Mildly Explosive Autoregression with Anti-persistent Errors*

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Abstract

An asymptotic distribution is derived for the least squares (LS) estimate of a first-order autoregression with a mildly explosive root and anti-persistent errors. While the sample moments depend on the Hurst parameter asymptotically, the Cauchy limiting distribution theory remains valid for the LS estimates in the model without intercept and a model with an asymptotically negligible intercept. Monte Carlo studies are designed to check the precision of the Cauchy distribution in finite samples. An empirical study based on the monthly NASDAQ index highlights the usefulness of the model and the new limiting distribution.

JEL classification: C22
Keywords: Anti-persistence, unit root, mildly explosive, sequential limit theory, bubble, fractional integration

1 Introduction

The autoregressive (AR) model with an explosive root was first studied in White (1958) and Anderson (1959) where the following process was considered:

$$y_t = \rho y_{t-1} + u_t,\ \ \rho > 1,\ \ t = 1, 2, ..., n. \tag{1}$$

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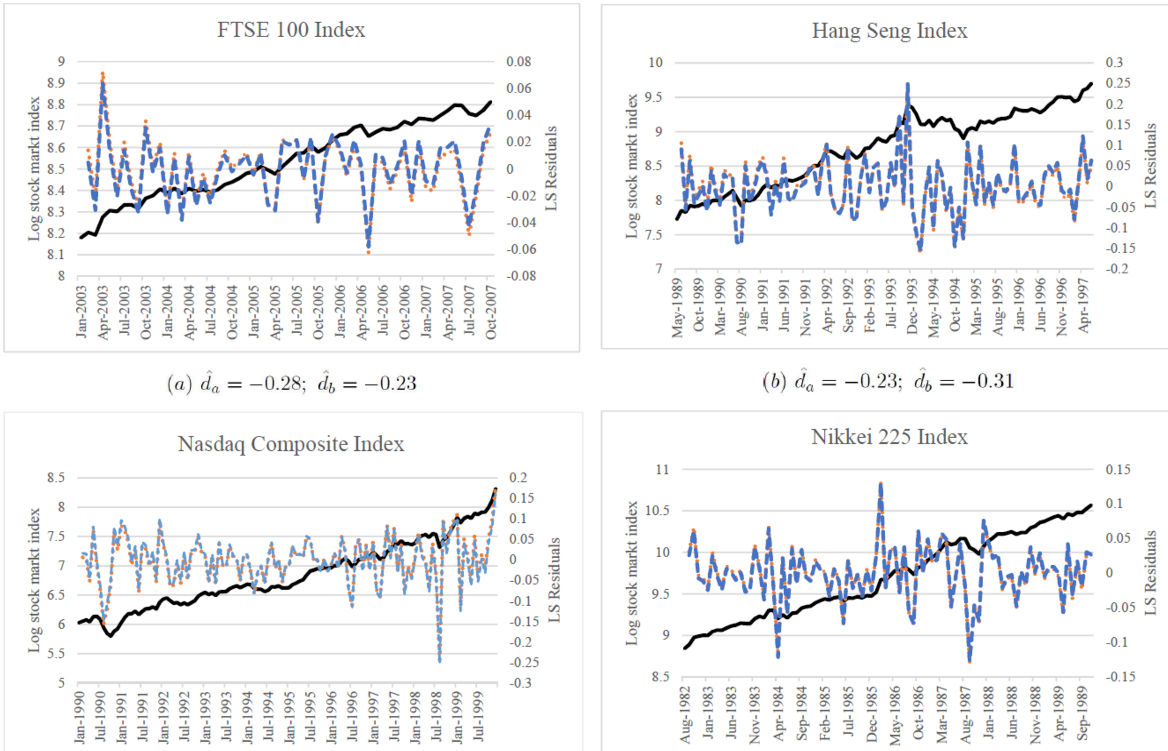

(a) $\hat{d}_a = -0.28$; $\hat{d}_b = -0.23$ (b) $\hat{d}_a = -0.23$; $\hat{d}_b = -0.31$

(c) $\hat{d}_a = -0.32$; $\hat{d}_b = -0.36$ (d) $\hat{d}_a = -0.31$; $\hat{d}_b = -0.22$

Figure 1: Time series plot of four logarithmic stock market indices (left axis) and their residuals obtained from the fitted AR(1) model with and without intercept by LS (right axis).

Under the assumptions of independent and identically distributed (iid) Gaussian errors (i.e. $u_t \overset{iid}{\sim} N(0, \sigma^2)$) and the zero initial condition (i.e. $y_0 = 0$), White (1958) and Anderson (1959) showed that the least squares (LS) estimate of ρ (denoted by $\hat{\rho}$) has the following Cauchy limiting distribution:

$$\frac{\rho^n}{\rho^2 - 1}(\hat{\rho} - \rho) \overset{as}{\to} C, \text{ as } n \to \infty, \tag{2}$$

where $\overset{as}{\to}$ denotes the convergence almost surely and C is a standard Cauchy variate.

It is noteworthy that the above limit theory is not obtained from an invariance principle because the distributional assumption $u_t \overset{iid}{\sim} N(0, \sigma^2)$ cannot be relaxed. To relax the assumption of Gaussian errors, and, in the meantime, to allow for a non-zero initial condition, Phillips and Magdalinos (2007a) (PM hereafter) and Phillips, Magdalinos and Giraitis (2010) (PMG hereafter) considered two variations which are analogous to Model (1). PM designed a mildly explosive AR model by letting $\rho = \rho_n = 1 + c/n^\alpha$, $c > 0$,

$\alpha \in (0,1)$, while PMG introduced a mildly explosive model by letting $\rho = \rho_{m,n} = 1 + cm/n$, $c > 0$. Under some suitable assumptions but without the requirements of Gaussian errors and the zero initial condition, PM and PMG obtained the limit theory which is analogous to (2):

$$\frac{\rho_n^n}{\rho_n^2 - 1}(\hat{\rho} - \rho_n) \quad \Rightarrow \quad C, \text{ as } n \to \infty; \tag{PM}$$

$$\frac{\rho_{m,n}^n}{\rho_{m,n}^2 - 1}\left(\hat{\rho} - \rho_{m,n}\right) \quad \Rightarrow \quad C, \text{ as } n \to \infty \text{ followed by } m \to \infty. \tag{PMG}$$

The pivotalness of the Cauchy distribution suggests that it is easy to test a hypothesis about the AR coefficient. Not surprisingly, it has been used in the literature to test the presence of rational bubbles in asset prices; see Phillips, Wu and Yu (2011). Moreover, considerable efforts have been made in the literature to explore the explosive-type AR models with dependent errors. The errors could be weakly dependent as in Phillips and Magdalinos (2007b), or strongly dependent as in Magdalinos (2012), or could involve conditional heteroskedasticity as in Arvanitis and Magdalinos (2018). These generalizations are important as the explosive-type model with dependent errors can potentially better describe the movement of real data than the pure explosive AR(1) model. A number of related studies in the literature allow for m-dependent errors (Pedersen and Schütte, 2017), errors with deterministic time-varying volatilities (Harvey, Leybourne and Zu, 2019a, 2019b).

To the best of our knowledge, no limit theory has been developed to cover any explosive-type AR model with anti-persistent errors. The goal of this paper is to fill the gaps in the context of the explosive-type AR model of PMG. Why are the gaps are important? To see the empirical relevance of an explosive model with anti-persistent errors, Figure 1 presents time series plots of four logarithmic stock market indices (left axis) and the residuals obtained from the fitted AR(1) model with and without intercept (right axis). In particular, we consider four monthly indices over different sampling periods, namely FTSE 100 Index from January 2003 to October 2007, Hang Seng Index from May 1989 to June 1997, NASDAQ Composite Index from January 1990 to December 1999, and Nikkei 225 Index from August 1982 to November 1989. The sampling periods are selected as these markets experienced exuberance over the respective periods, as it can be seen from the solid black lines in Figure 1. After fitting the AR(1) model with and without intercept to each time series by LS, we obtain two residual series with and without intercept and plot them in the blue and red dotted lines in Figure 1. These plots show that there is strong anti-persistence in the residuals.[1] When we apply the local Whittle (LW) method of Robinson (1994) to estimate the memory parameter d in the residuals, we find that the estimated d is always in the range $(-0.5, 0)$ in all cases. The estimated d is reported in Figure 1 with $\hat{d}_a$ and $\hat{d}_b$ corresponding to the model without and with intercept, respectively. These exercises strongly suggest that the explosive-type AR model with anti-persistent errors is not only of theoretical interest but also of empirical realism, making important the development of limit theory for an explosive-type AR model with anti-persistent errors.

The paper is organized as follows. Section 2 briefly reviews several forms of serially dependent error processes and mildly explosive AR models. Section 3 studies the mildly explosive AR model of PMG but with anti-persistent errors and develops the limiting distribution for the LS estimate of the AR coefficient under a sequential limit. Simulation studies are carried out in Section 4 to check the precision of the limiting distribution in finite samples. Section 5 provides an empirical study of a rational bubble in the NASDAQ index. Proofs of the main results in the paper are given in the Appendix.

We use the following notations throughout the paper: $\overset{p}{\to}, \overset{as}{\to}, \Rightarrow, \overset{a}{\sim}, \overset{d}{=}$ and $\overset{iid}{\sim}$ denote convergence in probability, convergence almost surely, weak convergence, asymptotic equivalence, equivalence in distribution, and iid, respectively.

2 Literature Reviews

2.1 A review of serially correlated errors

Although our paper focuses on anti-persistent errors, to facilitate discussion and comparison, we first review the concepts of weakly dependent errors and strongly dependent errors. Suppose that the error process admits a Wold-decomposition such that

$$u_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j}, c_0 = 1, \ \epsilon_t \overset{iid}{\sim} (0, \sigma^2), \tag{3}$$

where $\{c_j\}_{j=0}^{\infty}$ are real coefficients. Denote $\psi(k)$ the k^{th} order autocovariance function of u_t, that is, $\psi(k) := E(u_t u_{t-k})$.

Weakly dependent errors require $\sum_{j=0}^{\infty} |c_j| < \infty$ and $\sum_{j=0}^{\infty} c_j \neq 0$. These conditions imply that $\sum_{k=-\infty}^{\infty} |\psi(k)| \in (0, \infty)$ and $\sum_{k=-\infty}^{\infty} \psi(k) \neq 0$. For strongly dependent errors,

[1] A detailed discussion on anti-persistence is provided in the next section where we also relate anti-persistence to the memory parameter d.

it is assumed that c_j in (3) has a slow decay rate, such as $c_j \sim j^{-1+d}$ with $d \in (0, 0.5)$ when j is large. This leads to a violation of the summability condition of the linear coefficients and the autocovariance function as $\sum_{j=0}^{\infty} c_j = \infty$ and $\sum_{k=-\infty}^{\infty} |\psi(k)| = \infty$.

Anti-persistent errors are remarkably different from weakly dependent errors and strongly dependent errors. First, they are different from strongly dependent errors as c_j has a fast decay rate for anti-persistent errors, such as $c_j \sim j^{-1+d}$ with $d \in (-0.5, 0)$ when j is large. Second, they are different from weakly dependent errors in the sense that $\sum_{j=0}^{\infty} c_j = 0$ and $\sum_{k=-\infty}^{\infty} \psi(k) = 0$. Moreover, for any $k \neq 0$, $\psi(k)$ has a negative sign (see Proposition 3.2.1 (3) in Giraitis, Koul, and Surgailis (2012)), giving rise to the name of anti-persistence. These properties make the interpretation of corresponding stochastic integrals different from that when the errors are weakly dependent or strongly dependent. From the theoretical viewpoint, therefore, it is important to develop the limit theory for anti-persistent errors.

We now formally introduce the definition of anti-persistence.

Assumption 1 (AP) *Under (3) and let γ be a constant. Assume $c_j \overset{a}{\sim} \gamma j^{-1+d}$ for $j \to \infty$ with $d \in (-0.5, 0)$, $\sum_{j=0}^{\infty} c_j = 0$ and $\sum_{k=-\infty}^{\infty} \psi(k) = 0$.*

Assumption AP is general enough to include stationary ARFIMA(p, d, q) processes where $u_t = (1-L)^{-d}\phi(L)^{-1}\theta(L)\epsilon_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j}$, $\phi(L) = 1 - \sum_{j=0}^{p} \phi_j L^j$, $\theta(L) = 1 + \sum_{j=0}^{p} \theta_j L^j$ and L is the lag operator. We can show that c_j can be asymptotically approximated by $\frac{\theta(1)}{\phi(1)\Gamma(d)} j^{-1+d}$, where $\Gamma(\cdot)$ is a gamma function. When $d \in (-0.5, 0)$, the stationary ARFIMA process has the zero-sum for the linear coefficients, that is, $\sum_{j=0}^{\infty} c_j = 0$. It is well-known that u_t corresponds to a fractional Brownian motion (fBM) with the Hurst parameter $H = 1/2 + d$; see Giraitis, Koul and Surgailis (2012). When $H = 0.5$, an fBM becomes the standard Brownian motion. When $H \in (0, 0.5)$ which corresponds to the case of interest in the present paper, an fBM has a rough sample path and is anti-persistent. When $H \in (0.5, 1)$, an fBM has a smooth sample path in the sense that it is $1/2 - \epsilon$-Hölder continuous for any $\epsilon > 0$. The empirical relevance of anti-persistent processes in financial time series was recently documented in Gatheral, Jaisson, and Rosenbaum (2018) and Wang, Xiao, and Yu (2019). The empirical relevance of anti-persistent errors in an explosive model was shown earlier in Figure 1. Assuming a continuous record of observations is available, Xiao and Yu (2019a, 2019b) recently developed the limit theory for the persistence parameter in the fractional Vasicek model which corresponds to the AR coefficient in the discrete-time representation.

2.2 A mildly explosive model

PMG considered the following mildly explosive model:

$$y_t = \left(1 + \frac{cm}{n}\right) y_{t-1} + u_t, c > 0, u_t \overset{iid}{\sim} (0, \sigma^2), y_0 = O_p(1). \tag{4}$$

As suggested in PMG, one way of thinking of the model specification is that the total number of observations (n) is partitioned into m blocks with K samples so that $n = m \times K$. Thus, the chronological time for y_t becomes $t = \lfloor Kj \rfloor + k$, for $k \in \{1, ..., K\}$ and $j \in \{0, 1, ...m-1\}$. This model is closely related to the model proposed in Park (2003) where it was assumed that $c = -1 < 0$.

It is easy to see that as $n \to \infty$ with fixed m, Model (4) is a local-to-unity model with the noncentrality parameter cm and hence, the standard local-to-unity asymptotic theory is applicable. That is,

$$n(\hat{\rho} - \rho_{n,m}) \Rightarrow \int_0^1 J_{cm}(s) dW(s) / \int_0^1 J_{cm}^2(s) ds,$$

where $J_{cm}(s) = \int_0^s e^{cm(s-r)} dW(r)$ and $W(\cdot)$ denotes a standard Brownian motion.

However, since $c > 0$, if one assumes $n \to \infty$ followed by $m \to \infty$, Model (4) is akin to a mildly explosive AR model of PM whose root is in a larger neighborhood of unity than a local-to-unit-root. The second asymptotic ($m \to \infty$) creates a departure from the local-to-unit-root region; see Park (2003) and PMG for detailed discussions. With this sequential asymptotic scheme, we have

$$\begin{aligned} \frac{1}{2c} \frac{n}{m} e^{cm} \left(\hat{\rho} - \rho_{n,m}\right) &\Rightarrow \frac{e^{-cm} \int_0^1 J_{cm}(s) dW(s)}{2ce^{-2cm} \int_0^1 J_{cm}^2(s) ds}, \text{ as } n \to \infty \text{ with fixed } m \\ &= \frac{e^{-cm} \int_0^m \widetilde{J}_c(s) d\widetilde{W}(s)}{2ce^{-2cm} \int_0^m \widetilde{J}_c^2(s) ds} \\ &\Rightarrow C, \text{ as } m \to \infty, \end{aligned} \tag{5}$$

where $\widetilde{W}(t) = \sqrt{m} W(t/m)$ and $\widetilde{J}_c(t) = \int_0^t e^{c(t-s)} d\widetilde{W}(s)$. To see the link between this sequential asymptotic result in (5) and the asymptotic results in (2) and (PM), note that $e^{cm} = \exp\left(\frac{cm}{n}\right)^n \overset{a}{\sim} \rho_{n,m}^n$ and $\rho_{n,m}^2 - 1 \overset{a}{\sim} 2c\frac{m}{n}$.[2]

[2]Although the limiting distribution in PM is the same as that in PMG, the techniques used to develop

3 Mildly Explosive Model with Anti-persistent Errors

We now extend the model of PMG to the following model:

$$y_t = \mu_n + \rho y_{t-1} + u_t,\ t = 1, ..., n, \tag{6}$$

where $y_0 = o_p(n^{1/2+d})$, $\mu_n = \mu/n^{\vartheta}$, $\rho = \rho_{n,m} = \left(1 + \frac{cm}{n}\right)$, $\vartheta > 1/2 - d$, and u_t satisfies Assumption AP.

Model (6) is different from Model (4) in two aspects. First, instead of assuming an iid error process, we allow for anti-persistent errors in Model (6). Second, when $\mu \neq 0$, a non-zero intercept μ_n, which is asymptotically negligible, enters the model. Similar to Phillips, Shi and Yu (2014), we impose a restriction on ϑ so that the localized drift μ_n cannot dominate the random component introduced by u_t. However, if $\mu = 0$, then $\mu_n = 0$ and the intercept vanishes.

In this section, we aim to develop the limiting distribution for the centered LS estimate with and without intercept. To be more precise, we define the LS estimate without intercept by $\hat{\rho}_a$ and the LS estimate with intercept by $\hat{\rho}_b$. Thus, we can express the centered LS estimates as

$$\hat{\rho}_a - \rho = \frac{\sum_{t=1}^{n} y_{t-1} u_t}{\sum_{t=1}^{n} y_{t-1}^2}, \tag{7}$$

and

$$\hat{\rho}_b - \rho = \frac{\sum_{t=1}^{n} y_{t-1} u_t - \frac{1}{n} \sum_{t=1}^{n} y_{t-1} \sum_{t=1}^{n} u_t}{\sum_{t=1}^{n} y_{t-1}^2 - \frac{1}{n} \left(\sum_{t=1}^{n} y_{t-1}\right)^2}. \tag{8}$$

Before we develop the asymptotic theory, we first review the functional central limit theorem due to Giraitis, Koul, Surgailis (2012) which extends Donsker's theorem.

Lemma 3.1 (Corollary 4.4.1 in Giraitis, Koul and Surgailis (2012)) *Let u_t be as in (3). Assume $c_j \overset{a}{\sim} \gamma j^{-1+d}$ as $j \to \infty$ with γ being a constant and $d \in (-0.5, 0)$, $E|\epsilon_t|^p < \infty$ with $p > (0.5 + d)^{-1}$ and $\sum_{j=0}^{\infty} c_j = 0$. Then, as $n \to \infty$,*

$$n^{-H} \sum_{t=1}^{\lfloor nr \rfloor} u_t \Rightarrow \varsigma B^H(r), \tag{9}$$

in $\mathcal{D}[0,1]$ with the uniform metric, where $\varsigma = \sqrt{\sigma^2 \gamma^2 \frac{\mathcal{B}(d,1-2d)}{d(1+2d)}}$ with $\mathcal{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $H = \frac{1}{2} + d$, $B^H(r)$ is an fBM with the Hurst parameter H.

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the limiting distribution are different in these two studies. PM uses a Lindeberg-Feller CLT while PMG uses the local-to-unit-root theory together with the martingale convergence theorem. Our proof follows that of PMG, but there are technical difficulties that we need to deal with in our proof.

An fBM with the Hurst parameter $H \in (0,1)$ is a Gaussian process with zero mean and the following covariance,

$$E(B^H(r)B^H(s)) = \frac{1}{2}\left(|r|^{2H} + |s|^{2H} - |r-s|^{2H}\right).$$

Clearly, if $H = 1/2$, $B^H(t)$ becomes the standard Brownian motion $W(t)$. Unlike $W(t)$, $B^H(t)$ is not a semi-martingale if $H \neq 1/2$. Therefore, we cannot interpret the stochastic integral with respect to fBM as an Itô integral. In this paper, we interpret the stochastic integral with respect to fBM as a Young integral when we study the asymptotic theory for the error process under Assumption AP, where the mathematical techniques are related to those used in EI Machkouri, Es-Sebaiy and Ouknine (2016) and Xiao and Yu (2019a, 2019b). This interpretation is in contrast to PMG where $\widetilde{J}_c(t) = \int_0^t e^{c(t-s)} d\widetilde{W}(s)$ is viewed as an Itô integral. Moreover, we need a different asymptotic theory to obtain a sequential limit. The following lemma obtains the asymptotic behavior of the sample moments.

Lemma 3.2 *In Model (6) with $\{u_t\}$ satisfying Assumption AP, we assume $E|\epsilon_t|^p < \infty$ with $p > (0.5+d)^{-1}$. As $n \to \infty$ with m fixed, we have the local-to-unit-root asymptotic results:*

1. $\frac{1}{n^{1/2+d}} y_{\lfloor nr \rfloor} \Rightarrow \varsigma J_{cm}^H(r)$;
2. $\frac{1}{n^{3/2+d}} \sum_{t=1}^n y_t \Rightarrow \varsigma \int_0^1 J_{cm}^H(r)dr$;
3. $\frac{1}{n^{2+2d}} \sum_{t=1}^n y_t^2 \Rightarrow \varsigma^2 \int_0^1 (J_{cm}^H(r))^2 dr$;
4. $\frac{1}{n^{1+2d}} \sum_{t=1}^n y_{t-1}u_t + \frac{1}{n^{1+2d}} \frac{1}{2} \sum_{t=1}^n u_t^2 \Rightarrow \varsigma^2 \left[cmZ(1) \int_0^1 e^{cms} dB^H(s) + R(1)\right]$,

   where

$$\begin{aligned}
\varsigma &= \sqrt{\sigma^2 \gamma^2 \frac{\mathcal{B}(d, 1-2d)}{d(1+2d)}}, \\
J_{cm}^H(r) &= \int_0^r e^{cm(r-s)} dB^H(s), \; Z(1) = \int_0^1 e^{-cms} B^H(s) ds, \\
R(1) &= \frac{1}{2}\left[B^H(1)\right]^2 - cm \int_0^1 (B^H(s))^2 ds + (cm)^2 \int_0^1 \int_0^s e^{cm(r-s)} B^H(r) B^H(s) dr ds.
\end{aligned}$$

Since $B^H(s)$ is not a semi-martingale, in the present paper, we treat $J_{cm}^H(r)$ as a Young integral. For details about the Young integral, see (A.1) in El Machkouri, Es-Sebaiy and Ouknine (2016).

Remark 3.1 *The results in Lemma 3.2 are closely related to Lemma 1 in Phillips (1987), which can be used to show that for Model (6) with weakly dependent errors, when $n \to \infty$ with m fixed,*

$$\frac{1}{n^{1/2}} y_{\lfloor nr \rfloor} \Rightarrow \sigma J_{cm}(r),$$

$$\begin{aligned} \frac{1}{n^{3/2}} \sum_{t=1}^{n} y_t &\Rightarrow \sigma \int_0^1 J_{cm}(r) dr, \\ \frac{1}{n^2} \sum_{t=1}^{n} y_t^2 &\Rightarrow \sigma^2 \int_0^1 (J_{cm}(r))^2 dr, \end{aligned}$$

$$\frac{1}{n} \sum_{t=1}^{n} y_{t-1} u_t \Rightarrow \frac{1}{2} \left[\sigma^2 J_{cm}(1)^2 - 2cm\sigma^2 \int_0^1 (J_{cm}(r))^2 dr - E(u_t^2) \right],$$

where $J_{cm}(r) = \int_0^r e^{(r-s)cm} dW(s)$.

Remark 3.2 *For Model (6) with strongly dependent errors, the first three claims in Lemma 3.2 remain valid, while for the last claim, we have*

$$\frac{1}{n^{1+2d}} \sum_{t=1}^{n} y_{t-1} u_t \Rightarrow \varsigma^2 \left[cmZ(1) \int_0^1 e^{cms} dB^H(s) + R(1) \right].$$

where the term $\frac{1}{n^{1+2d}} \frac{1}{2} \sum_{t=1}^{n} u_t^2$ asymptotically vanishes as $n \to \infty$. This difference makes the development of the limiting distribution in the mildly explosive model with anti-persistent errors more difficult. In particular, when $n \to \infty$ with m fixed, the centered LS involves an additional term where $\frac{1}{n^{1+2d}} \frac{1}{2} \sum_{t=1}^{n} u_t^2$ appears in the numerator. Additional rate condition is needed to make sure this additional term vanishes asymptotically, as shown in the following theorem.

Theorem 3.1 *Let $c > 0$ in Model (6), under the same set of assumptions as in Lemma 3.2, if $n \to \infty$ followed by $m \to \infty$ with $m = \delta \ln n$ and $\delta > -\frac{2d}{c}$, we have*

$$\frac{1}{2c} \frac{n}{m} e^{cm} \left(\hat{\rho}_j - \rho \right) \Rightarrow C \,, \; \frac{\rho^n}{\rho^2 - 1} \left(\hat{\rho}_j - \rho \right) \Rightarrow C \,, \; j \in \{a, b\} \,. \tag{10}$$

Theorem 3.1 suggests that the centered LS estimates $\hat{\rho}_a$ and $\hat{\rho}_b$ in Model (6) have the Cauchy limiting distribution upon the correct normalization. Since the Cauchy distribution is pivotal and ρ can be consistently estimated by either $\hat{\rho}_a$ or $\hat{\rho}_b$, the limit theory provides a convenient way for hypothesis testing for ρ.

Remark 3.3 *The rate condition $m = \delta \ln n$ with $\delta > -\frac{2d}{c}$ suggests that m cannot go to infinity too slowly relative to n. This condition ensures that $\frac{1}{n^{1+2d}} \frac{1}{2} \sum_{t=1}^{n} u_t^2$ is dominated by $\frac{1}{n^{1+2d}} \sum_{t=1}^{n} y_{t-1} u_t$ as $m \to \infty$.*

Remark 3.4 *As in Phillips, Wu and Yu (2011), Theorem 3.1 suggests that a confidence interval (CI) for ρ can be constructed as*

$$\left\{ \hat{\rho}_j \pm \frac{\hat{\rho}_j^2 - 1}{\hat{\rho}_j^n} C_a \right\}, \; j \in \{a, b\}, \tag{11}$$

where C_a is the critical value for the two-tailed test with the significance level α and $C_{0.1} = 6.315, C_{0.05} = 12.7, C_{0.01} = 63.65674$.

Remark 3.5 *The Cauchy limiting distribution also holds when we have weakly/strongly dependent errors in Model (4). For example, suppose u_t is weakly dependent with $\sum_{j=0}^{\infty} |c_j| < \infty$, and $\sum_{j=0}^{\infty} c_j \neq 0$, $y_0 = o_p(n^{1/2})$ and $E|\epsilon_t|^{\beta+\varepsilon} < \infty$ for some $\beta > 2$ and $\varepsilon > 0$. With the sequential asymptotic, we have*

$$\begin{aligned} \frac{1}{2c} \frac{n}{m} e^{cm} (\hat{\rho}_a - \rho) \quad &\Rightarrow \quad \frac{e^{-cm} \int_0^1 J_{cm}(s) dW(s) + e^{-cm} \frac{1}{2} \left(1 - \frac{\upsilon}{\lambda^2}\right)}{2c e^{-2cm} \int_0^1 J_{cm}^2(s) ds}, \text{ as } n \to \infty \text{ with fixed } m \\ &= \quad \frac{e^{-cm} \int_0^m \widetilde{J}_c(s) d\widetilde{W}(s)}{2c e^{-2cm} \int_0^m \widetilde{J}_c^2(s) ds} + O_p(e^{-cm}) \\ &\Rightarrow \quad C, \text{ as } m \to \infty. \end{aligned} \tag{12}$$

The first convergence follows from Theorem 1 of Phillips (1987), where $\upsilon = \sigma^2 \sum_{j=0}^{\infty} c_j^2$ and $\lambda = \sigma \sum_{j=0}^{\infty} c_j$. The second convergence follows from the martingale convergence theorem.

Remark 3.6 *Suppose that $\rho_n = 1 + c/n^{\alpha}$ with $\alpha \in (0,1)$, $c > 0$, and $u_t = \epsilon_t \overset{iid}{\sim} N(0, \sigma^2)$. According to Theorem 4.3 of PM (2007a),*

$$\rho_n^{-n}/n^{\alpha} \sum_{t=1}^{n} y_{t-1} u_t \Rightarrow \omega_0 \eta_0, \quad \rho_n^{-2n}/n^{2\alpha} \sum_{t=1}^{n} y_{t-1}^2 \Rightarrow \eta_0^2,$$

where ω_0 and η_0 are independent $N(0, \sigma^2/2c)$ random variables. In our model, we have $\rho = \rho_{n,m} = 1 + cm/n$ and anti-persistent errors. Under the sequential asymptotic scheme, we have

$$\frac{e^{-cm}}{m} \frac{1}{n^{1+2d}} \frac{1}{\zeta^2} \sum_{t=1}^{n} y_{t-1} u_t \Rightarrow \omega_d \eta_d, \quad 2c e^{-2cm} \frac{1}{n^{2+2d}} \frac{1}{\zeta^2} \sum_{t=1}^{n} y_{t-1}^2 \Rightarrow \eta_d^2, \tag{13}$$

where ω_d and η_d are independent $N(0, H\Gamma(2H)/2c)$ random variables. We complement the results of PM and PMG to the model with anti-persistent errors.

Remark 3.7 *When u_t is strongly dependent, using the similar arguments in proving Theorem 3.1, we can obtain the results of (10) and (13). In this case, the assumption $m = \delta \ln n$ with $\delta > -\frac{2d}{c}$, which is used to eliminate $\frac{1}{n^{1+2d}} \frac{1}{2} \sum_{t=1}^{n} u_t^2$ as $m \to \infty$, is not needed.*

4 Monte Carlo Studies

In this section, we design several Monte Carlo experiments to evaluate the precision of the derived asymptotic distribution in finite samples. In all experiments, we simulate data from the following data generating process (DGP):

$$y_t = \mu_n + \rho y_{t-1} + u_t, t = 1, 2, ..., n, \tag{14}$$

where $\rho = \left(1 + \frac{cm}{n}\right)$, $y_0 = 0, c > 0, \mu_n = \mu/n^{\vartheta}$, $u_t = (1 - L)^d \epsilon_t$ with $\epsilon_t \overset{iid}{\sim} N(0, 1)$. We consider the following parameter settings:

$$\begin{aligned} (n, m) &\in \{(100, 10), (500, 15), (1000, 20)\}, \\ d &\in \{-0.45, -0.4, -0.3, -0.2, -0.1, -0.01\}, \\ c &\in \{0.5, 1\}, \mu = 1, \vartheta = \frac{1}{2} - d + 0.1. \end{aligned} \tag{15}$$

The number of replications is always set at 10,000.

Under the parameter settings (15), we first obtain the LS estimates $\hat{\rho}_a$ and $\hat{\rho}_b$, and then apply the Cauchy distribution to construct the 95% CI (CI$_a$ and CI$_b$) based on (11) for $\rho_{n,m}$. We calculate the empirical coverage of the true value ρ, i.e., $\frac{1}{10000} \sum_{l=1}^{10000} 1 \left(\rho_L^{(l)} \le \rho \le \rho_U^{(l)}\right)$, where $\rho_L^{(l)}$ and $\rho_U^{(l)}$ are the two bounds of the CI in the l^{th} replication, and $1(\cdot)$ is the indicator function.

Tables 1 reports the empirical coverage of 95% CIs for alternative parameter settings in (15). With $n = 100$, $m = 10$ and $c = 0.5$, there is an obvious over coverage problem for both CI$_a$ and CI$_b$. This problem is less severe as c increases to 1 or as both m and n increase. Moreover, the CIs have good finite sample performance when c is relatively large and d is between -0.01 and -0.3. When $c = 1$, it can be seen that both CI$_a$ and CI$_b$ provide the empirical coverage which is close to the nominal coverage 95%. Finally, the empirical coverage obtained from CI$_a$ and CI$_b$ are similar.

Table 1: Empirical coverage of 95% CI of ρ

	$(n=100, m=10)$				$(n=500, m=15)$				$(n=1000, m=20)$			
	$c=0.5$		$c=1$		$c=0.5$		$c=1$		$c=0.5$		$c=1$	
d	CI_a	CI_b	CI_a	CI_b	CI_a	CI_b	CI_a	CI_b	CI_a	CI_b	CI_a	CI_b
-0.45	.995	.995	.928	.92	.905	.889	.923	.917	.915	.905	.922	.917
-0.4	.996	.995	.933	.924	.917	.903	.928	.924	.925	.915	.929	.925
-0.3	.996	.994	.945	.937	.934	.923	.939	.933	.936	.928	.937	.935
-0.2	.995	.995	.948	.943	.949	.939	.944	.941	.944	.937	.947	.942
-0.1	.991	.992	.947	.943	.95	.946	.95	.945	.948	.943	.950	.947
-0.01	.988	.99	.951	.947	.952	.946	.952	.949	.950	.946	.952	.950

5 An Empirical Study

To highlight the usefulness of the proposed model and the derived limiting distribution in practice, we now conduct an empirical study of a rational bubble based on Model (4) and the asymptotic theory in Theorem 3.1. The standard no-arbitrage condition suggests that

$$P_t = \frac{1}{1+r_f} E_t \left[P_{t+1} + D_{t+1} \right], \tag{16}$$

where P_t, r_f, D_t and E_t denote the price of asset, the discount rate, the dividend, and the expectation based on information at time t, respectively. Equation (16) can be solved by forward substitutions, giving rise to the following expressions:

$$P_t = P_t^f + B_t, \tag{17}$$

$$P_t^f = \sum_{i=1}^{\infty} \left(\frac{1}{1+r_f} \right)^i E_t \left(D_{t+i} \right), \tag{18}$$

$$B_t = \frac{1}{1+r_f} E_t \left(B_{t+1} \right). \tag{19}$$

Equation (17) expresses price as a sum of two components: the fundamental price P_t^f which summarizes all the expected future discounted dividend and a bubble component B_t which is not related to the fundamentals.

If the transversality condition is imposed, then $B_t = 0$ and hence, $P_t = P_t^f$. Note that B_t is an explosive process since $(1 + r_f) > 1$. Therefore, when P_t^f is not explosive, testing the existence of a bubble is equivalent to examining the explosiveness in P_t. That is why in the literature looking for an explosive behavior in the price-dividend ratio (P_t/D_t) has been widely used; see, for example, Phillips, Shi and Yu (2015a, 2015b).

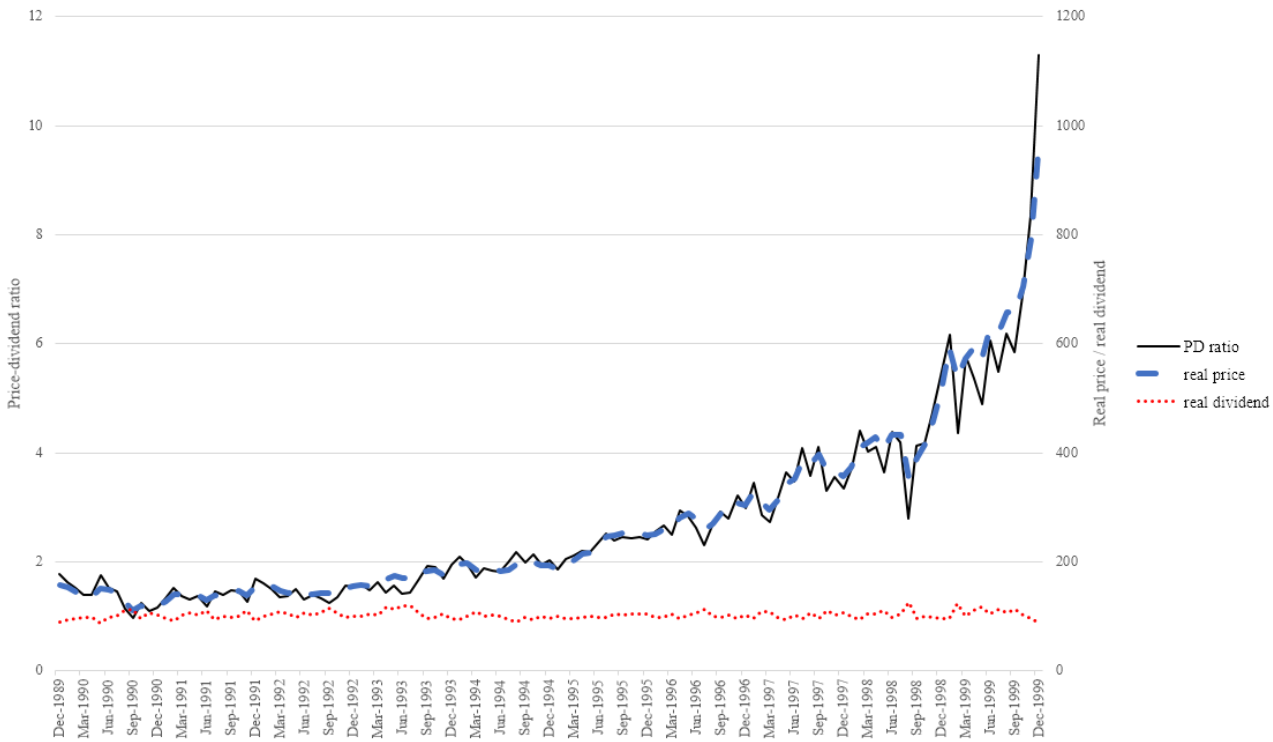

Figure 2: Price-dividend ratio in NASDAQ from December 1989 to December 1999

Our paper studies the price-dividend ratio in the NASDAQ composite index, we obtain the data set from Phillips, Wu and Yu (2011), which contains the monthly real price and real dividend series from February 1973 to June 2005. We then construct the price-dividend (PD) ratio based on the two time series. After obtaining the PD ratio, we focus on the sample period from December 1989 to December 1999.

In Figure 2, the PD ratio, the real price, and the real dividend are plotted in the black solid line, the blue dash line, and the red dotted line, respectively. We fit Model (4) with and without intercept to the PD ratio by LS, and then estimate the memory parameter (d) in the residuals by the LW method of Robinson (1994). The point estimate ("estimate" should be "estimates") of the intercept ($\hat{\mu}$), the AR coefficient ($\hat{\rho}$), and the memory parameter ($\hat{d}$) are reported in Panel A of Table 2. We use the subscript a and b to denote the LS estimate without and with intercept, respectively. Since the estimates of the AR coefficient are greater than 1 and $\hat{d} \in (-0.5, 0)$, Model (4) is relevant and the asymptotic theory developed in Theorem 3.1 is applicable. We then use the Cauchy limiting distribution to form the 95% CI of ρ which is reported in Panel A of Table 2. As the 95% CI excludes the unity, suggesting that there is strong evidence of explosiveness in the PD ratio and, hence, strong evidence of the presence of a bubble. In Panel B, we report the empirical results based on a subsample of the NASDAQ index, namely, January 1993 to December 1999. We continue to find that $\hat{\rho} > 1$, $\hat{d} \in (-0.5, 0)$, and that the 95% CI suggests the strong evidence of the presence of a bubble in the subsample.

Table 2: Empirical results for the NASDAQ Index

Panel A: Sample Period: December 1989 to December 1999, $n = 120$							
	$\hat{d}_a$	$\hat{\rho}_a$	95% CI$_a$	$\hat{d}_b$	$\hat{\mu}$	$\hat{\rho}_b$	95% CI$_b$
P_t/D_t	-0.084	1.0437	$[1.0370, 1.0504]$	-0.060	-0.1445	1.0862	$[1.0860, 1.0863]$
Panel B: Sample Period: January 1993 to December 1999, $n = 83$							
	$\hat{d}_a$	$\hat{\rho}_a$	95% CI$_a$	$\hat{d}_b$	$\hat{\mu}$	$\hat{\rho}_b$	95% CI$_b$
P_t/D_t	-0.079	1.0478	$[1.0220, 1.0736]$	-0.066	-0.1865	1.0969	$[1.0957, 1.0981]$

6 Conclusion

In this paper, we have made two contributions to the rapidly growing literature on explosive time series. First, we show that in empirical data, it is very plausible that we may have to use a mildly explosive model with anti-persistent errors to describe the movement of financial assets. Second, we show that, when anti-persistent errors are in a first-order autoregression with a mildly explosive root, the Cauchy limiting distribution remains valid for the LS estimate. To develop the limiting distribution, we following PMG's setup by assuming the autoregressive parameter is $\rho_{n,m} = 1 + \frac{cm}{n}$ and by adopting a sequential limit with $n \to \infty$ followed by $m \to \infty$. When the errors are anti-persistent, an extra rate condition $m = \delta \ln n$ with $\delta > -\frac{2d}{c}$ is needed.

We also discuss how to obtain a feasible confidence interval for the AR coefficient. Empirical coverage of CI based on the Cauchy limiting distribution is presented in the Monte Carlo studies, suggesting that the limiting distribution works well in finite samples. Finally, an empirical study of a rational bubble in the NASDAQ index is provided, highlighting the usefulness of the proposed model and the derived asymptotic theory.

A Appendix

Lemma A.1 (Lemma 2.3 in El Machkouri, Es-Sebaiy and Ouknine (2016)) *Suppose we have the following stochastic differential equation:*

$$dX(t) = cX(t)dt + dG(t), X(0) = X_0 = 0,$$

where $G(t)$ is a Gaussian process and $c > 0$. Further assume the following two assumptions hold for $G = (G(t), t \geq 0)$.

1. *The process G has Hölder continuous paths of order $\delta \in (0, 1]$;*

2. *For every $t \geq 0$, $E(G^2(t)) \leq ct^{2\gamma}$ for some positive constants c and γ.*

Then, for every $t \geq 0$, we have

$$\frac{1}{2}X^2(t) = c\int_0^t X^2(s)ds + cZ(t)\int_0^t e^{cs}dG(s) + R(t),$$

where

$$Z(t) = \int_0^t e^{-cs}G(s)ds,$$

$$R(t) = \frac{1}{2}G^2(t) - c\int_0^t G^2(s)ds + c^2\int_0^t\int_0^s e^{-c(s-r)}G(s)G(r)drds.$$

Proof of Lemma 3.2 Throughout the proof, we assume $n \to \infty$ with m fixed. By backward substitutions, we can write

$$y_{\lfloor nr\rfloor} = \frac{1-\rho_{n,m}^{\lfloor nr\rfloor}}{1-\rho_{n,m}}\mu_n + \rho_{n,m}^{\lfloor nr\rfloor}y_0 + \sum_{j=1}^{\lfloor nr\rfloor}\rho_{n,m}^{\lfloor nr\rfloor-j}u_j.$$

Note that $\rho_{n,m} = \exp(\frac{cm}{n}) + R_\rho$, with $R_\rho = -\sum_{k=2}^{\infty}\left(\frac{cm}{n}\right)^k/k! = O(n^{-2})$. Applying the binomial expansion, we have

$$\begin{aligned}
\rho_{n,m}^{\lfloor nr\rfloor} &= \left(\exp\left(\frac{cm}{n}\right) + R_\rho\right)^{\lfloor nr\rfloor} \\
&= \sum_{k=0}^{\lfloor nr\rfloor}\binom{\lfloor nr\rfloor}{k}\exp\left(\frac{cm}{n}\right)^{\lfloor nr\rfloor-k}R_\rho^k \\
&= \exp\left(\frac{cm}{n}\right)^{\lfloor nr\rfloor} + \sum_{k=1}^{\lfloor nr\rfloor}\binom{\lfloor nr\rfloor}{k}\exp\left(\frac{cm}{n}\right)^{\lfloor nr\rfloor-k}R_\rho^k.
\end{aligned}$$

We will show for any $k \geq 1$,

$$\binom{\lfloor nr \rfloor}{k} \exp\left(\frac{cm}{n}\right)^{\lfloor nr \rfloor - k} R_\rho^k \to 0. \tag{20}$$

To do so, note that $\binom{\lfloor nr \rfloor}{k} = O(n^k)$, $\exp(\frac{cm}{n})^{\lfloor nr \rfloor - k} = O(1)$, and $R_\rho^k = \exp(k \ln R_\rho) = \exp\left(k \ln O(n^{-2})\right) = \exp\left(-2k \ln(O(n))\right)$. Hence,

$$\binom{nr}{k} \exp\left(\frac{cm}{n}\right)^{nr-k} R_\rho^k = O[n^k \exp(-2k \ln(O(n)))].$$

Moreover,

$$\ln[n^k \exp(-2k \ln(n))] = k \ln(n) - 2k \ln(n) = -k \ln(n) \to -\infty.$$

This proves (20).

Letting $k^* = \arg\max_{k \in \{2,\ldots,n\}} \binom{nr}{k} \exp\left(\frac{cm}{n}\right)^{nr-k} R_\rho^k$, we have

$$\sum_{k=2}^{\lfloor nr \rfloor} \binom{\lfloor nr \rfloor}{k} \exp\left(\frac{cm}{n}\right)^{\lfloor nr \rfloor - k} R_\rho^k = O[n^{1+k^*} \exp(-2k^* \ln(O(n)))] \to 0, \tag{21}$$

because

$$\begin{aligned} \ln[n^{1+k^*} \exp(-2k^* \ln(n))] &= (k^* + 1 - 2k^*) \ln(n) \\ &= (1 - k^*) \ln n \to -\infty \text{ since } k^* \geq 2. \end{aligned}$$

From (20) and (21), we have

$$\sum_{k=1}^{\lfloor nr \rfloor} \binom{\lfloor nr \rfloor}{k} \exp\left(\frac{cm}{n}\right)^{\lfloor nr \rfloor - k} R_\rho^k \to 0.$$

So $\rho_{n,m}^{\lfloor nr \rfloor} = \exp\left(\frac{cm}{n}\right)^{\lfloor nr \rfloor} + o(1)$. Since $\lfloor nr \rfloor / nr \to 1$, we can write

$$y_{\lfloor nr \rfloor} = \frac{1 - \exp(cmr)}{-cm/n} \mu_n + (\exp(cmr) + o(1)) y_0 + \sum_{j=1}^{\lfloor nr \rfloor} \rho_{n,m}^{\lfloor nr \rfloor - j} u_j + o(1). \tag{22}$$

For the third term in (22), we can show that $\rho_{n,m}^{\lfloor nr \rfloor - j} = \exp\left(\frac{cm}{n}\right)^{\lfloor nr \rfloor - j} + o(1)$ which allows us to express

$$\sum_{j=1}^{\lfloor nr \rfloor} \rho_{n,m}^{\lfloor nr \rfloor - j} u_j = \sum_{j=1}^{\lfloor nr \rfloor} \left(\exp\left(\frac{cm}{n}\right)^{\lfloor nr \rfloor - j} + o(1)\right) u_j$$

$$= \sum_{j=1}^{\lfloor nr \rfloor} \exp\left(\frac{cm}{n}\right)^{\lfloor nr \rfloor - j} u_j + o(1) \sum_{j=1}^{\lfloor nr \rfloor} u_j$$

$$= \sum_{j=1}^{\lfloor nr \rfloor} \exp\left(\frac{cm}{n}\right)^{\lfloor nr \rfloor - j} u_j + o_p(n^{1/2+d}).$$

We obtain the third equality by using (9) where $\sum_{j=1}^{\lfloor nr \rfloor} u_j = O_p(n^{1/2+d})$.

Eventually, we can rewrite (22) as

$$y_{\lfloor nr \rfloor} = nr\mu_n + \exp(cmr)\, y_0 + \sum_{j=1}^{\lfloor nr \rfloor} \exp\left(\frac{cm}{n}\right)^{\lfloor nr \rfloor - j} u_j + o_p(n^{1/2+d}). \tag{23}$$

Let $X_n(r) = \frac{1}{n^{1/2+d}\varsigma} S_{\lfloor nr \rfloor}$ with $S_{\lfloor nr \rfloor} = \sum_{j=1}^{\lfloor nr \rfloor} u_j$. Recall that under Model (6), $y_0 = o_p(n^{1/2+d})$, $\mu_n = \mu/n^{\vartheta}$ with $\vartheta > 1/2 - d$. The first two terms in (23) vanish as $n \to \infty$. If we multiply both sides in (23) by $n^{-1/2-d}$, we have

$$\begin{aligned} n^{-1/2-d} y_{\lfloor nr \rfloor} &= \varsigma \sum_{j=1}^{\lfloor nr \rfloor} e^{(\lfloor nr \rfloor - j)cm/n} \int_{(j-1)/n}^{j/n} dX_n(s) + o_p(1) \\ &= \varsigma \sum_{j=1}^{\lfloor nr \rfloor} \int_{(j-1)/n}^{j/n} e^{(r-s)cm} dX_n(s) + o_p(1) \\ &= \varsigma \int_0^r e^{(r-s)cm} dX_n(s) + o_p(1) \\ &\Rightarrow \varsigma \int_0^r e^{(r-s)cm} dB^H(s) := \varsigma J_{cm}^H(r). \end{aligned}$$

We have applied Lemma 3.1 with the continuous mapping theorem (Billingsley, 1968, p. 30) to obtain the last result.

For the terms involving $\sum_{t=1}^n y_t$ and $\sum_{t=1}^n y_{t-1}^2$, note that we can write

$$n^{-3/2-d} \sum_{t=1}^n y_t = \frac{1}{n} \sum_{t=1}^n \left(n^{-1/2-d} y_t\right),$$

$$n^{-2-2d} \sum_{t=1}^n y_t^2 = \frac{1}{n} \sum_{t=1}^n \left(n^{-1/2-d} y_t\right)^2.$$

By applying the continuous mapping theorem, we obtain the second claim and the third claim in Lemma 3.2.

For the last claim, after squaring y_t and summing over t, we have

$$\sum_{t=1}^n y_t^2 = \rho_{n,m}^2 \sum_{t=1}^n y_{t-1}^2 + 2\rho_{n,m} \sum_{t=1}^n y_{t-1} u_t + \sum_{t=1}^n u_t^2$$

$$+n\mu_n^2 + 2\mu_n\rho_{n,m}\sum_{t=1}^n y_{t-1} + 2\mu_n\sum_{t=1}^n u_t,$$

which leads to

$$\begin{aligned} y_n^2 &= \frac{2cm}{n}\sum_{t=1}^n y_{t-1}^2 + 2\rho_{n,m}\sum_{t=1}^n y_{t-1}u_t + \sum_{t=1}^n u_t^2 \\ &\quad + \frac{(cm)^2}{n^2}\sum_{t=1}^n y_{t-1}^2 + n\mu_n^2 + 2\mu_n\rho_{n,m}\sum_{t=1}^n y_{t-1} + 2\mu_n\sum_{t=1}^n u_t. \end{aligned}$$

Thus, we have

$$\begin{aligned} 2\rho_{n,m}\sum_{t=1}^n y_{t-1}u_t &= y_n^2 - \frac{2cm}{n}\sum_{t=1}^n y_{t-1}^2 - \sum_{t=1}^n u_t^2 - \frac{(cm)^2}{n^2}\sum_{t=1}^n y_{t-1}^2 \\ &\quad -n\mu_n^2 - 2\mu_n\rho_{n,m}\sum_{t=1}^n y_{t-1} - 2\mu_n\sum_{t=1}^n u_t, \end{aligned}$$

$$\begin{aligned} \frac{2}{n^{1+2d}}\sum_{t=1}^n y_{t-1}u_t &= \frac{1}{n^{1+2d}}y_n^2 - \frac{2cm}{n^{2+2d}}\sum_{t=1}^n y_{t-1}^2 - \frac{1}{n^{2d}}\frac{1}{n}\sum_{t=1}^n u_t^2 \\ &\quad -\frac{n\mu_n^2}{n^{1+2d}} - 2\frac{\mu_n}{n^{1+2d}}\sum_{t=1}^n y_{t-1} - 2\frac{\mu_n}{n^{1+2d}}\sum_{t=1}^n u_t + o_p(1), \end{aligned}$$

and

$$\begin{aligned} \frac{2}{n^{1+2d}}\sum_{t=1}^n y_{t-1}u_t + \frac{1}{n^{2d}}\frac{1}{n}\sum_{t=1}^n u_t^2 &= \frac{1}{n^{1+2d}}y_n^2 - \frac{2cm}{n^{2+2d}}\sum_{t=1}^n y_{t-1}^2 - \frac{n\mu_n^2}{n^{1+2d}} \\ &\quad -2\frac{\mu_n}{n^{1+2d}}\sum_{t=1}^n y_{t-1} - 2\frac{\mu_n}{n^{1+2d}}\sum_{t=1}^n u_t + o_p(1), \end{aligned}$$

as $\rho_{n,m} \to 1$, and $\frac{(cm)^2}{n^2}\sum_{t=1}^n y_{t-1}^2 = O_p(n^{2d})$ is dominated by $\frac{2cm}{n}\sum_{t=1}^n y_{t-1}^2 = O_p(n^{1+2d})$ when m is fixed.

Note that $\frac{1}{n}\sum_{t=1}^n u_t^2 \overset{as}{\to} E[u_t^2]$ by the ergodic theorem and

$$\frac{n\mu_n^2}{n^{1+2d}} = \frac{n^{-2d}}{n^{2\theta}}\mu^2 < \frac{n^{1-2d}}{n^{2\theta}}\mu^2 = \left(\frac{n^{1/2-d}}{n^{\vartheta}}\right)^2\mu^2 \to 0,$$

$$\frac{\mu_n}{n^{1+2d}}\sum_{t=1}^n y_{t-1} = \mu\frac{1}{n^{\vartheta}}\frac{n^{3/2+d}}{n^{1+2d}}\left(\frac{1}{n^{3/2+d}}\sum_{t=1}^n y_{t-1}\right) = \mu\frac{n^{1/2-d}}{n^{\vartheta}}O_p(1) = o_p(1),$$

$$\frac{\mu_n}{n^{1+2d}}\sum_{t=1}^n u_t = \mu\frac{1}{n^{\vartheta}}\frac{n^{1/2+d}}{n^{1+2d}}\left(\frac{1}{n^{1/2+d}}\sum_{t=1}^n u_t\right)$$

$$= \mu n^{-1/2-d-\vartheta} O_p(1) = o_p(1) \text{ since } \vartheta > 0, \text{ and } d < 1/2.$$

These results lead to

$$\frac{2}{n^{1+2d}} \sum_{t=1}^{n} y_{t-1} u_t + \frac{1}{n^{2d}} \frac{1}{n} \sum_{t=1}^{n} u_t^2 \Rightarrow \varsigma^2 \left[\left(J_{cm}^H(1) \right)^2 - 2cm \int_0^1 \left(J_{cm}^H(r) \right)^2 dr \right].$$

So we have

$$\begin{aligned} \frac{1}{n^{1+2d}} \sum_{t=1}^{n} y_{t-1} u_t + \frac{1}{n^{2d}} \frac{1}{2n} \sum_{t=1}^{n} u_t^2 &\Rightarrow \varsigma^2 \left[\frac{1}{2} \left(J_{cm}^H(1) \right)^2 - cm \int_0^1 (J_{cm}^H(r))^2 dr \right] \\ &= \varsigma^2 \left[cmZ(1) \int_0^1 e^{cms} dB^H(s) + R(1) \right]. \end{aligned}$$

where the last step follows from Lemma A.1. This completes the proof of Lemma 3.2.

To analyze the asymptotics when $m \to \infty$, we introduce the following lemma, which documents some results of distributional equivalence. By the self-similarity property of fBM, we have $B^H\left(\frac{t}{m}\right) \overset{d}{=} \left(\frac{1}{m}\right)^H B^H(t)$. Let $\widetilde{B}^H(t) := m^H B^H\left(\frac{t}{m}\right)$.

Lemma A.2 *Applying the self-similarity property of fBM, we can obtain the following:*

1. $\int_0^1 J_{cm}^H(r) dr B^H(1) \overset{d}{=} \frac{1}{m^{2H+1}} \int_0^m \widetilde{J}_c^H(s) ds \widetilde{B}^H(m)$;
2. $\left(\int_0^1 J_{cm}^H(r) dr\right)^2 \overset{d}{=} \frac{1}{m^{2H+2}} \left(\int_0^m \widetilde{J}_c^H(s) ds\right)^2$;
3. $\int_0^1 (J_{cm}^H(r))^2 dr \overset{d}{=} \frac{1}{m^{2H+1}} \int_0^m \left(\widetilde{J}_c^H(s)\right)^2 ds$;
4. $cmZ(1) \int_0^1 e^{cms} dB^H(s) + R(1) \overset{d}{=} \frac{1}{m^{2H}} \left(c\tilde{Z}(m) \int_0^m e^{cs} d\widetilde{B}^H(s) + \widetilde{R}(m)\right)$,

where

$$\begin{aligned} \widetilde{J}_c^H(r) &= \int_0^r e^{c(r-s)} d\widetilde{B}^H(s), \\ \tilde{Z}(m) &= \int_0^m e^{-cs} \widetilde{B}^H(s) ds, \\ \widetilde{R}(m) &= \frac{1}{2} \left(\widetilde{B}^H(m)\right)^2 - c \int_0^m \left(\widetilde{B}^H(s)\right)^2 ds + c^2 \int_0^m \int_0^r e^{c(r-s)} \widetilde{B}^H(r) \widetilde{B}^H(s) dr ds. \end{aligned}$$

Proof of Lemma A.2

We only need to show the following results are correct:

1. $Z(1) \overset{d}{=} \frac{1}{m^{H+1}} \tilde{Z}(m)$;

2. $\int_0^1 e^{cms} dB^H(s) \overset{d}{=} \frac{1}{m^H} \int_0^m e^{cs} d\widetilde{B}^H(s)$;

3. $\int_0^1 \left(B^H(s)\right)^2 ds \overset{d}{=} \frac{1}{m^{2H+1}} \int_0^m \left(\widetilde{B}^H(s)\right)^2 ds$;

4. $\int_0^1 J_{cm}^H(s) ds \overset{d}{=} \frac{1}{m^{H+1}} \int_0^m \widetilde{J}_c^H(s) ds$;

5. $\int_0^1 (J_{cm}^H(s))^2 ds \overset{d}{=} \frac{1}{m^{2H+1}} \int_0^m \left(\widetilde{J}_c^H(s)\right)^2 ds$;

6. $m^2 \int_0^1 \int_0^s e^{cm(r-s)} B^H(r) B^H(s) dr ds \overset{d}{=} \frac{1}{m^{2H}} \int_0^m \int_0^s e^{c(r-s)} \widetilde{B}^H(r) \widetilde{B}^H(s) dr ds$.

As the steps to prove the above results are similar, we shall only prove the last two claims. For the fifth claim, we have

$$\begin{aligned}
\int_0^1 \left(J_{cm}^H(r)\right)^2 dr &= \int_0^1 \left(\int_0^r e^{cm(r-s)} dB^H(s)\right)^2 dr \\
&= \int_0^1 e^{2cmr} \left(\int_0^r e^{-cms} dB^H(s)\right)^2 dr \\
&= \int_0^1 e^{2cmr} \left(\int_0^{mr} e^{-cv} dB^H\left(\frac{v}{m}\right)\right)^2 dr \\
&= \frac{1}{m^{2H}} \int_0^1 e^{2cmr} \left(\int_0^{mr} e^{-cv} d\left(m^H B^H\left(\frac{v}{m}\right)\right)\right)^2 dr \\
&= \frac{1}{m^{2H}} \int_0^m e^{2cu} \left(\int_0^u e^{-cv} d\widetilde{B}^H(v)\right)^2 d\left(\frac{u}{m}\right) \\
&= \frac{1}{m^{2H+1}} \int_0^m \left(\int_0^u e^{c(u-v)} d\widetilde{B}^H(v)\right)^2 du \\
&= \frac{1}{m^{2H+1}} \int_0^m \left(\widetilde{J}_c^H(u)\right)^2 du.
\end{aligned}$$

For the sixth result, we have

$$\begin{aligned}
m^2 \int_0^1 \int_0^s e^{cm(r-s)} B^H(r) B^H(s) dr ds &= m^2 \int_0^1 e^{-cms} \left(\int_0^s e^{cmr} B^H(r) dr\right) B^H(s) ds \\
&= m^2 \int_0^1 e^{-cms} \left(\int_0^{ms} e^{cr} B^H\left(\frac{r}{m}\right) d\left(\frac{r}{m}\right)\right) B^H(s) ds \\
&= \frac{m}{m^H} \int_0^m e^{-cv} \left(\int_0^{ms} e^{cr} \widetilde{B}^H(r) dr\right) B^H\left(\frac{v}{m}\right) d\left(\frac{v}{m}\right) \\
&= \frac{1}{m^{2H}} \int_0^m e^{-cv} \left(\int_0^v e^{cr} \widetilde{B}^H(r) dr\right) \widetilde{B}^H(v) dv \\
&= \frac{1}{m^{2H}} \int_0^m \int_0^v e^{c(r-v)} \widetilde{B}^H(r) \widetilde{B}^H(v) dr dv.
\end{aligned}$$

Proof of Theorem 3.1

To avoid confusion, we now refer $n \to \infty$ with m fixed as the "fix-m asymptotics", and $n \to \infty$ followed by $m \to \infty$ as the "sequential asymptotics".

From (7) and (8), we can have the following expressions for the normalized centered LS estimates

$$
\begin{aligned}
\frac{e^{cm}}{m} n(\hat{\rho}_a - \rho) &= \frac{e^{cm}}{m} n \left(\frac{\sum_{t=1}^n y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^n u_t^2}{\sum_{t=1}^n y_{t-1}^2} - \frac{\frac{1}{2}\sum_{t=1}^n u_t^2}{\sum_{t=1}^n y_{t-1}^2} \right) \\
&= \frac{e^{cm}}{m} n \frac{\sum_{t=1}^n y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^n u_t^2}{\sum_{t=1}^n y_{t-1}^2} - \frac{e^{cm}}{m} B_n^a \\
&:= \frac{e^{cm}}{m} A_n^a - \frac{e^{cm}}{m} B_n^a, \qquad (24)
\end{aligned}
$$

where $A_n^a = n\frac{\sum_{t=1}^n y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^n u_t^2}{\sum_{t=1}^n y_{t-1}^2}$ and $B_n^a = \frac{n}{2}\sum_{t=1}^n u_t^2 / \sum_{t=1}^n y_{t-1}^2$.

Similarly, we can express

$$
\begin{aligned}
\frac{e^{cm}}{m} n(\hat{\rho}_b - \rho) &= \frac{e^{cm}}{m} n \left(\begin{array}{c} \frac{\sum_{t=1}^n y_{t-1}u_t - \frac{1}{n}\sum_{t=1}^n y_{t-1}\sum_{t=1}^n u_t + \frac{1}{2}\sum_{t=1}^n u_t^2}{\sum_{t=1}^n y_{t-1}^2 - \frac{1}{n}\left(\sum_{t=1}^n y_{t-1}\right)^2} \\ - \frac{\frac{1}{2}\sum_{t=1}^n u_t^2}{\sum_{t=1}^n y_{t-1}^2 - \frac{1}{n}\left(\sum_{t=1}^n y_{t-1}\right)^2} \end{array} \right) \\
&:= \frac{e^{cm}}{m} A_n^b - \frac{e^{cm}}{m} B_n^b, \qquad (25)
\end{aligned}
$$

where

$$
A_n^b = n \frac{\sum_{t=1}^n y_{t-1}u_t - \frac{1}{n}\sum_{t=1}^n y_{t-1}\sum_{t=1}^n u_t + \frac{1}{2}\sum_{t=1}^n u_t^2}{\sum_{t=1}^n y_{t-1}^2 - \frac{1}{n}\left(\sum_{t=1}^n y_{t-1}\right)^2}, \; B_n^b = n \frac{\frac{1}{2}\sum_{t=1}^n u_t^2}{\sum_{t=1}^n y_{t-1}^2 - \frac{1}{n}\left(\sum_{t=1}^n y_{t-1}\right)^2}.
$$

Since the proofs of the sequential asympotics for $\frac{e^{cm}}{m} n(\hat{\rho}_a - \rho)$ are very similar to those for $\frac{e^{cm}}{m} n(\hat{\rho}_b - \rho)$, we shall only prove the later. In fact, the only difference between the two estimates is the extra terms induced by the inclusion of an intercept in the LS regression. As we proceed, we will see the extra terms vanish in the sequential asympotics.

We first show the sequential limit of $\frac{e^{cm}}{m} A_n^b$ in (25). Applying Lemma 3.2 and Lemma A.2, as $n \to \infty$ with fixed m,

$$
\begin{aligned}
\frac{e^{cm}}{m} A_n^b &\Rightarrow \frac{e^{cm}}{m} \frac{cmZ(1)\int_0^1 e^{cms} dB^H(s) + R(1) - \int_0^1 J_{cm}^H(r)dr B^H(1)}{\int_0^1 (J_{cm}^H(r))^2 dr - \left(\int_0^1 J_{cm}^H(r)dr\right)^2} \\
&\overset{d}{=} \frac{e^{cm}}{m} \frac{\frac{1}{m^{2H}}\left(c\tilde{Z}(m)\int_0^m e^{-cr}\widetilde{B}^H(r)dr + \widetilde{R}(m) - \frac{1}{m}\int_0^m J_c^H(s)ds\widetilde{B}^H(m)\right)}{\frac{1}{m^{2H+1}}\left(\int_0^m \tilde{J}_c^H(s)^2 ds - \frac{1}{m}\left(\int_0^m \tilde{J}_c^H(s)ds\right)^2\right)} \\
&= e^{cm} \frac{c\tilde{Z}(m)\int_0^m e^{-cr}\widetilde{B}^H(r)dr + \widetilde{R}(m) - \frac{1}{m}\int_0^m J_c^H(s)ds\widetilde{B}^H(m)}{\int_0^m \tilde{J}_c^H(s)^2 ds - \frac{1}{m}\left(\int_0^m \tilde{J}_c^H(s)ds\right)^2}. \qquad (26)
\end{aligned}
$$

For the sake of notational simplicity, we now introduce the following process with $m \geq 0$,

$$\xi(m) = \int_0^m e^{-cr} d\widetilde{B}^H(r), \tag{27}$$

where the integral is interpreted in the Young sense.

From Lemma 2.1 of El Machkouri, Es-Sebaiy and Ouknine (2016), we obtain a well-defined limit $\tilde{Z}(\infty) = \int_0^\infty e^{-cr} \widetilde{B}^H(r)dr$. As $m \to \infty$, we have

$$\tilde{Z}(m) \xrightarrow{as} \tilde{Z}(\infty) \text{ and } \xi(m) \xrightarrow{as} \xi(\infty) = c\tilde{Z}(\infty). \tag{28}$$

These two results are similar to those obtained by the martingale convergence theorem used in PMG when $m \to \infty$.

By the definition of the Young integral, we obtain $\widetilde{B}^H(0) = 0$. By the definition of $\tilde{Z}(m)$, we have

$$\xi(m) = e^{-cm}\widetilde{B}^H(m) + c\int_0^m e^{-cr}\widetilde{B}^H(r)dr = e^{-cm}\widetilde{B}^H(m) + c\tilde{Z}(m),$$

$$\tilde{J}_c^H(r) = \int_0^r e^{c(r-s)} d\widetilde{B}^H(s) = e^{cr}\int_0^r e^{-cs} d\widetilde{B}^H(s) = e^{cr}\xi(r).$$

So we can express (26) as

$$\begin{aligned}
& e^{cm}\frac{\left(c\tilde{Z}(m)\int_0^m e^{cs} d\tilde{B}^H(s) + \tilde{R}(m)\right) - e^{cs}\xi(s)ds\frac{1}{m}\tilde{B}^H(m)}{\int_0^m e^{2cs}\xi^2(s)ds - \frac{1}{m}\left(e^{-cm}\int_0^m e^{cs}\xi(s)ds\right)^2} \\
= \quad & \frac{e^{-cm}\left[\left(c\tilde{Z}(m)\int_0^m e^{cs} d\tilde{B}^H(s) + \tilde{R}(m)\right) - e^{cs}\xi(s)ds\frac{1}{m}\tilde{B}^H(m)\right]}{e^{-2cm}\left[\int_0^m e^{2cs}\xi^2(s)ds - \frac{1}{m}\left(e^{-cm}\int_0^m e^{cs}\xi(s)ds\right)^2\right]} \\
= \quad & \frac{e^{-cm}\left(c\tilde{Z}(m)\int_0^m e^{cs} d\tilde{B}^H(s) + \tilde{R}(m)\right) - \varphi_1'}{e^{-2cm}\left[\int_0^m e^{2cs}\xi^2(s)ds\right] - \varphi_2'},
\end{aligned} \tag{29}$$

where $\varphi_1' = \left(e^{-cm}\int_0^m e^{cs}\xi(s)ds\right)\left(\frac{1}{m}\tilde{B}^H(m)\right) := \varphi_{1a}' \times \varphi_{1b}'$, $\varphi_2' = \frac{1}{m}\left(e^{-cm}\int_0^m e^{cs}\xi(s)ds\right)^2$.

By applying (28) and L'Hospital's rule, as $m \to \infty$,

$$\varphi_{1a}' = e^{-cm}\int_0^m e^{cs}\xi(s)ds \xrightarrow{as} \tilde{Z}(\infty) \ . \tag{30}$$

Since $E\left(\frac{1}{m}\tilde{B}^H(m)\right) = 0$, $Var\left(\frac{1}{m}\tilde{B}^H(m)\right) = \frac{m^{2H}}{m^2} \to 0$, as $m \to \infty$,

$$\varphi_{1b}' = \frac{1}{m}\tilde{B}^H(m) \xrightarrow{p} 0.$$

Moreover, the continuous mapping theorem and Equation (30) imply that, as $m \to \infty$,

$$\left(e^{-cm}\int_0^m e^{cs}\xi(s)ds\right)^2 \xrightarrow{as} \tilde{Z}^2(\infty).$$

As $\frac{1}{m} \to 0$, $\varphi_2' \xrightarrow{p} 0$. Note that φ_1' and φ_2' are the extra terms due to the inclusion of the intercept in the LS regression. As they vanish, $\frac{e^{cm}}{m}A_n^a$ and $\frac{e^{cm}}{m}A_n^b$ are asymptotically equivalent in the sequential asymptotics. Therefore, as $m \to \infty$, we can write (29) as,

$$\frac{e^{-cm}\left(c\tilde{Z}(m)\int_0^m e^{cs}d\tilde{B}^H(s) + \tilde{R}(m)\right)}{e^{-2cm}\int_0^m e^{2cs}\xi^2(s)ds} + o_p(1) \tag{31}$$

To derive the sequential limit of $\frac{e^{cm}}{m}A_n^b$, we need the following lemma.

Lemma A.3 *let ω and η be two independent standard normal random variables. Then, as $m \to \infty$, we obtain:*

1. $e^{-2cm}\int_0^m e^{2cs}\xi^2(s)ds \xrightarrow{as} \frac{c}{2}\tilde{Z}^2(\infty)$;
2. $c\tilde{Z}(m)\left(e^{-cm}\int_0^m e^{cs}d\widetilde{B}^H(s)\right) \Rightarrow c\tilde{Z}(\infty)\sqrt{\frac{H\Gamma(2H)}{c^{2H}}}\eta$;
3. $\xi(m) \xrightarrow{as} \xi(\infty) = \sqrt{\frac{H\Gamma(2H)}{c^{2H}}}\omega$;
4. $e^{-cm}\widetilde{R}(m) \xrightarrow{p} 0$.

The first result is immediate after applying (28) and L'Hospital's rule. The last three results can be obtained by applying Lemma 2.1, Lemma 2.2 and Lemma 2.4 of El Machkouri, Es-Sebaiy and Ouknine (2016). Hence, as $m \to \infty$ and using $\xi(\infty) = c\tilde{Z}(\infty)$, we have

$$\begin{aligned}
\frac{e^{-cm}\left(c\tilde{Z}(m)\int_0^m e^{cs}d\tilde{B}^H(s) + \tilde{R}(m)\right)}{e^{-2cm}\int_0^m e^{2cs}\xi^2(s)ds} + o_p(1) \quad &\Rightarrow \quad \frac{c\tilde{Z}(\infty)\sqrt{\frac{H\Gamma(2H)}{c^{2H}}}\eta}{\frac{c}{2}\tilde{Z}^2(\infty)} \\
&= \quad \frac{\xi(\infty)\sqrt{\frac{H\Gamma(2H)}{c^{2H}}}\eta}{\frac{1}{2c}\xi^2(\infty)} \\
&= \quad 2c \times \frac{\eta}{\omega} = 2c \times C,
\end{aligned} \tag{32}$$

where C is the standard Cauchy variate.

We now analyze the sequential limit of $\frac{e^{cm}}{m}B_n^b$ in (25). A standard calculation shows

$$\frac{e^{cm}}{m}B_n^b \quad = \quad \frac{e^{cm}}{m}n\frac{\frac{1}{2}\sum_{t=1}^n u_t^2}{\sum_{t=1}^n y_{t-1}^2 - \frac{1}{n}\left(\sum_{t=1}^n y_{t-1}\right)^2}$$

$$
\begin{aligned}
&= \frac{e^{cm}}{m} \frac{n^{-1-2d}\frac{1}{2}\sum_{t=1}^{n} u_t^2}{n^{-2-2d}\left(\sum_{t=1}^{n} y_{t-1}^2 - \frac{1}{n}\left(\sum_{t=1}^{n} y_{t-1}\right)^2\right)} \\
&= \frac{e^{-cm}}{e^{-2cm}} \frac{m^{2H}}{m^{2H+1}} \frac{O_p(n^{-2d})}{O_p\left(\frac{e^{2cm}}{m^{2H+1}}\right)} \text{ as } n \to \infty \\
&= O_p\left(\frac{m^{2H} n^{-2d}}{e^{cm}}\right).
\end{aligned}
$$

The third equality is established by $\frac{1}{n}\sum_{t=1}^{n} u_t^2 = O_{as}(1)$, Lemma 3.2, Lemma A.2 and Lemma A.3. The assumption $m = \delta \ln n$, with $\delta > -\frac{2d}{c}$ implies that $\frac{m^{2H} n^{-2d}}{e^{cm}} \to 0$. To see this,

$$
\begin{aligned}
\ln \frac{m^{2H} n^{-2d}}{e^{cm}} &= 2H \ln m - 2d \ln n - cm \\
&= 2H\left(\ln \delta + \ln \ln n\right) - 2d \ln n - c\delta \ln n \\
&= -(c\delta + 2d)\ln n + 2H\left(\ln \delta + \ln \ln n\right) \\
&\to -\infty.
\end{aligned}
$$

Hence,

$$
\frac{e^{cm}}{m} B_n^b = o_p(1). \tag{33}
$$

This suggests that when $n \to \infty$ followed by $m \to \infty$ and when $m = \delta \ln n$ with $\delta > -\frac{2d}{c}$, $\frac{1}{n^{1+2d}}\frac{1}{2}\sum_{t=1}^{n} u_t^2$ is dominated by $\frac{1}{n^{1+2d}}\sum_{t=1}^{n} y_{t-1}u_t$.

Equations (25), (32) and (33) imply that the sequential limit of $\frac{1}{2c}\frac{e^{cm}}{m} n(\hat{\rho}_b - \rho)$ is the standard Cauchy random variable C.

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