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Bias in the estimation of mean reversion in continuous-time Lévy processes ☆

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HIGHLIGHTS

- This paper studies the bias issue in continuous-time Lévy processes.
- The approximate bias of the least squares estimator of κ is derived.
- Both known and unknown long-run mean cases are considered.
- We consider both fixed and random initial conditions in approximating bias.
- Simulations are conducted to examine the performance of the bias approximation and correction.

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ABSTRACT

This paper develops the approximate bias of the ordinary least squares estimator of the mean reversion parameter in continuous-time Lévy processes. Several cases are considered, depending on whether the long-run mean is known or unknown and whether the initial condition is fixed or random. The approximate bias is used to construct a bias corrected estimator. The performance of the approximate bias and the bias corrected estimator is examined using simulated data.

1. Introduction

There is an extensive literature of using diffusion processes to model the dynamic behavior of financial asset prices, including Black and Scholes (1973), Vasicek (1977) and Cox et al. (1985), among others. Many processes considered in the literature are based on the Brownian motion. In recent years, however, strong evidence of jumps in financial variables has been reported. To capture jumps, continuous-time Lévy processes have become increasingly popular and various Lévy models have been developed in the asset pricing literature; see, for example, Barndorff-Nielsen (1998) and Carr and Wu (2003).

In practice, one can only obtain the observations at discrete points from a finite time span. Based on discrete-time observations, different methods have been used to estimate continuous-time models. Phillips and Yu (2009) provided an overview of some widely used estimation methods. When the drift function is linear and the process is slowly mean reverting, it is found that there exists serious bias in estimating the mean reversion parameter

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(say κ) by almost all the methods (Phillips and Yu, 2005). Because the mean reversion parameter has important implications for asset pricing, risk management and forecasting, accurate estimation of it has received considerable attentions in the literature. For example, Yu (2012) approximated the bias of the maximum likelihood estimator (MLE) of κ when the long-run mean is known and the initial value is random for the Gaussian Ornstein–Uhlenbeck (OU) process. Tang and Chen (2009) approximated the bias of the MLE of κ when the long-run mean is unknown for the Gaussian OU process and the Cox–Ingersoll–Ross (CIR) model.

While the bias in estimating κ has been studied in continuous-time diffusion processes, to the best of our knowledge, nothing has been reported on the analytical bias issue in continuous-time Lévy processes. The objective of this paper is to develop the approximate bias of the least squares (LS) estimator of κ under the Lévy measure. The proof of the results in this paper can be found in Bao et al. (2013).

2. Models and the bias

A Lévy-driven OU process is

$$\mathrm{d}x(t) = \kappa(\mu - x(t))\mathrm{d}t + \sigma \mathrm{d}L(t), \qquad x(0) = x_0, \tag{2.1}$$

where $L(t),\ t \geq 0$, is a Lévy process with $L(0) = 0$ a.s. In the special case when $L(t)$ is a Brownian motion, the process is the Gaussian OU process used by Vasicek (1977) to model interest rate data. When $\kappa > 0$, the process is stationary with μ being the long run mean and κ captures the speed of mean reversion.

It is well known that the LS estimator of κ is

$$\hat{\kappa} = -\frac{\ln(\hat{\phi})}{h}, \tag{2.2}$$

where $\hat{\phi}$ is the LS estimator of the autoregressive coefficient ϕ from the discretized AR(1) model

$$x_{th} = \alpha + \phi x_{(t-1)h} + \varepsilon_{th}, \tag{2.3}$$

in which $\alpha = \mu(1 - e^{-\kappa h}), \phi = e^{-\kappa h}, \varepsilon_{th} = \sigma \int_{(t-1)h}^{th} e^{-\kappa(th-s)}\mathrm{d}L(s)$, h is the sampling interval, $t = 1, \ldots, n$ such that the observed data are discretely recorded at $(0, h, 2h, \ldots, nh)$ in the time interval $[0, T]$ and $nh = T$. By the properties of Lévy process, the sequence of $\{\varepsilon_{th}\}_{t=1}^n$ consists of i.i.d. random variables. We assume that the moments of ε_{th} exist, up to order 4, with variance σ_ε^2, and skewness and excess kurtosis coefficients γ_1 and γ_2, respectively.[1]

We are interested in studying the properties of $\hat{\kappa}$ estimated from the discrete sample via $\hat{\phi}$. As it is expected, the properties of $\hat{\kappa}$ depend on how we spell out the initial observation $x(0) = x_0$: it can be fixed at a constant or can be random, independent of $(\varepsilon_1, \ldots, \varepsilon_n)$, such that the time series $(x_0, x_1, \ldots, x_n)$ is stationary.

For notational convenience, we drop the subscript h, and throughout, $\boldsymbol{x} = (x_1, \ldots, x_n)'$, $\boldsymbol{x}_{-1} = (x_0, \ldots, x_{n-1})'$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)'$. For a given ϕ, $\boldsymbol{f}_1$ is an $n \times 1$ vector with $f_{1,i} = \phi^i$, $\boldsymbol{f}_2 = \boldsymbol{f}_1/\phi$, $\boldsymbol{C}_1$ is a lower-triangular matrix with $c_{1,ij} = \phi^{i-j},\ i \geq j$, $\boldsymbol{C}_2$ is a strict lower-triangular matrix with $c_{2,ij} = \phi^{i-j-1},\ i > j$. Note that by definition, $\boldsymbol{C}_2 = \phi^{-1}(\boldsymbol{C}_1 - \boldsymbol{I})$. The dimensions of vectors/matrices are to be read from the context, and thus we suppress the dimension subscripts in what follows.

To derive the analytical bias of $\hat{\kappa}$, we follow the framework of Bao (2013). Let $\hat{\boldsymbol{\theta}}$ be a $\sqrt{n}$-consistent estimator of $\boldsymbol{\theta}$ identified by the moment condition $\psi(\hat{\boldsymbol{\theta}}) = \boldsymbol{0}$ from a sample of size n. Typically, $\psi(\hat{\boldsymbol{\theta}})$ denotes the moment condition. In finite samples, $\hat{\boldsymbol{\theta}}$ is usually biased and one may approximate the bias $\mathbb{E}(\hat{\boldsymbol{\theta}}) - \boldsymbol{\theta}$ to the second order, namely, $\mathbb{E}(\hat{\boldsymbol{\theta}}) - \boldsymbol{\theta} = \mathrm{B}(\hat{\boldsymbol{\theta}}) + o(n^{-1})$, where $\mathrm{B}(\hat{\boldsymbol{\theta}})$ is defined as the second-order bias. Bao (2013) showed that $\mathrm{B}(\hat{\boldsymbol{\theta}})$ can be written as

$$\begin{aligned}\mathrm{B}(\hat{\boldsymbol{\theta}}) &= \boldsymbol{\Sigma}^{-1}\mathbb{E}(\boldsymbol{H}_1 \otimes \boldsymbol{\psi}')\mathrm{vec}(\boldsymbol{\Sigma}^{-1}) \\ &\quad + \frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbb{E}\left(\boldsymbol{H}_2\right)\left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}\right)\mathrm{vec}\left[\mathbb{E}\left(\boldsymbol{\psi}\boldsymbol{\psi}'\right)\right],\end{aligned} \tag{2.4}$$

where $\boldsymbol{\psi} = \psi(\boldsymbol{\theta}),\ \boldsymbol{H}_l = \nabla^l\boldsymbol{\psi},\ l = 1, 2$, ∇ denotes the derivative with respect to $\boldsymbol{\theta}$, and $\boldsymbol{\Sigma}^{-1} = -[\mathbb{E}(\boldsymbol{H}_1)]^{-1}$. For the scalar case, it becomes

$$\mathrm{B}(\hat{\theta}) = \frac{1}{[\mathbb{E}(H_1)]^2}\mathbb{E}(H_1\psi) - \frac{1}{2[\mathbb{E}(H_1)]^3}\mathbb{E}(H_2)\mathbb{E}(\psi^2). \tag{2.5}$$

Note that (2.4) and (2.5) are derived for $\sqrt{n}$-consistent $\boldsymbol{\theta}$, so the bias approximations we will derive in the following are only for κ being strictly positive and correspondingly the discrete AR(1) model does not contain a unit root.

2.1. μ is known

When μ is known *a priori* ($= 0$, without loss of generality), we can write $\boldsymbol{x} = x_0\boldsymbol{f}_1 + \boldsymbol{C}_1\boldsymbol{\varepsilon}, \boldsymbol{x}_{-1} = x_0\boldsymbol{f}_2 + \boldsymbol{C}_2\boldsymbol{\varepsilon},\ \boldsymbol{\varepsilon} = \boldsymbol{x} - \exp(-\kappa h)\boldsymbol{x}_{-1}$.[2] The moment condition, up to some scaling constant, for estimating κ is

$$\psi(\kappa) = \frac{1}{n}\boldsymbol{x}_{-1}'\boldsymbol{\varepsilon}. \tag{2.6}$$

Upon taking derivatives, we have

$$H_l = \frac{-(-h)^l\phi}{n}\boldsymbol{x}_{-1}'\boldsymbol{x}_{-1}, \quad l = 1, 2. \tag{2.7}$$

By substituting (2.6) and (2.7) into (2.5), we derive the approximate bias of $\hat{\kappa}$, when x_0 is fixed,

$$\begin{aligned}\mathrm{B}(\hat{\kappa}) &= \frac{1 + 3e^{-2\kappa h} + 4e^{-2n\kappa h}}{2Te^{-2\kappa h}} - \frac{\left(1 - e^{-2n\kappa h}\right)\left(1 + 7e^{-2\kappa h}\right)}{2Tne^{-2\kappa h}\left(1 - e^{-2\kappa h}\right)} \\ &\quad - \frac{4e^{-2n\kappa h}\left(1 - e^{-2\kappa h}\right)x_0^2}{2T\sigma_\varepsilon^2 e^{-2\kappa h}} + \frac{\left(1 + 3e^{-2\kappa h}\right)\left(1 - e^{-2n\kappa h}\right)x_0^2}{2Tn\sigma_\varepsilon^2 e^{-2\kappa h}} \\ &\quad + \frac{2(1 + e^{-\kappa h})\left(1 - e^{-n\kappa h}\right)\left(e^{-\kappa h} - e^{-n\kappa h}\right)x_0\gamma_1}{2Tn\sigma_\varepsilon e^{-2\kappa h}},\end{aligned} \tag{2.8}$$

and when x_0 is random,

$$\mathrm{B}(\hat{\kappa}) = \frac{1}{2T}(3 + e^{2\kappa h}) - \frac{2(1 - e^{-2n\kappa h})}{Tn(1 - e^{-2\kappa h})}. \tag{2.9}$$

Remark 1. The skewness parameter γ_1 matters for the bias of $\hat{\kappa}$. Its effect, however, disappears for the special case of $x_0 = 0$, where the bias expression simplifies to

$$\mathrm{B}(\hat{\kappa}) = \frac{1 + 3e^{-2\kappa h} + 4e^{-2n\kappa h}}{2Te^{-2\kappa h}} - \frac{\left(1 - e^{-2n\kappa h}\right)\left(1 + 7e^{-2\kappa h}\right)}{2Tne^{-2\kappa h}\left(1 - e^{-2\kappa h}\right)}.$$

Remark 2. Eq. (2.9) suggests that the result in Yu (2012) is in fact robust to nonnormality.

[1] This might rule out some Lévy processes. Also, in general, the moments of ε_{th} depend on the parameters κ, σ, and the sampling frequency h.

[2] When μ is known but may not be 0, one just needs to define $y_t = x_t - \mu$ and work with y_t.

2.2. μ is unknown

When μ is unknown and has to be estimated, we put $\boldsymbol{x} = x_0\boldsymbol{f}_1+\alpha\boldsymbol{C}_1\boldsymbol{\iota}+\boldsymbol{C}_1\boldsymbol{\varepsilon}, \boldsymbol{x}_{-1} = x_0\boldsymbol{f}_2+\alpha\boldsymbol{C}_2\boldsymbol{\iota}+\boldsymbol{C}_2\boldsymbol{\varepsilon}$, $\alpha = \mu(1-\exp(-\kappa h))$, and $\boldsymbol{\varepsilon} = \boldsymbol{x} - \alpha\boldsymbol{\iota} - \exp(-\kappa h)\boldsymbol{x}_{-1}$, where $\boldsymbol{\iota}$ is an $n \times 1$ vector of unit elements. Since the pairs (α, ϕ), (α, κ), and (μ, κ) have one-to-one mapping into each other, and we focus on deriving the finite-sample bias of $\hat{\kappa}$, the reparametrized model $x_t = \alpha + \exp(-\kappa h)x_{t-1} + \varepsilon_t$ with parameter vector $\boldsymbol{\theta} = (\alpha, \kappa)$ gives exactly the same $\hat{\kappa}$ as that estimated from the original model $x_t = \mu(1-\exp(-\kappa h)) + \exp(-\kappa h)x_{t-1} + \varepsilon_t$ with parameter vector (μ, κ). Thus, we define the moment condition, up to some scaling constant, as

$$\psi(\boldsymbol{\theta}) = \frac{1}{n}\begin{pmatrix} \boldsymbol{\iota}'\boldsymbol{\varepsilon} \\ -h\phi\boldsymbol{x}'_{-1}\boldsymbol{\varepsilon} \end{pmatrix}. \tag{2.10}$$

By taking derivatives, we have $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ as given in Eq. (2.11) (see Box I):

The approximate bias of $\hat{\kappa}$, when x_0 is fixed, is in Eq. (2.12), (see Box II): and when x_0 is random,

$$\mathrm{B}(\hat{\kappa}) = \frac{5+2e^{h\kappa}+e^{2h\kappa}}{2T} - \frac{2e^{-h\kappa}\left(1-e^{-nh\kappa}\right)\left(1-e^{2h\kappa}\right)^2\mu^2}{Tn\sigma_\varepsilon^2} + \frac{\left(1-e^{-nh\kappa}\right)\left[e^{h\kappa}+4e^{2h\kappa}+e^{3h\kappa}+2e^{-(n-2)h\kappa}\right]}{\left(1-e^{2h\kappa}\right)Tn}. \tag{2.13}$$

Remark 3. The leading term (of order $O(T^{-1})$) in (2.13) gives the result derived in Tang and Chen (2009). Moreover, (2.13) suggests that the approximate bias of $\hat{\kappa}$ under the case of random x_0 is robust to nonnormality.

Remark 4. Similar as before, the skewness matters for the approximate bias. In contrast, for the special case when x_0 is fixed at 0, its effect does not disappear (see Box III):

Remark 5. When x_0 is fixed at μ, however, the effect of skewness disappears on the approximate bias (see Box IV):

Remark 6. For the random case, if further $\mu = 0$ (i.e., the true model has no drift term but we still estimate the discrete AR model with an intercept), the result reduces to

$$\mathrm{B}(\hat{\kappa}) = \frac{5+2e^{h\kappa}+e^{2h\kappa}}{2T} + \frac{\left(1-e^{-nh\kappa}\right)\left[e^{h\kappa}+4e^{2h\kappa}+e^{3h\kappa}+2e^{-(n-2)h\kappa}\right]}{\left(1-e^{2h\kappa}\right)Tn}.$$

3. Numerical results

Our bias formulae (2.8), (2.9), (2.12), and (2.13) involve unknown population parameters, but we can make them feasible by replacing the unknown parameters with their consistent estimates. That is, we may replace κ by $\hat{\kappa}$, μ by $\hat{\mu} = \hat{\alpha}/(1-\hat{\phi})$, σ_ε^2 and γ_1 by their sample analogues from the LS residuals, and denote the feasible bias by $\hat{\mathrm{B}}(\hat{\kappa})$. An immediate application of our bias results is to construct a bias corrected estimator of κ. Here we follow the indirect inference method introduced in Phillips and Yu (2009) to design the bias corrected estimator of κ as follows:

$$\hat{\kappa}_{bc} = \arg\min_{\kappa} \|\hat{\kappa} - \kappa - \hat{\mathrm{B}}(\kappa)\|, \tag{3.1}$$

where $\hat{\mathrm{B}}(\kappa)$ is $\hat{\mathrm{B}}(\hat{\kappa}(\kappa))$ with $\hat{\kappa}(\kappa)$ being the LS estimate of κ when its true value is κ. In (3.1) $\kappa + \hat{\mathrm{B}}(\kappa)$ is the approximate mean function of $\hat{\kappa}$ when the true value is κ. Unlike what is done for the indirect inference method that relies on simulations to obtain the mean function, we construct $\hat{\kappa}_{bc}$ without invoking simulations to approximate the mean of $\hat{\kappa}$, as we utilize directly our analytical bias.

Table 1
Bias and bias correction, known μ.

κ	*Bias*	$\hat{\mathrm{B}}(\hat{\kappa})$	$\hat{\kappa}$	$std(\hat{\kappa})$	$\hat{\kappa}_{bc}$	$std(\hat{\kappa}_{bc})$
$\mu = 0$ and x_0 fixed						
0.1	0.0402	0.029	0.1402	0.0897	0.1112	0.0839
0.3	0.0413	0.0373	0.3413	0.128	0.3040	0.1266
0.5	0.0420	0.0390	0.5420	0.1586	0.5030	0.1578
0.7	0.0426	0.0399	0.7426	0.1850	0.7026	0.1844
0.9	0.0431	0.0406	0.9431	0.2088	0.9025	0.2082
$\mu = 0$ and x_0 random						
0.1	0.0372	0.0335	0.1372	0.0841	0.1037	0.0801
0.3	0.0402	0.0389	0.3402	0.1251	0.3013	0.1243
0.5	0.0414	0.0400	0.5414	0.1564	0.5014	0.1558
0.7	0.0421	0.0406	0.7421	0.1832	0.7015	0.1827
0.9	0.0428	0.0411	0.9428	0.2072	0.9016	0.2067

We conduct Monte Carlo simulations to demonstrate the performance of our bias formulae and the bias corrected estimator in finite samples. In practice we observe only the discrete sample $\{x_0, \ldots, x_n\}$. So we simulate discrete time observations from the continuous time model (2.1) with the driving process being the skew normal process of Azzalini (1985) with the shape parameter $\alpha = 5$ (and correspondingly $\gamma_1 = 0.8510$ and $\gamma_2 = 0.7053$). We set $\mu = 0.1$ when it is unknown, $x_0 = \mu$ when it is fixed, $h = 1/12$, $\sigma = 1$. Tables 1 and 2 report our feasible bias $\hat{\mathrm{B}}(\hat{\kappa})$ and the bias corrected estimator $\hat{\kappa}_{bc}$, in comparison with the actual bias (denoted by "*Bias*" in the tables) and the LS estimator $\hat{\kappa}$, for the cases of known $\mu(=0)$ and unknown μ, respectively. The data span is set at $T = 50$. The results are averaged over 10,000 replications and the standard deviations (across the simulations) of $\hat{\kappa}$ and $\hat{\kappa}_{bc}$ are also reported. We observe that our bias approximation formulae work well to capture the true bias of $\hat{\kappa}$. The bias corrected estimator $\hat{\kappa}_{bc}$ performs much better than the uncorrected $\hat{\kappa}$, without the trade-off of bias reduction and increased variance. Similar findings have been recorded in Phillips and Yu (2009) regarding this feature of bias reduction based on the indirect inference approach. We note that when κ is small (0.1), the feasible bias does not capture well the true bias. (This is most pronounced when μ is unknown, but still $\hat{\kappa}_{bc}$ is much less biased than $\hat{\kappa}$.) Recall that $\phi = \exp(-\kappa h)$, so this corresponds to a discrete AR(1) process with $\phi = \exp(-0.1/12) = 0.9917$. Upon carefully examining the simulation results when μ is unknown, we find that in this near unit-root case, while the variance of $\hat{\phi}$ is small, the variance of the estimated intercept $\hat{\alpha}$ is very big in small samples, resulting in a very wide range of $\hat{\mu}$, which in turn substantially distorts the performance of our bias formulae.

4. Conclusions

Lévy processes have found increasing applications in economics and finance. It has been documented, however, that the typical quasi maximum likelihood estimation procedure tends to over estimate the mean reversion parameter in continuous-time Lévy processes. Based on the technique of Bao (2013), we have derived several analytical formulae to approximate the finite-sample bias of the estimated mean reversion parameter under different cases: known or unknown long-run mean, fixed or random initial condition. Our simulation results indicate in general good performance of the approximate bias formulae in capturing the true bias behaviors of the mean reversion estimator and good performance of our feasible bias corrected estimator. When the

$$\boldsymbol{H}_1 = \frac{1}{n}\begin{pmatrix} -n & h\phi\boldsymbol{\iota}'\boldsymbol{x}_{-1} \\ h\phi\boldsymbol{\iota}'\boldsymbol{x}_{-1} & h^2\phi\boldsymbol{x}'_{-1}\boldsymbol{\varepsilon} - h^2\phi^2\boldsymbol{x}'_{-1}\boldsymbol{x}_{-1}\end{pmatrix},$$

$$\boldsymbol{H}_2 = \frac{1}{n}\begin{pmatrix} 0 & 0 & 0 & -h^2\phi\boldsymbol{\iota}'\boldsymbol{x}_{-1} \\ 0 & -h^2\phi\boldsymbol{\iota}'\boldsymbol{x}_{-1} & -h^2\phi\boldsymbol{\iota}'\boldsymbol{x}_{-1} & -h^3\phi\boldsymbol{x}'_{-1}\boldsymbol{\varepsilon} + 3h^3\phi^2\boldsymbol{x}'_{-1}\boldsymbol{x}_{-1}\end{pmatrix} \tag{2.11}$$

Box I.

$$\begin{aligned}\mathrm{B}(\hat{\kappa}) = {} & \frac{5 + 2e^{h\kappa} + e^{2h\kappa} + 4e^{-2(n-1)h\kappa}}{2T} + \frac{2[e^{-2nh\kappa} - e^{-2(n-1)h\kappa}](x_0-\mu)^2}{T\sigma_\varepsilon^2} \\ & + \frac{\left(1-e^{-nh\kappa}\right)[2e^{h\kappa} + 13e^{2h\kappa} + 4e^{3h\kappa} + e^{4h\kappa} + e^{-(n-4)h\kappa} + 2e^{-(n-3)h\kappa} + 9e^{-(n-2)h\kappa}]}{2\left(1-e^{2h\kappa}\right)Tn} \\ & + \frac{\left(1-e^{-nh\kappa}\right)[e^{h\kappa} + 5e^{-(n-1)h\kappa}]\left(x_0^2+\mu^2\right)}{Tn\sigma_\varepsilon^2} + \frac{\left(1-e^{-nh\kappa}\right)[5 + e^{2h\kappa} + 5e^{-(n-2)h\kappa} + 9e^{-nh\kappa}](x_0-\mu)^2}{2Tn\sigma_\varepsilon^2} \\ & - \frac{2\left(1-e^{-nh\kappa}\right)[e^{-h\kappa} - e^{h\kappa} + e^{3h\kappa} + 5e^{-(n-1)h\kappa}]x_0\mu}{Tn\sigma_\varepsilon^2} - \frac{\gamma_1\left(1-e^{-nh\kappa}\right)[e^{-(n-1)h\kappa} + e^{-(n-2)h\kappa}](x_0-\mu)}{Tn\sigma_\varepsilon}\end{aligned} \tag{2.12}$$

Box II.

$$\begin{aligned}\mathrm{B}(\hat{\kappa}) = {} & \frac{5 + 2e^{h\kappa} + e^{2h\kappa} + 4e^{-2(n-1)h\kappa}}{2T} + \frac{2[e^{-2nh\kappa} - e^{-2(n-1)h\kappa}]\mu^2}{T\sigma_\varepsilon^2} \\ & + \frac{\left(1-e^{-nh\kappa}\right)[2e^{h\kappa} + 13e^{2h\kappa} + 4e^{3h\kappa} + e^{4h\kappa} + e^{-(n-4)h\kappa} + 2e^{-(n-3)h\kappa} + 9e^{-(n-2)h\kappa}]}{2\left(1-e^{2h\kappa}\right)Tn} \\ & + \frac{\left(1-e^{-nh\kappa}\right)[5 + 2e^{h\kappa} + e^{2h\kappa} + 10e^{-(n-1)h\kappa} + 5e^{-(n-2)h\kappa} + 9e^{-nh\kappa}]\mu^2}{2Tn\sigma_\varepsilon^2} + \frac{\gamma_1\left(1-e^{-nh\kappa}\right)[e^{-(n-1)h\kappa} + e^{-(n-2)h\kappa}]\mu}{Tn\sigma_\varepsilon}\end{aligned}$$

Box III.

$$\begin{aligned}\mathrm{B}(\hat{\kappa}) = {} & \frac{5 + 2e^{h\kappa} + e^{2h\kappa} + 4e^{-2(n-1)h\kappa}}{2T} - \frac{2\left(1-e^{-nh\kappa}\right)(e^{-h\kappa} - 2e^{h\kappa} + e^{3h\kappa})\mu^2}{Tn\sigma_\varepsilon^2} \\ & + \frac{\left(1-e^{-nh\kappa}\right)[2e^{h\kappa} + 13e^{2h\kappa} + 4e^{3h\kappa} + e^{4h\kappa} + e^{-(n-4)h\kappa} + 2e^{-(n-3)h\kappa} + 9e^{-(n-2)h\kappa}]}{2\left(1-e^{2h\kappa}\right)Tn}\end{aligned}$$

Box IV.

Table 2
Bias and bias correction, unknown μ.

κ	*Bias*	$\hat{\mathrm{B}}(\hat{\kappa})$	$\hat{\kappa}$	$std(\hat{\kappa})$	$\hat{\kappa}_{bc}$	$std(\hat{\kappa}_{bc})$
$\mu = 0.1$ and x_0 fixed						
0.1	0.1000	0.0586	0.2000	0.1122	0.1414	0.1005
0.3	0.0908	0.0732	0.3908	0.1438	0.3176	0.1402
0.5	0.0889	0.0772	0.5889	0.1709	0.5117	0.1689
0.7	0.0885	0.0792	0.7885	0.195	0.7092	0.1935
0.9	0.0886	0.0807	0.9886	0.2172	0.9079	0.2158
$\mu = 0.1$ and x_0 random						
0.1	0.1255	0.0687	0.2255	0.1136	0.1568	0.1067
0.3	0.1114	0.0768	0.4114	0.1414	0.3346	0.1393
0.5	0.1091	0.0792	0.6091	0.1690	0.5299	0.1677
0.7	0.1087	0.0807	0.8087	0.1935	0.7280	0.1924
0.9	0.1088	0.0818	1.0088	0.2159	0.9270	0.2148

mean reversion parameter is near its lower bound 0, which corresponds to the near unit-root case, however, our simulation results reveal that the feasible bias does not approximate well the true bias.

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