.996$ and d = -0.4 ((c) and (d))



Notes. (a) MPL with $\alpha = 0, d = 0.4$. (b) Whittle with $\alpha = 0, d = 0.4$. (c) MPL with $\alpha = 0.996, d = -0.4$. (d) Whittle with $\alpha = 0.996, d = -0.4$.

	$(d,\alpha)=(0.4,0)$	$(d,\alpha)=(0.4,0.3)$	$(d, \alpha) = (-0.4, 0.99)$	$(d, \alpha) = (-0.4, 0.996)$
Whittle Whittle (correct) Whittle (false) Whittle: PFI MPL MPL (correct) MPL (false) MPL (false)	$ \begin{bmatrix} 0.29 & (0.28), 0.11 & (0.31) \\ [0.39 & (0.04), 0.01 & (0.05) \\ [-0.50 & (0.00), 0.97 & (0.01)] \\ 0.110 \\ [0.29 & (0.28), 0.11 & (0.31)] \\ [0.39 & (0.04), 0.00 & (0.05)] \\ [-0.50 & (0.00), 0.98 & (0.01)] \\ 0.110 \\ \end{bmatrix} $	$ \begin{bmatrix} 0.33 & (0.17), 0.36 & (0.17) \end{bmatrix} \\ \begin{bmatrix} 0.38 & (0.06), 0.32 & (0.07) \end{bmatrix} \\ \begin{bmatrix} -0.29 & (0.06), 0.94 & (0.04) \end{bmatrix} \\ 0.062 \\ \begin{bmatrix} 0.35 & (0.17), 0.35 & (0.17) \end{bmatrix} \\ \begin{bmatrix} 0.39 & (0.06), 0.31 & (0.07) \end{bmatrix} \\ \begin{bmatrix} -0.29 & (0.07), 0.94 & (0.04) \end{bmatrix} \\ 0.061 \\ \end{bmatrix} $	$\begin{bmatrix} -0.38 & (0.13), 0.97 & (0.12) \end{bmatrix} \\ \begin{bmatrix} -0.40 & (0.03), 0.98 & (0.01) \end{bmatrix} \\ \begin{bmatrix} 0.47 & (0.03), 0.15 & (0.03) \end{bmatrix} \\ \begin{bmatrix} 0.022 \\ [-0.38 & (0.13), 0.97 & (0.12) \end{bmatrix} \\ \begin{bmatrix} -0.40 & (0.03), 0.99 & (0.01) \end{bmatrix} \\ \begin{bmatrix} 0.47 & (0.03), 0.15 & (0.03) \end{bmatrix} \\ \begin{bmatrix} 0.47 & (0.03), 0.15 & (0.03) \end{bmatrix} \\ \end{bmatrix}$	$\begin{bmatrix} -0.38 & (0.12), & 0.97 & (0.11) \\ [-0.39 & (0.03), & 0.99 & (0.01)] \\ [0.49 & (0.02), & 0.14 & (0.03)] \\ 0.018 \\ [-0.38 & (0.12), & 0.98 & (0.11)] \\ [-0.40 & (0.03), & 0.99 & (0.01)] \\ [0.49 & (0.02), & 0.14 & (0.03)] \\ 0.018 \end{bmatrix}$

Table 6. Means and Standard Errors (in Parentheses) of \hat{d} and $\hat{\alpha}$ of Each Cluster and All Replications and the PFI

Note. The sample size T = 1,024.

(Model 1). The LWE(diff) is the opposite of LWE. It works well under Model 2 but fails to yield a satisfactory estimation outcome for *d* when α is far away from unity.

The ML methods work well in general. However, interestingly and new to the literature, we found that when the true DGP is either Model 1 or Model 2, the log-likelihood surfaces of the two ML methods have two modes. One mode corresponds to Model 1, and the other corresponds to Model 2. Although the mode around the true parameter values is generally higher than the other mode, the other way around is possible. That is why the ML methods cannot separate Model 1 and Model 2. Thus, not surprisingly, research papers employing the two parametric ML methods have found both long memory and roughness in log volatilities. Suppose one finds Model 1 (Model 2) by the ML methods in empirical applications. It is very likely that the true model is Model 1 (Model 2), but there is also a nonnegligible probability that the true DGP is Model 2 (Model 1).

5.4. Discussions: Model 1 Vs. Model 2

Model 1 can be formulated as a fractional process with a

local to zero AR coefficient such that Model 1: $y_t = -\frac{c}{T}y_{t-1} + \sigma(1-L)^{-\tilde{d}}\varepsilon_t$ with constant *c* and $\tilde{d} > 0$. (20)

Model 2 can be understood as a local to unity AR(1)model with fractionally integrated errors. That is,

Model 2:
$$y_t = \left(1 - \frac{c}{T}\right)y_{t-1} + \sigma(1 - L)^{-d}\varepsilon_t$$

with $c \ge 0$ and $d \in (-0.5, 0)$, (21)

which can be rewritten as

$$\Delta y_t = -\frac{c}{T} y_{t-1} + \sigma (1-L)^{-d} \varepsilon_t.$$
(22)

When c = 0, Model 1 and Model 2 become AR1FI(0, d) and AR1FI(1, d), respectively. When d = d + 1, these two models are observationally equivalent. Hence, Model 1 and Model 2 can be viewed as local alternatives of these

Figure 5. (Color online) The Kernel Densities (Solid Lines) of \hat{d} and $\hat{\alpha}$ for MPL and Whittle When d = 0.4, $\alpha = 0$



Notes. The two segments of the dotted line around each mode form the 95% highest density interval. (a) MPL: d. (b) MPL: α. (c) Whittle: d. (d) Whittle: α .

two observationally equivalent models, respectively, with the same local deviation quantity (i.e., $-\frac{c}{T}y_{t-1}$). That is why Model 1 and Model 2 can generate similar sample paths, and it is difficult to distinguish them in finite samples when the local deviation (*c*) is small.

Although Model 1 and Model 2 deviate from the two observationally equivalent models by the same quantity, they have different asymptotic properties, unless c=0. Multiplying both sides of Model 1 by (1 - L) leads to

$$\Delta y_t = -\frac{c}{T} \Delta y_{t-1} + \sigma (1-L)^{-d} \varepsilon_t.$$

As $T \to \infty$, the first term on the right-hand side is dominated by the second term. The process is asymptotically equivalent to AR1FI(1, *d*)—a model employed by LWE(diff). From theorem 2 of Davydov (1970), as $T \to \infty$,

$$\frac{\delta_d \Gamma(1+d)}{T^{d+0.5}\sigma} y_{\lfloor Tr \rfloor} \Longrightarrow B_H(r), \tag{23}$$

where $B_H(r)$ is the fBm with Hurst parameter $H = d + 0.5 \in (0, 0.5)$ and $r \in [0, 1]$. The fBm process is used in Gatheral et al. (2018) to model and forecast log RV where H = 0.14 is assumed.

For Model 2, when c > 0, from Tanaka (2013), as $T \rightarrow \infty$,

$$\frac{\delta_d \Gamma(1+d)}{T^{d+0.5} \sigma} y_{\lfloor Tr \rfloor} \Longrightarrow J_c^H(r), \tag{24}$$

where $\delta_d = \sqrt{\frac{2(d+0.5)\Gamma(1-d)}{\Gamma(1+d)\Gamma(1-2d)}}$ and $J_c^H(r) := \exp(cr) \int_0^r \exp(-cs) dB_H(s)$ is a fOU process. Equation (21) can be rewritten as

$$y_t = -\frac{c}{T}(1-L)^{-1}y_{t-1} + \sigma(1-L)^{-\tilde{d}}\varepsilon_t.$$
 (25)

The first term on the right-hand side of (25) represents the difference between Model 2 with $\alpha < 1$ and the LWE(diff) model ($\alpha = 1$) and has the following limiting:

$$-\frac{\delta_d \Gamma(1+d)}{T^{d+0.5}\sigma} \frac{c}{T} (1-L)^{-1} y_{t-1} \Longrightarrow J_c^H(r) - B_H(r).$$

Because $-\frac{c}{T}(1-L)^{-1}y_{t-1} = O_p(T^{d+0.5})$, which is of the same order of magnitude as the second quantity in (22), one cannot ignore this term even asymptotically

whenever $c \neq 0$. As shown in the forecasting exercise in the application section, it matters more for long horizon forecasting.

6. Empirical Applications to RV

In this section, we investigate the dynamics of log RVs for the Standard and Poor's (S&P) 500 index exchange traded fund (ETF) and the nine industry ETFs over the past decade. The sample period starts from January 5, 2010 and ends on May 25, 2021. The QML realized volatility data are obtained from the Risk Lab. Assets under consideration are listed in Table 7, along with the number of observations and summary statistics of log RV (QMLE) of each data series.

6.1. Estimation Results

The AR1FI(α , d) model is fitted to each (demeaned) log RV series using the two semiparametric and two parametric methods. The bandwidth $m = \lfloor T^{0.65} \rfloor$ for LWE and $m = \lfloor T^{0.85} \rfloor$ for LWE(diff). For the Whittle method, we use the same grid searching method as in the simulation studies. For MPL, we use the estimation results of the Whittle method as the initial value.

The estimated parameters are reported in Table 8. The two ML methods provide almost identical results for all assets. The estimated autoregressive coefficients are close to unity, and the estimated fractional parameters are close to those of LWE(diff). These estimates point to Model 2 and are consistent with the findings of Gatheral et al. (2018) and Wang et al. (2021), where the fBm and fOU processes are fitted to log RV. They are also consistent with those of Liu et al. (2022), and Fukasawa et al. (2022), where log spot volatility is treated as latent but assumed to follow an AR(1) process with fractionally integrated errors or fBm.

On the contrary, LWE suggests that the memory parameter is between 0.54 and 0.70, implying that all the log RV series have a long memory. The autoregressive coefficient is always close to zero, suggesting Model 1. The estimated fractional parameters from LWE(diff) are between -0.41 and -0.36. The difference between the two estimates of *d* by LWE and LWE(diff) is almost one,

Table 7. Summary Statistics of the Log Realized Volatility (QMLE) of Various Financial Assets

Name (Ticker)	Observations	Mean	Standard deviation	Skewness	Kurtosis
S&P 500 ETF (SPY)	2,757	-2.40	0.50	0.66	4.04
Material Sector ETF (XLB)	2,723	-2.06	0.44	0.64	4.14
Energy Sector ETF (XLE)	2,724	-1.81	0.44	0.78	3.92
Energy Sector ETF (XLE)	2,724	-2.04	0.43	0.95	5.31
Financial Sector ETF (XLF)	2,724	-2.18	0.45	0.75	4.32
Healthcare Sector ETF (XLV)	2,723	-2.18	0.46	0.78	4.65
Industrial Sector ETF (XLI)	2,723	-2.42	0.41	1.28	7.31
Material Sector ETF (XLB)	2,723	-2.09	0.37	1.04	6.82
Technology Sector ETF (XLK)	2,723	-2.25	0.41	0.98	5.27
Utilities Sector ETF (XLU)	2,722	-2.22	0.46	0.73	4.01

Table 8. Model Estimation Results

	MPL		Whittle		LWE		T TATE (1:00)
Name (Ticker)	â	â	â	â	â	â	\hat{d}
S&P 500 ETF (SPY)	-0.382	0.995	-0.382	0.995	0.602	-0.006	-0.356
Material Sector ETF (XLB)	-0.446	0.999	-0.445	0.998	0.643	-0.105	-0.413
Energy Sector ETF (XLE)	-0.387	0.998	-0.385	0.997	0.698	-0.093	-0.373
Financial Sector ETF (XLF)	-0.427	0.998	-0.427	0.997	0.607	-0.053	-0.396
Industrial Sector ETF (XLI)	-0.422	0.998	-0.423	0.997	0.611	-0.056	-0.390
Technology Sector ETF (XLK)	-0.440	0.998	-0.439	0.997	0.540	0.004	-0.412
Consumer staples Sector ETF (XLP)	-0.468	0.997	-0.467	0.996	0.582	-0.110	-0.410
Utilities Sector ETF (XLU)	-0.470	0.998	-0.469	0.997	0.632	-0.162	-0.415
Healthcare Sector ETF (XLV)	-0.426	0.995	-0.426	0.994	0.601	-0.061	-0.397
Consumer discretionary Sector ETF (XLY)	-0.434	0.998	-0.434	0.997	0.607	-0.070	-0.400

as expected. The results from LWE(diff) are close to those of the ML methods.

The seemingly contradictory results between LWE and other methods are consistent with our simulation results. Suppose the true DGP is Model 2 as the ML methods suggested. According to our simulations, LWE will lead to the false conclusion of Model 1. LWE(diff) performs well under this setting and hence, leads to similar outcomes as the parametric methods. Suppose the estimation results of the ML methods are incorrect, and the true DGP is Model 1. Our simulations suggest that LWE performs well in this case, and employing LWE(diff) will lead to a false conclusion of Model 2.

Based on the simulation studies, the two ML methods are relatively more reliable than LWE. As such, it is more likely that the log RVs of the ETFs are generated from Model 2 than from Model 1. However, there is still a nonnegligible probability of false identification as the log-likelihood surfaces have two modes with similar values. As an illustration, we show in Figure 6 contour plots of the log-likelihood surfaces of MPL and Whittle for the log RV of SPY. Again, we remove log-likelihood values that are far from the peak to enable better visibility of the modes. Evidently, there are two modes in the likelihood surface. One is around Model 1, and the other one is around Model 2. The color around Model 2 is brighter, implying higher likelihood values in the region and hence, the estimation outcome.

6.2. Model Forecasting

The one-step-ahead linear prediction of $AR1FI(\alpha, d)$ can be written as

$$\hat{y}_{t+1} = \phi_{t,1} y_t + \phi_{t,2} y_{t-1} + \dots + \phi_{t,t} y_1 \text{ with } t > 1.$$
(26)

We employ the popular Durbin–Levinson algorithm for the computation of the forecasting coefficients $\phi_{t,j}$ with j = 1, 2, ..., t. Specifically, under the assumption that $\{y_t\}$ is a zero mean stationary process with autocovariance function $\gamma_y(.)$ such that $\gamma_y(0) > 0$ and $\gamma_y(k) \rightarrow 0$ as $k \rightarrow \infty$, $\phi_{1,1} = \gamma_y(1)/\gamma_y(0)$, $v_0 = \gamma_y(0)$,

$$v_{t} = v_{t-1} \left[1 - \phi_{t,t}^{2} \right] \text{ for } t = 1, 2, \dots,$$

$$\phi_{t,t} = \left[\gamma_{y}(t) - \sum_{j=1}^{t-1} \phi_{t-1,j} \gamma_{y}(t-j) \right] v_{t-1}^{-1},$$

$$\begin{bmatrix} \phi_{t,1} \\ \vdots \\ \phi_{t,t-1} \end{bmatrix} = \begin{bmatrix} \phi_{t-1,1} \\ \vdots \\ \phi_{t-1,t-1} \end{bmatrix} - \phi_{t,t} \begin{bmatrix} \phi_{t-1,t-1} \\ \vdots \\ \phi_{t-1,1} \end{bmatrix}.$$

Figure 6. Contour Plots of the Log-Likelihood Surfaces of MPL and Whittle for SPY

(a) 3830 0.4 3820 0.3 3810 0.2 0.1 3800 0 σ 3790 -0.1 -0.2 3780 -0.3 3770 -0.4 -0.4 -0.2 0 0.2 0.4 0.6 0.8



Notes. (a) MPL. (b) Whittle.

See Brockwell and Davis (1987, chapter 5). The two-stepahead recursive forecasting is computed recursively as

$$\hat{y}_{t+2} = \phi_{t+1,1}\hat{y}_{t+1} + \phi_{t+1,2}y_t + \dots + \phi_{t+1,t+1}y_1,$$

and the *k*-step-ahead forecast can be obtained in a similar fashion.

The computation of $\gamma_y(t-j)$ (and hence, $\phi_{t,j}$) of AR1FI(α , d) requires three model parameters: d, α , σ^2 . The σ^2 parameter is, however, not estimated directly by the four methods. Here, we estimate it by

$$\hat{\sigma}^2 = \frac{1}{T} \sum_t e_t^2$$
 with $e_t = (1 - L)\hat{d}(y_t - \hat{\alpha}_1 y_{t-1}).$

We conduct the forecasting exercise on the log RV (QML) of the 10 financial assets. The out-of-sample forecasting period starts from January 5, 2018. The model parameters are estimated from a rolling window with a size of eight years. The forecasting horizon ranges from 1 to 50 periods.

6.2.1. Forecasting Evaluation. The second step is to evaluate the forecasting accuracy. We consider two loss functions: mean squared forecast error (MSFE) and mean absolute forecast error (MAFE). They are defined as

$$MSFE_{k} = \frac{1}{(T - T_{0} + 1)} \sum_{t=T_{0}+1}^{T} (\hat{y}_{t+k} - y_{t+k})^{2},$$
$$MAFE_{k} = \frac{1}{(T - T_{0} + 1)} \sum_{t=T_{0}+1}^{T} |\hat{y}_{t+k} - y_{t+k}|,$$

where T_0 is the total number of observations in the training sample.

To assess whether the competing models are statistically different, we employ the model confidence set (MCS) approach proposed by Hansen et al. (2011). The approach aims to provide a model confidence set that contains the best models with probability greater than or equal to a prespecified level, say 5%. The MCS is expected to be large when the data do not contain sufficient information to tell the models apart. It also provides "*p*-value" for each individual model.

Let the set of competing models be \mathcal{M}_0 with objects indexed by i = 1, ..., M. We have M = 4 in our setting. The loss function, denoted by $L_{i,t}$, can either be the squared error $(\hat{y}_{t+k}^{(i)} - y_{t+k})^2$ or be the absolute error $|\hat{y}_{t+k}^{(i)} - y_{t+k}|$, where $\hat{y}_{t+k}^{(i)}$ is the *k*-step-ahead forecast from model *i*. The relative performance is measured by $d_{i,j,t} =$ $L_{i,t} - L_{j,t}$ for all $i, j \in \mathcal{M}_0$. The MCS procedure applies tests of

$$H_{0,\mathcal{M}}: E(d_{ij,t}) = 0$$
 for all $i, j \in \mathcal{M} \subset \mathcal{M}_0$,

against the alternative

$$H_{A,\mathcal{M}}$$
: $E(d_{ij,t}) \neq 0$ for some $i, j \in \mathcal{M}$.

The procedure is based on a model equivalence test for the null $H_{0,\mathcal{M}}$ for any $\mathcal{M} \subset \mathcal{M}_0$ and an elimination rule that identifies the object to be removed from \mathcal{M} if $H_{0,\mathcal{M}}$ is rejected. The algorithm is implemented as follows. The initial model set is $\mathcal{M} = \mathcal{M}_0$.

Step 1. Test $H_{0,M}$ using a model equivalence test at level α .

Step 2. If $H_{0,\mathcal{M}}$ is "accepted," set $\hat{\mathcal{M}}_{1-\alpha} = \mathcal{M}$; otherwise, use an elimination rule to remove objects from \mathcal{M} , and repeat Steps 1 and 2.

The set $\mathcal{M}_{1-\alpha}$ is referred to as the model confidence set. Let \mathcal{M}_i be the model set tested in the *i*th iteration, $P_{H_{0,\mathcal{M}_i}}$ be the *p*-value associated with the null hypothesis H_{0,\mathcal{M}_i} , and $e_{\mathcal{M}_i}$ be the element to be eliminated from set \mathcal{M}_i in the event that H_{0,\mathcal{M}_i} is rejected. The MCS *p*-value for model $e_{\mathcal{M}_i}$ is defined by

$$\hat{p}_{e_{\mathcal{M}_i}} = \max_{j \le i} P_{H_{0,\mathcal{M}_j}},$$

where $\mathcal{M}_1 \supset \mathcal{M}_2 \ldots \supset \mathcal{M}_i$.

For the model equivalence test, we employed the $T_{max,M}$ statistic with a bootstrapped implementation as recommended by Hansen et al. (2011). The block length of the bootstraps is set to be 20. See Hansen et al. (2011) for details of the test statistic and its associated elimination rule.

6.2.2. Forecasting Results. We compare the out-of-sample performance of the four estimation methods in forecasting the log RVs. The general conclusions for all of the data series are the same. That is, the Whittle method provides the best out-of-sample forecasting results (especially in the long run), followed by MPL.

For ease of presentation, we take the S&P 500 ETF (SPY) and the material industry ETF (XLB) as two examples. We employ a rolling window algorithm for the outof-sample forecast, where the model is reestimated for each subsample. Figure 7 displays the rolling window estimate of *d* and α from SPY and XLB. Figure 7, (a) and (b) shows estimation results from all four methods, and Figure 7, (c) and (d) displays only those from LWE(diff), Whittle, and MPL for SPY. Consistent with the wholesample analysis, results from LWE(diff), Whittle, and MPL are close to each other, with the estimated *d* fluctuating around -0.4 and the estimated α being either one or very close to one. However, the LWE method suggests that α is close to zero and d is greater than 0.5. Furthermore, from Figure 7 (c) and (d), we can see that the Whittle estimates of *d* seem to be between those of MPL and LWE(diff) for most of the rolling windows, whereas the Whittle estimates of α are consistently the lowest among the three methods.

The rolling window estimates of *d* and α from XLB are displayed in Figure 7, (e) and (f). For most of the sample periods, the estimated results from the two parametric methods are consistent with their whole-sample estimates





Notes. (a) SPY: \hat{a} . (b) SPY: \hat{a} . (c) SPY: \hat{a} (excluding LWE). (d) SPY: \hat{a} (excluding LWE). (e) XLB: \hat{a} . (f) XLB: \hat{a} .

(Model 2). However, interestingly, in the second half of 2018, the parameter estimates switch from Model 2 to Model 1. This result is consistent with our argument that there is a nonnegligible probability that the ML methods point to Model 1. This partially explains the diverse empirical findings that we have in the current literature. Although it appears to be a sample-sensitive outcome, the fundamental cause of the problem is the bimodality of the log likelihood of the two ML methods, as discussed earlier.

From Figure 7, there seems to be a small structural break in the model parameters in March 2020 at the onset of coronavirus disease 2019, although the changes are not dramatic. To avoid its impact on the forecasting evaluation, we compute the MSFEs and MAFEs from January 5, 2018 to February 29, 2020. The MSFEs and MAFEs of the four estimation methods for forecasting

horizons from 1 to 50 days are presented in Figure 8. Figure 8, (a) and (c) (Figure 8, (b) and (d)) is based on MSFE (MAFE). Evidently, when the forecasting horizon is short, there are no substantial differences among the four estimation methods. The gaps among the lines increase substantially as the forecasting horizon extends to 50. The gaps are more visible for SPY than XLB. Over the longer horizons, for SPY, the loss function of the Whittle method is consistently the smallest, followed by MPL. For XLB, the loss function of the two ML methods follows closely of each other across all horizons. The performances of LWE and LWE(diff) are similar to each other and not as good as those of the parametric methods.

To investigate the statistical significance of those gaps, we conduct the MCS test. The MCS *p*-values are displayed in Figure 9. For SPY, the Whittle method



Figure 8. (Color online) Mean Squared Forecast Error and Mean Absolute Forecast Error

Notes. (a) SPY: MSFE. (b) SPY: MAFE. (c) XLB: MSFE. (d) XLB: MAFE.

always has the *p*-value of one, suggesting that it is the best model of the four competing ones. With a significance level of 10%, we cannot reject all models being in the 'best model set' at the short forecasting horizon (up to five periods). Nevertheless, the Whittle method is the only survival model for horizons beyond five with the squared error loss function. With the absolute errors, at the 10% level, both MPL and Whittle survive the test, and we do not have sufficient information to distinguish between these two methods. For XLB, at the 10%, all four methods survive from one period ahead to a very long horizon forecast (45 periods ahead). The p-values of the Whittle method have almost consistently been the highest. For the remaining forecasting horizons, both MPL and Whittle have *p*-value close to one, whereas the *p*-values of the two semiparametric methods are below 10%. The overall conclusion is that the Whittle method yields the most accurate one-step-ahead forecasts, followed by MPL. This superiority of the Whittle estimation method in forecasting is expected. One can see from Figure 7 that, compared with the MPL estimated coefficients, the Whittle estimated coefficients are more responsive to short-term changes in the data.

7. Conclusion

In this paper, we first examine the finite sample properties of four alternative methods in estimating the AR1FI(α , d) model, including the two parametric ML



(MPL and Whittle) methods and the two semiparametric (LWE and LWE(diff)) methods. Special attention is paid to the part of the parameter space where *d* is close to -0.5 and α is close to unity (Model 2) and where d is close to 0.5 and α is close to zero (Model 1). These choices of parameter settings are motivated by the empirical findings in the RV literature. Via simulations, we find that all four methods have finite sample problems, although the problem associated with the ML methods is less severe. Specifically, LWE and LWE(diff) are significantly biased when the autoregressive coefficient α deviates far from zero and unity, respectively. Moreover, when the DGP is Model 2, LWE always points to Model 1. When the DGP is Model 1, LWE(diff) always points to Model 2. The two ML methods generally perform well. However, there exists a nonnegligible probability that the ML methods mix up Model 1 and Model 2. This problem of the ML methods has never been discovered in the literature. The source of the problem is that the AR1FI(0, d) model is observationally equivalent to the AR1FI(1, d-1) model, leading to a weak identification problem. These simulation findings explain the contradicting empirical evidence documented in the literature.

We apply the four estimation methods to the log RVs of 10 financial assets. The two ML methods and LWE-(diff) always suggest Model 2, whereas LWE always suggests Model 1. Based on what we learn from the

Figure 9. (Color online) The MCS p-Values



Notes. (a) SPY: squared error. (b) SPY: absolute error. (c) XLB: squared error. (d) XLB: absolute error.

simulation studies, we conclude that the true DGP is more likely to be Model 2 than Model 1. Unfortunately, because of the aforementioned finite sample issue of the ML methods, we cannot draw definitive conclusions regarding the DGP. Despite the uncertainty, we find that the estimated model from the Whittle method can generate the most accurate out-of-sample forecasts for the log RV series, especially at long horizons.

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Appendix A. Tapering

One pitfall of the periodogram $I(\lambda_j)$ is that there is leakage effect. In finite samples, when there are high peaks in the spectrum, the nonparametric estimator $I(\lambda_j)$ might significantly overestimate the spectrum at other frequencies and fail to discover spectrums with low peaks.

A.1. Whittle Estimator with Tapering

Dahlhaus (1988) proposes using tapering adapted from nonparametric spectral density estimation (Tukey 1967) for the Whittle estimator. A tapered series is define as

$$y_t^T = h_t y_t,$$

where h_t is the data taper satisfying certain time series properties (Dahlhaus 1988). The tapered periodogram is

$$I_T(\lambda_j) = \frac{1}{2\pi \sum_{t=0}^{T-1} h_t^2} \left| \sum_{t=0}^T h_t y_t \exp\left(-it\lambda_j\right) \right|^2.$$

Replacing $I(\lambda_j)$ in the Whittle estimator (19) by $I_T(\lambda_j)$ yields the tapered Whittle estimator. Dahlhaus (1988) show that the tapered Whittle estimator is \sqrt{T} consistent and asymptotically normal.

There are many tapers satisfying the conditions outlined in Dahlhaus (1988). One example is the Tukey–Hanning taper specified as

$$h_{\rho}(x) = \begin{cases} \frac{1}{2} \left[1 - \cos\left(2\pi x/\rho\right) \right] & x \in [0, \rho/2) \\ 1 & x \in [\rho/2, 1/2] \\ h_{\rho}(1-x) & x \in (1/2, 1] \end{cases}$$

and $h_t = h_{\rho}(t/T)$. For practical implementation, one could set $\rho = T^{-\kappa/3}$ with $\kappa \in [0, 1/2)$. Here, we set $\kappa = 1/4$.

A.2. Local Whittle Estimator with Tapering

One popular tapering method in the local Whittle content is proposed by Velasco (1999). For each positive integer p, there is a Kolmogorov taper, which is of order p in the sense of Velasco (1999). A taper with order p, if applied to the raw data, yields a tapered periodogram that is invariant to polynomial trends of order p - 1, provided that the periodogram is evaluated on the grid λ_{ip} with $i = 1, 2, ..., \lfloor m/p \rfloor$. The objective function of the tapered LW estimator becomes

$$(\hat{C}_p, \hat{d}_p) = \arg \max_{C, d} \frac{p}{m} \sum_{i=1}^{\lfloor m/p \rfloor} \left[-\log C + 2d \log \lambda_{ip} - \frac{1}{C} \lambda_{ip}^{2d} I^T(\lambda_{ip}) \right],$$
(A.1)

where

$$\begin{split} \hat{d}_p &= \arg \max \left\{ -\log \hat{C}_p(d) + 2d \frac{p}{m} \sum_{i=1}^{\lfloor m/p \rfloor} \log \lambda_{ip} \right\} \\ \hat{C}_p(d) &= \frac{p}{m} \sum_{i=1}^{\lfloor m/p \rfloor} \lambda_{ip}^{2d} I_T(\lambda_{ip}). \end{split}$$

The discrete sums include only frequencies λ_{ip} with $i = 1, 2, ..., \lfloor m/p \rfloor$.

The tapered local Whittle estimator is asymptotic normal with a variance of $p\Phi/(4m)$, where

$$\Phi = \lim_{T \to \infty} \left(\sum_{t=1}^{T} h_t^2 \right)^{-2} \sum_{k=0, p, 2p, \dots}^{n=p} \left\{ \sum_{t=1}^{n} h_t^2 \cos\left(t\lambda_k\right) \right\}^2.$$

Suppose we employ the full cosine bell taper (Tukey 1967)

$$h_t = 0.5 \bigg[1 - \cos \bigg(\frac{2\pi t}{T} \bigg)$$

and regard this taper as of order p = 3; the tapered local Whittle estimator is asymptotic normal with variance $pm\Phi/4$ with $\Phi = 1$, when $\mu = 0$ and d < 1.5. However, if we use all the Fourier frequencies from λ_2 to λ_m (i.e., p = 1), then $\Phi = 35/18$. In the simulation studies, we use the cosine bell taper with p = 3. Although the tapered local Whittle methods are invariant to trends and asymptotically normal, they lead to inflated asymptotic variance of the estimator.

Appendix B. Robustness Check: Alternative Estimators of Volatility

Taking the S&P 500 market index ETF as an example, we investigate the estimation robustness with respect to different measures of volatility. In addition to the QML estimator, we consider the popular realized volatility estimator as defined in (1) and the jump-robust volatility estimator bipower variation of Barndorff-Nielsen and Shephard (2004). The data are

Figure B.1. (Color online) Volatility Dynamics of the S&P 500 Market ETF



Notes. (a) The logarithmic of RV and BV. (b) The logarithmic of QML.

Table B.1. Model Estimation Results: S&P 500 ETF

MPL		Whittle		L	WE	I WE (diff)
â	â	â	â	â	â	â
-0.480 -0.472 -0.382	0.998 0.997 0.995	-0.480 -0.472 -0.382	0.997 0.997 0.995	0.542 0.539 0.602	-0.058 -0.043 -0.006	-0.445 -0.438 -0.356
	MP <i>â</i> -0.480 -0.472 -0.382	MPL â â -0.480 0.998 -0.472 0.997 -0.382 0.995	$\begin{array}{c c} \text{MPL} & \text{Whi} \\ \hline \\ $	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{array}{c c c c c c c c c c c c c c c c c c c $

downloaded from Refinitive Tick History at the one-second frequency and sampled every five minutes. The five-minute data are cleaned following the standard practice (Brownlees and Gallo 2006, Barndorff-Nielsen et al. 2009). Figure B.1 displays the estimated log volatilities over the sample period.

Estimation results from the three data series are presented in Table B.1. As for QML, the LWEs of *d* for the logarithmic RV and BV are higher than 0.5. The two parametric methods lead to similar estimation results. For both data series, the estimated *d* is negative and around -0.475, which is smaller than that from log RV (QML). For all data series, the estimated autoregressive coefficients are all very close to unity, with the one from log RV (QML) being the smallest. The LWE(diff) method leads to similar estimation results. As expected, the estimated fractional parameters are slightly higher than those from the parametric methods. In summary, the parametric methods point to Model 2 for all three volatility series.

Endnotes

¹ Long memory is typically defined within the class of stationary models and refers to the case of $d \in (0, 1/2)$. Our definition of long memory here is broader. It refers to the case of d > 0 as in Phillips and Shimotsu (2004).

² In fact, AR1FI(0, *d*) is observationally equivalent to AR1FI(1, d - 1).

³ Similar models have been considered in other papers. For example, Magdalinos (2012) proposes a mildly explosive autoregressive process with long memory errors (i.e., $d \in (0, 0.5)$). Yu (2021) considers a latent local to unity model with fractionally integrated errors.

⁴ Unreported simulations show that results remain the same when using other popular semiparametric methods (such as LPE and the exact local Whittle method of Shimotsu and Phillips 2006) or when tapering (Dahlhaus 1988, Velasco 1999) is applied. The tapering technique employed is described in the appendix.

⁵ See Smith et al. (1997), Nielsen and Frederiksen (2005), and Nadarajah et al. (2021).

⁶ Other noise-robust volatility estimators include the traditional RV obtained from returns sampled at the five-minute frequency, the preaveraging method of Jacod et al. (2009), and the flat-top realized kernel estimator of Varneskov (2017).

⁷ See https://dachxiu.chicagobooth.edu/#risklab.

⁸ Despite its popularity in modeling the log RV, there are two limitations in the AR1FI(α , d) model. First, it fails to take into account of estimation errors when the log RV is regarded as an estimator of the log quadratic variation. The estimation error necessitates an MA component; see Meddahi (2003) and Yu (2021). Second, it does not allow for jumps in the log volatility dynamic.

⁹ The instantaneous variance of u_t is $E(u_t^2) = \Gamma(1-2d)/(\Gamma(1-d))^2$.

¹⁰ Shimotsu and Phillips (2006) propose an exact local Whittle (ELW) estimation method, which can be applied to both stationary and

nonstationary variables. Unlike the conventional local Whittle estimator, which approximates $I_u(\lambda_j)$ by $\lambda_j^{2d}I_y(\lambda_j)$, the exact local Whittle method is based on the relationship that $I_u(\lambda_j) = I_{\Delta^d y}(\lambda_j)$, where $I_{\Delta^d y}(\lambda_j)$ is the periodogram of $\Delta^d y = (1-L)^d y_t$. Shimotsu (2010) proposes a two-stage approach, which uses a tapered local Whittle estimator (Velasco 1999) in the first stage and a modified ELW objective function in the second stage. The two-stage ELW method is designed to improve the performance of ELW when the mean (initial value) of the process is unknown. Unreported simulations show that both the ELW and the two-stage ELW perform similar to the LWE method under our model setting.

¹¹ For example, when $\alpha = 0.93$, d = 0.4, and k = 1, 500, $F(d + k, 1; 1 - d + k; \alpha) = -1.2143 \times 10^{19}$.

¹² Unreported simulations confirm that the estimation is more accurate than those presented here when d < 0.45.

¹³ Coursol and Dacunha-Castelle (1982) study the approximation error $\log L_N - \log L_W$.

¹⁴ The fractionally integrated process in (4) is simulated with the fracdiff function provided by Katsumi Shimotsu.

¹⁵ We set the sample size to be the power of two to ensure the accuracy of Fourier transformation. Moreover, the finite sample properties reported here remain qualitatively unchanged when T is increased to 2,408 and 4,096.

¹⁶ There is no need to estimate α using LWE(diff) as it assumes $\alpha = 1$.

¹⁷ Tapering has been shown to be capable of removing deterministic time trends (e.g., Žurbenko 1979, Robinson 1986, Dahlhaus 1988, Hurvich and Ray 1995, Velasco 1999, Hurvich and Chen 2000). The tapering methods are detailed in the appendix.

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