Online Supplement on "Asymptotic Theory for Rough Fractional Vasicek Models"

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This supplement provides proofs of Lemma 3.1, Theorem 3.1 and Theorem 3.2 in Xiao and Yu (2019).

1 Proof of Lemma 3.1

For $t \geq 0$, we define

$$Y_t = \sigma \int_{-\infty}^{t} e^{-\kappa(t-s)} dB_s^H \,. \tag{1}$$

Cheridito et al. (2003) showed that Y_t is Gaussian, stationary, and ergodic when $\kappa > 0$. The integral with respect to fBm exists as a path-wise Riemann-Stieltjes integral, and can be calculated using integration by parts (see Prop. A.1 in Cheridito et al. (2003)). To avoid integration with respect to fBm for $0 < H < 1/2$, using integration by parts and (1), we write the solution of fVm as

$$X_t = Y_t + \left(1 - e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t} + \sigma\kappa e^{-\kappa t}\int_{-\infty}^{0} e^{\kappa s} B_s^H ds \,. \tag{2}$$

Using (2), we have

$$\begin{aligned} \frac{1}{T}\int_0^T X_t dt &= \frac{1}{T}\int_0^T Y_t dt + \frac{\mu}{T}\int_0^T \left(1 - e^{-\kappa t}\right) dt + \frac{X_0}{T}\int_0^T e^{-\kappa t} dt \\ &\quad + \frac{\sigma\kappa}{T}\int_0^T e^{-\kappa t}\int_{-\infty}^{0} e^{\kappa s} B_s^H ds dt \,. \end{aligned} \tag{3}$$

For the first term in (3), using the ergodic theorem and the fact $\mathbb{E}\left[Y_0\right] = 0$, we obtain

$$\frac{1}{T}\int_0^T Y_t dt \overset{a.s.}{\to} \mathbb{E}\left(Y_0\right) = 0 \,. \tag{4}$$

For the second term in (3), it is obvious that

$$\frac{\mu}{T}\int_0^T \left(1-e^{-\kappa t}\right)dt \to \mu\,. \tag{5}$$

Using the fact that $X_0 = o_{a.s.}(\sqrt{T})$, we have

$$\frac{X_0}{T}\int_0^T e^{-\kappa t}dt \overset{a.s.}{\to} 0\,. \tag{6}$$

Moreover, a straightforward calculation shows that

$$\begin{aligned}
& \mathbb{E}\left[\frac{\sigma\kappa}{T}\int_0^T e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H dsdt\right]^2 \\
= & \mathbb{E}\left[\frac{\sigma^2\kappa^2}{T^2}\int_0^T e^{-\kappa t}dt\int_0^T e^{-\kappa v}dv\int_{-\infty}^0 e^{\kappa s}B_s^H ds\int_{-\infty}^0 e^{\kappa u}B_u^H du\right] \\
= & \frac{\sigma^2}{T^2}\left(1-e^{-\kappa T}\right)^2\kappa^{-2H-2}H\Gamma(2H).
\end{aligned} \tag{7}$$

From (7), we can deduce that

$$\left\|\frac{\sigma\kappa}{T}\int_0^T e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H dsdt\right\|_{L^2(\Omega)} \le CT^{-1}, \tag{8}$$

where C denotes a suitable positive constant. Consequently, from (8) and Lemma 2.1 of Kloeden and Neuenkirch (2007), we obtain

$$\frac{\sigma\kappa}{T}\int_0^T e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H dsdt \overset{a.s.}{\to} 0\,. \tag{9}$$

Substituting (4), (5), (6) and (9) into (3), we obtain the first claim in Lemma 3.1.

Next, using (2), we obtain

$$\begin{aligned}
\frac{1}{T}\int_0^T X_t^2 dt = & \frac{1}{T}\int_0^T \left[Y_t + \left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t} + \sigma\kappa e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H ds\right]^2 dt \\
= & \frac{1}{T}\int_0^T Y_t^2 dt + \frac{1}{T}\int_0^T \left[\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t}\right]^2 dt \\
& + \frac{\sigma^2\kappa^2}{T}\int_0^T \left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H ds\right)^2 dt + \frac{2}{T}\int_0^T Y_t\left[\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t}\right]dt \\
& + \frac{2\sigma\kappa}{T}\int_0^T Y_t\left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H ds\right)dt \\
& + \frac{2\sigma\kappa}{T}\int_0^T \left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H ds\right)\left[\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t}\right]dt\,.
\end{aligned} \tag{10}$$

Using the ergodic theorem, we obtain

$$\frac{1}{T}\int_0^T Y_t^2 dt \overset{a.s.}{\to} \mathbb{E}\left(Y_0^2\right) . \tag{11}$$

Integrating by parts together with similar arguments as in (7) yields

$$\begin{aligned}
\mathbb{E}\left(Y_0^2\right) &= \mathbb{E}\left[\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)^2\right] \\
&= \lim_{T\to-\infty} \mathbb{E}\left[\sigma^2\left(-e^{\kappa T}B_T^H - \kappa\int_T^0 e^{\kappa s}B_s^H ds\right)^2\right] \\
&= \sigma^2 \lim_{T\to\infty}\left[e^{-2\kappa T}\left(-T\right)^{2H} + \kappa^2\int_0^T\int_0^T e^{-\kappa s}e^{-\kappa t}\mathbb{E}\left[B_{-s}^H B_{-t}^H\right] dsdt\right. \\
&\quad \left. +2\kappa e^{-\kappa T}\int_0^T e^{-\kappa s}\mathbb{E}\left[B_{-T}^H B_{-s}^H\right] ds\right] \\
&= \sigma^2\kappa^{-2H}H\Gamma(2H). \qquad (12)
\end{aligned}$$

Combining (11) and (12), we obtain

$$\frac{1}{T}\int_0^T Y_t^2 dt \overset{a.s.}{\to} \sigma^2\kappa^{-2H}H\Gamma(2H) . \tag{13}$$

A straightforward calculation shows that

$$\frac{1}{T}\int_0^T \left[\mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right]^2 dt \overset{a.s.}{\to} \mu^2 . \tag{14}$$

Using similar arguments as in (7), we obtain

$$\frac{\sigma^2\kappa^2}{T}\int_0^T \left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H ds\right)^2 dt \overset{a.s.}{\to} 0 . \tag{15}$$

Moreover, by the Cauchy-Schwarz inequality and the same arguments as in (12),

$$\begin{aligned}
\frac{2}{T}\int_0^T Y_t\left[\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t}\right] dt &\overset{a.s.}{\to} 0 , && (16) \\
\frac{2\sigma\kappa}{T}\int_0^T Y_t\left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H ds\right) dt &\overset{a.s.}{\to} 0 , && (17) \\
\frac{2\sigma\kappa}{T}\int_0^T \left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H ds\right)\left[\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t}\right] dt &\overset{a.s.}{\to} 0 . && (18)
\end{aligned}$$

Substituting (13)-(18) into (10), we can obtain the second claim in Lemma 3.1.

Now, using the fact $dX_t = \kappa(\mu - X_t)\,dt + \sigma dB_t^H$, we can write

$$\frac{1}{T}\int_0^T X_t dX_t = \frac{\kappa\mu}{T}\int_0^T X_t dt - \frac{\kappa}{T}\int_0^T X_t^2 dt + \frac{\sigma}{T}\int_0^T X_t dB_t^H\,. \tag{19}$$

Using the relationship between the divergence integral and the Stratonovich integral (see Proposition 5.2.4 in Nualart (2006)), we have

$$\begin{aligned}\frac{\sigma}{T}\int_0^T X_t dB_t^H &= \frac{\sigma}{T}\int_0^T X_t \circ dB_t^H - \mathbb{E}\left[\frac{\sigma}{T}\int_0^T X_t \circ dB_t^H\right] \qquad (20)\\ &= \frac{1}{T}\int_0^T X_t \circ \left[dX_t - \kappa(\mu - X_t)\,dt\right] - \mathbb{E}\left[\frac{\sigma}{T}\int_0^T X_t \circ dB_t^H\right]\\ &= \frac{X_T^2}{2T} - \frac{\kappa\mu}{T}\int_0^T X_t dt + \frac{\kappa}{T}\int_0^T X_t^2 dt - \mathbb{E}\frac{\sigma}{T}\int_0^T \left(1 - e^{-\kappa t}\right)\mu \circ dB_t^H\\ &\quad -\mathbb{E}\left[\frac{\sigma}{T}\int_0^T \left[X_0 e^{-\kappa t} + \sigma\left(B_t^H - \kappa\int_0^t B_s^H e^{-\kappa(t-s)} ds\right)\right] \circ dB_t^H\right],\end{aligned}$$

where $\int_0^T X_t \circ dB_t^H$ denotes the Stratonovich integral.

From Eq. (3.7) of Hu et al. (2018), we can see that

$$\lim_{T\to\infty}\mathbb{E}\left[\frac{\sigma}{T}\int_0^T \left[\sigma\left(B_t^H - \kappa\int_0^t B_s^H e^{-\kappa(t-s)} ds\right)\right] \circ dB_t^H\right] = \sigma^2 H \kappa^{1-2H}\Gamma(2H). \tag{21}$$

A straightforward calculation shows that

$$\lim_{T\to\infty}\mathbb{E}\left[\frac{\sigma}{T}\int_0^T \left[\left(1 - e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t}\right] \circ dB_t^H\right] = 0\,. \tag{22}$$

Using Lemma 18 of Hu et al. (2018), we can see that, for any $\epsilon > 0$,

$$\frac{X_T}{T^\epsilon} \overset{a.s.}{\to} 0\,. \tag{23}$$

Substituting the first claim in Lemma 3.1, the second claim in Lemma 3.1, (21)-(23) into (20), we have

$$\frac{\sigma}{T}\int_0^T X_t dB_t^H \overset{a.s.}{\to} 0\,. \tag{24}$$

Finally, combining the first claim in Lemma 3.1, the second claim in Lemma 3.1, (19) and (24), we obtain the third claim in Lemma 3.1.

2 Proof of Theorem 3.1

First, we can write $\hat{\kappa}_{LS}$ as

$$\hat{\kappa}_{LS} = \frac{\frac{X_T - X_0}{T}\frac{1}{T}\int_0^T X_t dt - \frac{1}{T}\int_0^T X_t dX_t}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}\,. \tag{25}$$

By Lemma 3.1, (23), (25) and the arithmetic rule of convergence, we obtain the almost sure convergence of $\hat{\kappa}_{LS}$, i.e., $\hat{\kappa}_{LS} \overset{a.s.}{\to} \kappa$. Now, we can rewrite $\hat{\mu}_{LS}$ as

$$\hat{\mu}_{LS} = \frac{\frac{X_T - X_0}{T} \frac{1}{T} \int_0^T X_t^2 dt - \frac{1}{T} \int_0^T X_t dX_t \frac{1}{T} \int_0^T X_t dt}{\frac{X_T - X_0}{T} \frac{1}{T} \int_0^T X_t dt - \frac{1}{T} \int_0^T X_t dX_t} . \tag{26}$$

Similarly, using Lemma 3.1, (23) and (26), we obtain the strong consistency of $\hat{\mu}_{LS}$, i.e., $\hat{\mu}_{LS} \overset{a.s.}{\to} \mu$. This proves the first part of the theorem.

To prove the second part, let us first consider the asymptotic law of $\hat{\kappa}_{LS}$. Based on (2) and integration by parts, we can write

$$\sqrt{T} \left(\hat{\kappa}_{LS} - \kappa \right) = I_1 + I_2 + I_3 , \tag{27}$$

where

$$\begin{aligned}
I_1 &= \frac{\sigma \left(\frac{\mu - X_0}{\sqrt{T}} \left(e^{-\kappa T} B_T^H + \kappa \int_0^T B_t^H e^{-\kappa t} dt \right) - \frac{\sigma}{\sqrt{T}} \int_0^T \int_0^t e^{-\kappa(t-s)} dB_s^H dB_t^H \right)}{\frac{1}{T} \int_0^T X_t^2 dt - \left(\frac{1}{T} \int_0^T X_t dt \right)^2} , \\
I_2 &= \frac{\left(\frac{X_T - X_0}{\sqrt{T}} + \frac{\kappa(X_0 - \mu)}{\sqrt{T}} \int_0^T e^{-\kappa t} dt - \frac{\sigma}{\sqrt{T}} B_T^H + \frac{\sigma\kappa}{\sqrt{T}} e^{-\kappa T} \int_0^T B_s^H e^{\kappa s} ds \right) \frac{1}{T} \int_0^T X_t dt}{\frac{1}{T} \int_0^T X_t^2 dt - \left(\frac{1}{T} \int_0^T X_t dt \right)^2} , \\
I_3 &= \frac{\left(-\mu\sigma + \frac{\sigma}{T} \int_0^T X_t dt \right) \frac{B_T^H}{\sqrt{T}}}{\frac{1}{T} \int_0^T X_t^2 dt - \left(\frac{1}{T} \int_0^T X_t dt \right)^2} .
\end{aligned}$$

By the law of the iterated logarithm for fBm (see e.g. Corollary A1 in Taqqu (1977)), we have

$$\frac{\sigma \left(\mu - X_0 \right)}{\sqrt{T}} e^{-\kappa T} B_T^H \overset{a.s.}{\to} 0 .$$

Using similar arguments as those in (7), we have

$$\frac{\sigma\mu\kappa}{\sqrt{T}} \int_0^T B_t^H e^{-\kappa t} dt \overset{p}{\to} 0 .$$

Similarly, using the assumption $X_0/\sqrt{T} = o_p(1)$, we obtain

$$\frac{\sigma X_0 \kappa}{\sqrt{T}} \int_0^T B_t^H e^{-\kappa t} dt \overset{p}{\to} 0 .$$

Moreover, from Theorem 5 of Hu et al. (2018), we can obtain

$$\frac{\sigma^2}{\sqrt{T}} \int_0^T \int_0^t e^{-\kappa(t-s)} dB_s^H dB_t^H \overset{\mathcal{L}}{\to} \mathcal{N} \left(0, \sigma^4 H^2 \kappa^{1-4H} \Gamma^2(2H) \delta_{LS}^2 \right) .$$

Combining all these convergency results, Lemma 3.1 and applying Slutsky's theorem

$$I_1 \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \kappa\delta_{LS}^2\right) . \tag{28}$$

Using (23) and by the law of the iterated logarithm for fBm, we obtain

$$\frac{X_T}{\sqrt{T}} \xrightarrow{a.s.} 0\,, \quad \frac{X_0}{\sqrt{T}} \xrightarrow{p} 0\,, \quad \frac{\kappa\left(X_0-\mu\right)}{\sqrt{T}} \int_0^T e^{-\kappa t} dt \xrightarrow{p} 0\,, \quad \frac{\sigma}{\sqrt{T}} B_T^H \xrightarrow{a.s.} 0\,. \tag{29}$$

Furthermore, since $\int_0^\infty e^{-\kappa z} z^{2H-1} dz = \kappa^{-2H}\Gamma(2H)$, we have

$$\begin{aligned}
& \frac{\sigma^2\kappa^2}{T} e^{-2\kappa T} \int_0^T \int_0^T \mathbb{E}\left[B_s^H B_u^H\right] e^{\kappa s} e^{\kappa u} ds du \\
= & \frac{\sigma^2\kappa^2}{T} \frac{e^{-2\kappa T}}{2} \int_0^T \int_0^T e^{\kappa(s+u)} \left(s^{2H} + u^{2H} - |s-u|^{2H}\right) ds du \\
= & \frac{\sigma^2}{T} \kappa e^{-2\kappa T} \left(e^{\kappa T} - 1\right) \int_0^T e^{\kappa s} s^{2H} ds - \frac{\sigma^2}{T} \frac{\kappa^2 e^{-2\kappa T}}{2} \int_0^T \int_0^T e^{\kappa(s+u)} |s-u|^{2H} ds du \\
= & \frac{\sigma^2\kappa\left(e^{-\kappa T} - e^{-2\kappa T}\right)}{T} \int_0^T e^{\kappa s} s^{2H} ds - \frac{\sigma^2\kappa}{2T} \int_0^T e^{-\kappa v} v^{2H} dv + \frac{\sigma^2\kappa}{2T} e^{-2\kappa T} \int_0^T e^{\kappa v} v^{2H} dv \\
\to & \; 0\,.
\end{aligned}$$

The result above implies

$$\frac{\sigma\kappa}{\sqrt{T}} e^{-\kappa T} \int_0^T B_s^H e^{\kappa s} ds \xrightarrow{p} 0\,. \tag{30}$$

Combining Lemma 3.1, (29) and (30), we have

$$I_2 \xrightarrow{p} 0\,. \tag{31}$$

Finally, using Lemma 3.1 again and by the law of the iterated logarithm for fBm (see e.g. Corollary A1 in Taqqu (1977)), we have

$$I_3 \xrightarrow{p} 0\,. \tag{32}$$

By (27), (28), (31), (32) and Slutsky's theorem, we obtain the asymptotic law of $\hat{\kappa}_{LS}$.

Next, using (2), we have

$$\begin{aligned}
\frac{1}{T^H} \int_0^T X_t dt - T^{1-H}\mu &= \frac{1}{T^H} \int_0^T \left[\left(1 - e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dB_s^H\right] dt - T^{1-H}\mu \\
&= \frac{X_0-\mu}{T^H} \int_0^T e^{-\kappa t} dt + \frac{\sigma}{T^H} \int_0^T \int_0^t e^{-\kappa(t-s)} dB_s^H dt \\
&= \frac{X_0-\mu}{T^H} \int_0^T e^{-\kappa t} dt + \frac{\sigma B_T^H}{\kappa T^H} - \frac{\sigma}{\kappa T^H} \int_0^T e^{-\kappa(T-s)} dB_s^H\,. \qquad (33)
\end{aligned}$$

A straightforward calculation shows that

$$\frac{X_0-\mu}{T^H}\int_0^T e^{-\kappa t}dt \overset{a.s.}{\to} 0\,, \tag{34}$$

$$\frac{\sigma B_T^H}{\kappa T^H} \overset{\mathcal{L}}{\to} \mathcal{N}\left(0,\frac{\sigma^2}{\kappa^2}\right)\,. \tag{35}$$

Moreover, from Lemma 18 of Hu et al. (2018), we can see that

$$\frac{\sigma}{\kappa T^H}\int_0^T e^{-\kappa(T-s)}dB_s^H \overset{a.s.}{\to} 0\,. \tag{36}$$

On the other hand, a straightforward calculation shows that

$$T^{1-H}\left(\hat{\mu}_{LS}-\mu\right)=\frac{\frac{X_T-X_0}{T^H}\frac{1}{T}\int_0^T X_t^2dt-\frac{1}{T}\int_0^T X_tdX_t\frac{1}{T^H}\int_0^T X_tdt}{\frac{X_T-X_0}{T}\frac{1}{T}\int_0^T X_tdt-\frac{1}{T}\int_0^T X_tdX_t}-T^{1-H}\mu\,. \tag{37}$$

Combining (23), Lemma 3.1, (33)-(37) and Slutsky's theorem, we obtain the asymptotic law of $\hat{\mu}_{LS}$.

3 Proof of Theorem 3.2

Consistency of $\hat{\kappa}_{HN}$ and $\hat{\mu}_{HN}$ can be easily obtained by Lemma 3.1. Moreover, the asymptotic law of $\hat{\mu}_{HN}$ can be obtained by using Slutsky's theorem and (33)-(36). Hence, we only consider the asymptotic law of $\hat{\kappa}_{HN}$ here.

Using the expressions of $\hat{\kappa}_{HN}$ and $\hat{\kappa}_{LS}$, we have

$$\begin{aligned}\sqrt{T}\left(\hat{\kappa}_{HN}-\kappa\right) &= \sqrt{T}\left(\hat{\kappa}_{HN}-\kappa^{1-\frac{1}{2H}}\hat{\kappa}_{LS}^{\frac{1}{2H}}+\kappa^{1-\frac{1}{2H}}\hat{\kappa}_{LS}^{\frac{1}{2H}}-\kappa\right)\\ &= \sqrt{T}\hat{\kappa}_{LS}^{\frac{1}{2H}}\left[\left(\frac{\sigma^2H\Gamma\left(2H\right)}{\frac{X_T-X_0}{T}\frac{1}{T}\int_0^T X_tdt-\frac{1}{T}\int_0^T X_tdX_t}\right)^{\frac{1}{2H}}-\kappa^{1-\frac{1}{2H}}\right]\\ &\quad+\sqrt{T}\kappa^{1-\frac{1}{2H}}\left(\hat{\kappa}_{LS}^{\frac{1}{2H}}-\kappa^{\frac{1}{2H}}\right)\,. \end{aligned} \tag{38}$$

By Theorem 3.1 and the delta method, we get

$$\sqrt{T}\left(\hat{\kappa}_{LS}^{\frac{1}{2H}}-\kappa^{\frac{1}{2H}}\right)\overset{\mathcal{L}}{\to}\mathcal{N}\left(0,\kappa^{\frac{1}{H}-1}\delta_{HN}^2\right)\,. \tag{39}$$

Using Lemma 3.1, (23), (38), (39) and Slutsky's theorem, we obtain the asymptotic law of $\hat{\kappa}_{HN}$.

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