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An Improved Bayesian Unit Root Test in Stochastic Volatility Models*

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A new posterior odds analysis is developed to test for a unit root in volatility dynamics in the context of stochastic volatility models. Our analysis extends the Bayesian unit root test of So and Li (1999) in two important ways. First, a mixed informative prior distribution with a random weight is introduced for the Bayesian unit root testing in volatility. Second, a numerically more stable algorithm is introduced to compute Bayes factor, taking into account the special structure of the competing models. It can be shown that the approach introduced overcomes the problem of the diverging "size" in the marginal likelihood approach by So and Li (1999) and improves the "power" of the unit root test. A simulation study is used to investigate the finite sample performance of the improved method and an empirical study implements the proposed method and the unit root hypothesis in volatility is rejected.

Key Words: Bayes factor; Markov chain Monte Carlo; Posterior odds ratio; Stochastic volatility models; Unit root testing.

1. INTRODUCTION

Whether or not there is a unit root in volatility of financial assets has been a long-standing topic of interest to econometricians and empirical economists. There are several reasons for this attention. First, the property of unit root has important implications for the risk premium and asset allocations. For example, compared to a stationary volatility, volatility with a unit root implies a stronger negative relation between the return and the volatility (Chou, 1988). When there is a unit root in volatility, a ra-

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tional investor should constantly and permanently change the weighting of assets whenever a volatility shock arrives. Second, motivated from the fact that volatility of financial assets is typically highly persistent, econometric models which allow for a unit root in volatility have appeared. Leading examples include the IGARCH model of Engle and Bollerslev (1986) and the log-normal stochastic volatility (SV) model of Harvey et al. (1994). However, there is mixed empirical evidence as to whether non-stationarity exists in volatility. Third, if there is a unit root in volatility, the frequentist's inference, which is often based on asymptotic theory, is often more much complicated; see, for example, Park and Phillips (2001) and Bandi and Phillips (2003) for the development of asymptotic theory for nonlinear models with a unit root.

In a log-normal SV model, the volatility is often assumed to follow an AR(1) model with the autoregressive coefficient ϕ. The test of unit root amounts to testing $\phi = 1$. The estimation of ϕ is complicated by the fact that volatility is latent. In recent years, numerous estimation methods have been developed to estimate SV model; see, Shephard (2005) for a review. It is possible to test for a unit root in volatility without estimating the entire SV model, however. Harvey et al.(1994) suggested a classical unit root test by estimating ϕ in the log-squared return process. There are two problems with such a test. First, ϕ is less efficiently estimated. Second, all the classical unit root tests suffer from large size distortions because the log-squared return process follows an ARMA(1,1) model with a large negative MA root. This problem is well known in the unit root literature; see, for example, Schwert (1989). To overcome the second problem, Wright (1999) proposed to use the unit root test of Perron and Ng (1996). The severe distortion in size is nicely mitigated although there are still some distortions left in some parameter settings.

To deal with the first problem, So and Li (1999, SL hereafter) proposed a Bayesian unit root test approach based on the Bayes factor (BF). The test is implemented in two stages. At stage 1, the two competing models are estimated by the Bayesian MCMC method. As a full likelihood-based method, MCMC provides a more efficient estimate of ϕ than the least squares estimate and other estimates of ϕ in the log-squared return process, provided the model is corrected specified, see Andersen et al. (1999). At stage 2, the BF is obtained from the MCMC samples. The BF is an important statistic in the Bayesian literature and has served as the gold standard for Bayesian model testing and comparison for a long time (Kass and Raftery, 1995; Geweke, 2007). However, it is necessary to point out that the impact of prior specifications on BF is different from that on estimation. For estimation, it is well-known that in large samples, prior distributions can be picked for convenience because their effects on posterior distributions are insignificant (Kass and Raftery, 1995). For BF, standard improper noninformative priors can not be applied since such priors are defined only up to a constant, hence the resulting BF is a multiple of an arbitrary constant. In fact, as pointed out by Kass and Raftery (1995), if a prior with a very large spread is used on some parameter under a model to make it "noninformative", this behavior will force the BF to favor its competitive model. This problem is well-known as Jeffreys-Lindley-Bartlett's paradox in the Bayesian literature. Consequently, it should be very careful to apply the noninformative prior for a unit root testing problem.

To avoid the difficulty, the prior distributions are generally taken to be proper and not have too big a spread. Moreover, it is often suggested that for Bayesian model comparison, an equal model prior should be used. This practice was followed by SL. However, it is now known in the unit root literature that if a proper prior is adopted for parameters and an equal weight is used to represent the prior model ignorance, there is a bias toward stationary models; see, for example, Phillips (1991) and Ahking (2009). To alleviate this problem, our main contribution of the paper is to propose a mixed prior distribution with a random weight for the unit root test. The main idea is that when the prior information is not available, we can obtain an estimate for the random weight when a vague prior is assigned. If the data are generated from a unit root process, it can be expected that a larger weight is assigned to the unit root process. In other words, we use it to adjust the bias towards stationarity in the posterior odds analysis for unit root with the estimated weight. This idea is related to what was proposed by Kalaylioglu and Ghosh (2009). However, a key difference between our work and theirs is that we use the BF to compare the competing models while Kalaylioglu and Ghosh used the Bayesian credible interval.

In the literature, the computation of the BF often involves high-dimensional integration and hence numerically demanding. SL applied the marginal likelihood approach proposed by Chib (1995) to estimate the BF for the unit root test. This approach is very general and has a very wide applicability. However, for the SV models, the dimension of the parameters and the latent volatilities is very high, the marginalization of the joint probability density over the parameters and the latent variable poses a formidable computational challenge. In this paper, instead of calculating the marginal likelihood, we derive a novel form for the BF by taking into account the special structure of the competing models. In the new form, no marginalization is needed and hence numerically it is more stable. It is shown that this evaluation of the BF in the new form is a by-product of Bayesian MCMC estimation and hence it is trivial to compute. This idea is related to Jacquier et al. (2004), Kou et al. (2005), Nicholae et al.(2008), Liu and Li (2014), etc.

The remainder of this paper is organized as follows. In Section 2, we describe the simple log-normal SV model and the problem of the unit root test. In Section 3, the new approach for the posterior odds analysis of unit root is discussed. The performance of the proposed unit root test procedure is examined using simulation data in Section 4. Section 5 considers some empirical applications. This paper is concluded in Section 6.

2. STOCHASTIC VOLATILITY MODELS

The simple log-normal SV model is of the form:

$$y_t = \exp(h_t/2)u_t,\ u_t \sim N(0,1), \tag{1}$$

$$h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t,\ v_t \sim N(0,1), \tag{2}$$

where $t = 1, 2, \cdots, n$, y_t is the continuously compounded return, h_t the unobserved log-volatility, $h_0 \sim N\left(\tau, \frac{\sigma^2}{1-\phi^2}\right)$ when $|\phi| < 1$, $h_0 \sim N(\tau, \sigma^2)$ when $\phi = 1$, and (u_t, η_t) independently standard normal variables for all t. This model can explain several important stylized facts in the financial time series including volatility clustering, and its continuous time version has been used to price options.

The primary concern of our paper is to test $\phi = 1$ against $|\phi| < 1$. SL (1999) proposed a test by first estimating two competing models by a powerful MCMC algorithm – Gibbs sampler. This Bayesian simulation based method generates samples from the joint posterior distribution of the parameters and the latent volatility (so the data augmentation technique is adopted here). After that, the posterior odds ratio was calculated using the marginal likelihood method of Chib (1995).

To fixed the idea, let $p(\boldsymbol{\theta})$ be the prior distribution of the unknown parameter θ ($:= (\tau, \sigma, \phi)$ or (τ, σ) in the unit root case), $\mathbf{y} = (y_1, \cdots, y_n)$ the observation vector, $\mathbf{h} = (h_1, \cdots, h_n)$ the vector of the latent variables. Exact maximum likelihood methods are not possible because the likelihood $p(\mathbf{y}|\theta)$ does not have a closed-form expression. Bayesian methods overcome this difficulty by the data-augmentation strategy (Tanner and Wong, 1987), namely, the parameter space is augmented from θ to $(\theta, \mathbf{h})$. By successive conditioning and assuming prior independence in θ, the joint prior density is

$$p(\tau, \sigma, \phi, \mathbf{h}) = p(\tau)p(\sigma)p(\phi)p(h_0)\prod_{t=1}^{n} p(h_t|h_{t-1}, \theta). \tag{3}$$

The likelihood function is

$$p(\mathbf{y}|\theta, \mathbf{h}) = \prod_{t=1}^{n} p(y_t|h_t). \tag{4}$$

Obviously, both the joint prior density and the likelihood function are available analytically provided analytical expressions for the prior distributions of θ are supplied. By Bayes' theorem, the joint posterior distribution of the unobservables given the data is given by,

$$p(\tau, \sigma, \phi, \mathbf{h}|\mathbf{y}) \propto p(\tau)p(\sigma)p(\phi)p(h_0)\prod_{t=1}^{n} p(h_t|h_{t-1}, \theta)\prod_{t=1}^{n} p(x_t|h_t). \tag{5}$$

Gibbs sampler was used by SL to generate correlated samples from the joint posterior distribution (5). In particular, it samples each variate, one at a time, from (5). When all the variates are sampled in a cycle, we have one sweep. The algorithm is then repeated for many sweeps with the variates being updated with the most recent samples, producing draws from Markov chains. With regularity conditions, the draws converge to the posterior distribution at a geometric rate. By the ergodic theorem for Markov chains, the posterior moments and marginal densities may be estimated by averaging the corresponding functions over the sample. For example, one may estimate the posterior mean by the sample mean, and obtain the credible interval from the marginal density. When the simulation size is very large, the marginal densities can be regarded as the exact, enabling exact finite sample inferences.

To explain the unit root test of SL, let M_0 be the model formulated in the null hypothesis (i.e. $\phi = 1$), M_1 the model formulated under the alternative hypothesis (i.e. ϕ is an unknown parameter), $\pi(M_k)$ the prior model probability density, $p(\mathbf{y}|M_k)$ the marginal likelihood of model k, and $p(M_k|\mathbf{y})$ the posterior probability densities, where $k = 0, 1$. Under the Bayesian framework, testing the null hypothesis versus the alternative is equivalent to comparing the two competing models, M_0 versus M_1. Given the prior model probability density $\pi(M_0)$ and $\pi(M_1) = 1-\pi(M_0)$, the data $\mathbf{y}$ produce a posterior model density, $p(M_0|\mathbf{y})$ and $p(M_1|\mathbf{y}) = 1 - p(M_0|\mathbf{y})$.

Bayes' theorem gives rise to

$$\frac{p(M_0|\mathbf{y})}{p(M_1|\mathbf{y})} = \frac{p(\mathbf{y}|M_0)}{p(\mathbf{y}|M_1)} \times \frac{\pi(M_0)}{\pi(M_1)} \tag{6}$$

that is

$$\text{Posterior Odds Ratio (POR)} = \text{Bayes Factor (BF)} \times \text{Prior Odds Ratio} \tag{7}$$

or

$$\log_{01}(\text{POR}) = \log_{01}\ (\text{BF}) + \log_{01}(\text{Prior Odds Ratio}), \tag{8}$$

where the BF is defined as the ratio of the marginal likelihood of the competing models. If the prior odds is set to 1, as it is done in much of the

Bayesian literature, the posterior odds takes the same value as the BF. When the posterior odds is larger than 1, M_0 is favored over M_1 and vice versus. In SL, the sign of $\log_{01}$(BF) was checked. If it is positive, M_0 is favored over M_1. In general, one has to check the sign of $\log_{01}$(POR).

The marginal likelihood, $p(\mathbf{y}|M_k)$, can be expressed as

$$p(\mathbf{y}|M_k) = \int_{\Omega_k \cup \Omega_h} p(\mathbf{y}, \mathbf{h}|\theta_k, M_k) p(\theta_k|M_k) d\mathbf{h} d\boldsymbol{\theta}_k, \tag{9}$$

where Ω_k and Ω_h are the support of θ_k and $\mathbf{h}$, respectively. Alternatively, the marginal likelihood can be expressed as

$$p(\mathbf{y}|M_k) = \int_{\Omega_k} p(\mathbf{y}|\theta_k, M_k) p(\theta_k|M_k) \mathbf{d}\theta_k. \tag{10}$$

As solving the integrals in (9) and (10) requires high-dimensional numerical integration, Chib (1995) suggested evaluating the marginal likelihood by rearranging Bayes' theorem

$$p(\mathbf{y}|M_k) = \frac{p(\mathbf{y}|\theta_k, M_k) p(\theta_k|M_k)}{p(\theta_k|\mathbf{y}, M_k)}.$$

Thus, the log-marginal likelihood may be calculated by

$$\ln p(\mathbf{y}|\theta_k, M_k) + \ln p(\theta_k|M_k) - \ln p(\theta_k|\mathbf{y}, M_k) \tag{11}$$

where θ_k is an appropriately selected high density point in estimated M_k and Chib suggested using the posterior mean, $\overline{\theta}_k$. The first term of Equation (11) is the log-likelihood evaluated at $\overline{\theta}_k$. Since it is marginalized over the latent volatilities, $\mathbf{h}$, it is computationally demanding and possibly numerically unstable. The second term is the log prior density evaluated at $\overline{\theta}_k$ and has to be specified by the econometrician. The third quantity involves the posterior density which is only known up to a normality constant. The approximation can be obtained by using a multivariate kernel density estimate based on the posterior MCMC sample of θ_k.

To estimate θ, SL used the flat normal prior for τ, an inverse Gamma prior for σ^2. For ϕ, four different priors were used – uniform on the interval (0,1), truncated normal on (0,1), two truncated Beta on (0,1). For the unit root test, the prior odds is set to 1. This choice was argued to reflect prior ignorance. Simulation studies were conducted by SL to check the performances of their Bayesian unit root test. While in general, their test perform reasonably well, we identify some problems. It is noted that the "size" diverges with the sample size. Namely, when the sample size gets larger, the probability for the test to pick M_0 when the data are simulated from M_0 is getting smaller. Since their empirical results suggest that

M_1 is favored over M_0, concerns about the diverged "size" are especially important. Second, when ϕ is very close to 1, the test does not seem to have good "power" properties.

We argue that there is an obvious inconsistency between the choice of the prior of ϕ and the choice of the prior odds. On the one hand, using a prior density whose support exclude $\phi = 1$ means that the researcher has no prior confidence about M_0. On the other hand, setting the prior odds to 1 implies that the researcher is equally confident about the two competing model. It is well known in the unit root literature that the posterior distribution is sensitive to the prior specification; see, for example, Phillips (1991), and the discussion and the rejoinder in the same issue. From Equation (6) it is obvious that the prior odds is important. As a result, it is reasonable to believe that the diverged "size" may be due to the choice of the priors.

Consequently, we suggest two ways to improve the unit root test of SL. First, a computationally easier and numerically more stable algorithm is introduced to compute the BF, taking into account the special structure of the competing models. Our method completely avoids the calculation of marginal likelihood. Second, different priors for ϕ and the model specification are employed. Our priors of ϕ allow for a positive mass at unity. More important, a mixed model prior with random weights is used.[1]

3. IMPROVED BAYESIAN UNIT ROOT TESTING

3.1. A Set of Hierarchial Priors

Since we are concerned about the suitability of a prior for ϕ over $(-1, 1)$ for the unit root test, we first broaden the support of the prior distribution. In particular, we consider the prior densities that assign a positive mass at unity. To be more specific, the prior is set to

$$f(\phi) = \pi I(\phi = 1) + (1 - \pi) f_C(\phi) I(-1 < \phi < 1), \tag{12}$$

where $I(x)$ is indicator function such that $I(x) = 1$ if x is true and 0 otherwise, π the weight that represents the prior probability for model M_0, and $f_C(\phi)$ a proper distribution that will be specified later. When $\pi > 0$, a

[1]We need point out that approaches that serve as alternatives to BF have been proposed for hypothesis testing in the literature. For example, Bernardo and Rueda (2002) demonstrated that for the point null hypothesis testing, the BF can be regarded as a decision problem with a simple zero-one loss function. The idea was followed by Li and Yu (2012), Li, Zeng and Yu (2014) and Li, Liu and Yu (2015) where different continuous loss functions or net loss functions were proposed. The justification of these extensions is made by large sample theory under repeated sampling. However, these approaches are difficult to apply because the traditional large sample theory are not held in the model under the null hypothesis due to the presence of unit root.

positive mass is assignment to model M_0.[2]The mixed prior of this kind has been widely used in the unit root literature; see, for example, Sim (1988) and Schotman and van Dijk (1991). In the SV literature, the same prior was used in Kalayliglu and Ghosh (2009).

As discussed before, when $\pi(M_0) = \pi(M_1) = 0.5$, POR takes the same value as the BF, justifying the use of the BF for Bayesian model comparison. However, since we assign probability π to model M_0, when we specify the prior for ϕ, we have to assign $\pi(M_0) = \pi$ to be logically consistent. In this case, the prior odds is $\pi/(1 - \pi)$. One choice for π is to set $\pi = 1/2$. If so, POR is the same as the BF and we cannot improve the power of the unit root test of SL. It is known in the unit root literature this prior tends favor stationary or trend-stationary hypothesis; see, for example, Ahking (2009).

Alternatively, a uniform distribution over $[0, 1]$ may be used for the hierarchical specification of π to represent the prior ignorance. Based on the mixture prior specification, Kalaylioglu and Ghosh (2009) used the posterior confidence interval for unit root testing. Although the credible interval approach is simple to implement, it has some practical difficulties, as pointed out in Robert (2002). First, the confidence interval is not unique. Second, the credible interval approach typically does not have good test behavior. Kalaylioglu and Ghosh used the 95% symmetric posterior confidence interval for unit root testing. Under the uniform hierarchical prior specification, it can be found that, when the sample size was 500 and 1000, the "size" of the test is 0.21 and 0.11, suggesting the test is seriously distorted. Perhaps a better choice for credible intervals is the highest posterior density (HPD) credible region. The computation of the HPD credible region is usually more demanding; see Chen et al. (2000). In this paper, we deviate from Kalaylioglu and Ghosh by using the posterior odds for unit root testing.

Ideally, a training sample should be selected to help determine the mean of π (denoted by $\overline{\pi}$), that may be used to compute the prior odds $\overline{\pi}/(1-\overline{\pi})$.

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[2]In the unit root literature, for the autoregressive coefficient, an "objective" ignorance prior is the so-called Jeffreys or reference prior of Jeffreys (1961) and Berger and Bernardo (1992). As shown in Phillips (1991) these priors are intended to represent a state ignorance about the value of the autoregression coefficient and are very different from flat priors in the unit root testing problem. Unfortunately, these priors are improper and $p(\theta_k|M_k) = C_k f(\theta_k)$ where $f(\theta_k)$ is a nonintegrable function and C_k is an arbitrary positive constant. As a result the posterior odds can be rewritten as:

$$\mathrm{POR} = \mathrm{BF} = \frac{C_0}{C_1} \frac{\int_{\Omega_0 \cup \Omega_h} p(\mathbf{y}, \mathbf{h}|\theta_0, M_0) f(\theta_0) d\mathbf{h} d\theta_0}{\int_{\Omega_1 \cup \Omega_h} p(\mathbf{y}, \mathbf{h}|\theta_1, M_1) f(\theta_1) d\mathbf{h} d\theta_1} \tag{13}$$

Thus, the posterior odds and the BF are not well defined since they both depend on the arbitrary constants C_0/C_1. This is the reason why we decide not to use the Jeffrey's prior to do the posterior odds analysis for unit root.

When $\overline{\pi} \neq 0.5$, the POR no longer takes the same value as the BF. If $\overline{\pi} > 0.5$, $\log_{01}(\overline{\pi}/(1-\overline{\pi})) > 0$ and more weight will be assigned to the positive mass at unity. In this case, compared with the BF, the POR will be more in favor of the unit root hypothesis. It is expected that this feature should improve the power of the test because if data indeed come from a unit root model, it is expected that $\overline{\pi} > 0.5$. When data are generated from a stationary model, it is expected that $\overline{\pi} < 0.5$. Instead of splitting the entire sample into the training sample and the sample for estimation, we estimate π from the entire sample in order to get a precise estimate of π. By using the same data to estimate π and the prior odds ratio as well as calculate the BF, strictly speaking, our approach is not a full Bayesian method. Our idea of estimating π, however, was partly inspired by Aitkin (1991) and Schotman and van Dijk (1991). In Aitkin (1991) the data are re-used to get the prior distributions for the parameters while in Schotman and van Dijk (1991) the threshold parameter of the defined interval for ϕ is dependent on the data.

3.2. Computing Posterior Odds

Although the marginal likelihood approach proposed by Chib (1995) is very general and has been applied in various studies (Kim, et al 1998; Chib et al, 2002; Berg et al, 2004), it requires one to calculate the log-likelihood functions $\ln p(\mathbf{y}|\theta_k, M_k)$, $k = 0, 1$. For the SV models, this is a challenging task. In this paper, we acknowledge that unit root testing is a special model comparison problem which has the special structure to link the competing models. The structure is that the two marginal likelihood functions have the common latent variable which may be exploited to facilitate the computation of BF. Instead of calculating the two marginal likelihood functions as suggested in Chib (1995), in our method we only need to compute BF directly.

In the literature, Jacquier et al. (2004) proposed an efficient method to compute BF for comparing the basic SV model with the fat-tailed SV model. They showed that in the case the BF can be written as the expectation of the ratio of un-normalized posteriors with respect to the posterior under the fat-tailed SV model. In addition, Kou et al. (2005) and Nicolae et al. (2008) showed that for nested models, BF can be written as the posterior mean of the likelihood ratio between the two competing models. Liu and Li (2014) also generalize these ideas by showing that the BF for unit rooting testing also can be written as the complete likelihood ratio of posterior quantities by introducing an appropriate weight function for SV models with jumps. More details about BF for unit root testing, one can refer to Liu and Li (2014) and reference therein.

Following their idea, let $\boldsymbol{\theta}_0 = (\mu, \sigma^2), \boldsymbol{\theta}_1 = (\mu, \phi, \sigma^2)$ and note that

$$\begin{aligned} B_{01} &= \int_{\Omega_0 \cup \Omega_h} \frac{p(\boldsymbol{\theta}_0|M_0)p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_0, M_0)}{p(\mathbf{y}|M_1)} \mathrm{d}\boldsymbol{\theta}_0 \mathrm{d}\mathbf{h} \\ &= \int_{\Omega_1 \cup \Omega_h} \frac{p(\boldsymbol{\theta}_0|M_0)p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_0, M_0)w(\phi|\boldsymbol{\theta}_0)}{p(\mathbf{y}|M_1)} \mathrm{d}\phi \mathrm{d}\boldsymbol{\theta}_0 \mathrm{d}\mathbf{h} \\ &= \int_{\Omega_1 \cup \Omega_h} p(\boldsymbol{\theta}_0|M_0)p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_0, M_0)w(\phi|\boldsymbol{\theta}_0) \frac{p(\mathbf{h}, \boldsymbol{\theta}_1|\mathbf{y}, M_1)}{p(\mathbf{y}, \mathbf{h}, \boldsymbol{\theta}_1|M_1)} \mathrm{d}\phi \mathrm{d}\boldsymbol{\theta}_1 \mathrm{d}\mathbf{h} \\ &= \int_{\Omega_1 \cup \Omega_h} \frac{p(\boldsymbol{\theta}_0|M_0)w(\phi|\theta_0)p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_0, M_0)}{p(\boldsymbol{\theta}_1|M_1)p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_1, M_1)} p(\mathbf{h}, \boldsymbol{\theta}_1|\mathbf{y}, M_1) \mathrm{d}\phi \mathrm{d}\boldsymbol{\theta}_1 \mathrm{d}\mathbf{h} \end{aligned}$$

where $w(\boldsymbol{\phi}|\boldsymbol{\theta}_0)$ is the an arbitrary weight function of ϕ conditional on $\boldsymbol{\theta}_0$ such that

$$\int w(\phi|\boldsymbol{\theta}_0)\mathrm{d}\phi = 1$$

In practice, the prior distribution of the common parameter vector $\boldsymbol{\theta}_0$ under two models is often specified as the same, that is $p(\boldsymbol{\theta}_0|M_0) = p(\boldsymbol{\theta}_0|M_1)$. Furthermore, for the purpose of the posterior odds analysis, $p(\phi|\boldsymbol{\theta}_0, M_1)$ is required to be a proper conditional prior distribution. This distribution can be regarded as a weight function, then, $p(\phi|\boldsymbol{\theta}_0, M_1)p(\boldsymbol{\theta}_0|M_1) = p(\boldsymbol{\theta}_1|M_1)$, hence,

$$\begin{aligned} B_{01} &= \int_{\Omega_1 \cup \Omega_h} \frac{p(\boldsymbol{\theta}_0|M_0)p(\phi|\boldsymbol{\theta}_0, M_1)p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_0, M_0)}{p(\boldsymbol{\theta}_1|M_1)p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_1, M_1)} p(\boldsymbol{h}, \boldsymbol{\theta}_1|\boldsymbol{y}, M_1) \mathrm{d}\phi \mathrm{d}\boldsymbol{\theta}_1 \mathrm{d}\mathbf{h} \\ &= \int_{\Omega_1 \cup \Omega_h} \frac{p(\boldsymbol{\theta}_0|M_1)p(\phi|\boldsymbol{\theta}_0, M_1)p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_0, M_0)}{p(\boldsymbol{\theta}_1|M_1)p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_1, M_1)} p(\mathbf{h}, \boldsymbol{\theta}_1|\mathbf{y}, M_1) \mathrm{d}\phi \mathrm{d}\boldsymbol{\theta}_1 \mathrm{d}\mathbf{h} \\ &= \int_{\Omega_1 \cup \Omega_h} \frac{p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_0, M_0)}{p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_1, M_1)} p(\mathbf{h}, \boldsymbol{\theta}_1|\mathbf{y}, M_1) \mathrm{d}\phi \mathrm{d}\boldsymbol{\theta}_1 \mathrm{d}\mathbf{h} \\ &= E\left\{\frac{p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_0, M_0)}{p(\mathbf{y}, \mathbf{h}|\boldsymbol{\theta}_1, M_1)}\right\} \end{aligned} \tag{14}$$

where the expectation is with respect to the posterior distribution $p(\mathbf{h}, \boldsymbol{\theta}_1|\mathbf{y}, M_1)$.

From (14), it can be seen that the BF is only a by-product of Bayesian estimation of the SV model in the alternative hypothesis, namely, under the stationary case. Once draws from Markov chains are available, the BF can be approximated conveniently and efficiently by averaging over the MCMC draws. In fact, only one line of code is needed to compute the BF. In detail, let $\{\boldsymbol{h}^{(s)}, \boldsymbol{\theta}_1^{(s)}\}$, $s = 1, 2, \cdots, S$, be the draws, generated by the MCMC technique, from the posterior distribution $p(\mathbf{h}, \boldsymbol{\theta}_1|\mathbf{y}, M_1)$. The BF is approximated by:

$$\hat{B}_{01} = \frac{1}{S}\sum_{s=1}^{S}\left\{\frac{p(\mathbf{y},\boldsymbol{h}^{(s)}|\boldsymbol{\theta}_0^{(s)},M_0)}{p(\mathbf{y},\boldsymbol{h}^{(s)}|\boldsymbol{\theta}_1^{(s)},M_1)}\right\}$$

When the prior odds ratio is known, one can easily obtain the posterior odds ratio as in (6) for the unit root test.

In the context of the simple log-normal SV model, suppose $\theta^{(1)},...,\theta^{(S)}$ and $h^{(1)},...,h^{(S)}$ are the MCMC draws, then

$$\hat{B}_{01} = \frac{1}{S}\sum_{s=1}^{S}\exp\left\{-\frac{\sum_{t=2}^{n}(1-\phi^{(s)})(\mu^{(s)}-h_{t-1}^{(s)})(2h_t^{(s)}-h_{t-1}^{(s)}(1+\phi^{(s)})-\mu^{(s)}(1-\phi^{(s)}))}{2\left(\tau^{(s)}\right)^2}\right\}. \tag{15}$$

Hence, the posterior odds can be given by

$$\frac{p(M_0|\mathbf{y})}{p(M_1|\mathbf{y})} \approx \hat{B}_{01}\times\frac{\hat{\pi}}{1-\hat{\pi}} \tag{16}$$

where $\hat{\pi}$ is the plug-in estimate using the uniform hierarchical prior specification.

4. A SIMULATION STUDY

In this section, we check the reliability of the proposed Bayesian unit root test procedure using simulated data. For the purposes of comparison, the same design as in SL is adopted. In particular, for ϕ, three true values are considered, 1, 0.98, 0.95, corresponding to the nonstationary case, the nearly nonstationary case, and the stationary case. The other two parameters are set at $\tau = -9, \sigma^2 = 0.1$. These values are empirically reasonable for daily equity returns. Three different sample sizes have been considered, 500, 1000 and 1500. The number of replications is always fixed at 100.

For the mixed prior of ϕ, three distributions have been considered for $f_C(\phi)$ in (12), namely, $U(0,1)$, $Beta(10,1)$, $Beta\ (20,2)$.[3] These three distributions were used as the priors for ϕ in SL. A key difference is that we mix them with a point mass at unity with probability π and estimate π from actual data. Both the pure priors and the mixed prior are implemented in combination with our new way of computing the posterior odds. Denote the Bayesian estimator in association with a pure prior by $\widetilde{\phi}$ and that in association with the mixed prior of the form (12) by $\widehat{\phi}$.

[3]SL used four prior distributions for ϕ. When implementing them in WinBUGS, unfortunately, we found there was a trap error with the truncated normal prior. As a result, the truncated normal is not considered here.

It is important to emphasize that our proposed unit root approach involves two steps. In the first step, the uniform prior defined in the interval (0,1) is assigned to the weight π and a MCMC algorithm is implemented to fit the stationary model and to produce a Bayesian estimate for π. In the second step, based on the estimated weight, we compute $\log_{01}(\text{POR})$ for the unit root test using the same MCMC output.

TABLE 1.

Posterior mean of π and ϕ and $\log_{01}(POR)$ from simulated data. $\hat{\pi}$, $\hat{\phi}$, and $SE(\hat{\phi})$ are obtained using the mixed prior with f_C being $U(0,1)$. $\tilde{\phi}$, $SE(\tilde{\phi})$ are obtained using the pure prior $U(0,1)$.

n		$\phi = 1$	$\phi = 0.98$	$\phi = 0.95$
500	$\hat{\pi}$	0.660398	0.594336	0.453336
	$SE(\hat{\pi})$	0.239676	0.263729	0.269372
	$\hat{\phi}$	0.999672	0.992187	0.957713
	$SE(\hat{\phi})$	0.001221	0.011537	0.029694
	$\log_{01}(\text{POR})$	0.660653	−0.465388	−1.767029
500	$\tilde{\phi}$	0.994510	0.972956	0.942451
	$SE(\tilde{\phi})$	0.003729	0.013400	0.026063
1000	$\hat{\pi}$	0.657433	0.489338	0.354721
	$SE(\hat{\pi})$	0.239337	0.271226	0.244838
	$\hat{\phi}$	0.999496	0.985573	0.952437
	$SE(\hat{\phi})$	0.000821	0.010954	0.017481
	$\log_{01}(\text{POR})$	0.571005	−1.552973	−3.679288
1000	$\tilde{\phi}$	0.996557	0.977271	0.949703
	$SE(\tilde{\phi})$	0.002026	0.008692	0.016123
1500	$\hat{\pi}$	0.659380	0.410694	0.335646
	$SE(\hat{\pi})$	0.238388	0.259901	0.236373
	$\hat{\phi}$	0.999708	0.982465	0.951085
	$SE(\hat{\phi})$	0.000428	0.008408	0.012562
	$\log_{01}(\text{POR})$	0.621857	−2.522120	−5.839594
1500	$\tilde{\phi}$	0.997878	0.978930	0.950531
	$SE(\tilde{\phi})$	0.001234	0.006725	0.012459

Following the suggestion of Meyer and Yu (2000), we make the use of a freely available Bayesian software, WinBUGS, to do the Gibbs sampling. WinBUGS provides an easy and efficient implementation of the Gibbs sampler. It has been extensively used to estimate various univariate and multivariate SV models in the literature; see for example, Yu (2005) and Yu and Meyer (2006). In each case, we simulated 15000 samples with 10000 discarded as burn-in samples. The simulation studies are implemented using R2WinBUGS (Sturtz, Ligges, and Gelman, 2005).

TABLE 2.

Posterior mean of π and ϕ and $\log_{01}(POR)$ from simulated data. $\hat{\pi}$, $\hat{\phi}$, and $SE(\phi)$ are obtained using the mixed prior with f_C being $Beta(10,1)$. $\tilde{\phi}$, $SE(\tilde{\phi})$ are obtained using the pure prior $Beta(10,1)$.

n		$\phi = 1$	$\phi = 0.98$	$\phi = 0.95$
500	$\hat{\pi}$	0.613521	0.462330	0.368495
	$SE(\hat{\pi})$	0.257304	0.273583	0.253513
	$\hat{\phi}$	0.997468	0.978772	0.952226
	$SE(\hat{\phi})$	0.003738	0.015827	0.025402
	$\log_{01}$(POR)	0.436139	−0.870776	−1.798431
500	$\tilde{\phi}$	0.992543	0.971148	0.947974
	$SE(\tilde{\phi})$	0.004852	0.014535	0.023358
1000	$\hat{\pi}$	0.632616	0.393827	0.336252
	$SE(\hat{\pi})$	0.251075	0.260068	0.237112
	$\hat{\phi}$	0.999229	0.980816	0.948146
	$SE(\hat{\phi})$	0.001390	0.010063	0.016493
	$\log_{01}$(POR)	0.496278	−1.646592	−3.905835
1000	$\tilde{\phi}$	0.996644	0.978345	0.948466
	$SE(\tilde{\phi})$	0.002078	0.008649	0.016126
1500	$\hat{\pi}$	0.645280	0.361362	0.333793
	$SE(\hat{\pi})$	0.246506	0.249318	0.235888
	$\hat{\phi}$	0.999686	0.981888	0.948119
	$SE(\hat{\phi})$	0.000668	0.007208	0.012976
	$\log_{01}$(POR)	0.578791	−2.339415	−6.285648
1500	$\tilde{\phi}$	0.998080	0.980844	0.947987
	$SE(\tilde{\phi})$	0.001183	0.006389	0.013045

Tables 1-3 report the estimates of ϕ (obtained as the posterior mean of ϕ), the standard errors of ϕ (SE hereafter, defined as the mean of the standard errors of ϕ, averaged across the replications), the estimate of π, and the mean values of $\log_{01}$(POR) when the mixed priors are used. When the pure priors are used, we reports the estimates of ϕ and the SE of ϕ. The three tables correspond to the three different priors, respectively, and are compared to Table 1 in SL where the BF is calculated using the marginal likelihood method.

The following conclusions may be drawn after we examine the three tables and compare them to Table 1 in SL. First, the estimates of ϕ are always close to the true value and the SEs are always small, suggesting MCMC provides reliable estimates on ϕ with both sets of priors. Furthermore, the behavior of estimates improves (smaller bias and SE) when the sample size increases. Second, when data are generated from a unit root model, using

TABLE 3.

Posterior mean of π and ϕ and $\log_{01}(POR)$ from simulated data. $\hat{\pi}$, $\hat{\phi}$, and $SE(\phi)$ are obtained using the mixed prior with f_C being $Beta(20, 2)$. $\tilde{\phi}$, $SE(\tilde{\phi})$ are obtained using the pure prior $Beta(20, 2)$.

n		$\phi = 1$	$\phi = 0.98$	$\phi = 0.95$
500	$\hat{\pi}$	0.637654	0.504941	0.376864
	$SE(\hat{\pi})$	0.247124	0.273527	0.253784
	$\hat{\phi}$	0.997752	0.983746	0.947044
	$SE(\hat{\phi})$	0.003408	0.014309	0.025526
	$\log_{01}$(POR)	0.773400	−0.551413	−1.960837
500	$\tilde{\phi}$	0.989874	0.972451	0.942477
	$SE(\tilde{\phi})$	0.005208	0.012176	0.022994
1000	$\hat{\pi}$	0.653888	0.425385	0.336909
	$SE(\hat{\pi})$	0.241840	0.264469	0.238225
	$\hat{\phi}$	0.999518	0.981596	0.948867
	$SE(\hat{\phi})$	0.001055	0.010266	0.015704
	$\hat{log}_{01}$(POR)	0.954405	−1.579565	−3.945060
1000	$\tilde{\phi}$	0.995418	0.976659	0.948180
	$SE(\tilde{\phi})$	0.002178	0.008084	0.015282
1500	$\hat{\pi}$	0.656273	0.366009	0.333473
	$SE(\hat{\pi})$	0.239154	0.249150	0.235813
	$\hat{\phi}$	0.999561	0.979079	0.949505
	$SE(\hat{\phi})$	0.000560	0.007606	0.012501
	$\log_{01}$(POR)	0.999422	−2.668887	−6.001346
1500	$\tilde{\phi}$	0.997143	0.977704	0.949588
	$SE(\tilde{\phi})$	0.001303	0.006501	0.012410

a mixed prior always leads to better estimates of ϕ than using a pure prior. The bias is smaller and the SE is also reduced. Third, in the two stationary cases, no prior dominates the other although the pure priors tend to lead to a slightly smaller SE. There is no pattern in the bias, however. Fourth, when 500 observations are generated from a stationary model with $\phi = 0.98$ and a pure uniform prior is used, SL found that $\log_{01}(POR)$ took a wrong sign, suggesting that on average a unit root model cannot be rejected even though data are simulated from the stationary model. When the mixed priors are used, the sign of $\log_{01}(POR)$ becomes negative which is the correct sign. This piece of evidence suggests that the mixed priors improve the power of the test. Fifth, when data are generated from a unit root model, our estimate of π is always larger than 0.5. This result is encouraging and, as it will be shown below, helps improve the “size” and “power” performances of our test relative to the test of SL.

TABLE 4.

Proportion of correct decisions by the two methods

ϕ	Prior	$n = 500$	$n = 1000$	$n = 1500$
1	Uniform	83 (96)	83 (90)	82 (84)
	Mixed Uniform	96	91	91
	Beta1	70 (86)	78 (75)	84 (73)
	Mixed Beta1	76	87	90
	Beta2	82 (85)	86 (84)	89 (82)
	Mixed Beta2	88	90	92
0.98	Uniform	91 (36)	99 (64)	100 (73)
	Mixed Uniform	79	97	100
	Beta1	92 (60)	98 (66)	100 (89)
	Mixed Beta1	90	97	100
	Beta2	86 (50)	99 (80)	100 (85)
	Mixed Beta2	77	96	100
0.95	Uniform	100 (82)	100 (98)	100 (100)
	Mixed Uniform	100	100	100
	Beta1	100 (89)	100 (97)	100 (100)
	Mixed Beta1	100	100	100
	Beta2	100 (93)	100 (100)	100 (100)
	Mixed Beta2	98	100	100

Both the pure priors and mixed priors are used in conjunction with the proposed method of computing the BF. The numbers are obtained from 100 replications. The numbers in parentheses are extracted from Table 2 of SL where the marginal likelihood method is used to compute the BF.

Table 4 reports the proportion of the correct decision over the 100 replications when both the mixed priors and the pure priors are used in conjunction with the BF (15). The results for the pure priors are compared to those reported in Table 2 of SL where the marginal likelihood method was used. Several results emerge from Table 4 and the comparison of Table 4 with Table 2 of SL. First, when the marginal likelihood method is used to compute the BF, the "size" of the unit root test diverges. For example, the test of SL chooses the correct model 96%, 86% and 85% of the time when 500 observations are used but only 84%, 73% and 82% of the time when 1500 observations are used for the three priors, respectively. This result is no way satisfactory because it suggests that more data does one have, less reliable the unit root test is. When the BF is computed using (15), without changing the priors of SL, we find the "size" does not diverge any more. The correct model is chosen 83%, 70%, and 82% of the time when 500 observations are used and 82%, 84%, and 89% of the time when 1500 observations are used. However, the "type I" errors are not in acceptable range.

Second, comparing the performance of the pure priors and the mixed priors, the pure priors seem to be have higher "power" than the mixed priors. However, when the sample size is large or ϕ is not so close to unity, the difference in power disappear. Moreover, the gain in "power" comes with the cost of lower "size". This is true even when the sample size is 1500. Third, formula (15) not only ensures an converged size, but also increases the power of the unit root tests, when either the pure priors or the mixed priors are used. For example, when $\phi = 0.98$ and the sample size is 1000, the marginal likelihood approach of SL has a power of 66% while the pure and the mixed Beta1 priors have a power of 98% and 97%, respectively. The gain is remarkable because there is also substantial gain in "size" at this sample size.

5. AN EMPIRICAL STUDY

In the empirical study, a direct comparison is done between this paper and SL. The data used by SL is also used for this paper.[4] To preserve space, however, we only report the empirical results for the Taiwan Stock Exchange Weighted Stock Index (TWSI). The empirical results for the other indices are qualitatively the same. In this case, we only use one common mixed prior for ϕ in which $f_C(\phi^*)$ is assumed to be $Beta(20, 1.5)$ where $\phi = 2\phi^* - 1$. We always simulated 35000 random samples with 10000 discarded as burn-in samples.

In addition to test for a unit root in the simple log-normal SV model, we also estimate the following SV-t model

$$y_t = \exp(h_t/2)u_t, \ u_t \sim t_{(k)}, \tag{17}$$

$$h_t = \tau + \phi(h_{t-1} - \tau) + \sigma v_t, \ v_t \sim N(0,1), \tag{18}$$

and test for a unit root under the more general setting. It is well known in the literature that the simple log-normal SV model cannot produce enough kurtosis as it is observed in actual data. This is the main motivation for introducing a fat-tailed conditional distribution of the error term u_t. Here we use a t distribution with k degrees of freedom that allows for Levy jumps in return process. The empirical importance of Levy jumps was shown in a recent study by Li, et al. (2009). Relative to the normal distribution, the t distribution will absorb some abnormal behavior in h_t, as a result, we expect that the volatility process, h_t, is smoother, making the unit root model more difficult to reject. Following much of the literature, we rewrite

[4] We wish to thank Mike So to share the data with us.

the t distribution with a normal scale mixture representation, namely,

$$u_t|w_t \sim N(0, w_t^{-1}), w_t \sim \Gamma(k/2, k/2).$$

It is easy to show that for the SV-t model, the BF has the same expression as in (15).

Table 5 report the posterior mean of ϕ, π, $\log_{01}(\text{BF})$ and $\log_{01}(\text{POR})$ for TWSI used by SL. The empirical results based on the simple log-normal SV model suggest that although the posterior mean of ϕ is so close to unity and the estimate of π is large than 0.5, we still reject the unit root hypothesis. The marginal likelihood of the estimated stationary model is so much larger than that of the estimated unit root model so that the adjustment from the estimated π is not able to change the sign of $\log_{01}(\text{BF})$ in $\log_{01}(\text{POR})$. This result perhaps explain why SL got the conflicting empirical results when different priors are used. Interestingly, when the SV-t model is estimated, the estimated degrees of freedom parameter is very large (29.17), suggesting that the t-distribution does not make much contribution to the model. Not surprisingly, the results for the unit root test remain nearly the same. However, the estimated volatility process is smoother in the SV-t model.

TABLE 5.

Empirical results from TWSI

Model	ϕ	π	k	$\log_{01}(\text{BF})$	$\log_{01}(\text{POR})$
SV	0.9994	0.6204	NA	−0.9335	−0.7109
SV-t	0.9997	0.6358	29.17	−0.7688	−0.5268

6. CONCLUSION

The main purpose of this paper is to provide an improved Bayesian approach for testing the unit root hypothesis in volatility in the context of SV models. The test procedure is based on the posterior odds. Unlike the parameter estimation which permits the use of objective and uninformative priors, the BF is ill-defined because it depends on the arbitrary constants. As a result, an informative prior has to be used in order to do the posterior odds analysis.

To overcome this difficulty, one simple method suggested in Kass and Raftery (1995) is to use part of the data as a training sample which is combined with the noninformative prior distribution to produce an informative prior distribution. The BF is then computed from the remainder of the data. However, the selection of the training sample may be arbitrary. Other empirical measures, such as intrinsic BF of Berger and Pericchi (1996) and fractional BF of O'Hagan (1995), also involve with theoretical or practical problems. To the best of our knowledge, there is no satisfactory method to solve this Jeffreys-Lindley-Bartlett's paradox. In this paper, we introduce a mixed informative prior distribution with a random weight for the Bayesian unit root testing. The improved method for computing the BF is numerically stable and easy to implement. We illustrate the method using both simulated data and real data. Simulations show that our method improve the performance of the unit root test of So and Li (1999) in terms of both the "size" and the "power". Empirical analysis, based the equity data of TWSI, shows that the unit root hypothesis is also rejected when our method is used.

Although we use the unit root in volatility to describe high volatility persistency, no mean the unit root model is the only way to produce high persistency in volatility. Other models, which can potentially explain high persistency in volatility, include the fractionally integrated SV models and the SV model with a shift in mean and/or a shift in persistency. Although we do not pursuit this direction of research here, our method can be adopted and modified to compare some of these alternative models.

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