ECON671
Factor Models: Kalman Filters

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Factor Models: Kalman Filters

Learning Objectives

1. Understand dynamic factor models using Kalman filters.
2. Estimation of the parameters by maximum likelihood.
3. Applications to
    (a) Ex ante real interest rates
    (b) Stochastic volatility
    (c) Term structure of interest rates

Background Reading

1. Previous lecture notes on factor models in finance.

EViews Computer Files

1. kalman_exante.wf1
2. stochastic_volatility.wf1
3. yields_us.wf1

Introduction

- The discussion so far has concentrated on specifying and estimating factor models based on contemporaneous relationships amongst the observed variables.
- In the case of the principal components estimator the aim is to decompose the covariance or correlation matrix of the N observable variables in terms of a set of K latent factors

$$s_{1,t}, s_{2,t}, \cdots s_{K,t}$$

- However, an important feature of many financial time series is that they exhibit dynamic patterns as the following example demonstrates.

Introduction

Example (**Term Structure of Interest Rates**)

The following table gives the autocorrelations for up to 10 lags on the 1-month, 1-year and 5-year U.S. Treasury yields.

Autocorr.	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8
1-month	0.977	0.948	0.921	0.887	0.852	0.819	0.778	0.731
1-year	0.980	0.950	0.917	0.883	0.849	0.815	0.779	0.739
5-year	0.936	0.855	0.786	0.727	0.670	0.630	0.600	0.576

Source: yields_us.wf1

The dynamics of the three series are very similar with the autocorrelations slowly decaying at an exponential rate. This suggests that a single factor could potentially capture the autocorrelation in all three yields.

Introduction

- As the previous example suggests that the dynamics of the interest rates can be explained by a common factor it is necessary to expand the factor structure as adopted in the principal components framework and replace the assumption that the factors are independent over time with a more dynamic specification.
- In the case of N variables and $K = 1$ factor, a potential specification is

$$\begin{aligned} y_{i,t} &= \alpha_i + \beta_i s_t + u_{i,t} \\ s_t &= \phi s_{t-1} + v_t, \end{aligned}$$

where $u_{i,t} \sim N(0, \sigma_i^2)$ and $v_t \sim N(0, 1)$ are independent disturbance terms and

$$\{\alpha_1, \alpha_2, \cdots, \alpha_N; \beta_1, \beta_2, \cdots, \beta_N; \sigma_1, \sigma_2, \cdots, \sigma_N; \phi\},$$

are the unknown parameters.
- Not only are the contemporaneous relationships captured by the factor s_t, but the dynamic relationships are as well.

Introduction

- An important special case is where there is no autocorrelation

$$\phi = 0$$

The factor s_t is now an *iid* disturbance term given by

$$s_t = v_t$$

which is the specification underlying the principal components framework.
- The expansion of the factor model to include a dynamic factor means that an alternative approach to the principal components estimator is needed.
- The approach presented here is based on the Kalman filter.

Historical Background: Rudolf Kalman (1930 -)

Rudy Kálmán was born in Hungary but educated in the U.S. where he spent most of his life. Is credited with inventing the filter commonly known as the Kalman filter, although others also contributed to the theory: often the filter is called the Kalman-Bucy filter.

The Kalman filter is applied in many areas, including econometrics, Bayesian learning and even the Apollo space program!

- To understand the Kalman filter a simple model is specified consisting of a single observable variable (y_t) and a single latent factor (s_t)

$$\begin{aligned} y_t &= \beta s_t + u_t \\ s_t &= \phi s_{t-1} + v_t \end{aligned}$$

where $u_t \sim N\left(0, \sigma^2\right)$ and $v_t \sim N(0, 1)$ are independent disturbances, and $\left\{\beta, \phi, \sigma^2\right\}$ are unknown parameters.

- This representation of the model is also known as a state-space system with the first equation representing the signal equation (the equation of the observable variable y_t) and the second representing the state equation (the equation of the unobservable variable s_t).

- Define the conditional mean of y_t based on information at time $t-1$

$$y_{t|t-1} = E_{t-1}\left[y_t\right]$$

with variance

$$V_{t|t-1} = E\left[\left(y_t - y_{t|t-1}\right)^2\right]$$

- As s_t is unknown the aim of the Kalman filter is to estimate the factor s_t using the available information on the observable variable y_t .
- The best estimator of the factor s_t based on information at time $t-1$, is the conditional mean

$$s_{t|t-1} = E_{t-1}\left[s_t\right]$$

with variance

$$P_{t|t-1} = E\left[\left(s_t - s_{t|t-1}\right)^2\right]$$

- But when information on y_t becomes available then a better estimator of s_t is given by the updated conditional mean

$$s_{t|t} = E_t\left[s_t\right]$$

with variance

$$P_{t|t} = E\left[\left(s_t - s_{t|t}\right)^2\right]$$

- This sequence of updating the estimate of s_t as more information on y_t becomes available is an important feature of the Kalman filter.
- To understand the recursive nature of the algorithm it is assumed that the parameters

$$\beta, \sigma, \phi$$

are known, or at least represent some starting values. Issues of estimation are discussed below.

- For the 1-factor model the Kalman filter equations are summarized as

Prediction:

$$s_{t|t-1} = \phi s_{t-1|t-1}$$
$$P_{t|t-1} = \phi^2 P_{t-1|t-1} + 1$$

Observation:

$$y_{t|t-1} = \beta s_{t|t-1}$$
$$V_{t|t-1} = \beta^2 P_{t|t-1} + \sigma^2$$

Updating:

$$s_{t|t} = s_{t|t-1} + \frac{\beta P_{t|t-1}}{V_{t|t-1}}(y_t - y_{t|t-1})$$
$$P_{t|t} = P_{t|t-1} - \frac{\beta^2 P_{t|t-1}^2}{V_{t|t-1}}$$

- At $t = 1$, starting values are needed for the two prediction equations

$$s_{1|0}, P_{1|0}.$$

- A typical choice of the mean of the factor is

$$s_{1|0} = 0$$

  although other values can be used. A typical choice of the variance of the factor is

$$P_{1|0} = 1/\left(1 - \phi^2\right)$$

  which is the variance of the unconditional distribution of an AR(1) process.

- For given values of the parameters, the filter is computed for $t = 1, 2, \cdots T$.

Example (Numerical Example of the Filter)

Suppose that there are $T = 2$ observations on the variable y_t given by $y_t = \{2, 5\}$. Assume that the parameters are $\beta = 0.5$, $\sigma = 0.1$, $\phi = 0.8$, and the initial estimate of the factor is chosen as $s_{1|0} = 0.1$. The first step $(t = 1)$ is

Prediction: $s_{1|0} = 0.1$

(initialization) $P_{1|0} = \dfrac{1}{1-\phi^2} = \dfrac{1}{1-0.8^2} = 2.7778$

Observation:

$$y_{1|0} = \beta s_{1|0} = 0.5 \times 0.1 = 0.05$$

$$V_{1|0} = \beta^2 P_{1|0} + \sigma^2 = 0.5^2 \times 2.7778 + 0.1^2 = 0.7045$$

Example (**Numerical Example of the Filter continued**)

Updating:

$$s_{1|1} = s_{1|0} + \frac{\beta P_{1|0}}{V_{1|0}}(y_1 - y_{1|0})$$

$$s_{1|1} = 0.1 + \frac{0.5 \times 2.7778}{0.7045} \times (2 - 0.05) = 3.9444$$

$$P_{1|1} = P_{1|0} - \frac{\beta^2 P_{1|0}^2}{V_{1|0}}$$

$$P_{1|1} = 2.7778 - \frac{0.5^2 \times 2.7778^2}{0.7045} = 0.0396$$

Intuitively, the initial estimate of 0.1 for the factor at $t = 1$, results in an underestimate of the observed variable, $0.05 < 2$. By updating the estimate of the factor to 3.9444 this yields a better estimate of y_1.

Example (**Numerical Example of the Filter continued**)

The second step $(t = 2)$ is

Prediction:

$$s_{2|1} = \phi s_{1|1}$$
$$s_{2|1} = 0.8 \times 3.9444 = 3.1555$$

$$P_{2|1} = \phi^2 P_{1|1} + 1$$
$$P_{2|1} = 0.8^2 \times 0.0396 + 1 = 1.0253$$

Observation:

$$y_{2|1} = \beta s_{2|1}$$
$$y_{2|1} = 0.5 \times 3.1555 = 1.5778$$

$$V_{2|1} = \beta^2 P_{2|1} + \sigma^2$$
$$V_{2|1} = 0.5^2 \times 1.0253 + 0.1^2 = 0.2663$$

Example (Numerical Example of the Filter continued)

The second step $(t = 2)$ is

Updating:

$$s_{2|2} = s_{2|1} + \frac{\beta P_{2|1}}{V_{2|1}}(y_2 - y_{2|1})$$

$$s_{2|2} = 3.1555 + \frac{0.5 \times 1.0253}{0.2663} \times (5 - 1.5778) = 9.7435$$

$$P_{2|2} = P_{2|1} - \frac{\beta^2 P_{2|1}^2}{V_{2|1}}$$

$$P_{2|2} = 1.0253 - \frac{0.5^2 \times 1.0253^2}{0.2663} = 0.03840$$

- Consider a model where $N = 3$ variables and $K = 2$ factors

$$
\begin{aligned}
y_{1,t} &= \alpha_1 + \beta_{1,1} s_{1,t} + \beta_{1,2} s_{2,t} + u_{1,t} \\
y_{2,t} &= \alpha_2 + \beta_{2,1} s_{1,t} + \beta_{2,2} s_{2,t} + u_{2,t} \\
y_{3,t} &= \alpha_3 + \beta_{3,1} s_{1,t} + \beta_{3,2} s_{2,t} + u_{3,t}
\end{aligned}
$$

$$
\begin{aligned}
s_{1,t} &= \phi_{1,1} s_{1,t-1} + v_{1,t} \\
s_{2,t} &= \phi_{2,2} s_{2,t-1} + v_{2,t}
\end{aligned}
$$

or in matrix notation

$$
\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \\ \beta_{3,1} & \beta_{3,2} \end{bmatrix} \begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{bmatrix}
$$

$$
\begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_{1,1} & 0 \\ 0 & \phi_{2,2} \end{bmatrix} \begin{bmatrix} s_{1,t-1} \\ s_{2,t-1} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}
$$

- For an extension the previous example, consider the case of N variables $\{y_{1,t}, y_{2,t} \cdots, y_{N,t}\}$ and K factors $\{s_{1,t}, s_{2,t} \cdots, s_{K,t}\}$. The multivariate version of the state-space system is

$$\begin{aligned} y_t &= A + Bs_t + u_t \\ s_t &= \Phi s_{t-1} + v_t \end{aligned}$$

  where the disturbances are distributed as

$$\begin{aligned} u_t &\sim N(0, R) \\ v_t &\sim N(0, Q) \end{aligned}$$

  where $E\left[u_t u_t'\right] = R$ and $E\left[v_t v_t'\right] = Q$ are respectively the covariances of u_t and v_t.
- The dimensions of the parameter matrices are as follows: A is $(N \times 1)$, B is $(N \times K)$, Φ is $(K \times K)$, R is $(N \times N)$ and Q is $(K \times K)$.

- The recursions of the multivariate Kalman filter are

Prediction:
$$s_{t|t-1} = \Phi s_{t-1|t-1}$$
$$P_{t|t-1} = \Phi P_{t-1|t-1} \Phi' + Q$$

Observation:
$$y_{t|t-1} = B s_{t|t-1}$$
$$V_{t|t-1} = B P_{t|t-1} B' + R$$

Updating:
$$s_{t|t} = s_{t|t-1} + P_{t|t-1} B' V_{t|t-1}^{-1} (y_t - y_{t|t-1})$$
$$P_{t|t} = P_{t|t-1} - P_{t|t-1} B' V_{t|t-1}^{-1} B P_{t|t-1}$$

- The formulae for the multivariate version of the Kalman filter contain the univariate formulae with $N = K = 1$.

- To start the recursion two cases are considered.
  1. Stationary Latent Factors
     The initial values $s_{1|0}$ and $P_{1|0}$ for the multivariate K factor model are given by
     $$\begin{aligned} s_{1|0} &= 0 \\ \mathrm{vec}(P_{1|0}) &= (I_{K\times K} - (\Phi \otimes \Phi))^{-1}\mathrm{vec}(Q) \end{aligned}$$
  2. Nonstationary Latent Factors
     In the case the starting values for the variance would be undefined if the previous approach is adopted. To circumvent this problem, starting values are chosen as
     $$\begin{aligned} s_{1|0} &= \psi \\ P_{1|0} &= \omega\,\mathrm{vec}(Q) \end{aligned}$$
     where ψ represents the best guess of starting value for the conditional mean and ω is a positive constant whereby larger values of ω correspond to the distribution of $s_{1|0}$ being more diffuse.

- The state-space model is under-identified unless some restrictions are imposed.
- The difficulty is seen by noting that the volatility in the factor is controlled by Q, but the impact of the factor on y_t is given by B.
- There is an infinite number of combinations of Q and B that will be consistent with the volatility of y_t ie in the case of $N = K = 1$, then

$$var(y_t) = \beta^2 var(s_t) + var(u_t)$$

Thus it is necessary to fix one of these quantities.

  - A common approach is to set

$$Q = I$$

  - Another approach is to place restrictions on B and allow Q to be estimated.

Maximum Likelihood Estimator

- The discussion so far has concentrated on extracting the factor s_t, assuming given values for the population parameters

$$\theta = \{A, B, \Phi, R, Q\}$$

  In general, however, it is necessary to estimate these parameters.
- If the factors are known, then the parameters are estimated by simply regressing y_t on s_t and regressing s_t on s_{t-1}. But as s_t is unobservable (latent), an alternative estimation strategy is needed.
- The natural estimator of the parameters is the maximum likelihood estimator which constructs the log-likelihood function based on

$$y_t \sim N(y_{t|t-1}, V_{t|t-1})$$

  As the likelihood is a nonlinear function of the parameters an iterative algorithm is required to obtain the maximum likelihood estimates.

Maximum Likelihood Estimator

- For a sample of $t = 1, 2, \cdots, T$ observations on y_t, the log-likelihood function for the t^{th} observation using the multivariate normal distribution is given by

$$\begin{aligned} \log L_t &= -\frac{N}{2}\log(2\pi) - \frac{1}{2}\log\left|V_{t|t-1}\right| \\ &\quad -\frac{1}{2}(y_t - y_{t|t-1})' V_{t|t-1}^{-1}(y_t - y_{t|t-1}) \end{aligned}$$

- For the entire sample, the log-likelihood function is

$$\log L = \frac{1}{T}\sum_{t=1}^{T}\log L_t$$

- This expression is a nonlinear function of the parameters

$$\theta = \{A, B, \Phi, R, Q\}$$

via $y_{t|t-1}$ and $V_{t|t-1}$ from the Kalman filter.

Maximum Likelihood Estimator

Using EViews

- Consider estimating a one-factor model of the spread between the one-year yield and the one-month yield

$$\begin{aligned} YIELD\_Y1_t - YIELD\_M1_t &= \alpha + \beta s_t + u_t, \quad u_t \sim N\left(0, \sigma^2\right) \\ s_t &= \phi s_{t-1} + v_t \quad v_t \sim N(0, 1) \end{aligned}$$

  with starting values $\{\alpha_{(0)} = 0.1, \beta_{(0)} = 0.1, \sigma^2_{(0)} = 0.1, \phi_{(0)} = 0.9\}$.
- The EViews commands are:

  Object / New Object... / SSpace / OK

  In the window type in the following commands

  ```
  @signal yield_y1-yield_m1 = c(1) + c(2)*s + [var = c(3)]
  @state s = c(4)*s(-1) + [var = 1]
  @param c(1) 0.1 c(2) 0.1 c(3) 0.1 c(4) 0.9
  ```

  Then click

  Estimate / OK

Factor Extraction

- Once the algorithm has converged estimates of the latent factor s_t at each point in time are available.
- In fact, three estimates can be calculated depending on the form of the conditioning information set used

$$\begin{array}{lcrcl} \text{One-step-ahead} & : & s_{t|t-1} & = & E_{t-1}\left[s_t\right] \\ \text{Filtered} & : & s_{t|t} & = & E_t\left[s_t\right] \\ \text{Smoothed} & : & s_{t|T} & = & E_T\left[s_t\right] \end{array}$$

  - The first two estimates, $s_{t|t-1}$ and $s_{t|t}$, are a by-product of the Kalman filter algorithm which are automatically available once the algorithm has converged.
  - The third estimator $s_{t|T}$, is effectively obtained by running the Kalman filter algorithm in the reverse direction (from T to $t-1$) once the maximum likelihood estimates are obtained.

Factor Extraction

Using EViews

1. The one-step-ahead estimate of the factor $s_{t|t-1} = E_{t-1}[s_t]$

   View / State Views / Graph State Series...
   / One-step-ahead: Predicted States / OK

2. The filtered estimate of the factor $s_{t|t} = E_t[s_t]$

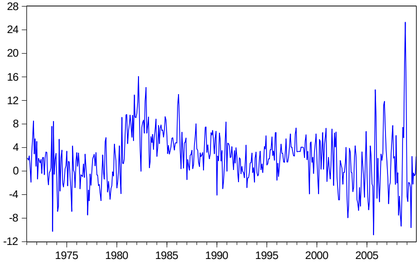
   View / State Views / Graph State Series...
   / Filtered: State Estimates / OK

3. The smoothed estimate of the factor $s_{t|T} = E_T[s_t]$

   View / State Views / Graph State Series...
   / Smoothed: State Estimates / OK

Estimating the Ex Ante Real Interest Rate

- There exist two broad types of real interest rates
  - (i) Ex post real interest rates (observed).
  - (ii) Ex ante real interest rates (unobserved).
- The ex post real interest rate is observed (as given in the following Figure which gives the U.S. ex post 1-month real interest rate), but the ex ante real interest rate is not.



Source: kalman_exante.wf1

Estimating the Ex Ante Real Interest Rate

- But it is the ex ante real interest rate that is important in finance and economics as it provides a measure of the real return on an asset between the present and the future.

How can the ex ante interest rate be measured?

- There are two strategies:
  (i) Proxy
  Use the ex post real interest rate as a proxy for the ex ante interest rate.
  (ii) Latent Factor
  Treat the ex ante real interest rate as unknown using a latent factor model.

Estimating the Ex Ante Real Interest Rate

- Formally the ex ante real interest rate is defined as

$$r_t^e = i_t - \pi_t^e$$

  where i_t is the nominal interest rate and π_t^e is the expected inflation rate defined as

$$\pi_t^e = \log p_{t+1} - \log p_t$$

- Whilst i_t is observed, π_t^e is not.
- So it is the expected inflation rate that makes the ex ante real interest rate unobservable.

Estimating the Ex Ante Real Interest Rate

- Consider the ex post real interest rate

$$r_t = i_t - \pi_t$$

which is observed where $\pi_t = \log p_t - \log p_{t-1}$ is the actual inflation rate. Expanding this expression to allow for expected inflation, π_t^e, gives

$$\begin{aligned} r_t &= i_t - \pi_t^e + \pi_t^e - \pi_t \\ &= i_t - \pi_t^e + u_t \end{aligned}$$

- Defining $s_t = i_t - \pi_t^e - \alpha$ as the ex ante real interest rate (adjusted by α) and $u_t = \pi_t^e - \pi_t$ as the inflation expectations error, this expression is written as a latent factor model as

$$r_t = \alpha + s_t + u_t\,, \qquad u_t \sim N(0, \sigma_u^2)$$

- The key advantage of this formulation of the model is that it avoids the measurement error from using realized inflation and not expected inflation.

Estimating the Ex Ante Real Interest Rate

- To estimate the ex ante real interest rate, monthly data starting in January 1971 and ending in December 2009 on the following U.S. series are used

$$\begin{array}{lcl} EURO\_1MTH & : & \text{1-month Eurodollar rate, (\%, p.a.)} \\ CPI & : & \text{Consumer price index} \end{array}$$

- The annualized percentage inflation rate is computed as

$$INF = 1200 \times DLOG\,(CPI)$$

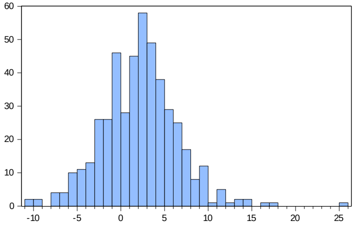
and the ex post real interest rate is computed as

$$R = EURO\_1MTH - INF$$

This is the ex post real interest rate given in the previous Figure.

Estimating the Ex Ante Real Interest Rate

- Some summary statistics are given in the Figure below.



Series: R	
Sample 1971M01 2009M12	
Observations 467	
Mean	2.173661
Median	2.258335
Maximum	25.28643
Minimum	-10.85508
Std. Dev.	4.333232
Skewness	0.376376
Kurtosis	4.984465
Jarque-Bera	87.65467
Probability	0.000000

Source: kalman_exante.wf1

- Here the average real ex post interest rate is 2.174% p.a. over the sample period.

Estimating the Ex Ante Real Interest Rate

- The autocorrelation of the real ex post interest rate if given in the following figure.

Date: 05/05/14 Time: 14:20
Sample: 1971M01 2009M12
Included observations: 467

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.551	0.551	142.61	0.000
		2	0.319	0.022	190.57	0.000
		3	0.211	0.039	211.66	0.000
		4	0.206	0.101	231.72	0.000
		5	0.177	0.023	246.53	0.000
		6	0.152	0.029	257.52	0.000
		7	0.193	0.115	275.34	0.000
		8	0.170	-0.001	289.07	0.000
		9	0.211	0.116	310.30	0.000
		10	0.276	0.144	346.88	0.000

Source: kalman_exante.wf1

- The correlogram shows strong evidence of first order autocorrelation. This result is important as identification of the parameters of the model require that there is significant autocorrelation.

Estimating the Ex Ante Real Interest Rate

- The factor model of the ex ante real interest rate is specified as

$$\begin{array}{lll} r_t = \alpha + s_t + u_t, & u_t \sim N\left(0, \sigma_u^2\right) & \text{[Signal equation]} \\ s_t = \phi s_{t-1} + v_t, & v_t \sim N\left(0, \sigma_v^2\right) & \text{[State equation]} \end{array}$$

  where the unknown parameters are $\theta = \left\{\alpha, \phi, \sigma_u^2, \sigma_v^2\right\}$.
- The starting values for the parameters are chosen as follows:
  - α is based on the sample mean of r_t, equal to 2.174.
  - ϕ is based on the first autocorrelation coefficient of r_t, equal to 0.551.
  - σ_u and σ_v are both set equal to half of the standard deviation of r_t, equal to 4.333/2.

- The EViews window to estimate the model is given below.

```
View | Proc | Object | Print | Name | Freeze | Spec | Estimate | Stats | Forecast
@signal r = c(1) + s + [var = c(3)^2]

@state s = c(2)*s(-1) + [var = c(4)^2]

@param c(1) 2.174 c(2) 0.551 c(3) 2.167 c(4) 2.167
```

Source: kalman_exante.wf1

where $C(1)$ corresponds to α, $C(2)$ corresponds to ϕ, $C(3)$ corresponds to σ_u, $C(4)$ corresponds to σ_v.

- Note that it is the standard deviations σ_u and σ_v that are being estimated and not the variance. This choice of parameterization has the advantage that the variance is guaranteed to be positive. If either of the estimates of σ_u and σ_v happen to be negative, it is appropriate to just change the sign and report a positive estimate.

Estimating the Ex Ante Real Interest Rate

- The parameter estimates are contained in the following window.

Sspace: EXANTE
Method: Maximum likelihood (Marquardt)
Date: 05/14/14 Time: 06:41
Sample: 1971M01 2009M12
Included observations: 468
Valid observations: 467
Convergence achieved after 18 iterations

	Coefficient	Std. Error	z-Statistic	Prob.
C(1)	2.173917	0.383756	5.664847	0.0000
C(2)	0.583044	0.046197	12.62082	0.0000
C(3)	1.037080	0.571235	1.815504	0.0694
C(4)	3.410909	0.231904	14.70830	0.0000

	Final State	Root MSE	z-Statistic	Prob.
S	0.065966	3.459736	0.019067	0.9848

Log likelihood	-1262.523	Akaike info criterion	5.424083
Parameters	4	Schwarz criterion	5.459598
Diffuse priors	0	Hannan-Quinn criter.	5.438059

Source: kalman_exante.wf1

- The estimated model is

$$\begin{aligned} r_t &= 2.174 + \widehat{s}_t + \widehat{u}_t \\ \widehat{s}_t &= 0.583\widehat{s}_{t-1} + \widehat{v}_t \end{aligned}$$

where $\widehat{\sigma}_u = 1.037$, $\widehat{\sigma}_v = 3.411$.

Estimating the Ex Ante Real Interest Rate

- As it is the ex ante estimate of the real interest rate that is required, the one-step ahead factor $s_{t|t-1}$, is the appropriate quantity as it provides an estimate of the interest rate in the future at time t, based on information at time $t-1$, without using current or future information.
- The Eviews commands to extract the estimate of the one-step-ahead estimate of the factor $s_{t|t-1} = E_{t-1}\left[s_t\right]$, are

Proc / Make State Series...

Choose **One-step-ahead: Predicted states**, then for **Series names** choose

S_HAT

Estimating the Ex Ante Real Interest Rate

- As the factor is defined as

$$s_t = i_t - \pi_t^e - \alpha$$

the ex ante real interest rate is given by rearranging this expression as

$$r_t^e = i_t - \pi_t^e = s_t + \alpha$$

- Given that $s_{t|t-1}$ is the apropriate conditional mean estimate of the factor, from the definition of the factor an estimate of the ex ante real interest rate is given by

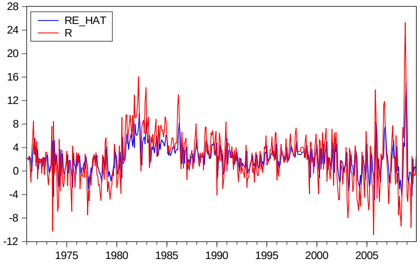
$$\widehat{r_t^e} = \widehat{s}_{t|t-1} + \widehat{\alpha}$$

- This quantity is computed using Genr as

RE_HAT = S_HAT + 2.174

Estimating the Ex Ante Real Interest Rate

- The estimate of the ex ante real interest rate $\left(\widehat{r}_t^e\right)$ and the ex post real interest rate (r_t) are compared in the following Figure.



Source: kalman_exante.wf1

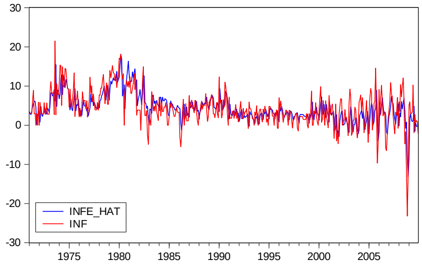
- The estimate of the ex ante real interest rate $\widehat{r}_t^e$ follows r_t closely but exhibits less volatility.

Estimating the Ex Ante Real Interest Rate

- Alternatively, as the ex ante real interest rate is a function of the expected inflation rate, then the latter can be estimated as

$$\widehat{\pi}_t^e = i_t - \widehat{r}_t^e$$

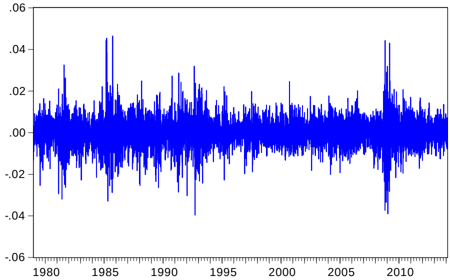
- Using the Genr command, $\widehat{\pi}_t^e$ is computed and plotted in the following Figure together with the actual inflation rate π_t.



Source: kalman_exante.wf1

A Stochastic Volatility Model of the Exchange Rate

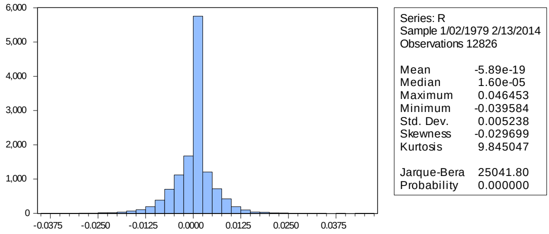
- Volatility is an important input into financial decision-making as it represents the risk of an asset.
- Consider the case where the asset is the UK/US exchange rate. The (demeaned) return on the UK/US exchange rate (r_t) is given in the following Figure from January 2nd 1979 to February 13th 2014.



Source: stochastic_volatility.wf1

A Stochastic Volatility Model of the Exchange Rate

- The aim is to extract a measure of the volatility of the exchange rate.
- One approach is to assume constant volatility. The following Figure yields an estimate of 0.005238.



Source: stochastic_volatility.wf1

- Another approach is to assume time-varying volatility by specifying a GARCH model where the volatility is assumed to be a function of lagged (squared) shocks.

A Stochastic Volatility Model of the Exchange Rate

- Another approach is the stochastic volatility model given by

$$\begin{aligned} r_t &= \sigma_t w_t && \text{[Mean equation]} \\ \log(\sigma_t^2) &= \alpha + \phi \log(\sigma_{t-1}^2) + v_t && \text{[Variance equation]} \end{aligned}$$

  where r_t is the (demeaned) exchange rate return, σ_t represents the exchange rate volatility, and w_t and v_t are disturbance terms with the properties $w_t \sim N(0,1)$ and $v_t \sim N\left(0, \sigma_v^2\right)$.
- An important feature of this model is the additional stochastic term given by v_t, in the variance equation. For this reason the model is called the stochastic volatility model.
- Estimating the stochastic volatility model is in general difficult arising from the presence of the additional disturbance term v_t as that now makes the volatility σ_t^2 stochastic as well.
- One solution is to express the model as a latent factor model and use the Kalman filter to estimate the model by maximum likelihood methods.

A Stochastic Volatility Model of the Exchange Rate

- The strategy consists of squaring both sides of the mean equation as

$$r_t^2 = \sigma_t^2 w_t^2$$

Now taking natural logarithms gives

$$\log r_t^2 = \log\left(\sigma_t^2\right) + \log\left(w_t^2\right)$$

- Redefine the variables as

$$\begin{aligned} y_t &= \log r_t^2 \\ s_t &= \log\left(\sigma_t^2\right) \\ u_t &= \log\left(w_t^2\right) + 1.27 \end{aligned}$$

where the term 1.27 in the equation for u_t appears as it can be shown that $E\left[\log\left(w_t^2\right)\right] = -1.27$, so $E\left[u_t\right] = 0$.

- Also, it can be shown that the variance of $\log\left(w_t^2\right)$ and hence u_t, is

$$E\left[u_t^2\right] = \frac{\pi^2}{2} = 4.9348$$

A Stochastic Volatility Model of the Exchange Rate

- The stochastic volatility model is rewritten as a latent factor model as

$$\begin{aligned} y_t &= -1.27 + s_t + u_t && \text{[Mean equation]} \\ s_t &= \alpha + \phi s_{t-1} + v_t && \text{[Variance equation]} \end{aligned}$$

  where $y_t = \log r_t^2$, the natural logarithm of the squared exchange rate.
- The variable y_t is constructed using Genr in EViews.
- To generate some starting values the following AR(1) model is estimated

$$y_t = \beta_1 + \beta_2 y_{t-1} + w_t$$

  where $w_t \sim N\left(0, \sigma_w^2\right)$.

A Stochastic Volatility Model of the Exchange Rate

- The parameter estimates are given in the following window.

Dependent Variable: Y
Method: Least Squares
Date: 05/04/14 Time: 13:51
Sample (adjusted): 1/04/1979 2/13/2014
Included observations: 12825 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-10.87204	0.134430	-80.87485	0.0000
Y(-1)	0.273568	0.008494	32.20855	0.0000

R-squared	0.074846	Mean dependent var	-14.96627
Adjusted R-squared	0.074774	S.D. dependent var	5.149445
S.E. of regression	4.953184	Akaike info criterion	6.038094
Sum squared resid	314599.9	Schwarz criterion	6.039258
Log likelihood	-38717.28	Hannan-Quinn criter.	6.038483
F-statistic	1037.391	Durbin-Watson stat	1.783787
Prob(F-statistic)	0.000000		

Source: stochastic_volatility.wf1

- The EViews window to estimate the model is given below.

View | Proc | Object | Print | Name | Freeze | Spec | Estimate | Stats | Forecast

```
@signal y = -1.27+ s + [var=4.9348]

@state s = c(1)+c(2)*s(-1)+[var=c(3)^2]

param c(1) -10.8720 c(2) 0.2736 c(3) 4.9532
```

Source: stochastic_volatility.wf1

where

$C(1)$ corresponds to α with the starting value based on $\widehat{\beta}_1 = -10.872$
$C(2)$ corresponds to ϕ with the starting value $\widehat{\beta}_2 = 0.2736$
$C(3)$ corresponds to σ_v with starting value based on $\widehat{\sigma}_w = 4.9532$

A Stochastic Volatility Model of the Exchange Rate

- The parameter estimates are contained in the following window.

Sspace: STVOL
Method: Maximum likelihood (Marquardt)
Date: 05/04/14 Time: 14:23
Sample: 1/02/1979 2/13/2014
Included observations: 12827
Valid observations: 12826
Convergence achieved after 9 iterations

	Coefficient	Std. Error	z-Statistic	Prob.
C(1)	-9.885800	0.179148	-55.18237	0.0000
C(2)	0.277877	0.011185	24.84375	0.0000
C(3)	4.475922	0.080611	55.52466	0.0000

	Final State	Root MSE	z-Statistic	Prob.
S	-12.86922	4.510047	-2.853457	0.0043

Log likelihood	-38801.18	Akaike info criterion	6.050862
Parameters	3	Schwarz criterion	6.052607
Diffuse priors	0	Hannan-Quinn criter.	6.051445

Source: stochastic_volatility.wf1

- The estimated model is

$$\begin{aligned} y_t &= -1.27 + \widehat{s}_t + \widehat{u}_t && \text{[Mean equation]} \\ \widehat{s}_t &= -9.8858 + 0.2779\widehat{s}_{t-1} + \widehat{v}_t && \text{[Variance equation]} \end{aligned}$$

where $\widehat{\sigma}_v = 4.4759$.

A Stochastic Volatility Model of the Exchange Rate

- As $s_t = \log\left(\sigma_t^2\right)$, an estimate of the volatility is

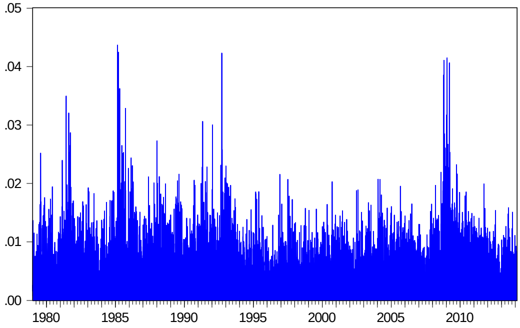
$$\widehat{\sigma}_t = \exp\left(\frac{\widehat{s}_t}{2}\right)$$

- If the strategy is to derive an historical estimate of the volatility the best estimates of the factor at each point in time is based on all of the sample information, namely $\widehat{s}_{t|T}$, which is the smoothed estimate. Hence the volatility estimate is based on

$$\widehat{\sigma}_t = \exp\left(\frac{\widehat{s}_{t|T}}{2}\right)$$

A Stochastic Volatility Model of the Exchange Rate

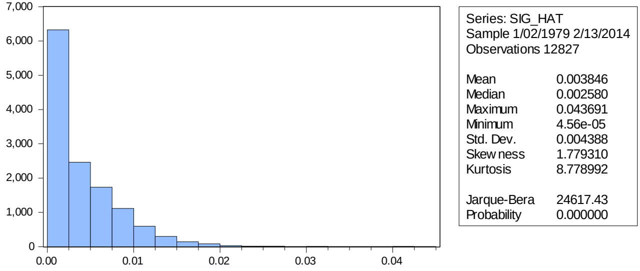
- The volatility estimate is given in the following figure.



Source: stochastic_volatility.wf1

- The increase in volatility during times of financial crises is clear where the estimates of volatility reach 0.04.

- Descriptive statistics on the volatility series are given below.



Source: stochastic_volatility.wf1

- An estimate of the mean of the volatility series is 0.0038 which is a little it smaller than the constant volatility estimate of 0.005238.

A Dynamic One-Factor Model of the Term Structure

- Factor models are widely used in finance to model the term structure of interest rates. An important example is Cox, Ingersoll and Ross (1985) who derive a 1-factor model of the term structure of interest rates where the unoserved factor is the instantaneous interest rate.
- Consider the following one-factor model of the term structure of interest rates

$$
\begin{aligned}
r_{i,t} &= \alpha_i + \beta_i s_t + u_{i,t}, \qquad i = 1, 2, \cdots, 9 \\
s_t &= \phi s_{t-1} + v_t \\
u_{i,t} &\sim N\left(0, \sigma_i^2\right), \qquad v_t \sim N\left(0, 1\right)
\end{aligned}
$$

- There are 28 parameters. The starting parameters are chosen as

$$
\begin{aligned}
\left\{\alpha_i, \beta_i, \sigma_i^2\right\} &= 0.1 \\
\phi &= 0.9
\end{aligned}
$$

A Dynamic One-Factor Model of the Term Structure

- The EViews window to estimate the model is given below.

View | Proc | Object | Print | Name | Freeze | Spec | Estimate | Stats | Forecast

```
@signal yield_m1 = c(1) + c(10)*s + [var = c(19)]
@signal yield_m3 = c(2) + c(11)*s + [var = c(20)]
@signal yield_m6 = c(3) + c(12)*s + [var = c(21)]

@signal yield_y1 = c(4) + c(13)*s + [var = c(22)]
@signal yield_y2 = c(5) + c(14)*s + [var = c(23)]
@signal yield_y3 = c(6) + c(15)*s + [var = c(24)]

@signal yield_y5 = c(7) + c(16)*s + [var = c(25)]
@signal yield_y7 = c(8) + c(17)*s + [var = c(26)]
@signal yield_y10 = c(9) + c(18)*s + [var = c(27)]

@state s = c(28)*s(-1) + [var = 1]
@param c(1) 0.1 c(2) 0.1 c(3) 0.1 c(4) 0.1 c(5) 0.1 c(6) 0.1 c(7) 0.1 c(8) 0.1 c(9) 0.1 c(10)
0.1 c(11) 0.1 c(12) 0.1 c(13) 0.1 c(14) 0.1 c(15) 0.1 c(16) 0.1 c(17) 0.1 c(18) 0.1 c(19) 0.1
c(20) 0.1 c(21) 0.1 c(22) 0.1 c(23) 0.1 c(24) 0.1 c(25) 0.1 c(26) 0.1 c(27) 0.1 c(28) 0.9
```

Source: yields_us.wf1

- The parameter estimates are contained in the following window.

Sspace: KALMAN
Method: Maximum likelihood (Marquardt)
Date: 04/29/14 Time: 18:06
Sample: 2001M07 2010M09
Included observations: 111
Convergence achieved after 622 iterations

	Coefficient	Std. Error	z-Statistic	Prob.
C(1)	-2.363971	13.09975	-0.180459	0.8568
C(2)	-2.358248	13.27509	-0.177645	0.8590
C(3)	-2.254918	13.42593	-0.167952	0.8666
C(4)	-1.857111	12.60488	-0.147333	0.8829
C(5)	-0.966897	10.96083	-0.088214	0.9297
C(6)	-0.180924	9.449453	-0.019146	0.9847
C(7)	1.181755	8.332502	0.141825	0.8872
C(8)	2.150556	5.078758	0.423441	0.6720
C(9)	2.933001	4.382718	0.669220	0.5034
C(10)	0.168607	0.015277	11.03670	0.0000
C(11)	0.171209	0.010041	17.05155	0.0000
C(12)	0.172858	0.009675	17.86644	0.0000
C(13)	0.162808	0.016945	9.607756	0.0000
C(14)	0.139998	0.056018	2.499155	0.0124
C(15)	0.120449	0.080705	1.492448	0.1356
C(16)	0.087738	0.157020	0.558769	0.5763
C(17)	0.064572	0.165674	0.389754	0.6967
C(18)	0.046940	0.078354	0.599074	0.5491

(continued on the next slide)

A Dynamic One-Factor Model of the Term Structure

C(19)	0.039699	0.014213	2.793184	0.0052
C(20)	0.013433	0.006783	1.980373	0.0477
C(21)	0.002336	0.002098	1.113649	0.2654
C(22)	0.012359	0.007456	1.657509	0.0974
C(23)	0.089894	0.113655	0.790937	0.4290
C(24)	0.149941	0.267096	0.561376	0.5745
C(25)	0.210447	0.437776	0.480718	0.6307
C(26)	0.231194	0.418256	0.552756	0.5804
C(27)	0.204069	0.262756	0.776648	0.4374
C(28)	0.999234	0.007354	135.8760	0.0000

	Final State	Root MSE	z-Statistic	Prob.
S	13.95279	1.025722	13.60290	0.0000

Log likelihood	-122.7663	Akaike info criterion	2.716509
Parameters	28	Schwarz criterion	3.399994
Diffuse priors	0	Hannan-Quinn criter.	2.993779

Source: yields_us.wf1

A Dynamic One-Factor Model of the Term Structure

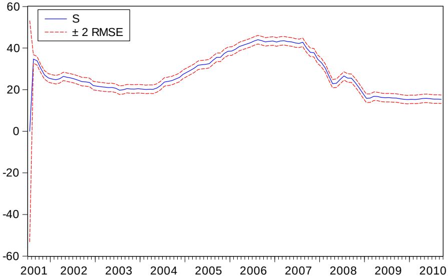
- The log-likelihood value is

$$\ln L\left(\widehat{\theta}\right) = -121.2888$$

- The estimated loadings (β), given by parameters 10 to 18, show that the latent factor has its greatest impact on the shorter maturities (less than one year) which progressively diminishes in importance across the maturity spectrum.
- The estimates of the idiosyncratic parameter $\left(\sigma^2\right)$, given by parameters 19 to 27, are smallest for the 6-month yield suggesting that this yield follows the factor more closely than the other yields.
- As the intercept estimates (α), given by parameters 1 to 9, increase over the maturity spectrum, this suggests an upward yield curve on average.
- The parameter estimate of ϕ is 0.999, suggesting that the latent factor is nonstationary.

A Dynamic One-Factor Model of the Term Structure

- The one-step ahead estimates of the latent factor $s_{t|t-1} = E_{t-1}[s_t]$, are given in the following Figure.



Source: yields_us.wf1

A Dynamic One-Factor Model of the Term Structure

- The confidence interval for the initial estimate of the factor is very wide representing a lack of information at this point in time. The confidence interval quickly narrows showing that the estimates for later points in time are more precise.
- The factor is relatively flat in the first part of the period, then rises reaching a peak around by the end of 2006. As the loadings of the factor are relatively larger on the smaller maturities than the longer maturities, this increase in the factor is associated with a narrowing of the spreads.
- From about mid-2007 the factor falls resulting in a widening of spreads, which eventually stabilize from 2009 onwards.

A Dynamic One-Factor Model of the Term Structure

- The prediction properties of the model are obtained by computing

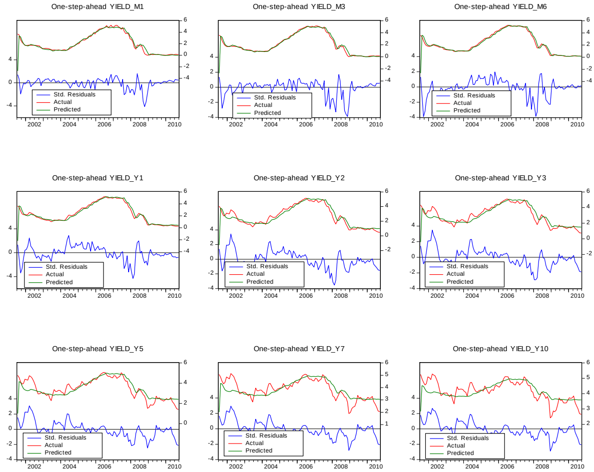
$$y_{i,\,t|t-1} = \widehat{\alpha}_i + \widehat{\beta}_i s_{t|t-1}$$

- The EViews commands are

View / Actual,Predicted,Residual Graph / OK

- This figure further highlights how the estimated factor follows the shorter maturities very closely, while the longer maturities tend to exhibit additional dynamics suggesting the need for a second factor.

A Dynamic One-Factor Model of the Term Structure



Source: yields_us.wf1

- The state-space model represents a flexible framework which can easily accommodate a number of extensions.
- Two important extensions are:
  1. Dynamics
     Have focussed on a AR(1) representations of s_t with the idiosyncratic disturbance u_t being white noise. These restrictions can e relaxed.
  2. Exogenous and Predetermined Variables
     Can allow for exogenous and predetermined variables in the signal and state equations.

An AR(2) Model of s_t

- Suppose that the latent factor is an $AR(2)$ process

$$s_t = \phi_1 s_{t-1} + \phi_2 s_{t-2} + v_t$$

- This equation can be written as a vector $AR(1)$ model

$$\begin{bmatrix} s_t \\ s_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ s_{t-2} \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \end{bmatrix}$$

- The Kalman filter proceeds as before except now there are two factors, s_t and s_{t-1}, with

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}$$

- To accommodate the additional lag the signal equation becomes

$$y_t = \begin{bmatrix} \beta & 0 \end{bmatrix} \begin{bmatrix} s_t \\ s_{t-1} \end{bmatrix} + u_t$$

Using EViews

- Consider estimating a one-factor model of the spread between the one-year yield and the one-month yield

$$
\begin{aligned}
YIELD\_Y1_t - YIELD\_M1_t &= \alpha + \beta s_t + u_t, \quad u_t \sim N\left(0, \sigma^2\right) \\
s_t &= \phi_1 s_{t-1} + \phi_2 s_{t-2} + v_t \quad v_t \sim N\left(0, 1\right)
\end{aligned}
$$

- The EViews window to estimate the model is given below.

View | Proc | Object | Print | Name | Freeze | Spec | Estimate | Stats | Forecast

```
@signal yield_y1-yield_m1 = c(1) + c(2)*s1 + [var = c(3)]

@state s1 = c(4)*s1(-1)  + c(5)*s2(-1) + [var = 1]

@state s2 = s1(-1)

@param c(1) 0.1 c(2) 0.1 c(3) 0.1 c(4) 0.9 c(5) 0.1
```

Source: yields_us.wf1

Note that the second factor $s2$, represents the lag of the first factor $s1$, and thus does not have a disturbance term.

An AR(p) Model of s_t

- Consider an $AR(p)$ model of s_t

$$s_t = \phi_1 s_{t-1} + \phi_2 s_{t-2} + \cdots + \phi_p s_{t-p} + v_t$$

- The state equation is written as

$$\begin{bmatrix} s_t \\ s_{t-1} \\ s_{t-2} \\ \vdots \\ s_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ s_{t-2} \\ s_{t-3} \\ \vdots \\ s_{t-p} \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- In this case, the model is viewed as having p factors

$$\{s_t, s_{t-1}, \cdots, s_{t-p+1}\}$$

although it is really just the first element of this set of factors that is of interest.

Idiosyncratic Dynamics

- Consider the model

$$\begin{aligned} y_{i,t} &= \beta_i s_t + \sigma_i u_{i,t}\,, \qquad i = 1, 2, \cdots, 4 \\ s_t &= \phi_1 s_{t-1} + \phi_2 s_{t-2} + v_t \\ u_{i,t} &= \delta_i u_{i,t-1} + w_{i,t} \end{aligned}$$

where $u_{i,t} \sim N(0, I)$ and $w_{i,t} \sim N(0, I)$.

- The state equation is now augmented to accommodate the dynamics in the idiosyncratic terms.

$$\begin{bmatrix} s_t \\ s_{t-1} \\ u_{1,t} \\ u_{2,t} \\ u_{3,t} \\ u_{4,t} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_4 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ s_{t-2} \\ u_{1,t-1} \\ u_{2,t-1} \\ u_{3,t-1} \\ u_{4,t-1} \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \\ w_{1,t} \\ w_{2,t} \\ w_{3,t} \\ w_{4,t} \end{bmatrix}$$

- In this case, the model is viewed as having six factors

$$\left\{ s_t, s_{t-1}, u_{1,t}, u_{2,t}, u_{3,t}, u_{4,t} \right\}$$

- In this scenario the idiosyncratic terms are redefined as factors.
- As there are now no disturbances terms, then the covariance matrix of the disturbances $\mathrm{E}\left[u_t u_t'\right] = R$, reduces to

$$R = 0$$

- The full model in this alternative parameterization is

$$\begin{aligned} y_t &= B s_t \\ s_t &= \Phi s_{t-1} + v_t , \qquad v_t \sim N(0, I) \end{aligned}$$

where s_t represents the vector of six factors.

- The state-space model is easily extended to include M exogenous or lagged dependent variables, x_t. These variables can be included in one of two different ways.
- The first approach is to include exogenous or predetermined variables in the signal equation

$$y_t = Bf_t + \Gamma x_t + u_t,$$

where Γ is $(N \times M)$ and x_t is $(M \times 1)$.

- This class of model is called a factor VAR model (F-VAR), where $x_t = y_{t-1}$ and Γ is now a $(N \times N)$ diagonal matrix.
- The second approach is to include the exogenous or predetermined variables in the state equation

$$s_t = \Phi s_{t-1} + \Gamma x_t + u_t,$$

where Γ is now a $(K \times M)$ matrix of parameters.

End of Lecture