Econometric Theory, **35**, 2019, 198–231.
doi:10.1017/S0266466618000051

ASYMPTOTIC THEORY FOR ESTIMATING DRIFT PARAMETERS IN THE FRACTIONAL VASICEK MODEL

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This article develops an asymptotic theory for estimators of two parameters in the drift function in the fractional Vasicek model when a continuous record of observations is available. The fractional Vasicek model with long-range dependence is assumed to be driven by a fractional Brownian motion with the Hurst parameter greater than or equal to one half. It is shown that, when the Hurst parameter is known, the asymptotic theory for the persistence parameter depends critically on its sign, corresponding asymptotically to the stationary case, the explosive case, and the null recurrent case. In all three cases, the least squares method is considered, and strong consistency and the asymptotic distribution are obtained. When the persistence parameter is positive, the estimation method of Hu and Nualart (2010) is also considered.

1. INTRODUCTION

The Vasicek model of Vasicek (1977) has found a wide range of applications in many fields, including but not limited to economics, finance, biology, physics, chemistry, medicine, and environmental studies. An intrinsic property implied by the standard Vasicek model is short-range dependence in the stochastic component of the model because the autocovariance decays at a geometric rate. This property is at odds with abundant empirical evidence that indicates long-range dependence or long memory in time series data (see, e.g., Lo, 1991; Comte and Renault, 1996; Granger and Hyung, 2004). As a result, stochastic models with long-range dependence have been used to describe the movement of time series data in hydrology, geophysics, climatology, telecommunications, economics, and finance.

We gratefully thank the editor, a co-editor, and two anonymous referees for constructive comments. All errors are our own. Xiao's research is supported by the Humanities and Social Sciences of Ministry of Education Planning Fund of China (No. 17YJA630114). Yu's research was supported by the Singapore Ministry of Education (MOE) Academic Research Fund Tier 3 grant MOE2013-T3-1-009. Address correspondence to Jun Yu, School of Economics and Lee Kong Chian School of Business, Singapore Management University, 90 Stamford Road, Singapore 178903, Singapore; e-mail: yujun@smu.edu.sg.

In continuous time, the fractional Brownian motion (fBm), with the Hurst parameter greater than one half, is an important stochastic process to characterize long-range dependence (see, e.g., Mandelbrot and Van Ness, 1968). An fBM can produce burstiness, self-similarity, and stationary increments in the sample path. Excellent surveys on fractional Brownian motions can be found in Biagini, Hu, Øksendal, and Zhang (2008) and Mishura (2008).

If the Brownian motion in the Vasicek model is replaced with an fBM, we get the following fractional Vasicek model (fVm)

$$dX_t = \kappa(\mu - X_t)\,dt + \sigma\,dB_t^H, \tag{1.1}$$

where σ is a positive constant, $\mu, \kappa \in \mathbb{R}$, and B_t^H is an fBM (which will be defined formally below) with the Hurst parameter $H \geq 1/2$. Long-range dependence in X_t is generated by B_t^H.

In Model (1.1), $\kappa(\mu - X_t)$ is the drift function and there are two unknown parameters in it, μ and κ. Parameter κ determines the persistence in X_t. Depending on the sign of κ, the model can capture stationary, explosive, and null recurrent behavior. The fVm was first used to describe the dynamics in volatility by Comte and Renault (1998). Other applications of the fVm can be found in Comte, Coutin, and Renault (2012), Chronopoulou and Viens (2012a, 2012b), Corlay, Lebovits, and Véhel (2014), Bayer, Friz, and Gatheral (2016) and references therein. Despite many applications of the fVm in practice, estimation and asymptotic theory for the fVm have received little attention in the literature. The main purpose of the present article is to propose estimators for μ and κ and to develop an asymptotic theory for these estimators based on a continuous record of observations over an increasing time span (i.e., the period of $[0, T]$ with $T \to \infty$) when the Hurst parameter H is known and $H \in [1/2, 1)$. This range of values for H is empirically relevant for much economic and financial data; see, for example, Cheung (1993), Baillie (1996).

A very important special case of fVm is the so-called fractional Ornstein–Uhlenbeck (fOU) process given by

$$dX_t = -\kappa X_t dt + \sigma\,dB_t^H,\ X_0 = 0. \tag{1.2}$$

The key difference between (1.1) and (1.2) is that μ is assumed to be zero and known in (1.2) while μ is unknown in (1.1). A smaller difference between (1.1) and (1.2) is that $X_0 = 0$ in (1.2) while X_0 may not be zero in (1.1). The order of the initial condition will be assumed when we develop the asymptotic theory.

In fact, the fOU process is closely related to the following discrete-time model

$$y_t = \left(1 - \frac{\kappa}{T}\right) y_{t-1} + u_t,\ (1-L)^d u_t = \varepsilon_t,\ y_0 = 0,\ (t = 1, \ldots, T), \tag{1.3}$$

where L is the lag operator, $d = H - 1/2$ is the fractional differencing parameter, and $\varepsilon_t \sim$ i.i.d.$(0, \sigma^2)$. When $H = 1/2$, $d = 0$, $u_t = \varepsilon_t$, and y_t follows a standard AR(1) model with an i.i.d. error term. When $1/2 < H < 1$, $0 < d < 1/2$, and u_t is a stationary long memory process given by

$$u_t = (1-L)^{-d}\,\varepsilon_t = (1-L)^{-(H-1/2)}\,\varepsilon_t = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)}\varepsilon_{t-j},$$

where $\Gamma(x)$ is the gamma function. Davydov (1970) and Sowell (1990) related the process in (1.3) to that in (1.2) by showing the following functional central limit theorem,

$$\frac{\delta_H \Gamma(H+1/2)}{\sigma T^H} y_{[Ts]} \Rightarrow X_s\,, \forall\, 0 \le s \le 1\,,$$

where $[z]$ denotes the smallest integer greater than or equal to z, $\delta_H = \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}$ (see also Tanaka, 2013, 2015). If $\kappa = 0$, y_t has a unit root; if $\kappa > 0$, y_t is asymptotically stationary; if $\kappa < 0$, y_t has an explosive root.

Depending on the sign of κ, alternative estimation methods have been proposed in the literature to estimate κ in the fOU model and the asymptotic theory for these estimators has been obtained. When $\kappa > 0$, Kleptsyna and Le Breton (2002), Tudor and Viens (2007), Tanaka (2014) studied the maximum likelihood (ML) estimator; Hu and Nualart (2010) studied the least squares (LS) estimator; Tanaka (2013) studied the minimum contrast (MC) estimator; Hu and Nualart (2010) introduced and studied an estimator based on the ergodic property of X_t. When $\kappa < 0$, two estimators have been studied, namely, the ML estimator (Tanaka, 2015) and the LS estimator (Belfadli, Es-Sebaiy, and Ouknine, 2011; El Machkouri, Es-Sebaiy, and Ouknine, 2016). When $\kappa = 0$, the ML method and the MC method were considered in Kleptsyna and Le Breton (2002), Tanaka (2013). Prakasa Rao (2010) is a textbook treatment of alternative methods and the asymptotic theory for estimating parameters in the fOU model.

In almost all empirically relevant cases, the parameter, μ, in the drift function of model (1.1) is unknown. Thus, it is important to estimate both κ and μ. This is the reason why we consider the problem of estimating both κ and μ in the fVm. As in the fOU model, asymptotic theory for κ critically depends on the sign of κ, namely whether $\kappa > 0$, $\kappa < 0$ or $\kappa = 0$. When $\kappa > 0$, two estimators are considered, i.e., the LS estimators and the estimators of Hu and Nualart (2010). The estimators of Hu and Nualart (2010) do not contain any stochastic integral and hence are easier to calculate. Our results suggest that, unless $H = 1/2$, the estimators of Hu and Nualart (2010) are asymptotically more efficient than the LS estimators. The relative asymptotic efficiency increases with H when $H \in [1/2, 3/4)$ and also when $H \in (3/4, 1)$. When $\kappa < 0$ or $\kappa = 0$, the LS estimators are considered. Strong consistency and asymptotic distributions are established for both κ and μ. The proof is based on the Malliavin calculus, the Itô–Skorohod integral and the Young integral for fractional Brownian motions. In particular, we use the Itô–Skorohod integral for the stationary case and the Young integral for the explosive case. To the best of our knowledge, this is the first article in the literature where an fVm is estimated and an asymptotic theory is developed.

A drawback of the model considered here is that H is assumed to be known *a priori*. In practice, H is always unknown unless a Brownian motion is used. It is possible to estimate H with a continuous record of observations by the generalized quadratic variation of Gradinaru and Nourdin (2006). Section 4 will provide more details about this estimator of H. However, the study of asymptotic properties of this estimator is beyond the scope of the present article.

A related drawback of our approach is that a continuous record of observations is required. In economics and finance, this assumption is too strong. However, as high-frequency data are now widely available, it is possible to approximate a continuous record of observations by discretely sampled high-frequency data and to approximate the fVm by the discrete-time model (1.3). The underlying assumption for these approximations to work well is to allow the sampling interval in discretely sampled data to go to zero. The asymptotic theory for H under the long-span asymptotic scheme is expected to correspond to that of d under the long-span and in-fill asymptotic scheme. It is important to point out that alternative estimation methods, such as log-periodogram regression and local Whittle estimation, have been suggested in the discrete-time literature to estimate d. The long-span asymptotic theory has been developed for d; see for example, Geweke and Porter-Hudak (1983), Robinson (1995a, 1995b), Shimotsu and Phillips (2005).

The third drawback of the model considered here is that the asymptotic theory developed here is only applicable for $H \in [1/2, 1)$. Some empirical studies based on economic and financial time series have found the estimated d to be smaller than 0 or larger than 1 (see, for example, Baillie, 1996; Baillie, Bollerslev, and Mikkelsen, 1996), which implies that $H \notin [1/2, 1)$. While the asymptotic theory for κ and μ can be extended to a wide range values of H by using the exact local Whittle method of Shimotsu and Phillips (2005), such an extension is complicated and will be reported in later work.

The rest of the article is organized as follows. Section 2 contains some basic facts about fractional Brownian motions and introduces the LS method and the method of Hu and Nualart (2010) for estimating the two parameters in the drift function of the fVm. In Section 3, we establish consistency and the asymptotic distributions for κ and μ. Section 4 contains some concluding remarks and gives directions of further research. All the proofs are collected in the Appendix.

We use the following notations throughout the article: $\stackrel{p}{\to}$, $\stackrel{a.s.}{\to}$, $\stackrel{\mathcal{L}}{\to}$, $\Rightarrow$, $\stackrel{d}{=}$, and $\sim$ denote convergence in probability, convergence almost surely, convergence in distribution, weak convergence, equivalence in distribution, and asymptotic equivalence, respectively, as $T \to \infty$.

2. THE ESTIMATION METHODS

Before introducing our estimation techniques, we first state some basic facts about fractional Brownian motions. For more complete treatments on the subject, see Nualart (2006), Biagini et al. (2008), Mishura (2008) and references therein.

An fBM with the Hurst parameter $H \in (0, 1)$, B_t^H for $t \in \mathbb{R}$, is a zero mean Gaussian process with covariance

$$\mathbb{E}(B_t^H B_s^H) = R_H(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{2.1}$$

This covariance function implies that the fBm is self-similar with the self-similarity parameter H, that is,

$$B_{\lambda t}^H \overset{d}{=} \lambda^H B_t^H. \tag{2.2}$$

For $t > 0$, Mandelbrot and Van Ness (1968) presented the following integral representation for B_t^H (see also in Davidson and De Jong, 2000):

$$B_t^H = \frac{1}{c_H}\left\{\int_{-\infty}^{0}\left[(t-u)^{H-1/2} - (-u)^{H-1/2}\right] dW_u + \int_0^t (t-u)^{H-1/2} dW_u\right\}, \tag{2.3}$$

where W_t is a standard Brownian motion, $c_H = \left[\frac{1}{2H} + \int_0^\infty \left((1+s)^{H-1/2} - s^{H-1/2}\right)^2 ds\right]^{1/2}$. If $H = 1/2$, B_t^H becomes the standard Brownian motion W_t. If $0 < H < 1/2$, B_t^H is negatively correlated. For $1/2 < H < 1$, B_t^H has long-range dependence in the sense that if $r(n) = \mathbb{E}\left(B_1^H(B_{n+1}^H - B_n^H)\right)$, then $\sum_{n=1}^\infty r(n) = \infty$. In this case, B_t^H is a persistent fBm, since the positive (negative) increments are likely to be followed by positive (negative) increments. Given that long-range dependence is empirically found in many financial time series, the fVm with $H \in [1/2, 1)$ is the focus of the present article. To estimate κ and μ in the fVm, we assume that one observes the whole trajectory of X_t for $t \in [0, T]$. The asymptotic theory is developed by assuming $T \to \infty$, which corresponds to a long-span scheme.

Motivated by the work of Hu and Nualart (2010), Belfadli et al. (2011), El Machkouri, Es-Sebaiy, and Ouknine (2016), we denote the LS estimators of κ and μ to be the minimizers of the following (formal) quadratic function

$$L(\kappa, \mu) = \int_0^T \left(\dot{X}_t - \kappa(\mu - X_t)\right)^2 dt, \tag{2.4}$$

where $\dot{X}_t$ denotes the differentiation of X_t with respect to t, although $\int_0^T \dot{X}_t^2 dt$ does not exist. Consequently, we obtain the following analytical expressions for the LS estimators of κ and μ (denoted by $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$, respectively),

$$\hat{\kappa}_{LS} = \frac{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}, \tag{2.5}$$

$$\hat{\mu}_{LS} = \frac{(X_T - X_0)\int_0^T X_t^2 dt - \int_0^T X_t dX_t \int_0^T X_t dt}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}. \tag{2.6}$$

When $H = 1/2$, it is well-known that we can interpret the stochastic integral $\int_0^T X_t dX_t$ as an Itô integral. When $H \in (1/2, 1)$, X_t is no longer a semimartingale. In this case, for $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ to consistently estimate κ and μ, we have to interpret the stochastic integral $\int_0^T X_t dX_t$ carefully. In fact, we interpret it differently when the sign of κ is different. If $\kappa > 0$, we interpret it as an Itô–Skorohod integral; if $\kappa < 0$, we interpret it as a Young integral; if $\kappa = 0$, we can interpret it as either an Itô–Skorohod integral or a Young integral. The asymptotic distributions of $\hat{\kappa}_{LS}$ are different across these three cases.

If $\kappa > 0$, we can consider alternative estimators of κ and μ (denoting them by $\hat{\kappa}_{HN}$ and $\hat{\mu}_{HN}$, respectively). The estimators are motivated by Hu and Nualart (2010) where the stationary and ergodic properties of a process were used to construct a new estimator for κ in the fOU model. To fix ideas, the strong solution of the fVm in (1.1) is given by

$$X_t = \mu + (X_0 - \mu)\exp(-\kappa t) + \sigma \int_{-\infty}^{t} e^{-\kappa(t-s)} dB_s^H, \tag{2.7}$$

which leads to the following discrete-time representation

$$X_t = \mu + e^{-\kappa}(X_{t-1} - \mu) + \sigma \int_{t-1}^{t} e^{-\kappa(t-s)} dB_s^H. \tag{2.8}$$

When $\kappa > 0$, X_t is asymptotically stationary and ergodic. When $\kappa = 0$, X_t has a unit root and is null recurrent. When $\kappa < 0$, X_t has an explosive root.

When $\kappa > 0$, by the ergodic theorem, $\frac{1}{T}\int_0^T X_t dt \overset{a.s.}{\to} \mu$. So an alternative estimator of μ is the continuous-time sample mean

$$\hat{\mu}_{HN} = \frac{1}{T}\int_0^T X_t dt. \tag{2.9}$$

Moreover, following Hu and Nualart (2010), we can show that when $\kappa > 0$,

$$\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2 \overset{a.s.}{\to} \sigma^2 \kappa^{-2H} H\Gamma(2H).$$

Hence, an alternative estimator of κ is

$$\hat{\kappa}_{HN} = \left(\frac{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}{T^2\sigma^2 H\Gamma(2H)}\right)^{-\frac{1}{2H}}. \tag{2.10}$$

Compared with the LS estimators in (2.5) and (2.6) which involve the stochastic integral $\int_0^T X_t dX_t$, $\hat{\mu}_{HN}$ and $\hat{\kappa}_{HN}$ in (2.9) and (2.10) do not contain any stochastic integral with respect to fBm but only involve quadratic integral functionals. Hence, they are conceptually easier to understand and numerically easier to compute than the LS estimators.

3. ASYMPTOTIC THEORY FOR κ AND μ

In the case of a Brownian motion-driven or a Lévy process-driven Vasicek model, it is known that the asymptotic theory for κ depends on the sign of κ (see, Wang and Yu, 2016). In the case of the fVm, we show below that the asymptotic theory for κ continues to depend on the sign of κ.

3.1. Asymptotic theory when $\kappa > 0$

In the context of the fVm in (1.1), we can represent the stochastic integral $\int_0^T X_t dX_t$ as

$$\int_0^T X_t dX_t = \kappa\mu \int_0^T X_t dt - \kappa \int_0^T X_t^2 dt + \sigma \int_0^T X_t dB_t^H .$$

When $H = 1/2$, the stochastic integral $\int_0^T X_t dB_t^H$, which can be interpreted as an Itô integral, is approximated by forward Riemann sums. When $H > 1/2$, we interpret $\int_0^T X_t dB_t^H$ as an Itô–Skorohod stochastic integral. In this case, following Duncan, Hu, and Pasik-Duncan (2000), $\int_0^T X_t dB_t^H$ is approximated by Riemann sums defined in terms of the Wick product, i.e.,

$$\int_0^T X_t dB_t^H = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} X_{t_i} \diamond \left(B_{t_{i+1}}^H - B_{t_i}^H \right), \tag{3.1}$$

where $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of $[0, T]$ with $|\pi| = \max_{0\le i\le n-1} (t_{i+1} - t_i)$.

Unfortunately, this approximation is less useful for computing the stochastic integral because the Wick product cannot be calculated just from the values of X_{t_i} and $B_{t_{i+1}}^H - B_{t_i}^H$. In other words, unless $H = 1/2$, there is no computable representation of the term $\int_0^T X_t dX_t$ given the observations $X_t, t \in [0, T]$.

Using the Itô–Skorohod integral for fBm and the Malliavin derivative for X_t, we can rewrite $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ as[1]

$$\hat{\kappa}_{LS} = \frac{\frac{X_T - X_0}{T} \frac{\int_0^T X_t dt}{T} - \left(\frac{X_T^2}{2T} - \frac{X_0^2}{2T} - \frac{\alpha_H \sigma^2}{T} \int_0^T \int_0^t s^{2H-2} e^{-\kappa s} ds dt \right)}{\frac{\int_0^T X_t^2 dt}{T} - \left(\frac{\int_0^T X_t dt}{T} \right)^2}, \tag{3.2}$$

$$\hat{\mu}_{LS} = \frac{\frac{X_T - X_0}{T} \frac{\int_0^T X_t^2 dt}{T} - \frac{\int_0^T X_t dt}{T} \left(\frac{X_T^2}{2T} - \frac{X_0^2}{2T} - \frac{\alpha_H \sigma^2}{T} \int_0^T \int_0^t s^{2H-2} e^{-\kappa s} ds dt \right)}{\frac{X_T - X_0}{T} \frac{\int_0^T X_t dt}{T} - \left(\frac{X_T^2}{2T} - \frac{X_0^2}{2T} - \frac{\alpha_H \sigma^2}{T} \int_0^T \int_0^t s^{2H-2} e^{-\kappa s} ds dt \right)}, \tag{3.3}$$

where $\alpha_H = H(2H - 1)$. Clearly, $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ in (3.2) and (3.3) are easier to compute than those in (2.5) and (2.6).

Before we prove consistency of $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$, we first obtain consistency of $\hat{\kappa}_{HN}$ and $\hat{\mu}_{HN}$ which follows directly by ergodicity.

THEOREM 3.1. *Let* $H \in [1/2, 1)$*,* $X_0/\sqrt{T} = o_{a.s.}(1)$*, and* $\kappa > 0$ *in* (1.1). Then we have $\hat{\kappa}_{HN} \stackrel{a.s.}{\rightarrow} \kappa$ and $\hat{\mu}_{HN} \stackrel{a.s.}{\rightarrow} \mu$.

Remark 3.1. Almost sure convergence of $\hat{\kappa}_{HN}$ in Theorem 3.1 extends that of Hu and Nualart (2010) from the fOU model to the fVm.

Remark 3.2. Applying the well-known result $\frac{1}{T}\int_0^T \int_0^t s^{2H-2} e^{-\kappa s} ds dt \rightarrow \kappa^{1-2H}\Gamma(2H-1)$ to (3.2) and (3.3) and using Lemma 5.2 in Hu and Nualart (2010), we can show that $\hat{\kappa}_{LS} \stackrel{a.s.}{\rightarrow} \kappa$ and $\hat{\mu}_{LS} \stackrel{a.s.}{\rightarrow} \mu$ for $H \in [1/2, 1)$.

To establish the asymptotic distributions for the two sets of estimators, we first consider $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$, and then use the asymptotic distributions of $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ to develop the asymptotic distributions of $\hat{\kappa}_{HN}$ and $\hat{\mu}_{HN}$.

THEOREM 3.2. *Let* $X_0/\sqrt{T} = o_p(1)$ *and* $\kappa > 0$ *in* (1.1). Then the following convergence results hold true.

(i) For $H \in [1/2, 3/4)$, we have

$$\sqrt{T}\left(\hat{\kappa}_{LS} - \kappa\right) \stackrel{\mathcal{L}}{\rightarrow} \mathcal{N}\left(0, \kappa C_H\right), \tag{3.4}$$

where $C_H = (4H-1)\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2H)\Gamma(2-2H)}\right)$.

(ii) For $H = 3/4$, we have

$$\frac{\sqrt{T}}{\sqrt{\log(T)}}\left(\hat{\kappa}_{LS} - \kappa\right) \stackrel{\mathcal{L}}{\rightarrow} \mathcal{N}\left(0, \frac{4\kappa}{\pi}\right). \tag{3.5}$$

(iii) For $H \in (3/4, 1)$, we have

$$T^{2-2H}\left(\hat{\kappa}_{LS} - \kappa\right) \stackrel{\mathcal{L}}{\rightarrow} \frac{-\kappa^{2H-1}}{H\Gamma(2H)} R(H), \tag{3.6}$$

where $R(H)$ is the Rosenblatt random variable whose characteristic function is given by

$$c(s) = \exp\left(\frac{1}{2}\sum_{k=2}^{\infty}\left(2is\sigma(H)\right)^k \frac{a_k}{k}\right), \tag{3.7}$$

with $i = \sqrt{-1}$, $\sigma(H) = \sqrt{H(H-1/2)}$ and

$$a_k = \int_0^1 \cdots \int_0^1 |x_1 - x_2|^{H-1} \cdots |x_{k-1} - x_k|^{H-1} |x_k - x_1|^{H-1} dx_1 \cdots dx_k.$$

Remark 3.3. A straightforward calculation shows that

$$T^{1-H}\left(\hat{\mu}_{LS}-\mu\right)=\frac{\frac{X_T-X_0}{T^H}\frac{1}{T}\int_0^T X_t^2dt-\frac{1}{T}\int_0^T X_tdX_t\frac{1}{T^H}\int_0^T X_tdt}{\frac{X_T-X_0}{T}\frac{1}{T}\int_0^T X_tdt-\frac{1}{T}\int_0^T X_tdX_t} -\frac{\mu\left(\frac{X_T-X_0}{T^H}\frac{1}{T}\int_0^T X_tdt-\frac{1}{T^H}\int_0^T X_tdX_t\right)}{\frac{X_T-X_0}{T}\frac{1}{T}\int_0^T X_tdt-\frac{1}{T}\int_0^T X_tdX_t}.$$

For $H\in[1/2,1)$ and $X_0/\sqrt{T}=o_p(1)$, we can easily obtain the following asymptotic distribution of $\hat{\mu}_{LS}$,

$$T^{1-H}\left(\hat{\mu}_{LS}-\mu\right)\xrightarrow{\mathcal{L}}\mathcal{N}\left(0,\frac{\sigma^2}{\kappa^2}\right). \tag{3.8}$$

THEOREM 3.3. *Let $H\in[1/2,1)$, $X_0/\sqrt{T}=o_p(1)$, and $\kappa>0$ in* (1.1). Then, we have

$$T^{1-H}\left(\hat{\mu}_{HN}-\mu\right)\xrightarrow{\mathcal{L}}\mathcal{N}\left(0,\frac{\sigma^2}{\kappa^2}\right). \tag{3.9}$$

Moreover, let $X_0/\sqrt{T}=o_p(1)$, and $\kappa>0$ in (1.1). Then the following convergence results hold true.

(i) For $H\in[1/2,3/4)$, we have

$$\sqrt{T}\left(\hat{\kappa}_{HN}-\kappa\right)\xrightarrow{\mathcal{L}}\mathcal{N}\left(0,\kappa\rho_H\right), \tag{3.10}$$

where $\rho_H=\frac{4H-1}{4H^2}\left(1+\frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2H)\Gamma(2-2H)}\right)=\frac{C_H}{4H^2}$.

(ii) For $H=3/4$, we have

$$\frac{\sqrt{T}}{\sqrt{\log(T)}}\left(\hat{\kappa}_{HN}-\kappa\right)\xrightarrow{\mathcal{L}}\mathcal{N}\left(0,\frac{16\kappa}{9\pi}\right). \tag{3.11}$$

(iii) For $H\in(3/4,1)$, we have

$$T^{2-2H}\left(\hat{\kappa}_{HN}-\kappa\right)\xrightarrow{\mathcal{L}}\frac{-\kappa^{2H-1}}{H\Gamma(2H+1)}R(H), \tag{3.12}$$

where $R(H)$ is the Rosenblatt random variable defined in (3.7).

Remark 3.4. Comparing the two sets of asymptotic theory for κ, we can draw a few conclusions. First, the rate of convergence of $\hat{\kappa}_{HN}$ is the same as that of $\hat{\kappa}_{LS}$ which is $\sqrt{T}$ and independent of H. Second, the two asymptotic variances depend on H. When $H=1/2$, the two estimators have the same asymptotic variance which is 2κ. In this case, the asymptotic distribution is identical to that in Feigin (1976), i.e., $\mathcal{N}(0, 2\kappa)$. When $1/2 < H < 3/4$, $4H^2 > 1$ and hence the asymptotic variance of $\hat{\kappa}_{HN}$ is smaller than that of $\hat{\kappa}_{LS}$, suggesting that the method of Hu and Nualart (2010) can estimate κ more efficiently. Third, the asymptotic distribution of $\hat{\kappa}_{LS}$ and $\hat{\kappa}_{HN}$ is the same as given in the fOU model; see p. 1034 and p. 1037 in Hu and Nualart (2010).

Remark 3.5. The two sets of asymptotic theory for μ are identical and the rate of convergence is T^{1-H}. These two features differ from those for κ.

Remark 3.6. The asymptotic variance of $\hat{\kappa}_{HN}$ and $\hat{\kappa}_{LS}$ depends on H. Figure 1 plots ρ_H and C_H as a function of H. Obviously, both ρ_H and C_H monotonically increase in H over the interval $[1/2, 3/4)$. They reach the minimum value of 2 when $H = 1/2$. As $H \to 3/4$, both diverge to infinity. Hence, both ρ_H and C_H have a singularity at $H = 3/4$. Since ρ_H diverges faster than C_H, the relative asymptotic efficiency of $\hat{\kappa}_{HN}$ to $\hat{\kappa}_{LS}$ increases in H.

Remark 3.7. If we interpret the integral $\int_0^T X_t dX_t$ in (2.5) as a Young integral, then we can obtain

$$\hat{\kappa}_{LS} = \frac{\frac{(X_T - X_0)}{T}\frac{\int_0^T X_t dt}{T} - \frac{1}{2}\frac{(X_T^2 - X_0^2)}{T}}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}, \tag{3.13}$$

which converges to zero, following (A.9), (A.16) and Lemma 5.2 in Hu and Nualart (2010). Hence, $\hat{\kappa}_{LS}$ defined by (3.13) is inconsistent. For this reason, we

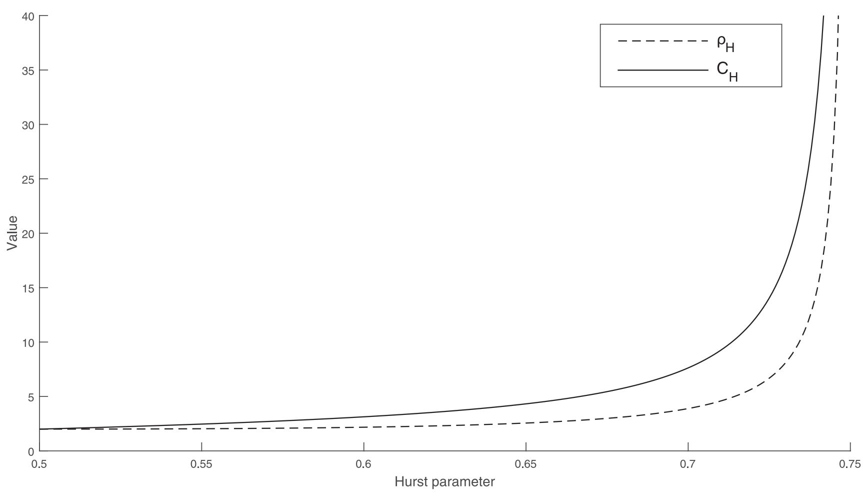

FIGURE 1. Plots of ρ_H and C_H.

have interpreted the stochastic integral $\int_0^T X_t dX_t$ in (2.5) as an Itô–Skorohod integral, which corresponds to the classical Itô integral when $H = 1/2$. For the same reason, $\int_0^T X_t dX_t$ in (2.6) should be interpreted as an Itô–Skorohod integral as well.

Remark 3.8. For $H = 3/4$, it is interesting to find that both $\hat{\kappa}_{LS}$ and $\hat{\kappa}_{HN}$ are still asymptotically normally distributed with rate of convergence of $\sqrt{T}/\sqrt{\log(T)}$. For $H > 3/4$, we established the noncentral limit theorem for both $\hat{\kappa}_{LS}$ and $\hat{\kappa}_{HN}$. In fact, we have identified the asymptotic distribution of $\hat{\kappa}_{LS}$ and $\hat{\kappa}_{HN}$ as a Rosenblatt random variable. The central limit theorem ($H \leq 3/4$) and the noncentral limit theorem ($H > 3/4$) of $\hat{\kappa}_{LS}$ and $\hat{\kappa}_{HN}$ share the spirit of a result in Breton and Nourdin (2008), where it was shown that the asymptotic distribution of the empirical quadratic variations of fBm is normal if $H \leq 3/4$ but non-normal if $H > 3/4$.

Remark 3.9. Comparing (3.5) with (3.11), we see that the asymptotic variance of $\hat{\kappa}_{LS}$ is 2.5 times as large as that of $\hat{\kappa}_{HN}$ when $H = 3/4$, suggesting $\hat{\kappa}_{LS}$ is asymptotically much less efficient. Moreover, since $\Gamma(2H + 1)/\Gamma(2H) = 2H \in (1.5, 2)$ when $H \in (3/4, 1)$, comparing (3.6) with (3.12) we see that $\hat{\kappa}_{LS}$ continues to be asymptotically less efficient than $\hat{\kappa}_{HN}$ when $H \in (3/4, 1)$.

Remark 3.10. Combining Remarks 3.4 and 3.9, we can conclude that $\hat{\kappa}_{HN}$ is asymptotically more efficient than $\hat{\kappa}_{LS}$ when $H \in (1/2, 1)$. When $H = 1/2$, the two estimators are asymptotically equivalent.

3.2. Asymptotic theory when $\kappa < 0$

When $\kappa < 0$, the model of (1.1) is explosive. In this case, the stochastic integral $\int_0^T X_t dX_t$ is interpreted as a Young integral (see Young, 1936). Indeed, using the Young integral, we can obtain strong consistency of the LS estimators, $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$. Moreover, it turns out that the pathwise approach is the preferred way to simulate numerically LS estimators, $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$. As a consequence, using the Young integral, we can easily obtain $\int_0^T X_t dX_t = \left(X_T^2 - X_0^2\right)/2$. The techniques used here are related to those in recent articles by Belfadli et al. (2011), El Machkouri et al. (2016).

Applying the Young integral to (2.5) and (2.6), we can rewrite $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ as

$$\begin{aligned}\hat{\kappa}_{LS} &= \frac{(X_T - X_0)\int_0^T X_t dt - \frac{T}{2}\left(X_T^2 - X_0^2\right)}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \\ &= \frac{\frac{X_T}{T}e^{\kappa T}e^{\kappa T}\int_0^T X_t dt - \frac{X_0}{T}e^{\kappa T}e^{\kappa T}\int_0^T X_t dt - \frac{1}{2}X_T^2 e^{2\kappa T} + \frac{1}{2}X_0^2 e^{2\kappa T}}{e^{2\kappa T}\int_0^T X_t^2 dt - e^{2\kappa T}\frac{1}{T}\left(\int_0^T X_t dt\right)^2}, \qquad \textbf{(3.14)}\end{aligned}$$

$$\hat{\mu}_{LS} = \frac{(X_T - X_0)\int_0^T X_t^2 dt - \frac{X_T^2 - X_0^2}{2}\int_0^T X_t dt}{(X_T - X_0)\int_0^T X_t dt - T\frac{X_T^2 - X_0^2}{2}}$$

$$= \frac{\frac{e^{\kappa T}}{T}\int_0^T X_t^2 dt - \frac{X_T + X_0}{2T}e^{\kappa T}\int_0^T X_t dt}{\frac{e^{\kappa T}}{T}\int_0^T X_t dt - \frac{X_T + X_0}{2}e^{\kappa T}}. \qquad \textbf{(3.15)}$$

Before considering strong consistency of $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$, we first introduce a lemma, which will be used to prove strong consistency.

LEMMA 3.1. *Let* $H \in [\frac{1}{2}, 1)$, $X_0 = O_p(1)$, *and* $\kappa < 0$ *in* (1.1). Then, as $T \to \infty$, we have

$$\frac{e^{\kappa T}}{T^H}\int_0^T X_t dB_t^H \xrightarrow{a.s.} 0.$$

THEOREM 3.4. *Let* $H \in [\frac{1}{2}, 1)$, $X_0 = O_p(1)$, *and* $\kappa < 0$ *in* (1.1). Then, as $T \to \infty$, $\hat{\kappa}_{LS} \xrightarrow{a.s.} \kappa$ and $\hat{\mu}_{LS} \xrightarrow{a.s.} \mu$.

The asymptotic distributions of $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ are developed in the following theorem.

THEOREM 3.5. *Let* $H \in [1/2, 1)$, $X_0 = O_p(1)$, *and* $\kappa < 0$ *in* (1.1). Then as $T \to \infty$,

$$\frac{e^{-\kappa T}}{2\kappa}\left(\hat{\kappa}_{LS} - \kappa\right) \xrightarrow{\mathcal{L}} \frac{\sigma\frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\nu}{X_0 - \mu + \sigma\frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\omega}, \qquad \textbf{(3.16)}$$

where ν and ω are two independent standard normal variables. Moreover, as $T \to \infty$,

$$T^{1-H}\left(\hat{\mu}_{LS} - \mu\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\sigma^2}{\kappa^2}\right). \qquad \textbf{(3.17)}$$

Remark 3.11. In (3.16), if we set $X_0 = \mu$, the limiting distribution of $\frac{e^{-\kappa T}}{2\kappa}\left(\hat{\kappa}_{LS} - \kappa\right)$ becomes ν/ω which is a standard Cauchy variate. This limiting distribution is the same as that in the fOU model (see, e.g., Belfadli et al., 2011; El Machkouri et al., 2016) and that in the Vasicek model driven by a standard Brownian motion (see, e.g., Feigin, 1976). Moreover, the asymptotic theory in (3.16) is similar to that in the explosive discrete-time and continuous-time models when discretely sampled data are available (see, e.g., White, 1958; Anderson, 1959; Phillips and Magdalinos, 2007; Wang and Yu, 2015, 2016).

Remark 3.12. In the context of the fOU model, Belfadli et al. (2011) showed that the LS estimator of κ is consistent and derived the asymptotic Cauchy distribution. Our result here not only extends their results on κ to a general model with an unknown μ and a general initial condition, but also includes the asymptotic theory for μ. The asymptotic distribution of $\hat{\mu}_{LS}$ is normal with rate of convergence of T^{1-H} and variance σ^2/κ^2. This asymptotic distribution is the same as that of $\hat{\mu}_{LS}$ and $\hat{\mu}_{HN}$ when $\kappa > 0$, as shown in (3.8) and (3.9).

3.3. Asymptotic theory when $\kappa = 0$

When $\kappa = 0$, the fVm is null recurrent. In this case, we have

$$X_t = X_0 + \sigma B_t^H,$$

and the parameter μ vanishes. By a simple calculation, we have

$$\hat{\kappa}_{LS} = \frac{\sigma B_T^H \int_0^T \left(X_0 + \sigma B_t^H\right) dt - T\sigma \int_0^T \left(X_0 + \sigma B_t^H\right) dB_t^H}{T \int_0^T \left(X_0 + \sigma B_t^H\right)^2 dt - \left(\int_0^T \left(X_0 + \sigma B_t^H\right) dt\right)^2} \tag{3.18}$$
$$= \frac{B_T^H \int_0^T B_t^H dt - T \int_0^T B_t^H dB_t^H}{T \int_0^T (B_t^H)^2 dt - \left(\int_0^T B_t^H dt\right)^2}.$$

On the one hand, if we interpret $\int_0^T B_t^H dB_t^H$ as the Itô–Skorohod integral, we can rewrite (3.18) as

$$\hat{\kappa}_{LS} = \frac{B_T^H \int_0^T B_t^H dt - \frac{T}{2}\left(\left(B_T^H\right)^2 - T^{2H}\right)}{T \int_0^T (B_t^H)^2 dt - \left(\int_0^T B_t^H dt\right)^2}. \tag{3.19}$$

On the other hand, if we interpret $\int_0^T B_t^H dB_t^H$ as a Young integral, we can also rewrite (3.18) as

$$\hat{\kappa}_{LS} = \frac{B_T^H \int_0^T B_t^H dt - \frac{T}{2}\left(B_T^H\right)^2}{T \int_0^T (B_t^H)^2 dt - \left(\int_0^T B_t^H dt\right)^2}. \tag{3.20}$$

Using the law of the iterated logarithm for fBm (see, for example, Taqqu, 1977) and the scaling properties of fBm (see, for example, Nualart, 2006), we develop the following strong consistency and asymptotic distribution for $\hat{\kappa}_{LS}$.

THEOREM 3.6. *Let $H \in [1/2, 1)$, $X_0 = O_p(1)$, and $\kappa = 0$ in* (1.1). Then, as $T \to \infty$, $\hat{\kappa}_{LS} \xrightarrow{a.s.} 0$. Moreover,

$$T\hat{\kappa}_{LS} \stackrel{d}{=} -\frac{\int_0^1 \overline{B}_u^H dB_u^H}{\int_0^1 \left(\overline{B}_u^H\right)^2 du}, \tag{3.21}$$

where $\overline{B}_u^H = B_u^H - \int_0^1 B_t^H dt$.

Remark 3.13. Interestingly, for $\hat{\kappa}_{LS}$ to consistently estimate κ, the stochastic integral $\int_0^T B_t^H dB_t^H$ can be interpreted as either an Itô–Skorohod integral or a Young integral.

Remark 3.14. This limiting distribution is neither a normal variate nor a mixture of normals. In addition, the distribution depends on H. If $H = 1/2$, the limiting distribution becomes a Dickey–Fuller–Phillips type of distribution (see, e.g., Phillips and Perron, 1988) which has been widely used for testing unit roots in autoregression with an intercept included. Tanaka (2013) derived the limiting distribution of the LS estimator of κ in the fOU model when $\kappa = 0$. His limiting distribution is another Dickey–Fuller–Phillips type of distribution (see, e.g., Phillips, 1987) and corresponds to that in autoregression without intercept.

4. CONCLUDING REMARKS AND FUTURE DIRECTIONS

Models with long-range dependence are growing in popularity due to their empirical success in practice. In the continuous-time setting, long-range dependence can be modeled with the help of an fBM when the Hurst parameter is greater than one half. Consequently, statistical inference for stochastic models driven by an fBM is important. This article considers the Vasicek model driven by an fBM and deals with the estimation problem of the two parameters in the drift function in the fVm and their asymptotic theory when a continuous record of observations is available.

As the time span goes to infinity, it is shown that the LS estimators of μ and κ are strongly consistent regardless of the sign of the persistence parameter κ. Moreover, the asymptotic distribution of the LS estimator of μ is asymptotically normal regardless of the sign of κ. However, the asymptotic distribution of the LS estimator of κ critically depends on the sign of κ. In particular, when $\kappa > 0$ and $H \in [1/2, 3/4)$, we have shown that the asymptotic distribution of the LS estimator of κ is normal with rate of convergence of $\sqrt{T}$. The asymptotic variance depends on H which monotonically increases in H. Moreover, when $\kappa > 0$ and $H = 3/4$, we have shown that the asymptotic distribution of the LS estimator of κ is also normal with rate of convergence of $\sqrt{T/\log(T)}$. However, a noncentral limit theorem for the LS estimator of κ is established for $H \in (3/4, 1)$. In this situation, we have established the asymptotic law as a Rosenblatt random variable. When $\kappa < 0$, it is shown that the limiting distribution is a Cauchy-type with rate of convergence of $e^{-\kappa T}$. If μ is the same as the initial condition, it becomes the standard Cauchy distribution. When $\kappa = 0$, the asymptotic distribution is neither normal nor a mixture of normals, but a Dickey–Fuller–Phillips type of distribution. The rate of convergence is T. In addition, we have considered an alternative estimation technique by exploiting the ergodic property of fVm when $\kappa > 0$. Borrowing the idea of Hu and Nualart (2010), we have studied the asymptotic properties of the ergodic type estimators. The asymptotic properties of these two alternative estimators are compared.

This study also suggests several important directions for future research. First, what are the asymptotic properties of the ML estimators for κ and μ? Given that the model is fully parametrically specified, one may wish to estimate the fVm using ML. Based on the fractional version of Girsanov's theorem, one can obtain the Radon–Nikodym derivative and the likelihood ratio function. Consequently, the ML estimators can be obtained. The asymptotic properties of ML estimators can be derived by using the Laplace transform and the properties of deterministic fractional operators determined by the Hurst parameter.

Second, the present study assumes that a continuous record is available for parameter estimation. This assumption is too strong in almost all empirically relevant cases. How to estimate parameters in an fVm from discrete-time observations and how to obtain the asymptotic theory are open questions. In fact, we can approximate an fVm by the Euler approximation and appeal to an in-fill asymptotic scheme. In this case, however, it is not clear how to get an explicit approximation for the increment of an fBM. To overcome this obstacle, we may replace the increment of an fBM by a disturbed random walk. Consequently, we can obtain the corresponding LS estimators and consider their asymptotic properties under both the long-span and the in-fill asymptotic schemes.

Third, the asymptotic theory developed in this article is valid for a narrow range of values for $H \in [1/2, 1)$. This corresponds to $d \in [0, 1/2)$ in the discrete-time model (1.3). Existing empirical studies have fitted the discrete-time model $(1-L)^d y_t = u_t$ with u_t being a stationary and ergodic process to financial and macroeconomic time series. Most studies obtained the estimated d in the range of $[0, 1/2)$. However, some studies found the estimated d to be smaller than 0 or larger than 1 and this implies that $H \notin [1/2, 1)$. There is a clear need to extend the asymptotic theory to a wider range of values for H. Such an extension will be considered in a separate study.

Lastly, in this article H is assumed to be known. In practice, H is almost always unknown. How to estimate H with a continuous record of observations is an open question. One possibility for estimating H is to use the generalized quadratic variation introduced by Gradinaru and Nourdin (2006). For $T > 0$, $\beta > 0$, $\gamma > 0$ and $\beta \neq \gamma$, assume X_t is observed continuously over the interval $[0, T + \max(\beta, \gamma)]$. Motivated by Gradinaru and Nourdin (2006), we can estimate H by

$$\hat{H} = \frac{1}{2}\log(\beta/\gamma)\log\left(\int_0^T \left(X_{t+\gamma} - X_t\right)^2 dt \Big/ \int_0^T \left(X_{t+\beta} - X_t\right)^2 dt\right).$$

The asymptotic properties of this estimator will be reported in later work.

NOTE

1. The definition of the Malliavin derivative is given in A.3 of Appendix.

REFERENCES

Anderson, T.W. (1959) On asymptotic distributions of estimates of parameters of stochastic difference equations. *Annals of Mathematical Statistics* 30, 676–687.

Baillie, R.T. (1996) Long memory processes and fractional integration in econometrics. *Journal of Econometrics* 73, 5–59.

Baillie, R.T., T. Bollerslev, & H.O. Mikkelsen (1996) Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 74, 3–30.

Bayer, C., P. Friz, & J. Gatheral (2016) Pricing under rough volatility. *Quantitative Finance* 16, 887–904.

Belfadli, R., K. Es-Sebaiy, & Y. Ouknine (2011) Parameter estimation for fractional Ornstein–Uhlenbeck processes: Non-ergodic case. *Frontiers in Science and Engineering* (An International Journal Edited by Hassan II Academy of Science and Technology) 1, 1–16.

Biagini, F., Y. Hu, B. Øksendal, & T. Zhang (2008) *Stochastic Calculus for Fractional Brownian Motion and Applications*. Springer.

Breton, J.C. & I. Nourdin (2008) Error bounds on the non-normal approximation of Hermite power variations of fractional Brownian motion. *Electronic Communications in Probability* 13, 482–493.

Cheung, Y.W. (1993) Long memory in foreign-exchange rates. *Journal of Business and Economic Statistics* 11, 93–101.

Chronopoulou, A. & F.G. Viens (2012a) Estimation and pricing under long-memory stochastic volatility. *Annals of Finance* 8, 379–403.

Chronopoulou, A. & F.G. Viens (2012b) Stochastic volatility and option pricing with long-memory in discrete and continuous time. *Quantitative Finance* 12, 635–649.

Comte, F., L. Coutin, & E. Renault (2012) Affine fractional stochastic volatility models. *Annals of Finance* 8, 337–378.

Comte, F. & E. Renault (1996) Long memory continuous time models. *Journal of Econometrics* 73, 101–149.

Comte, F. & E. Renault (1998) Long memory in continuous-time stochastic volatility models. *Mathematical Finance* 8, 291–323.

Corlay, S., J. Lebovits, & J.L. Véhel (2014) Multifractional stochastic volatility models. *Mathematical Finance* 24, 364–402.

Davidson, J. & R.M. De Jong (2000) The functional central limit theorem and weak convergence to stochastic integrals II: Fractionally integrated processes. *Econometric Theory* 16, 643–666.

Davydov, Y.A. (1970) The invariance principle for stationary processes. *Theory of Probability and its Applications* 15, 487–498.

Duncan, T., Y. Hu, & B. Pasik-Duncan (2000) Stochastic calculus for fractional Brownian motion I: Theory. *SIAM Journal on Control and Optimization* 38, 582–612.

El Machkouri, M., K. Es-Sebaiy, & Y. Ouknine (2016) Least squares estimator for non-ergodic Ornstein–Uhlenbeck processes driven by Gaussian processes. *Journal of the Korean Statistical Society* 45, 329–341.

Feigin, P.D. (1976) Maximum likelihood estimation for continuous-time stochastic processes. *Advances in Applied Probability* 8, 712–736.

Geweke, J. & S. Porter-Hudak (1983) The estimation and application of long memory time series models. *Journal of Time Series Analysis* 4, 221–238.

Gradinaru, M. & I. Nourdin (2006) Approximation at first and second order of m-order integrals of the fractional Brownian motion and of certain semimartingales. *Electronic Journal of Probability* 8, 1–26.

Granger, C.W. & N. Hyung (2004) Occasional structural breaks and long memory with an application to the S&P 500 absolute stock returns. *Journal of Empirical Finance* 11, 399–421.

Hu, Y. & D. Nualart (2010) Parameter estimation for fractional Ornstein–Uhlenbeck processes (with the online supplement for Section 5). *Statistics and Probability Letters* 80, 1030–1038.

Hu, Y., D. Nualart, & H. Zhou (2018) Parameter estimation for fractional Ornstein–Uhlenbeck processes of general Hurst parameter. *Statistical Inference for Stochastic Processes*, forthcoming. doi.org/10.1007/s11203-017-9168-2.

Isserlis, L. (1918) On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika* 12, 134–139.

Kleptsyna, M. & A. Le Breton (2002) Statistical analysis of the fractional Ornstein–Uhlenbeck type process. *Statistical Inference for Stochastic Processes* 5, 229–248.

Kloeden, P. & A. Neuenkirch (2007) The pathwise convergence of approximation schemes for stochastic differential equations. *LMS Journal of Computation and Mathematics* 10, 235–253.

Lo, A.W. (1991) Long-term memory in stock market prices. *Econometrica* 59, 1279–1313.

Mandelbrot, B.B. & J.W. Van Ness (1968) Fractional Brownian motions, fractional noises and applications. *Society for Industrial and Applied Mathematics Review* 10, 422–437.

Mishura, Y. (2008) *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. Springer.

Nualart, D. (2006) *The Malliavin Calculus and Related Topics*, 2nd ed. Springer.

Phillips, P.C.B. (1987) Time series regression with a unit root. *Econometrica* 55, 277–301.

Phillips, P.C.B. & T. Magdalinos (2007) Limit theory for moderate deviations from a unit root. *Journal of Econometrics* 136, 115–130.

Phillips, P.C.B. & P. Perron (1988) Testing for a unit root in time series regression. *Biometrika* 75, 335–346.

Prakasa Rao, B.L.S. (2010) *Statistical Inference for Fractional Diffusion Processes*. Wiley, Chichester.

Robinson, P. (1995a) Log-periodogram regression of time series with long-range dependence. *Annals of Statistics* 23, 1048–1072.

Robinson, P. (1995b) Gaussian semiparametric estimation of long-range dependence. *Annals of Statistics* 23, 1630–1661.

Shimotsu, K. & P.C.B. Phillips (2005) Exact local Whittle estimation for fractional integration. *Annals of Statistics* 33, 1890–1933.

Sowell, F. (1990) The fractional unit root distribution. *Econometrica* 58, 495–505.

Tanaka, K. (2013) Distributions of the maximum likelihood and minimum contrast estimators associated with the fractional Ornstein–Uhlenbeck process. *Statistical Inference for Stochastic Processes* 16, 173–192.

Tanaka, K. (2014) Distributions of quadratic functionals of the fractional Brownian Motion based on a martingale approximation. *Econometric Theory* 30, 1078–1109.

Tanaka, K. (2015) Maximum likelihood estimation for the non-ergodic fractional Ornstein–Uhlenbeck process. *Statistical Inference for Stochastic Processes* 18, 315–332.

Taqqu, M.S. (1977) Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence. *Probability Theory and Related Fields* 40, 203–238.

Tudor, C. & F. Viens (2007) Statistical aspects of the fractional stochastic calculus. *Annals of Statistics* 35, 1183–1212.

Vasicek, O. (1977) An equilibrium characterization of the term structure. *Journal of Financial Economics* 5, 177–188.

Wang, X. & J. Yu (2015) Limit theory for an explosive autoregressive process. *Economics Letters* 126, 176–180.

Wang, X. & J. Yu (2016) Double asymptotics for explosive continuous time models. *Journal of Econometrics* 193, 35–53.

White, J.S. (1958) The limiting distribution of the serial correlation coefficient in the explosive case. *Annals of Mathematical Statistics* 29, 1188–1197.

Young, L.C. (1936) An inequality of the Hölder type, connected with Stieltjes integration. *Acta Mathematica* 67, 251–282.

APPENDIX

A.1. Proof of Theorem 3.1

We first consider strong consistency of $\hat{\mu}_{HN}$. The solution of (1.1) is

$$X_t = \left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dB_s^H. \tag{A.1}$$

For $t \geq 0$, we define

$$Y_t = \sigma \int_{-\infty}^t e^{-\kappa(t-s)} dB_s^H. \tag{A.2}$$

Since $\kappa > 0$, $(Y_t, t \geq 0)$ is Gaussian, stationary, and ergodic, using the ergodic theorem and the fact $\mathbb{E}\left[Y_0\right] = 0$, we obtain

$$\frac{1}{T}\int_0^T Y_t dt \overset{a.s.}{\to} \mathbb{E}\left(Y_0\right) = 0. \tag{A.3}$$

Combining (A.1) and (A.2), we can rewrite Y_t as,

$$Y_t = X_t + \left(e^{-\kappa t}-1\right)\mu - X_0 e^{-\kappa t} + \sigma \int_{-\infty}^0 e^{-\kappa(t-s)} dB_s^H. \tag{A.4}$$

Hence,

$$\begin{aligned}\frac{1}{T}\int_0^T Y_t dt &= \frac{1}{T}\int_0^T \left[X_t + \mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t} + e^{-\kappa t}\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right] dt \\ &= \frac{1}{T}\int_0^T X_t dt + \frac{\mu}{T}\int_0^T \left(e^{-\kappa t}-1\right) dt - \frac{X_0}{T}\int_0^T e^{-\kappa t} dt \\ &\quad + \frac{\sigma}{T}\int_0^T e^{-\kappa t}\left(\int_{-\infty}^0 e^{\kappa s} dB_s^H\right) dt.\end{aligned} \tag{A.5}$$

For the second term in (A.5), it is obvious that

$$\frac{\mu}{T}\int_0^T \left(e^{-\kappa t}-1\right) dt \to -\mu.$$

Based on the assumption $X_0/\sqrt{T} = o_{a.s.}(1)$, we obtain

$$\frac{X_0}{T}\int_0^T e^{-\kappa t} dt \overset{a.s.}{\to} 0.$$

Using an argument similar to that in Lemma 5.1 of Hu and Nualart (2010), we have

$$\mathbb{E}\left[\int_{-\infty}^0 e^{\kappa s} dB_s^H\right]^2 = \kappa^{-2H} H\Gamma(2H). \tag{A.6}$$

Hence, $\int_0^T e^{-\kappa(T-s)} dB_s^H$ has the limiting (normal) distribution of $\int_{-\infty}^0 e^{\kappa s} dB_s^H$. Moreover, a standard calculation yields

$$\int_0^T e^{-\kappa t} dt \to \frac{1}{\kappa}. \tag{A.7}$$

It is now necessary to investigate the asymptotic behavior of the last term in (A.5). Denote $F_T = \frac{\sigma}{\sqrt{T}} \int_0^T e^{-\kappa t} \left(\int_{-\infty}^0 e^{\kappa s} dB_s^H \right) dt$. From (A.6) and (A.7), we see that $\sup_T \mathbb{E}\left[\left|F_T^2\right|\right] < \infty$ and $\sup_T \mathbb{E}\left[\left|F_T^4\right|\right] < \infty$. For any fixed $\varepsilon > 0$, it follows from Chebyshev's inequality that

$$\mathbb{P}\left(\left| \frac{\sigma}{T} \int_0^T e^{-\kappa t} \left(\int_{-\infty}^0 e^{\kappa s} dB_s^H \right) dt \right| > \varepsilon \right) = \mathbb{P}\left(|F_T| > \sqrt{T}\varepsilon \right) \le \frac{81}{T^2 \varepsilon^4} \mathbb{E}\left[\left|F_T^2\right|\right]^2.$$

Then, the Borel–Cantelli lemma implies that

$$\frac{\sigma}{T} \int_0^T e^{-\kappa t} \left(\int_{-\infty}^0 e^{\kappa s} dB_s^H \right) dt \overset{a.s.}{\to} 0. \tag{A.8}$$

Plugging all these convergency results to (A.5), we obtain

$$\hat{\mu}_{HN} = \frac{1}{T} \int_0^T X_t dt \overset{a.s.}{\to} \mu. \tag{A.9}$$

To establish strong consistency of $\hat{\kappa}_{HN}$ defined in (2.10), we need to consider strong consistency of $\frac{1}{T} \int_0^T X_t^2 dt$. From the expression of Y_t in (A.4), we obtain

$$\begin{aligned}
\frac{1}{T} \int_0^T Y_t^2 dt &= \frac{1}{T} \int_0^T \left[X_t + \mu\left(e^{-\kappa t} - 1\right) - X_0 e^{-\kappa t} + e^{-\kappa t} \left(\sigma \int_{-\infty}^0 e^{\kappa s} dB_s^H \right) \right]^2 dt \qquad \text{(A.10)} \\
&= \frac{1}{T} \int_0^T \left[X_t + \mu\left(e^{-\kappa t} - 1\right) - X_0 e^{-\kappa t} \right]^2 dt + \frac{1}{T} \int_0^T \left[e^{-\kappa t} \left(\sigma \int_{-\infty}^0 e^{\kappa s} dB_s^H \right) \right]^2 dt \\
&\quad + \frac{2}{T} \int_0^T \left[X_t + \mu\left(e^{-\kappa t} - 1\right) - X_0 e^{-\kappa t} \right] \left[e^{-\kappa t} \left(\sigma \int_{-\infty}^0 e^{\kappa s} dB_s^H \right) \right] dt \\
&= \frac{1}{T} \int_0^T \left[\mu\left(e^{-\kappa t} - 1\right) - X_0 e^{-\kappa t} \right]^2 dt + \frac{2}{T} \int_0^T X_t \left[\mu\left(e^{-\kappa t} - 1\right) - X_0 e^{-\kappa t} \right] dt \\
&\quad + \frac{1}{T} \int_0^T X_t^2 dt + \frac{1}{T} \int_0^T \left[e^{-\kappa t} \left(\sigma \int_{-\infty}^0 e^{\kappa s} dB_s^H \right) \right]^2 dt \\
&\quad + \frac{2}{T} \int_0^T \left[X_t + \mu\left(e^{-\kappa t} - 1\right) - X_0 e^{-\kappa t} \right] \left[e^{-\kappa t} \left(\sigma \int_{-\infty}^0 e^{\kappa s} dB_s^H \right) \right] dt.
\end{aligned}$$

By (A.8) and Lemma 3.3 in Hu and Nualart (2010), it is not hard to see that

$$\frac{\sigma^2}{T} \int_0^T \left[\int_0^t e^{-\kappa(t-s)} dB_s^H + e^{-\kappa t} \left(\int_{-\infty}^0 e^{\kappa s} dB_s^H \right) \right]^2 dt - \frac{\sigma^2}{T} \int_0^T \left[\int_0^t e^{-\kappa(t-s)} dB_s^H \right]^2 dt \overset{a.s.}{\to} 0.$$

Combining the above result and (A.8), we deduce that

$$\frac{2}{T} \int_0^T \left[\sigma \int_0^t e^{-\kappa(t-s)} dB_s^H \right] \left[e^{-\kappa t} \left(\sigma \int_{-\infty}^0 e^{\kappa s} dB_s^H \right) \right] dt \overset{a.s.}{\to} 0.$$

Using (A.1) and the result above, we obtain

$$\frac{2}{T}\int_0^T \left[X_t+\mu\left(e^{-\kappa t}-1\right)-X_0e^{-\kappa t}\right]\left[e^{-\kappa t}\left(\sigma\int_{-\infty}^0 e^{\kappa s}dB_s^H\right)\right]dt \overset{a.s.}{\to} 0. \tag{A.11}$$

A standard calculation yields

$$\frac{2}{T}\int_0^T X_t\left[\mu\left(e^{-\kappa t}-1\right)-X_0e^{-\kappa t}\right]dt \overset{a.s.}{\to} -2\mu^2, \tag{A.12}$$

$$\frac{1}{T}\int_0^T \left[\mu\left(e^{-\kappa t}-1\right)-X_0e^{-\kappa t}\right]^2 dt \overset{a.s.}{\to} \mu^2. \tag{A.13}$$

By (A.10)–(A.13) and the ergodic theorem, we obtain

$$\frac{1}{T}\int_0^T X_t^2 dt \overset{a.s.}{\to} \mathbb{E}\left(Y_0^2\right)+\mu^2. \tag{A.14}$$

Moreover, it is well-known that (see, e.g., Lemma 5.1 of Hu and Nualart, 2010)

$$\mathbb{E}(Y_0^2)=\alpha_H\sigma^2\int_0^\infty\int_0^\infty e^{-\kappa(s+u)}|u-s|^{2H-2}duds=\sigma^2\kappa^{-2H}H\Gamma(2H). \tag{A.15}$$

Combining (A.14) and (A.15), we have

$$\frac{1}{T}\int_0^T X_t^2 dt \overset{a.s.}{\to} \sigma^2\kappa^{-2H}H\Gamma(2H)+\mu^2. \tag{A.16}$$

By (A.9), (A.16), and the arithmetic rule of convergence, we obtain strong convergence of $\hat{\kappa}_{HN}$, i.e., $\hat{\kappa}_{HN}\overset{a.s.}{\to}\kappa$.

A.2. Proof of Theorem 3.2

First, we consider (3.4). Based on (2.5), (1.1), and (A.1), we can rewrite $\hat{\kappa}_{LS}$ as

$$\begin{aligned}
\hat{\kappa}_{LS} &= \frac{(X_T-X_0)\int_0^T X_t dt-\kappa\mu T\int_0^T X_t dt+\kappa T\int_0^T X_t^2 dt-\sigma T\int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt-\left(\int_0^T X_t dt\right)^2}\\
&=\kappa+\frac{(X_T-X_0)\int_0^T X_t dt-\kappa\mu T\int_0^T X_t dt-\sigma T\int_0^T X_t dB_t^H+\kappa\left(\int_0^T X_t dt\right)^2}{T\int_0^T X_t^2 dt-\left(\int_0^T X_t dt\right)^2}\\
&=\kappa-\frac{\sigma T\int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt-\left(\int_0^T X_t dt\right)^2}+\frac{\left(X_T-X_0-\kappa\mu T+\kappa\int_0^T X_t dt\right)\int_0^T X_t dt}{T\int_0^T X_t^2 dt-\left(\int_0^T X_t dt\right)^2}\\
&=\kappa-\frac{\sigma T\int_0^T\left(\left(1-e^{-\kappa t}\right)\mu+X_0e^{-\kappa t}+\sigma\int_0^t e^{-\kappa(t-s)}dB_s^H\right)dB_t^H}{T\int_0^T X_t^2 dt-\left(\int_0^T X_t dt\right)^2}\\
&\quad+\frac{\left(X_T-X_0+\kappa\int_0^T\left(X_0e^{-\kappa t}-\mu e^{-\kappa t}+\sigma\int_0^t e^{-\kappa(t-s)}dB_s^H\right)dt\right)\int_0^T X_t dt}{T\int_0^T X_t^2 dt-\left(\int_0^T X_t dt\right)^2}.
\end{aligned}$$

Thus, we have the following decomposition

$$\sqrt{T}\left(\hat{\kappa}_{LS}-\kappa\right) \tag{A.17}$$

$$= -\frac{\sigma\left(\mu\frac{B_T^H}{\sqrt{T}}+\frac{X_0-\mu}{\sqrt{T}}\int_0^T e^{-\kappa t}dB_t^H+\frac{\sigma}{\sqrt{T}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H\right)}{\frac{1}{T}\int_0^T X_t^2 dt-\left(\frac{1}{T}\int_0^T X_t dt\right)^2}$$

$$+\frac{\left(\frac{X_T-X_0}{\sqrt{T}}+\frac{\kappa(X_0-\mu)}{\sqrt{T}}\int_0^T e^{-\kappa t}dt-\frac{\sigma}{\sqrt{T}}e^{-\kappa T}\int_0^T e^{\kappa s}dB_s^H+\sigma\frac{B_T^H}{\sqrt{T}}\right)\frac{1}{T}\int_0^T X_t dt}{\frac{1}{T}\int_0^T X_t^2 dt-\left(\frac{1}{T}\int_0^T X_t dt\right)^2}$$

$$:= I_1+I_2+I_3\,,$$

where

$$I_1=\frac{\sigma\left(\frac{\mu-X_0}{\sqrt{T}}\int_0^T e^{-\kappa t}dB_t^H-\frac{\sigma}{\sqrt{T}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H\right)}{\frac{1}{T}\int_0^T X_t^2 dt-\left(\frac{1}{T}\int_0^T X_t dt\right)^2},$$

$$I_2=\frac{\left(\frac{X_T-X_0}{\sqrt{T}}+\frac{\kappa(X_0-\mu)}{\sqrt{T}}\int_0^T e^{-\kappa t}dt-\frac{\sigma}{\sqrt{T}}e^{-\kappa T}\int_0^T e^{\kappa s}dB_s^H\right)\frac{1}{T}\int_0^T X_t dt}{\frac{1}{T}\int_0^T X_t^2 dt-\left(\frac{1}{T}\int_0^T X_t dt\right)^2},$$

$$I_3=\frac{\left(-\mu\sigma+\frac{\sigma}{T}\int_0^T X_t dt\right)\frac{B_T^H}{\sqrt{T}}}{\frac{1}{T}\int_0^T X_t^2 dt-\left(\frac{1}{T}\int_0^T X_t dt\right)^2}.$$

We consider I_1 first. Using (A.15), we have

$$\mathbb{E}\left[\left(\frac{\mu\sigma}{\sqrt{T}}\int_0^T e^{-\kappa t}dB_t^H\right)^2\right]=\frac{\mu^2\sigma^2}{T}\alpha_H\int_0^T\int_0^T e^{-\kappa(s+u)}|u-s|^{2H-2}duds\to 0.$$

This implies

$$\frac{\mu\sigma}{\sqrt{T}}\int_0^T e^{-\kappa t}dB_t^H\xrightarrow{p}0. \tag{A.18}$$

Since $X_0=o_{a.s.}(\sqrt{T})$, we have

$$\frac{X_0\sigma}{\sqrt{T}}\int_0^T e^{-\kappa t}dB_t^H\xrightarrow{p}0. \tag{A.19}$$

Furthermore, from Theorem 3.4 of Hu and Nualart (2010), (A.9) and (A.16), we obtain

$$\frac{-\frac{\sigma^2}{\sqrt{T}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H}{\frac{1}{T}\int_0^T X_t^2 dt-\left(\frac{1}{T}\int_0^T X_t dt\right)^2}\xrightarrow{\mathcal{L}}\mathcal{N}(0,\kappa C_H)\,, \tag{A.20}$$

where $C_H = (4H-1)\left(1+\frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2H)\Gamma(2-2H)}\right)$. Combining (A.18), (A.19), and (A.20) and applying Slutsky's theorem, we have

$$I_1 \xrightarrow{\mathcal{L}} \mathcal{N}(0, \kappa C_H). \tag{A.21}$$

Next, we consider I_2. From Lemma 5.2 and equation (3.8) in Hu and Nualart (2010), we have

$$\frac{X_T}{\sqrt{T}} \xrightarrow{a.s.} 0, \quad \frac{X_0}{\sqrt{T}} \xrightarrow{p} 0, \quad \frac{\sigma}{\sqrt{T}} e^{-\kappa T}\left(\int_0^T \sigma e^{\kappa s} dB_s^H\right) \xrightarrow{a.s.} 0. \tag{A.22}$$

A straightforward calculation shows that

$$\frac{\kappa(X_0-\mu)}{\sqrt{T}} \int_0^T e^{-\kappa t} dt \xrightarrow{p} 0. \tag{A.23}$$

Combining (A.22), (A.23), (A.9), and (A.16), we have

$$I_2 \xrightarrow{p} 0. \tag{A.24}$$

Finally, we consider I_3. Based on (A.1), we have

$$\begin{aligned}
&\left(-\mu\sigma + \frac{\sigma}{T}\int_0^T X_t dt\right)\frac{B_T^H}{\sqrt{T}} \qquad \text{(A.25)}\\
&= \frac{\sigma}{T}\int_0^T \left((X_0-\mu)e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dB_s^H\right) dt \frac{B_T^H}{\sqrt{T}}\\
&= \left(\frac{\sigma(X_0-\mu)}{T^{\frac{3}{2}-H}}\int_0^T e^{-\kappa t} dt - \frac{\sigma^2}{\kappa T^{\frac{3}{2}-H}} e^{-\kappa T}\int_0^T e^{\kappa s} dB_s^H + \frac{\sigma^2}{\kappa}\frac{B_T^H}{T^{\frac{3}{2}-H}}\right)\frac{B_T^H}{T^H}.
\end{aligned}$$

It is easy to see that

$$\frac{\sigma(X_0-\mu)}{T^{\frac{3}{2}-H}}\int_0^T e^{-\kappa t} dt \xrightarrow{a.s.} 0. \tag{A.26}$$

From Lemma 5.2 and equation (3.8) in Hu and Nualart (2010), we obtain

$$\frac{\sigma^2}{\kappa T^{\frac{3}{2}-H}} e^{-\kappa T}\int_0^T e^{\kappa s} dB_s^H \xrightarrow{a.s.} 0. \tag{A.27}$$

Since $H \in [1/2, 3/4)$, we have

$$\mathbb{E}\left[\left(\frac{\sigma^2}{\kappa}\frac{B_T^H}{T^{\frac{3}{2}-H}}\right)^2\right] = \frac{\sigma^4}{\kappa^2} T^{4H-3},$$

which implies

$$\frac{\sigma^2}{\kappa}\frac{B_T^H}{T^{\frac{3}{2}-H}} \xrightarrow{p} 0. \tag{A.28}$$

By (A.25)–(A.28), we obtain

$$I_3 \xrightarrow{p} 0. \tag{A.29}$$

By (A.17), (A.21), (A.24), (A.29), and Slutsky's theorem, we obtain the desired result in (3.4).

Next, we deal with (3.5). Using an argument similar to (A.17), we have

$$\frac{\sqrt{T}}{\sqrt{\log(T)}}\left(\hat{\kappa}_{LS}-\kappa\right) := J_1 + J_2 + J_3, \tag{A.30}$$

where

$$J_1 = \frac{\sigma\left(\frac{\mu - X_0}{\sqrt{T\log(T)}}\int_0^T e^{-\kappa t}dB_t^H - \frac{\sigma}{\sqrt{T\log(T)}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H\right)}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2},$$

$$J_2 = \frac{\left(\frac{X_T - X_0}{\sqrt{T\log(T)}} + \frac{\kappa(X_0-\mu)}{\sqrt{T\log(T)}}\int_0^T e^{-\kappa t}dt - \frac{\sigma}{\sqrt{T\log(T)}}e^{-\kappa T}\int_0^T e^{\kappa s}dB_s^H\right)\frac{1}{T}\int_0^T X_t dt}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2},$$

$$J_3 = \frac{\left(-\mu\sigma + \frac{\sigma}{T}\int_0^T X_t dt\right)\frac{B_T^H}{\sqrt{T\log(T)}}}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}.$$

When $H = 3/4$, from Theorem 5 of Hu, Nualart, and Zhou (2018), (A.9), and (A.16), we obtain

$$\frac{-\frac{\sigma^2}{\sqrt{T\log(T)}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{4\kappa}{\pi}\right). \tag{A.31}$$

Combining (A.18), (A.19), and (A.31) and applying Slutsky's theorem, we have

$$J_1 \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{4\kappa}{\pi}\right). \tag{A.32}$$

Using arguments similar to I_2 and I_3, we can easily obtain

$$J_2 \xrightarrow{p} 0, \tag{A.33}$$

$$J_3 \xrightarrow{p} 0. \tag{A.34}$$

By (A.30), (A.32), (A.33), (A.34), and Slutsky's theorem, we obtain the desired result in (3.5).

Finally, we are left with (3.6) for $3/4 < H < 1$. By an argument similar to (A.17), we get

$$T^{2-2H}\left(\hat{\kappa}_{LS}-\kappa\right) := K_1 + K_2 + K_3, \tag{A.35}$$

where

$$K_1 = \frac{\sigma\left(\frac{\mu - X_0}{T^{2H-1}}\int_0^T e^{-\kappa t} dB_t^H - \frac{\sigma}{T^{2H-1}}\int_0^T \int_0^t e^{-\kappa(t-s)} dB_s^H dB_t^H\right)}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2},$$

$$K_2 = \frac{\left(\frac{X_T - X_0}{T^{2H-1}} + \frac{\kappa(X_0-\mu)}{T^{2H-1}}\int_0^T e^{-\kappa t} dt - \frac{\sigma}{T^{2H-1}} e^{-\kappa T}\int_0^T e^{\kappa s} dB_s^H\right)\frac{1}{T}\int_0^T X_t dt}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2},$$

$$K_3 = \frac{\left(-\mu\sigma + \frac{\sigma}{T}\int_0^T X_t dt\right)\frac{B_T^H}{T^{2H-1}}}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}.$$

For $H > 3/4$, from Theorem 5 of Hu et al. (2018), (A.9), and (A.16), we get

$$\frac{-\frac{\sigma^2}{T^{2H-1}}\int_0^T \int_0^t e^{-\kappa(t-s)} dB_s^H dB_t^H}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2} \xrightarrow{\mathcal{L}} \frac{2R(H)}{\kappa}, \tag{A.36}$$

where $R(H)$ is the Rosenblatt random variable defined in (3.7).

Using the fact $2H - 1 > 1/2$ for $H > 3/4$, Slutsky's theorem and arguments similar to (A.18) and (A.19), we have

$$K_1 \xrightarrow{\mathcal{L}} \frac{2R(H)}{\kappa}. \tag{A.37}$$

Using the fact $2H - 1 > 1/2$ for $H > 3/4$ and applying arguments similar to I_2 and I_3, we can easily obtain

$$K_2 \xrightarrow{p} 0, \tag{A.38}$$

$$K_3 \xrightarrow{p} 0. \tag{A.39}$$

By (A.35), (A.37), (A.38), (A.39), and Slutsky's theorem, we obtain the desired result of (3.6).

A.3. Proof of Theorem 3.3

We first consider the asymptotic distribution of $\hat{\mu}_{HN}$. Using (A.1), we obtain

$$\begin{aligned} & T^{1-H}\left(\frac{1}{T}\int_0^T X_t dt - \mu\right) \\ &= T^{1-H}\left[\frac{1}{T}\int_0^T \left((X_0 - \mu)e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)} dB_s^H\right) dt\right] \\ &= \frac{X_0 - \mu}{T^H}\int_0^T e^{-\kappa t} dt + \frac{\sigma}{T^H}\int_0^T \int_0^t e^{-\kappa(t-s)} dB_s^H dt. \end{aligned} \tag{A.40}$$

A simple calculation yields

$$\frac{X_0 - \mu}{T^H}\int_0^T e^{-\kappa t} dt \xrightarrow{a.s.} 0. \tag{A.41}$$

Moreover, a standard calculation together with Fubini's stochastic theorem (see, e.g., Nualart, 2006) yields

$$\begin{aligned}\frac{\sigma}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dt &= \frac{\sigma}{T^H}\int_0^T e^{\kappa s}\int_s^T e^{-\kappa t}dt dB_s^H \\ &= -\frac{\sigma}{\kappa T^H}\int_0^T e^{-\kappa(T-s)}dB_s^H + \frac{\sigma B_T^H}{\kappa T^H}.\end{aligned} \tag{A.42}$$

From equation (3.8) of Hu and Nualart (2010), we know that

$$\frac{\sigma}{\kappa T^H}\int_0^T e^{-\kappa(T-s)}dB_s^H \xrightarrow{a.s.} 0. \tag{A.43}$$

By (A.42), (A.43), and Slutsky's theorem, we have

$$\frac{\sigma}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dt \xrightarrow{\mathcal{L}} \mathcal{N}(0,\frac{\sigma^2}{\kappa^2}). \tag{A.44}$$

Combining (A.40), (A.41), and (A.44) and by Slutsky's theorem, we obtain

$$T^{1-H}\left(\frac{1}{T}\int_0^T X_t dt-\mu\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\frac{\sigma^2}{\kappa^2}). \tag{A.45}$$

Note that

$$T^{1-H}\left(\hat{\mu}_{HN}-\mu\right) = T^{1-H}\left(\frac{1}{T}\int_0^T X_t dt-\mu\right). \tag{A.46}$$

Using (2.9), (A.45), and (A.46), we obtain (3.9).

In what follows, we consider the asymptotic distribution of $\hat{\kappa}_{HN}$. First, we deal with (3.10) for $H \in [1/2, 3/4)$. We need to use a technique known as Malliavin calculus which we define now. For a time interval $[0,T]$, we denote by $\mathcal{H}$ the canonical Hilbert space associated to the fBm B^H. The construction and properties of $\mathcal{H}$ can be found in Nualart (2006). We use the following notation for Wiener integrals with respect to B^H:

$$B^H(\varphi) = \int_0^T \varphi(s)dB^H.$$

The Malliavin derivative D with respective to B^H, which is an $\mathcal{H}$-valued operator, is defined first by setting that

$$DB^H(\varphi) = \varphi,$$

for any $\varphi \in \mathcal{H}$. As a consequence, for a smooth and cylindrical random variable $F = f(x_1,\ldots,x_n) = f(B^H(\varphi_1),\ldots,B^H(\varphi_n))$, with any $\varphi_1,\ldots,\varphi_n \in \mathcal{H}$ and any $f \in C_b^\infty(\mathbb{R}^n,\mathbb{R})$ (infinitely differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ with bounded partial derivatives), we define its Malliavin derivative as the $\mathcal{H}$-valued random variable given by (see, equation (1.29) of Nualart, 2006)

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\varphi_1),\ldots,B^H(\varphi_n))\varphi_i.$$

Using (A.1) and applying the Malliavin calculus to X_t (see, equation (1.29) of Nualart, 2006), we have

$$D_s X_t = D_s \left[\left(1 - e^{-\kappa t}\right) \mu + X_0 e^{-\kappa t} + \sigma \int_0^t e^{-\kappa (t-s)} dB_s^H \right] = \sigma e^{-\kappa (t-s)} \mathbf{1}_{[0,t]}(s),$$

where $\mathbf{1}_{[\cdot]}$ is the indicator function. Consequently, we obtain

$$\int_0^T X_t dX_t = \frac{X_T^2 - X_0^2}{2} - \alpha_H \sigma^2 \int_0^T \int_0^t u^{2H-2} e^{-\kappa u} du dt. \tag{A.47}$$

Based on (2.5) and (2.10), we can rewrite $\hat{\kappa}_{HN}$ as

$$\begin{aligned} \hat{\kappa}_{HN} &= \left(\frac{T^2 \sigma^2 H \Gamma(2H)}{(X_T - X_0) \int_0^T X_t dt - T \int_0^T X_t dX_t} \right)^{\frac{1}{2H}} \\ &\quad \times \left(\frac{(X_T - X_0) \int_0^T X_t dt - T \int_0^T X_t dX_t}{T \int_0^T X_t^2 dt - \left(\int_0^T X_t dt \right)^2} \right)^{\frac{1}{2H}} \\ &= \left(\frac{T^2 \sigma^2 H \Gamma(2H)}{(X_T - X_0) \int_0^T X_t dt - T \int_0^T X_t dX_t} \right)^{\frac{1}{2H}} \hat{\kappa}_{LS}^{\frac{1}{2H}}. \end{aligned} \tag{A.48}$$

Substituting (A.47) into (A.48), we have

$$\begin{aligned} \hat{\kappa}_{HN} &= \left(\frac{T^2 \sigma^2 H \Gamma(2H)}{(X_T - X_0) \int_0^T X_t dt - T \left(\frac{1}{2} X_T^2 - \frac{1}{2} X_0^2 - \alpha_H \sigma^2 \int_0^T \int_0^t u^{2H-2} e^{-\kappa u} du dt \right)} \right)^{\frac{1}{2H}} \hat{\kappa}_{LS}^{\frac{1}{2H}} \\ &= \left(\frac{\sigma^2 H \Gamma(2H) \hat{\kappa}_{LS}}{\frac{X_T}{T} \frac{1}{T} \int_0^T X_t dt - \frac{X_0}{T} \frac{1}{T} \int_0^T X_t dt - \frac{1}{2T} X_T^2 + \frac{1}{2T} X_0^2 + \alpha_H \sigma^2 \frac{1}{T} \int_0^T \int_0^t u^{2H-2} e^{-\kappa u} du dt} \right)^{\frac{1}{2H}}. \end{aligned}$$

Hence,

$$\begin{aligned} &\sqrt{T} \left(\hat{\kappa}_{HN} - \kappa \right) \\ &= \sqrt{T} \left(\hat{\kappa}_{HN} - \kappa^{1-\frac{1}{2H}} \hat{\kappa}_{LS}^{\frac{1}{2H}} + \kappa^{1-\frac{1}{2H}} \hat{\kappa}_{LS}^{\frac{1}{2H}} - \kappa \right) \\ &= \sqrt{T} \left(\hat{\kappa}_{HN} - \kappa^{1-\frac{1}{2H}} \hat{\kappa}_{LS}^{\frac{1}{2H}} \right) + \sqrt{T} \kappa^{1-\frac{1}{2H}} \left(\hat{\kappa}_{LS}^{\frac{1}{2H}} - \kappa^{\frac{1}{2H}} \right) \\ &= \left[\left(\frac{\sigma^2 H \Gamma(2H)}{\frac{X_T}{T} \frac{1}{T} \int_0^T X_t dt - \frac{X_0}{T} \frac{1}{T} \int_0^T X_t dt - \frac{1}{2T} X_T^2 + \frac{1}{2T} X_0 + \alpha_H \sigma^2 \frac{1}{T} \int_0^T \int_0^t u^{2H-2} e^{-\kappa u} du dt} \right)^{\frac{1}{2H}} \right. \\ &\quad \left. - \kappa^{1-\frac{1}{2H}} \right] \sqrt{T} \hat{\kappa}_{LS}^{\frac{1}{2H}} + \sqrt{T} \kappa^{1-\frac{1}{2H}} \left(\hat{\kappa}_{LS}^{\frac{1}{2H}} - \kappa^{\frac{1}{2H}} \right). \end{aligned} \tag{A.49}$$

By Theorem 3.2 and the delta method, we get

$$\sqrt{T} \left(\hat{\kappa}_{LS}^{\frac{1}{2H}} - \kappa^{\frac{1}{2H}} \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left(0, \left(\frac{1}{2H} \kappa^{\frac{1-2H}{2H}} \right)^2 \kappa C_H \right). \tag{A.50}$$

By (A.9), equation (4.3) and Lemma 5.2 of Hu and Nualart (2010), we can obtain

$$\left(\frac{\sigma^2 H\Gamma(2H)}{\left(\frac{X_T}{T}-\frac{X_0}{T}\right)\frac{1}{T}\int_0^T X_t dt - \frac{1}{2T}X_T^2 + \frac{1}{2T}X_0^2 + \alpha_H \sigma^2 \frac{1}{T}\int_0^T \int_0^t u^{2H-2}e^{-\kappa u}dudt}\right)^{\frac{1}{2H}}$$
$$= \kappa^{1-\frac{1}{2H}} + o\left(\frac{1}{\sqrt{T}}\right). \qquad \textbf{(A.51)}$$

By Slutsky's theorem, Remark 3.2, (A.49), (A.50), and (A.51), we obtain the desired asymptotic distribution in (3.10).

Next, we consider (3.11) in the case of $H = 3/4$. Applying arguments similar to those in (A.49) and using (A.51), we have

$$\frac{\sqrt{T}}{\sqrt{\log(T)}}(\hat{\kappa}_{HN}-\kappa) = o\left(\frac{1}{\sqrt{T}}\right)\frac{\sqrt{T}}{\sqrt{\log(T)}}\hat{\kappa}_{LS}^{\frac{2}{3}} + \frac{\sqrt{T}}{\sqrt{\log(T)}}\kappa^{\frac{1}{3}}\left(\hat{\kappa}_{LS}^{\frac{2}{3}}-\kappa^{\frac{2}{3}}\right). \qquad \textbf{(A.52)}$$

For $H = 3/4$, using Theorem 3.2 and the delta method, we get

$$\frac{\sqrt{T}}{\sqrt{\log(T)}}\left(\hat{\kappa}_{LS}^{\frac{2}{3}}-\kappa^{\frac{2}{3}}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{16}{9\pi}\kappa^{\frac{1}{3}}\right). \qquad \textbf{(A.53)}$$

By Slutsky's theorem, (A.52), and (A.53), we obtain (3.11).

Finally, for $H \in (3/4, 1)$, similar arguments together with the delta method yield the asymptotic law for $\hat{\kappa}_{HN}$ in (3.12).

A.4. Proof of Lemma 3.1

Using (A.1), we obtain

$$\begin{aligned}
&\frac{e^{\kappa T}}{T^H}\int_0^T X_t dB_t^H \\
&= \frac{e^{\kappa T}}{T^H}\int_0^T \left[\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)}dB_s^H\right]dB_t^H \\
&= \frac{\mu e^{\kappa T}}{T^H}B_T^H + \frac{X_0-\mu}{T^H}e^{\kappa T}\int_0^T e^{-\kappa t}dB_t^H + \frac{\sigma e^{\kappa T}}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H. \qquad \textbf{(A.54)}
\end{aligned}$$

First, it is easy to see that

$$\frac{\mu e^{\kappa T}}{T^H}B_T^H \xrightarrow{a.s.} 0. \qquad \textbf{(A.55)}$$

For $H \in (1/2, 1)$, by Lemma 6 of Belfadli et al. (2011), we have

$$\frac{X_0-\mu}{T^H}e^{\kappa T}\int_0^T e^{-\kappa t}dB_t^H \xrightarrow{a.s.} 0. \qquad \textbf{(A.56)}$$

Let us mention that (A.56) also follows obviously when $H = 1/2$.

Next, we consider the last term of (A.54). If $H = 1/2$, a simple calculation yields

$$\mathbb{E}\left[\frac{\sigma e^{\kappa T}}{T^{\frac{1}{4}}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s dB_t\right]^2 = \frac{\sigma^2 e^{2\kappa T}}{T^{\frac{1}{2}}}\int_0^T\int_0^t e^{-2\kappa(t-s)}dsdt \tag{A.57}$$

$$= \frac{\sigma^2}{2\kappa}T^{\frac{1}{2}}e^{2\kappa T} + \frac{\sigma^2}{4\kappa^2\sqrt{T}} - \frac{\sigma^2 e^{2\kappa T}}{4\kappa^2\sqrt{T}}.$$

If $H \in (1/2, 1)$, by the isometry property of the double stochastic integral, we have

$$\mathbb{E}\left[\frac{\sigma e^{\kappa T}}{T^{\frac{H}{2}}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H\right]^2 = \sigma^2\alpha_H^2\frac{I_T}{e^{-2\kappa T}T^H},$$

where

$$I_T = \int_{[0,T]^4} e^{-\kappa|v-s|}e^{-\kappa|u-r|}|u-v|^{2H-2}|r-s|^{2H-2}dudvdrds.$$

Taking the derivative of I_T and $e^{-2\kappa T}T^H$ with respect to T, we have

$$\frac{dI_T}{d\left(e^{-2\kappa T}T^H\right)} = \frac{4\int_{[0,T]^3} e^{-\kappa(T-s)}e^{-\kappa|u-r|}(T-u)^{2H-2}|r-s|^{2H-2}dudrds}{HT^{H-1}e^{-2\kappa T} - 2\kappa T^H e^{-2\kappa T}}.$$

By changing variables $T - s = x_1, T - r = x_2, T - u = x_3$, we get

$$\frac{dI_T}{d\left(e^{-2\kappa T}T^H\right)} = \frac{4\int_{[0,T]^3} e^{-\kappa x_1}e^{-\kappa|x_2-x_3|}x_3^{2H-2}|x_1-x_2|^{2H-2}dx_1dx_2dx_3}{HT^{H-1}e^{-2\kappa T} - 2\kappa T^H e^{-2\kappa T}}.$$

Indeed, we can decompose the above integral into integrals over six disjoint regions $\{x_{\tau(1)} < x_{\tau(2)} < x_{\tau(3)}\}$, where τ runs over all permutations of indices $\{1,2,3\}$. In the case $x_1 < x_3 < x_2$, making the change of variables as $x_1 = a$, $x_3 - x_1 = b$ and $x_2 - x_3 = c$ (other cases can be handled in a similar way), we obtain

$$\frac{dI_T}{d\left(e^{-2\kappa T}T^H\right)} = \frac{4\int_{[0,T]^3} e^{-\kappa a}e^{-\kappa c}\,(a+b)^{2H-2}\,(b+c)^{2H-2}\,dadbdc}{HT^{H-1}e^{-2\kappa T} - 2\kappa T^H e^{-2\kappa T}}.$$

Thus,

$$\frac{dI_T}{d\left(e^{-2\kappa T}T^H\right)} \le \frac{4\int_{[0,T]^3} e^{-\kappa(a+c)}b^{4H-4}dadbdb}{HT^{H-1}e^{-2\kappa T} - 2\kappa T^H e^{-2\kappa T}}. \tag{A.58}$$

Then, from (A.57)–(A.58), we obtain

$$\left\|\frac{\sigma e^{\kappa T}}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H\right\|_{L^2(\Omega)} \le CT^{-\frac{H}{2}}, \tag{A.59}$$

with $H \in [1/2, 1)$ and C denotes a suitable positive constant. Consequently, we deduce from (A.59) and Lemma 2.1 of Kloeden and Neuenkirch (2007) that

$$\frac{\sigma e^{\kappa T}}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H \stackrel{a.s.}{\to} 0. \tag{A.60}$$

Finally, the result in Lemma 3.1 follows by combining (A.54), (A.55), (A.56), and (A.60).

A.5. Proof of Theorem 3.4

We prove the convergence of $\hat{\kappa}_{LS}$ first. For the sake of simple notations, we introduce the two processes with $T \geq 0$

$$Z_T = \int_0^T e^{\kappa s} B_s^H ds, \tag{A.61}$$

$$\xi_T = \int_0^T e^{\kappa s} dB_s^H. \tag{A.62}$$

By the definition of the Young integral, $B_0^H = 0$. By the definition of Z_T, we have

$$\xi_T = e^{\kappa T} B_T^H - \kappa \int_0^T e^{\kappa s} B_s^H ds = e^{\kappa T} B_T^H - \kappa Z_T. \tag{A.63}$$

By Lemma 2.1 of El Machkouri et al. (2016), we obtain $Z_\infty = \int_0^\infty e^{\kappa s} B_s^H ds$ which is well-defined and

$$Z_T \overset{a.s.}{\to} Z_\infty, \tag{A.64}$$

$$\xi_T \overset{a.s.}{\to} \xi_\infty := -\kappa Z_\infty. \tag{A.65}$$

Using (A.61) and the Young integral, we can rewrite the solution of (1.1) as

$$\begin{aligned} X_t &= X_0 e^{-\kappa t} + (1-e^{-\kappa t})\mu + e^{-\kappa t}\sigma \int_0^t e^{\kappa s} dB_s^H \\ &= X_0 e^{-\kappa t} + (1-e^{-\kappa t})\mu + e^{-\kappa t}\sigma \xi_t \\ &= X_0 e^{-\kappa t} + (1-e^{-\kappa t})\mu + e^{-\kappa t}\sigma \left[e^{\kappa t} B_t^H - \int_0^t B_s^H e^{\kappa s} \kappa ds \right] \\ &= X_0 e^{-\kappa t} + (1-e^{-\kappa t})\mu + \sigma B_t^H - \sigma e^{-\kappa t} \kappa \int_0^t B_s^H e^{\kappa s} ds \\ &= X_0 e^{-\kappa t} + (1-e^{-\kappa t})\mu + \sigma B_t^H - \sigma e^{-\kappa t} \kappa Z_t. \end{aligned} \tag{A.66}$$

To prove strong consistency of $\hat{\kappa}_{LS}$, we will analyze separately the numerator and the denominator of the estimator (3.14). First, we consider the term $e^{\kappa T}\int_0^T X_t dt$. Using L'Hôspital's rule, (A.64), (A.65), and (A.66), we obtain

$$\begin{aligned} e^{\kappa T}\int_0^T X_t dt &= e^{\kappa T}\int_0^T \left[X_0 e^{-\kappa t} + \left(1-e^{-\kappa t}\right)\mu + \sigma e^{-\kappa t}\xi_t\right] dt \\ &= -\frac{X_0}{\kappa}\left(1-e^{\kappa T}\right) + e^{\kappa T}\mu T + \frac{\mu}{\kappa} e^{\kappa T}\left(e^{-\kappa T} - 1\right) + \sigma \frac{\int_0^T e^{-\kappa t}\xi_t dt}{e^{-\kappa T}} \\ &\overset{a.s.}{\to} -\frac{X_0}{\kappa} + \frac{\mu}{\kappa} + \sigma Z_\infty. \end{aligned} \tag{A.67}$$

Combining (A.64), (A.65), and (A.66), we deduce that

$$\begin{aligned} \frac{1}{T} e^{\kappa T} X_T &= \frac{e^{\kappa T}}{T}\left[X_0 e^{-\kappa T} + \left(1-e^{-\kappa T}\right)\mu + \sigma e^{-\kappa T}\xi_T\right] \\ &= \frac{1}{T}\left[X_0 + \mu e^{\kappa T} - \mu + \sigma \xi_T\right] \\ &\overset{a.s.}{\to} 0. \end{aligned} \tag{A.68}$$

By (A.64) and (A.65), we have

$$\begin{aligned} X_T^2 e^{2\kappa T} &= e^{2\kappa T}\left[X_0 e^{-\kappa T} + \left(1-e^{-\kappa T}\right)\mu + \sigma e^{-\kappa T}\xi_T\right]^2 \qquad \textbf{(A.69)} \\ &= e^{2\kappa T}\left[\left(X_0 e^{-\kappa T}\right)^2 + \left(1-e^{-\kappa T}\right)^2\mu^2 + \sigma^2 e^{-2\kappa T}\xi_T^2 + 2X_0 e^{-\kappa T}\sigma e^{-\kappa T}\xi_T \right.\\ &\quad \left. +2\mu\left(1-e^{-\kappa T}\right)\sigma e^{-\kappa T}\xi_T + 2X_0 e^{-\kappa T}\left(1-e^{-\kappa T}\right)\mu\right] \\ &= X_0^2 + \left(e^{\kappa T}-1\right)^2\mu^2 + \sigma^2\xi_T^2 + 2X_0\sigma\xi_T + 2\mu\sigma\xi_T(e^{\kappa T}-1) + 2\mu X_0\left(e^{\kappa T}-1\right) \\ &\overset{a.s.}{\to} X_0^2 + \mu^2 + \sigma^2\kappa^2 Z_\infty^2 - 2\sigma X_0\kappa Z_\infty + 2\mu\sigma\kappa Z_\infty - 2X_0\mu\,. \end{aligned}$$

By (A.64) and (A.65) again, we obtain

$$\begin{aligned} e^{2\kappa T}\int_0^T X_t^2 dt &= e^{2\kappa T}\int_0^T \left[X_0 e^{-\kappa t} + \left(1-e^{-\kappa t}\right)\mu + \sigma e^{-\kappa t}\xi_t\right]^2 dt \\ &= e^{2\kappa T}X_0\int_0^T e^{-2\kappa t}dt + e^{2\kappa T}\int_0^T \mu^2(1-e^{-\kappa t})^2 dt + \sigma^2 e^{2\kappa T}\int_0^T e^{-2\kappa t}\xi_t^2 dt \\ &\quad +2e^{2\kappa T}\mu X_0\int_0^T e^{-\kappa t}\left(1-e^{-\kappa t}\right)dt + 2e^{2\kappa T}X_0\sigma\int_0^T e^{-2\kappa t}\xi_t dt \\ &\quad +2e^{2\kappa T}\mu\sigma\int_0^T\left(1-e^{-\kappa t}\right)e^{-\kappa t}\xi_t dt \\ &= \frac{X_0}{2\kappa}\left(e^{2\kappa T}-1\right) + \mu^2\left[e^{2\kappa T}T - \frac{1}{2\kappa}(1-e^{2\kappa T}) + \frac{2}{\kappa}(e^{\kappa T}-e^{2\kappa T})\right] \\ &\quad +\sigma^2 e^{2\kappa T}\int_0^T e^{-2\kappa t}\xi_t^2 dt + 2\mu X_0\left[\frac{1}{2\kappa}(1-e^{2\kappa T}) - \frac{1}{\kappa}\left(e^{\kappa T}-e^{2\kappa T}\right)\right] \\ &\quad +2\sigma X_0\frac{\int_0^T e^{-2\kappa t}\xi_t dt}{e^{-2\kappa T}} + 2\mu\sigma\left(\frac{\int_0^T e^{-\kappa t}\xi_t dt}{e^{-2\kappa T}} - \frac{\int_0^T e^{-2\kappa t}\xi_t dt}{e^{-2\kappa T}}\right) \\ &\overset{a.s.}{\to} -\frac{X_0}{2\kappa} - \frac{\mu^2}{2\kappa} - \frac{\sigma^2}{2}\kappa Z_\infty^2 + \frac{\mu X_0}{\kappa} + X_0\sigma Z_\infty - \mu\sigma Z_\infty\,. \qquad \textbf{(A.70)} \end{aligned}$$

A standard calculation together with (A.67) yields

$$\frac{e^{2\kappa T}}{T}\left(\int_0^T X_t dt\right)^2 = \frac{1}{T}\left(e^{\kappa T}\int_0^T X_t dt\right)^2 \overset{a.s.}{\to} 0. \qquad \textbf{(A.71)}$$

Combining (A.67), (A.68), (A.69), (A.70), (A.71), and (3.14), we obtain strong consistency of $\hat{\kappa}_{LS}$.

It remains to show strong consistency of $\hat{\mu}_{LS}$. From (1.1) and the fact that $B_0^H = 0$, we can rewrite X_t as

$$X_t = X_0 + \mu\kappa t - \kappa\int_0^t X_s ds + \sigma B_t^H\,. \qquad \textbf{(A.72)}$$

By (1.1), (3.15), (A.72), and the Young integral, we can rewrite $\hat{\mu}_{LS}$ as

$$\hat{\mu}_{LS} = \frac{(X_T - X_0)\int_0^T X_t^2 dt - \int_0^T X_t dX_t \frac{X_0 + \mu\kappa T + \sigma B_T^H - X_T}{\kappa}}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t} \tag{A.73}$$

$$= \frac{(X_T - X_0)\int_0^T X_t^2 dt - \mu T\int_0^T X_t dX_t - \frac{X_0 + \sigma B_T^H - X_T}{\kappa}\int_0^T X_t \left[\kappa(\mu - X_t)dt + \sigma dB_t^H\right]}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}$$

$$= \mu + \frac{\frac{X_T - X_0}{\kappa}\sigma\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa}\int_0^T X_t dX_t}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}$$

$$= \mu + \frac{e^{\kappa T}\frac{X_T - X_0}{\kappa}\frac{\sigma}{T}e^{\kappa T}\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa T}e^{2\kappa T}\frac{X_T^2 - X_0^2}{2}}{e^{\kappa T}\frac{X_T - X_0}{T}e^{\kappa T}\int_0^T X_t dt - e^{2\kappa T}\frac{X_T^2 - X_0^2}{2}}.$$

Finally, using (A.67), (A.68), (A.69), Lemma 3.1, and (A.73), we obtain strong consistency for $\hat{\mu}_{LS}$.

A.6. Proof of Theorem 3.5

Using (1.1), (A.72), and the Young integral, we can rewrite $\hat{\kappa}_{LS}$ as

$$\hat{\kappa}_{LS} = \frac{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t\left[\kappa(\mu - X_t)dt + \sigma dB_t^H\right]}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \tag{A.74}$$

$$= \frac{(X_T - X_0 - \kappa\mu T)\int_0^T X_t dt + \kappa T\int_0^T X_t^2 dt - \sigma T\int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}$$

$$= \kappa + \frac{\sigma B_T^H\int_0^T X_t dt - \sigma T\int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}.$$

Hence,

$$e^{-\kappa T}(\hat{\kappa}_{LS} - \kappa) = \frac{\sigma B_T^H e^{-\kappa T}\int_0^T X_t dt - \sigma T e^{-\kappa T}\int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \tag{A.75}$$

$$= \frac{\frac{\sigma B_T^H}{T}\frac{e^{\kappa T}\int_0^T X_t dt}{e^{2\kappa T}\int_0^T X_t^2 dt} - \frac{\sigma e^{\kappa T}\int_0^T X_t dB_t^H}{e^{2\kappa T}\int_0^T X_t^2 dt}}{1 - \frac{1}{T}\frac{\left(e^{\kappa T}\int_0^T X_t dt\right)^2}{e^{2\kappa T}\int_0^T X_t^2 dt}}.$$

A standard calculation yields

$$-\frac{\sigma e^{\kappa T}\int_0^T X_t dB_t^H}{e^{2\kappa T}\int_0^T X_t^2 dt} = -\sigma\frac{e^{\kappa T}\int_0^T\left[\mu + (X_0 - \mu)e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)}dB_s^H\right]dB_t^H}{e^{2\kappa T}\int_0^T X_t^2 dt} \tag{A.76}$$

$$= -\frac{\sigma}{e^{2\kappa T}\int_0^T X_t^2 dt}\left[\mu e^{\kappa T}B_T^H - \sigma e^{\kappa T}\int_0^T\int_0^s e^{-\kappa(t-s)}dB_t^H dB_s^H \right.$$
$$\left. + e^{\kappa T}\int_0^T e^{-\kappa t}dB_t^H\left[(X_0 - \mu) + \sigma\int_0^T e^{\kappa s}dB_s^H\right]\right].$$

By Lemmas 6 and 3 of Belfadli et al. (2011), we have

$$e^{\kappa T}\int_0^T e^{-\kappa s}dB_s^H \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{H\Gamma(2H)}{|\kappa|^{2H}}\right), \tag{A.77}$$

$$\int_0^T e^{\kappa s}dB_s^H \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{H\Gamma(2H)}{|\kappa|^{2H}}\right). \tag{A.78}$$

Moreover, it is easy to check

$$\mu e^{\kappa T} B_T^H \xrightarrow{a.s.} 0. \tag{A.79}$$

Obviously, both $e^{\kappa t}$ and $e^{\kappa s}$ are nonrandom Hölder continuous functions. According to Lemma 7 of Belfadli et al. (2011) and the relationship between the divergence integral and pathwise integral (see, e.g., equation (2.4) in Belfadli et al., 2011), we can deduce that

$$\sigma e^{\kappa T}\int_0^T \int_0^t e^{-\kappa s}dB_s^H e^{\kappa t}dB_t^H \xrightarrow{p} 0. \tag{A.80}$$

By (A.70), (A.76)–(A.80), and Slutsky's theorem, we have

$$-\frac{\sigma e^{\kappa T}\int_0^T X_t dB_t^H}{e^{2\kappa T}\int_0^T X_t^2 dt} \xrightarrow{\mathcal{L}} \frac{2\kappa\sigma\frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\nu}{X_0-\mu+\sigma\frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\omega}, \tag{A.81}$$

with ν and ω being two independent standard normal random variables. Combining (A.67), (A.70), (A.75), and (A.81), we obtain (3.16).

Let us now obtain the asymptotic distribution of $\hat{\mu}_{LS}$. From (A.73), we can rewrite $\hat{\mu}_{LS}$ as

$$\begin{aligned}
\hat{\mu}_{LS} &= \frac{(X_T-X_0)\int_0^T X_t^2 dt - \int_0^T X_t dX_t \frac{X_0+\mu\kappa T+\sigma B_T^H - X_T}{\kappa}}{(X_T-X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t} \\
&= \frac{(X_T-X_0)\int_0^T X_t^2 dt - \mu T\int_0^T X_t dX_t - \frac{X_0+\sigma B_T^H - X_T}{\kappa}\int_0^T X_t\left[\kappa(\mu-X_t)dt+\sigma dB_t^H\right]}{(X_T-X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t} \\
&= \mu + \frac{\frac{X_T-X_0}{\kappa}\sigma\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa}\int_0^T X_t dX_t}{(X_T-X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t} \\
&= \mu + \frac{\frac{X_T-X_0}{\kappa}\sigma\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa}\frac{X_T^2-X_0^2}{2}}{(X_T-X_0)\int_0^T X_t dt - T\frac{X_T^2-X_0^2}{2}}.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
T^{1-H}\left(\hat{\mu}_{LS}-\mu\right) &= \frac{\frac{2\sigma}{\kappa T^H X_T}\int_0^T X_t dB_t^H - \frac{2X_0\sigma}{\kappa T^H X_T^2}\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa T^H} + \frac{\sigma B_T^H}{\kappa T^H}\frac{X_0^2}{X_T^2}}{\frac{2}{TX_T}\int_0^T X_t dt - \frac{2X_0}{TX_T^2}\int_0^T X_t dt - 1 + \frac{X_0}{X_T^2}} \\
&= \frac{\frac{2\sigma}{\kappa e^{\kappa T}X_T}\frac{e^{\kappa T}}{T^H}\int_0^T X_t dB_t^H - \frac{2X_0\sigma}{\kappa X_T^2 e^{2\kappa T}}\frac{e^{2\kappa T}}{T^H}\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa T^H} + \frac{\sigma}{\kappa}\frac{e^{2\kappa T}X_0^2}{e^{2\kappa T}X_T^2}\frac{B_T^H}{T^H}}{\frac{2}{Te^{\kappa T}X_T}e^{\kappa T}\int_0^T X_t dt - \frac{2X_0}{Te^{2\kappa T}X_T^2}e^{2\kappa T}\int_0^T X_t dt - 1 + \frac{e^{2\kappa T}X_0^2}{e^{2\kappa T}X_T^2}}.
\end{aligned}$$

By (A.67)–(A.69), Lemma 3.1, and the above equation, we can obtain the desired result in (3.17).

A.7. Proof of Theorem 3.6

We first prove strong convergence of $\hat{\kappa}_{LS}$, which can be rewritten as (3.19) and (3.20), respectively. Using the isometry property of the double stochastic integral (see, e.g., equation (5.6) of Nualart, 2006), we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^1 B_u^H du\right)^2\right] &= \mathbb{E}\left[\left(\int_0^1\int_0^1 \mathbf{1}_{[0,1]}(s)\, dB_s^H du\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^1\int_0^1 \mathbf{1}_{[0,1]}(s)\, du dB_s^H\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^1 (1-u)\, dB_s^H\right)^2\right] \\
&= \alpha_H \int_0^1\int_0^1 (1-u)(1-s)|s-u|^{2H-2} ds du \\
&< \infty. \qquad \textbf{(A.82)}
\end{aligned}$$

From (A.82), we can see that $\int_0^1 B_u^H du$ is a Gaussian process with mean zero and bounded variance. Consequently, we have

$$\int_0^1 B_u^H du = O_p(1). \qquad \textbf{(A.83)}$$

Moreover, a standard calculation together with Isserlis' Theorem (see Isserlis, 1918) yields

$$\mathbb{E}\left[\int_0^1 \left(B_u^H\right)^2 du\right] = \frac{1}{2H+1}, \qquad \textbf{(A.84)}$$

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^1 \left(B_u^H\right)^2 du\right)^2\right] &= \int_0^1\int_0^1 \left[\mathbb{E}\left(B_u^H\right)^2 \mathbb{E}\left(B_s^H\right)^2 + 2\mathbb{E}\left(B_u^H B_s^H\right)\mathbb{E}\left(B_u^H B_s^H\right)\right] ds dt \\
&= \int_0^1\int_0^1 s^{2H} t^{2H} ds dt + \int_0^1\int_0^1 \left(|t|^{2H}+|s|^{2H}-|t-s|^{2H}\right)^2 ds dt \\
&< \infty. \qquad \textbf{(A.85)}
\end{aligned}$$

From (A.84) and (A.85), we can easily obtain

$$\int_0^1 \left(B_u^H\right)^2 du = O_p(1). \qquad \textbf{(A.86)}$$

By the law of the iterated logarithm for fBm (see, e.g., Corollary A.1 in Taqqu, 1977), for any $\epsilon > 0$, we can have

$$\frac{B_T^H}{T^{H+\epsilon}} \overset{a.s.}{\to} 0. \qquad \textbf{(A.87)}$$

Using (A.83), (A.86), and (A.87), we obtain

$$\frac{B_T^H \int_0^T B_t^H dt}{T\int_0^T (B_t^H)^2 dt} \overset{a.s.}{\to} 0. \qquad \textbf{(A.88)}$$

Using a similar argument, we have

$$\frac{\left(B_T^H\right)^2}{2\int_0^T (B_t^H)^2 dt} \overset{a.s.}{\to} 0, \tag{A.89}$$

$$\frac{T^{2H}}{2\int_0^T (B_t^H)^2 dt} \overset{a.s.}{\to} 0. \tag{A.90}$$

Now, using (3.19), (A.88), (A.89), and (A.90), we obtain

$$\hat{\kappa}_{LS} = \frac{\frac{B_T^H \int_0^T B_t^H dt}{T\int_0^T (B_t^H)^2 dt} - \frac{\left(B_T^H\right)^2}{2\int_0^T (B_t^H)^2 dt} + \frac{T^{2H}}{2\int_0^T (B_t^H)^2 dt}}{1 - \frac{\left(\int_0^T B_t^H dt\right)^2}{T\int_0^T (B_t^H)^2 dt}} \overset{a.s.}{\to} 0. \tag{A.91}$$

Similarly, using (3.20), (A.88), (A.89), and (A.90), we have

$$\hat{\kappa}_{LS} = \frac{\frac{B_T^H \int_0^T B_t^H dt}{T\int_0^T (B_t^H)^2 dt} - \frac{\left(B_T^H\right)^2}{2\int_0^T (B_t^H)^2 dt}}{1 - \frac{\left(\int_0^T B_t^H dt\right)^2}{T\int_0^T (B_t^H)^2 dt}} \overset{a.s.}{\to} 0. \tag{A.92}$$

By (A.91) and (A.92), we complete the proof of strong consistency of $\hat{\kappa}_{LS}$.

Finally, we need to prove (3.21). By the scaling properties of fBm of (2.2) (see also in Nualart, 2006), we have

$$\begin{cases} B_T^H \overset{d}{=} T^H B_1^H \\ B_T^H \int_0^T B_t^H dt \overset{d}{=} T^{2H+1} B_1^H \int_0^1 B_u^H du \\ T\int_0^T B_t^H dB_t^H \overset{d}{=} T^{2H+1} \int_0^1 B_u^H dB_u^H \\ T\int_0^T (B_t^H)^2 dt \overset{d}{=} T^{2H+2} \int_0^1 (B_u^H)^2 du \\ \left(\int_0^T B_t^H dt\right)^2 \overset{d}{=} T^{2H+2} \left(\int_0^1 B_u^H du\right)^2 \end{cases}. \tag{A.93}$$

Combining (3.18) and (A.93), we obtain the desired asymptotic distribution.