Econ671
Factor Models: Principal Components

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Factor Models: Principal Components

Learning Objectives

1. Show how models in finance have a factor structure interpretation.
2. Understand principal components.
3. Deriving and interpreting an eigen decomposition.
4. A factor model of the term structure of interest rates.

Background Reading

1. "*Matrices*", Handout on course website - especially the section on eigenvalues and eigenvectors.

EViews Computer Files

1. yields_us.wf1
2. yields_us_loadings.wf1
3. capm.wf1

Introduction

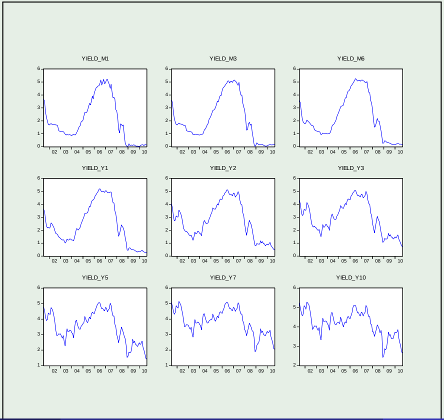
- An important feature of many asset markets is that key variables (returns, yields, spreads etc) often display similar features.

  1. Mean
     Asset markets tend to move together whereby the spreads between the key variables are stationary, even though these key variables may be nonstationary (cointegration).
  2. Variance
     Periods of turbulence and tranquility tend to coincide across asset markets both locally and internationally, with the global financial crisis representing an example of synchronized turbulence.
  3. Dynamics
     The dynamical behavior within classes of asset markets tend to exhibit very similar autocorrelation patterns.

Introduction

Example (Term Structure of Interest Rates)

The term structure of interest rates is the relationship between the yields of a bond for differing maturities The Figure shows that U.S. Treasury yields (%p.a.) for maturities from 1mthh to 10yrs, tend to move together.



Introduction

- Suggests that the time series characteristics of many financial variables can be summarized by a small set of factors.
- To formalise this model consider the following linear regression equation

  $$r_{i,t} = \alpha_i + \beta_i s_t + u_{i,t}$$

  where

  - $r_{i,t}$ is the return on the i^{th} asset
  - s_t is the factor (or set of factors) with parameters (vectors) α_i, β_i
  - $u_{i,t}$ is an unknown disturbance term representing idiosyncratic movements in $r_{i,t}$.

- It is important to distinguish between two types of factors.

  1. Observable s_t
  2. Unobservable s_t

Introduction

1. Observable s_t
   Time series are available on s_t, whereby the parameter β_i is estimated simply by regressing $r_{i,t}$ on s_t. Typical examples of this type of model are CAPM, and the Fama-French three-factor model.

Example (CAPM)

The CAPM

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i(r_{m,t} - r_{f,t}) + u_{i,t}$$

is estimated for 6 assets using monthly data for the U.S. beginning May 1990 and ending July 2004. The OLS parameter estimates are summarised below.

Asset:	Exxon	GE	Gold	IBM	Microsoft	Walmart
α_i :	0.012	0.016	-0.003	0.004	0.012	0.007
β_i :	0.502	1.144	-0.098	1.205	1.447	0.868

2. Unobservable s_t

   Time series are not directly available on s_t making the application of least squares to estimate α_i and β_i unworkable. There are two solutions:

   (a) Proxy variables

   A proxy variable is used instead of s_t, but this creates errors in variables problems. In fact, this is a common strategy, if adopted only implicitly, with the application of the CAPM being a typical example.

   (b) Latent variables

   The statistical solution is to still treat s_t as an unobservable variable, but to introduce additional structure on the model thereby providing additional information to enable the parameters to be estimated.

This is the main focus of this and the next lecture!

Introduction

- If s_t is latent it is not immediately obvious how the parameters of

$$r_{i,t} = \alpha_i + \beta_i s_t + u_{i,t}$$

can be identified when all of the terms on the right hand-side of the equation

(i)	Parameters	:	α_i, β_i
(ii)	Explanatory variables	:	s_t
(iii)	Disturbance term	:	$u_{i,t}$

are unknown.

- Two broad methods are investigated to show how to address this identification problem.
    1. Principal Components (this lecture)
    2. Kalman Filter (next lecture)

The Role of Factors in Finance

- To motivate the role of factors in finance various models are presented, ranging from theoretical to purely statistical models.
- An important feature of this discussion is the distinction between factors that are observable and factors that are unobservable, that is latent.
- The examples consist of
    1. Term Structure of Interest Rates
    2. Capital Asset Pricing Model (CAPM)
    3. Arbitrage Pricing Theory (APT)

- Inspection of the U.S. Treasury yields (%p.a.) for maturities from 1 month to 10 years, reveals two important characteristics.
    1. The levels of the yields tend to move together.
    2. Neighboring yields move together more closely than yields with greater differences in maturities. This latter empirical property of yields is highlighted by the following correlation matrix.

$$
corr = \begin{array}{c} \begin{array}{ccccccccc} \text{1-mth} & \text{3-mth} & \text{6-mth} & \text{1-yr} & \text{2-yrs} & \text{3-yrs} & \text{5-yrs} & \text{7-yrs} & \text{10-yrs} \end{array} \\ \begin{bmatrix} 1.000 & 0.998 & 0.992 & 0.985 & 0.960 & 0.933 & 0.869 & 0.787 & 0.702 \\ 0.998 & 1.000 & 0.997 & 0.992 & 0.968 & 0.940 & 0.872 & 0.787 & 0.700 \\ 0.992 & 0.997 & 1.000 & 0.997 & 0.974 & 0.945 & 0.874 & 0.784 & 0.697 \\ 0.985 & 0.992 & 0.997 & 1.000 & 0.987 & 0.964 & 0.901 & 0.816 & 0.734 \\ 0.960 & 0.968 & 0.974 & 0.987 & 1.000 & 0.993 & 0.952 & 0.885 & 0.812 \\ 0.933 & 0.940 & 0.945 & 0.964 & 0.993 & 1.000 & 0.979 & 0.929 & 0.867 \\ 0.869 & 0.872 & 0.874 & 0.901 & 0.952 & 0.979 & 1.000 & 0.984 & 0.949 \\ 0.787 & 0.787 & 0.784 & 0.816 & 0.885 & 0.929 & 0.984 & 1.000 & 0.987 \\ 0.702 & 0.700 & 0.697 & 0.734 & 0.812 & 0.867 & 0.949 & 0.987 & 1.000 \end{bmatrix} \end{array}.
$$

- These two characteristics suggest that a potential model to explain the term structure of interest rates is given by at least a two-factor model

$$r_{i,t} = \alpha_i + \beta_{1,i} s_{1,t} + \beta_{2,i} s_{2,t} + u_{i,t}, \qquad i = 1, 2, \cdots, 9,$$

where $u_{i,t}$ is the disturbance term and $s_{1,t}$ and $s_{2,t}$ are the factors designed to

Level factor $(s_{1,t})$: Capture the level of yields

Slope factor $(s_{2,t})$: Capture the correlation between neighbouring yields

- As the two factors are implied by the data, they are latent factors.

- The CAPM
$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i (r_{m,t} - r_{f,t}) + u_{i,t}$$
represents a single factor model given by the excess return on the market $r_{m,t} - r_{f,t}$, where the (market) factor is observable. Extending the model to allow for other observable factors including Fama-French factors, momentum, liquidity etc, generates a multi-factor CAPM.
- From a theoretical point of view, the use of $r_{m,t}$ in the model actually serves as a proxy for the excess return on all invested wealth. As this is an unobservable variable, the excess return on the market portfolio, $r_{m,t} - r_{f,t}$, is essentially serving as a proxy.
- This suggests that the more theoretically correct CAPM specification is
$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i s_t + u_{i,t}$$
where s_t is now a latent factor representing the excess return on all invested wealth.

- This is an alternative form of the CAPM equation where the (unknown) excess return on wealth is extended to the multi-factor version of the CAPM where all factors are unknown

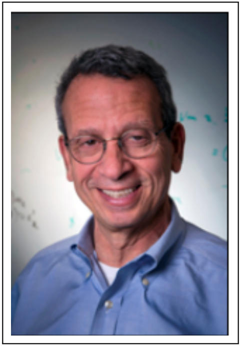
$$r_{i,t} = \alpha_i + \sum_{j=1}^{K} \beta_{i,j} s_{j,t} + u_{i,t}$$

where there are K latent factors $(s_{1,t}, s_{2,t}, \cdots, s_{K,t})$ and as before $u_{i,t}$ is a disturbance term representing the idiosyncratic factor (risk).

- This is the arbitrage pricing model (APT) of Ross (1976). The factors and the disturbance term are assumed to have the following properties

$$\begin{array}{lcll}
\text{Factors} & : & E[s_{j,t}] = 0, & E[s_{j,t} s_{k,t}] = \begin{cases} 1 & : \quad j = k \\ 0 & : \quad j \neq k \end{cases} \\
\text{Disturbance} & : & E[u_{i,t}] = 0, & E[u_{i,t} u_{k,t}] = \begin{cases} \sigma_i^2 & : \quad i = k \\ 0 & : \quad i \neq k \end{cases} \\
\text{Covariance} & : & E[u_{i,t} s_{j,t}] = 0 &
\end{array}$$

Historical Background: Stephen Ross (1943-)
He is the inventor of Arbitrage Pricing Theory and also very well known for his work in finance including the CIR model of interest rates (he is the "R").



- These assumptions imply a decomposition of the covariance matrix of $r_t = \{r_{1,t}, r_{2,t}, \cdots, r_{N,t}\}$.
- In the case of a $K = 1$ factor model (let $\beta_i = \beta_{i,1}$ for simplicity of notation)

$$r_{i,t} = \alpha_i + \beta_i s_{1,t} + u_{i,t}$$

the covariance matrix is of the form (see the next tutorial for the derivation of this result)

$$cov(r_t) = \begin{bmatrix} \beta_1^2 + \sigma_1^2 & \beta_1\beta_2 & \cdots & \beta_1\beta_N \\ \beta_2\beta_1 & \beta_2^2 + \sigma_2^2 & & \beta_2\beta_N \\ \vdots & & \ddots & \vdots \\ \beta_N\beta_1 & \beta_N\beta_2 & \cdots & \beta_N^2 + \sigma_N^2 \end{bmatrix}$$

- This result shows that the covariance matrix can be reorganized into a factor structure which may be more informative about the movements in returns.

- Consider the following model containing N asset returns $r_t = \{r_{1,t}, r_{2,t}, \cdots, r_{N,t}\}$ and $K \leq N$ factors $s_t = \{s_{1,t}, s_{2,t}, \cdots, s_{K,t}\}$

$$\begin{bmatrix} r_{1,t} \\ r_{2,t} \\ \vdots \\ r_{N,t} \end{bmatrix} - \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,K} \\ \beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N,1} & \beta_{N,2} & \cdots & \beta_{N,K} \end{bmatrix} \begin{bmatrix} s_{1,t} \\ s_{2,t} \\ \vdots \\ s_{K,t} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{N,t} \end{bmatrix}$$

or in matrix notation

$$r_t - \alpha\mu = \beta s_t + u_t$$

where

- α is a $(N \times 1)$ vector containing the means of r_t.
- s_t is a $(K \times 1)$ vector containing the factors.
- β is a $(N \times K)$ matrix containing the factor loadings.
- u_t is a $(N \times 1)$ vector of disturbances.

Principal Components
Specification

- The vector of latent factors and the disturbance vector have the properties

$$
\begin{aligned}
E\left[u_t\right] &= 0, \qquad E\left[u_t u_t'\right] = \Omega \\
E\left[s_t\right] &= 0, \qquad E\left[s_t s_t'\right] = I, \\
E\left[s_t u_t'\right] &= 0,
\end{aligned}
$$

where Ω is a $(N \times N)$ covariance matrix.

- As $E\left[s_t\right] = E\left[u_t\right] = 0$, the mean of the returns is

$$E\left[r_t\right] = E\left[\mu + \beta s_t + u_t\right] = \mu + \beta E\left[s_t\right] + E\left[u_t\right] = \mu$$

- The factor equation

$$r_t - \alpha = \beta s_t + u_t$$

shows that $r_t - \alpha$ can be decomposed into a systematic component (βs_t) and an idiosyncratic component (u_t).

- Given the properties of s_t and u_t, the covariance structure of r_t simplifies as

$$
\begin{aligned}
cov\left(r_t\right) &= E\left[\left(r_t-\alpha\right)\left(r_t-\alpha\right)'\right] \\
&= E\left[\left(\beta s_t+u_t\right)\left(\beta s_t+u_t\right)'\right] \\
&= \beta E\left[s_t s_t'\right]\beta' + \beta E\left[s_t u_t'\right] + E\left[u_t s_t'\right]\beta' + E\left[u_t u_t'\right] \\
&= \underbrace{\beta\beta'}_{\text{Systematic Risk}} + \underbrace{\Omega}_{\text{Idiosyncratic Risk}}
\end{aligned}
$$

- This equation shows that the covariance matrix can be decomposed in terms of two sources of factors/risks.
    1. Systematic risk $\left(\beta\beta'\right)$.
    2. Idiosyncratic risk $\left(\Omega\right)$.

Principal Components
Specification

- In the special case where the number of variables matches the number of factors $(N = K)$, there is an exact decomposition of the covariance matrix of r_t as $\Omega = 0$ in this case, with the covariance matrix $cov\,(r_t)$ reducing to

$$cov\,(r_t) = \beta\beta' = \beta_1\beta_1' + \beta_2\beta_2' + \cdots \beta_N\beta_N'$$

- An important property of principal components is that they exhibit the same features as this equation

$$cov\,(r_t) = \lambda_1 P_1 P_1' + \lambda_2 P_2 P_2' + \cdots + \lambda_N P_N P_N'$$

where λ_i is the i^{th} eigenvalue with $(N \times 1)$ orthonormal eigenvector P_i, that is, $P_i' P_i = 1$ and $P_i P_j = 0$, $\forall i \neq j$.
- A comparison of these equations suggests that the loading parameter vector β_i be chosen as

$$\beta_i = \sqrt{\lambda_i} P_i, \qquad i = 1, 2, \cdots, N$$

- Now consider the N variances of $r_t = \{r_{1,t}, r_{2,t}, \cdots, r_{N,t}\}$, corresponding to the diagonal elements of $cov(r_t)$

$$\begin{bmatrix} var(r_{1,t}) \\ var(r_{2,t}) \\ \vdots \\ var(r_{N,t}) \end{bmatrix} = \lambda_1 \begin{bmatrix} P_{1,1}^2 \\ P_{2,1}^2 \\ \vdots \\ P_{N,1}^2 \end{bmatrix} + \lambda_2 \begin{bmatrix} P_{1,2}^2 \\ P_{2,3}^2 \\ \vdots \\ P_{N,2}^2 \end{bmatrix} + \cdots + \lambda_1 \begin{bmatrix} P_{1,N}^2 \\ P_{2,N}^2 \\ \vdots \\ P_{N,N}^2 \end{bmatrix}$$

- Given that the eigenvectors are normalized as $P_i' P_i = 1$, then the sum of the elements of each of the column vectors on the right hand-side all equal unity, that is

$$\begin{array}{rcl} P_{1,1}^2 + P_{2,1}^2 + \cdots + P_{N,1}^2 & = & 1 \\ P_{1,2}^2 + P_{2,2}^2 + \cdots + P_{N,2}^2 & = & 1 \\ \vdots & \vdots & \vdots \\ P_{1,N}^2 + P_{2,N}^2 + \cdots + P_{N,N}^2 & = & 1 \end{array}$$

- Thus a measure of the total volatility of all asset returns

$$\sum_{i=1}^{N} var\left(r_{i,t}\right) = var\left(r_{1,t}\right) + var\left(r_{2,t}\right) + \cdots + var\left(r_{N,t}\right)$$

is achieved by combining all of the variance equations as

$$\begin{aligned}\sum_{i=1}^{N} var\left(r_{i,t}\right) \quad = \quad & \lambda_1\left(P_{1,1}^2 + P_{2,1}^2 + \cdots + P_{N,1}^2\right) \\ & +\lambda_2\left(P_{1,2}^2 + P_{2,2}^2 + \cdots + P_{N,2}^2\right) \\ & + \cdots + \lambda_N\left(P_{1,N}^2 + P_{2,N}^2 + \cdots + P_{N,N}^2\right)\end{aligned}$$

Given the normalization of the eigenvectors, this equation simplifies to

$$\sum_{i=1}^{N} var\left(r_{i,t}\right) = \lambda_1 + \lambda_2 + \cdots + \lambda_N = \sum_{i=1}^{N} \lambda_i$$

That is, the total volatility of all r_t equals the sum of all eigenvalues.

- Instead of performing an eigen decomposition on the covariance matrix $cov(r_t)$, to circumvent scaling issues when variables are measured in different units for example, the correlation matrix $cor(r_t)$ can be used instead.
- In this case as the correlation matrix contains unity on the main diagonal by definition, then the sum of correlations becomes

$$\sum_{i=1}^{N} \lambda_i = N$$

- By inspecting the relative magnitude of the largest eigenvalues it is possible to quantify the proportion of the total variance of the data that is explained by $K < N$ principal components. Bai and Ng (2002, *Econometrica*) propose a formal test of the number of factors based on information criteria.

Historical Background: Karl Pearson (1857 – 1936)
Inventor of Principal Components plus many other well-used and well-loved statistical techniques.


Portrait of Karl Pearson, by Elliott & Fry, 1890.

- In practice it is necessary to estimate the eigenvalues and the eigenvectors from the data.
- As an example consider the following correlation matrix of the 1-month, 1-year and 5-year U.S. Treasury yields given earlier

$$cor\left(r_t\right) = \begin{bmatrix} 1.000 & 0.985 & 0.869 \\ 0.985 & 1.000 & 0.901 \\ 0.869 & 0.901 & 1.000 \end{bmatrix}$$

- The EViews commands to estimate the eigenvalues and eigenvectors of $cor\left(r_t\right)$, highlight the 3 series and double click the shaded region and then click

  Open Group / View / Principal Components... / Calculation

  For **Type:** choose

  Correlation / OK

  The output is given in the following window.

Principal Components

Estimation

Principal Components Analysis
Date: 03/23/14 Time: 15:57
Sample: 2001M07 2010M09
Included observations: 111
Computed using: Ordinary correlations
Extracting 3 of 3 possible components

Eigenvalues: (Sum = 3, Average = 1)

Number	Value	Difference	Proportion	Cumulative Value	Cumulative Proportion
1	2.837472	2.686832	0.9458	2.837472	0.9458
2	0.150640	0.138751	0.0502	2.988111	0.9960
3	0.011889	---	0.0040	3.000000	1.0000

Eigenvectors (loadings):

Variable	PC 1	PC 2	PC 3
YIELD_M1	0.581282	-0.490485	0.649258
YIELD_Y1	0.587635	-0.298881	-0.751901
YIELD_Y5	0.562847	0.818593	0.114492

Ordinary correlations:

	YIELD_M1	YIELD_Y1	YIELD_Y5
YIELD_M1	1.000000		
YIELD_Y1	0.985507	1.000000	
YIELD_Y5	0.868743	0.900611	1.000000

Source: EViews yields_us.wf1

Principal Components
Estimation

- The estimated eigenvalues and eigenvectors of the correlation matrix are respectively

$$\widehat{\lambda} = \begin{bmatrix} 2.837 \\ 0.151 \\ 0.012 \end{bmatrix}, \qquad \widehat{P} = \begin{bmatrix} 0.581 & -0.490 & 0.649 \\ 0.588 & -0.299 & -0.752 \\ 0.563 & 0.819 & 0.114 \end{bmatrix}.$$

with the columns of $\widehat{P}$ representing the three eigenvectors

$$\widehat{P}_1 = \begin{bmatrix} 0.581 \\ 0.588 \\ 0.563 \end{bmatrix}, \qquad \widehat{P}_2 = \begin{bmatrix} -0.490 \\ -0.299 \\ 0.819 \end{bmatrix}, \qquad \widehat{P}_3 = \begin{bmatrix} 0.649 \\ -0.752 \\ 0.114 \end{bmatrix},$$

corresponding to the order of the three eigenvalues in $\widehat{\lambda}$.

- The use of the "^" emphasises that the population parameters λ and P are estimated from the data.

Recovering the Correlation Matrix from the Decomposition

- The correlation matrix is recovered as follows:
  1. Diagonal elements (ie own correlations)

$$
\begin{aligned}
cor\left(r_{1,t}\right) &= 2.837 \times (0.581)^2 + 0.151 \times (-0.490)^2 \\
&\quad +0.012 \times (0.649)^2 \\
&= 1.000
\end{aligned}
$$

$$
\begin{aligned}
cor\left(r_{2,t}\right) &= 2.837 \times (0.588)^2 + 0.151 \times (-0.299)^2 \\
&\quad +0.012 \times (-0.752)^2 \\
&= 1.000
\end{aligned}
$$

$$
\begin{aligned}
cor\left(r_{3,t}\right) &= 2.837 \times (0.563)^2 + 0.151 \times (0.819)^2 \\
&\quad +0.012 \times (0.144)^2 \\
&= 1.000
\end{aligned}
$$

2. Off-diagonal elements

$$
\begin{aligned}
cor\left(r_{1,t}, r_{2,t}\right) &= 2.837 \times (0.581) \times (0.588) \\
&\quad +0.151 \times (-0.490) \times (-0.299) \\
&\quad +0.012 \times (0.649) \times (-0.752) \\
&= 0.985 \\
cor\left(r_{1,t}, r_{3,t}\right) &= 2.837 \times (0.581) \times (0.563) \\
&\quad +0.151 \times (-0.490) \times (0.819) \\
&\quad +0.012 \times (0.649) \times (0.144) \\
&= 0.869 \\
cor\left(r_{2,t}, r_{3,t}\right) &= 2.837 \times (0.588) \times (0.563) \\
&\quad +0.151 \times (-0.299) \times (0.819) \\
&\quad +0.012 \times (-0.752) \times (0.144) \\
&= 0.901
\end{aligned}
$$

Properties of the Estimates

- Some key properties of the parameter estimates are as follows.
  1. *Normalization of eigenvectors*

$$\begin{aligned} 0.581^2 + 0.588^2 + 0.563^2 &= 1.000 \\ (-0.490)^2 + (-0.299)^2 + 0.819^2 &= 1.000 \\ 0.649^2 + (-0.752)^2 + 0.114^2 &= 1.000 \end{aligned}$$

  2. *Orthogonality of eigenvectors*

$$\begin{aligned} \widehat{P}_1' \widehat{P}_2 &= 0.581 \times (-0.490) + 0.588 \times (-0.299) + 0.563 \times 0.819 \\ &= 0 \\ \widehat{P}_1' \widehat{P}_3 &= 0.581 \times 0.649 + 0.588 \times (-0.752) + 0.563 \times 0.114 \\ &= 0 \\ \widehat{P}_2' \widehat{P}_3 &= (-0.490) \times (0.649) + (-0.299) \times (-0.752) + 0.819 \times 0.114 \\ &= 0 \end{aligned}$$

3. *Eigenvalues*
   As it is the correlation matrix that is being used in the eigen decomposition, the eigenvalues sum to $N = 3$

$$\sum_{i=1}^{3} \widehat{\lambda}_i = 2.837 + 0.151 + 0.012 = 3$$

The normalized eigenvalues are

$$\frac{2.837}{3} + \frac{0.151}{3} + \frac{0.012}{3} = 0.946 + 0.050 + 0.004 = 1$$

The first eigenvalue explains 94.6% of the total variance, the second explains an additional 5%, while the contribution of the third and last eigenvalue is 0.4%.

4. *Factor loadings*

$$\widehat{\beta}_1 = \sqrt{\widehat{\lambda}_1}\widehat{P}_1 = \sqrt{2.837}\begin{bmatrix} 0.581 \\ 0.588 \\ 0.563 \end{bmatrix} = \begin{bmatrix} 0.979 \\ 0.990 \\ 0.948 \end{bmatrix}$$

$$\widehat{\beta}_2 = \sqrt{\widehat{\lambda}_2}\widehat{P}_2 = \sqrt{0.151}\begin{bmatrix} -0.490 \\ -0.299 \\ 0.819 \end{bmatrix} = \begin{bmatrix} -0.190 \\ -0.116 \\ 0.318 \end{bmatrix}$$

$$\widehat{\beta}_3 = \sqrt{\widehat{\lambda}_3}\widehat{P}_3 = \sqrt{0.012}\begin{bmatrix} 0.649 \\ -0.752 \\ 0.114 \end{bmatrix} = \begin{bmatrix} 0.071 \\ -0.082 \\ 0.012 \end{bmatrix}$$

5. Intercepts
As $E[r_t] = \alpha$, an estimate of α is given by the sample mean of r_t given by

$$\widehat{\alpha} = \bar{r} = \begin{bmatrix} 2.038 & 2.393 & 3.472 \end{bmatrix}'$$

The $K = 1$ Estimated Factor Model

- The estimated model is

$$
\begin{aligned}
r_{1,t} &= \widehat{\alpha}_1 + \widehat{\beta}_{1,1} s_{1,t} + \widehat{u}_{1,t} \\
&= 2.038 + 0.979\, s_{1,t} + \widehat{u}_{1,t} \\
r_{2,t} &= \widehat{\alpha}_2 + \widehat{\beta}_{2,1} s_{1,t} + \widehat{u}_{2,t} \\
&= 2.393 + 0.990\, s_{1,t} + \widehat{u}_{3,t} \\
r_{3,t} &= \widehat{\alpha}_3 + \widehat{\beta}_{3,1} s_{1,t} + \widehat{u}_{3,t} \\
&= 3.472 + 0.948\, s_{1,t} + \widehat{u}_{3,t}
\end{aligned}
$$

- An increase in the factor $s_{1,t}$, results in all 3 yields increasing by similar amounts.
- This suggests that $s_{1,t}$ is a LEVEL factor.

- As $E\left[s_{1,t}^2\right] = 1$, the systematic risks are

$$\begin{aligned}
\widehat{h}_1 &= \widehat{\beta}_{1,1}^2 = 0.979^2 = 0.958 \\
\widehat{h}_2 &= \widehat{\beta}_{2,1}^2 = 0.990^2 = 0.980 \\
\widehat{h}_3 &= \widehat{\beta}_{3,1}^2 = 0.948^2 = 0.899
\end{aligned}$$

- The idiosyncratic risks are

$$\begin{aligned}
var\left(\widehat{u}_{1,t}\right) &= var\left(r_{1,t}\right) - \widehat{h}_{1,t} = 1.000 - 0.958 = 0.042 \\
var\left(\widehat{u}_{2,t}\right) &= var\left(r_{2,t}\right) - \widehat{h}_{2,t} = 1.000 - 0.980 = 0.020 \\
var\left(\widehat{u}_{3,t}\right) &= var\left(r_{3,t}\right) - \widehat{h}_{3,t} = 1.000 - 0.899 = 0.101
\end{aligned}$$

The $K = 2$ Estimated Factor Model

- The estimated model is

$$
\begin{aligned}
r_{1,t} &= 2.038 + 0.979 s_{1,t} - 0.190 s_{2,t} + \widehat{u}_{1,t} \\
r_{2,t} &= 2.393 + 0.990 s_{1,t} - 0.116 s_{2,t} + \widehat{u}_{2,t} \\
r_{3,t} &= 3.472 + 0.948 s_{1,t} + 0.318 s_{2,t} + \widehat{u}_{3,t}
\end{aligned}
$$

- An increase in the factor $s_{2,t}$, widens the spreads between all 3 yields as $r_{1,t}$ falls, as does $r_{2,t}$ but by a smaller amount, whilst $r_{3,t}$ increases.
- This suggests that $s_{2,t}$ is a SLOPE factor as it changes the slope of the yield curve.

- As $E\left[s_{1,t}^2\right] = E\left[s_{2,t}^2\right] = 1$, the systematic risks are

$$
\begin{aligned}
\widehat{h}_1 &= \widehat{\beta}_{1,1}^2 + \widehat{\beta}_{1,2}^2 = 0.979^2 + (-0.190)^2 = 0.994 \\
\widehat{h}_2 &= \widehat{\beta}_{2,1}^2 + \widehat{\beta}_{2,2}^2 = 0.990^2 + (-0.116)^2 = 0.993 \\
\widehat{h}_3 &= \widehat{\beta}_{3,1}^2 + \widehat{\beta}_{3,2}^2 = 0.948^2 + 0.318^2 = 0.999
\end{aligned}
$$

- The idiosyncratic risks are

$$
\begin{aligned}
var\left(\widehat{u}_{1,t}\right) &= var\left(r_{1,t}\right) - \widehat{h}_1 = 1.000 - 0.994 = 0.006 \\
var\left(\widehat{u}_{2,t}\right) &= var\left(r_{2,t}\right) - \widehat{h}_2 = 1.000 - 0.993 = 0.007 \\
var\left(\widehat{u}_{3,t}\right) &= var\left(r_{3,t}\right) - \widehat{h}_3 = 1.000 - 0.999 = 0.001
\end{aligned}
$$

Principal Components

Estimation

The $K = 3$ Estimated Factor Model

- The estimated model is

$$
\begin{aligned}
r_{1,t} &= 2.038 + 0.979 s_{1,t} - 0.190 s_{2,t} + 0.071 s_{3,t} \\
r_{2,t} &= 2.393 + 0.990 s_{1,t} - 0.116 s_{2,t} - 0.082 s_{3,t} \\
r_{3,t} &= 3.472 + 0.948 s_{1,t} + 0.318 s_{2,t} + 0.012 s_{3,t}.
\end{aligned}
$$

There are no idiosyncratic terms in this model as 3 factors perfectly explain the movements of the 3 interest rates.

Nonetheless it is sometimes hard to diversify all latent factors!

- As $E\left[s_{1,t}^2\right] = E\left[s_{2,t}^2\right] = E\left[s_{3,t}^2\right] = 1$, the systematic risks now equal the diagonal components of the correlation matrix as

$$
\begin{aligned}
\widehat{h}_1 &= \widehat{\beta}_{1,1}^2 + \widehat{\beta}_{1,2}^2 + \widehat{\beta}_{1,3}^2 = 0.979^2 + (-0.190)^2 + 0.071^2 = 1.000 \\
\widehat{h}_2 &= \widehat{\beta}_{2,1}^2 + \widehat{\beta}_{2,2}^2 + \widehat{\beta}_{2,3}^2 = 0.990^2 + (-0.116)^2 + (-0.082)^2 = 1.000 \\
\widehat{h}_3 &= \widehat{\beta}_{3,1}^2 + \widehat{\beta}_{3,2}^2 + \widehat{\beta}_{3,3}^2 = 0.948^2 + 0.318^2 + 0.012^2 = 1.000
\end{aligned}
$$

- In which case the idiosyncratic risks are all zero

$$
\begin{aligned}
var\left(\widehat{u}_{1,t}\right) &= var\left(r_{1,t}\right) - \widehat{h}_1 = 1.000 - 1.000 = 0.000 \\
var\left(\widehat{u}_{2,t}\right) &= var\left(r_{2,t}\right) - \widehat{h}_2 = 1.000 - 1.000 = 0.000 \\
var\left(\widehat{u}_{3,t}\right) &= var\left(r_{3,t}\right) - \widehat{h}_3 = 1.000 - 1.000 = 0.000
\end{aligned}
$$

Method

- Since $r_{i,t} = \alpha_i + \beta_i s_{1,t} + u_{i,t}$, $i = 1, ..., N$, having estimated the factor model, estimates of the vector of factor loadings at time t are obtained as
$$\widehat{s}_t = (\widehat{\beta}'\widehat{\Omega}^{-1}\widehat{\beta})^{-1}\widehat{\beta}'\widehat{\Omega}^{-1}(r_t - \widehat{\mu})$$
where $\widehat{s}_t$ is a $(K \times 1)$ vector of the estimated factors at time t, $\widehat{\beta}$ is a $(N \times K)$ estimated matrix of factor loadings, $\widehat{\Omega}$ is the $(N \times N)$ covariance matrix of $\widehat{u}_t$, r_t is a $(N \times 1)$ vector of interest rates and $\widehat{\mu}$ is a $(N \times 1)$ vector of sample means corresponding to the vector of interest rates.

- If $\widehat{\Omega} = \sigma^2 I$, $K = 1$, then $\widehat{s}_t = \frac{\sum_{i=1}^{N} \widehat{\beta}_i (r_{it} - \widehat{\mu}_i)}{\sum_{i=1}^{N} \widehat{\beta}_i^2}$. The bigger the factor loading, the more importance of the variable in determining the factor.
- This expression shows that the factors at time t are a (weighted) linear function of the actual interest rates at time t.

EViews Commands

- The EViews commands to extract the factors, highlight the 3 series and double click the shaded region and then click

  Open Group / Proc / Make Principal Components...

  For **Scores series names:**, write in the window

  Level Slope Curvature

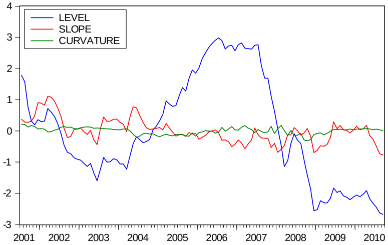
  Then click **Calculation** and for **Type:** choose

  Correlation / OK

  The output will be presented in a spreadsheet.

Interpretation

- The three factors are plotted over time.



Source: EViews file yields_us.wf1

- The results show that:
  (i) The LEVEL factor dominates the SLOPE and CURVATURE factors.
  (ii) As the LEVEL factor tracks the three yields this suggests that a 1-factor model suffices to explain the three yields.

A Multi-Factor Model of Interest Rates

- Consider the $N = 9$, U.S. Treasury yields (percentage, annualized) from July 2001 to September 2010, presented in the Figure earlier.
- The $N = 9$ eigenvalues of the correlation matrix from highest to lowest are

$$\begin{aligned} \widehat{\lambda} &= \{8.2339, 0.7123, 0.0427, 0.0078, 0.0018, \\ &\quad 0.0006, 0.0005, 0.0002, 0.0001\} \end{aligned}$$

- The proportionate contributions of the first three eigenvalues are

$$\widehat{\lambda}_1 = \frac{8.2339}{9} = 0.915,\ \widehat{\lambda}_2 = \frac{0.7123}{9} = 0.079,\ \widehat{\lambda}_3 = \frac{0.0427}{9} = 0.005$$

- This suggest that a $K = 3$ factor model explains the U.S. term structure with the first three eigenvalues explaining $0.915 + 0.079 + 0.005 = 0.999$, or 99.9% of the total variance/correlation in the yields.

A Multi-Factor Model of Interest Rates

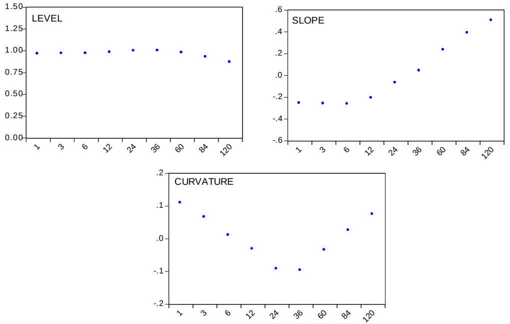
- The estimated factor loadings on the first factor are computed as

$$\begin{bmatrix} \widehat{\beta}_{1,1} \\ \widehat{\beta}_{2,1} \\ \widehat{\beta}_{3,1} \\ \widehat{\beta}_{4,1} \\ \widehat{\beta}_{5,1} \\ \widehat{\beta}_{6,1} \\ \widehat{\beta}_{7,1} \\ \widehat{\beta}_{8,1} \\ \widehat{\beta}_{9,1} \end{bmatrix} = \sqrt{8.2339} \begin{bmatrix} 0.3340 \\ 0.3351 \\ 0.3355 \\ 0.3400 \\ 0.3460 \\ 0.3464 \\ 0.3389 \\ 0.3214 \\ 0.3003 \end{bmatrix} = \begin{bmatrix} 0.9583 \\ 0.9617 \\ 0.9626 \\ 0.9757 \\ 0.9928 \\ 0.9941 \\ 0.9724 \\ 0.9221 \\ 0.8617 \end{bmatrix}$$

- The other factor loadings are computed in a similar way.

A Multi-Factor Model of Interest Rates

- The following Figure plots the loadings of the first three factors which are identified respectively as level, slope and curvature.



Source: EViews file yields_us.wf1

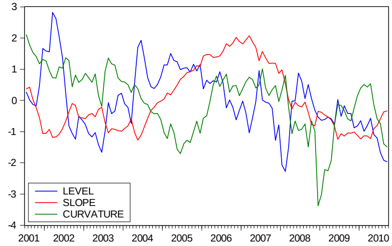
A Multi-Factor Model of Interest Rates

Factor Interpretation

1. *Level Factor*
   The first factor represents a levels effect as a shock to this factor raises all yields by approximately the same amount.
2. *Slope Factor*
   The second factor is a slope factor as a positive shock twists the yield curve by lowering short rates (negative loadings) and raising long rates (positive loadings).
3. *Curvature Factor*
   The third factor is a curvature factor which bends the yield curve by simultaneously raising the very short and long rates (positive loadings), while lowering (negative loadings) the intermediate rates at around 2 years (24 months) and 3 years (36 months).

A Multi-Factor Model of Interest Rates

- The three factors are plotted over time.



Source: EViews file yields_us.wf1

- During the recent crisis:
  (i) The LEVEL factor falls showing that all yields were falling.
  (ii) The SLOPE factor becomes negative suggesting an inverted yield curve.
  (iii) The CURVATURE factor becomes negative at the end of 2008, suggesting that short and long yields fall relative to intermediate yields.

A Latent Factor CAPM

- The latent multi-factor CAPM is
$$r_{i,t} - r_{f,t} = \alpha_i + \sum_{j=1}^{K} \beta_{i,j} s_{j,t} + u_{i,t}$$
where $s_{j,t}$ are unobserved latent factors.
- The data are the excess returns on 6 stocks consisting of Exxon, GE, Gold, IBM, Microsoft and Walmart. The data are monthly beginning in April 1990 and ending in July 2004.
- The covariance matrix is

$$cov\left(r_t\right) = \begin{bmatrix} 0.0019 & 0.0009 & -0.0002 & 0.0014 & 0.0009 & 0.0006 \\ 0.0009 & 0.0053 & -0.0004 & 0.0020 & 0.0032 & 0.0018 \\ -0.0002 & -0.0004 & 0.0009 & -0.0005 & -0.0006 & -0.0004 \\ 0.0014 & 0.0020 & -0.0005 & 0.0088 & 0.0048 & 0.0011 \\ 0.0009 & 0.0032 & -0.0006 & 0.0048 & 0.0113 & 0.0024 \\ 0.0006 & 0.0018 & -0.0004 & 0.0011 & 0.0024 & 0.0057 \end{bmatrix}$$

A Latent Factor CAPM

The EViews output from the principal components decomposition based on the covariance matrix is

Principal Components Analysis
Date: 04/22/14 Time: 13:53
Sample (adjusted): 1990M05 2004M07
Included observations: 171 after adjustments
Balanced sample (listwise missing value deletion)
Computed using: Ordinary covariances
Extracting 6 of 6 possible components

Eigenvalues: (Sum = 0.03397365, Average = 0.005662275)

Number	Value	Difference	Proportion	Cumulative Value	Cumulative Proportion
1	0.017418	0.011541	0.5127	0.017418	0.5127
2	0.005877	0.001127	0.1730	0.023294	0.6857
3	0.004750	0.001161	0.1398	0.028045	0.8255
4	0.003589	0.002078	0.1057	0.031634	0.9311
5	0.001511	0.000683	0.0445	0.033145	0.9756
6	0.000828	---	0.0244	0.033974	1.0000

Eigenvectors (loadings):

Variable	PC 1	PC 2	PC 3	PC 4	PC 5	PC 6
E_EXXON	0.122268	-0.018438	0.207953	0.105001	0.964524	0.011610
E_GE	0.330706	0.337231	0.232868	0.831073	-0.176505	0.029067
E_GOLD	-0.056703	-0.013811	-0.034159	0.003252	0.001925	0.997704
E_IBM	0.535891	-0.665271	0.475793	-0.114318	-0.171247	0.038241
E_MSOFT	0.720721	0.108083	-0.668593	-0.129414	0.068703	0.019855
E_WMART	0.256614	0.656869	0.477457	-0.518136	-0.067011	0.041842

Source: EViews file capm.wf1

A Latent Factor CAPM

- The $N = 6$ eigenvalues of the covariance matrix from highest to lowest are

$$\widehat{\lambda} = \{0.017418, 0.005877, 0.004750, 0.003589, 0.001511, 0.000828\}$$

- The total sum of the eigenvalues is

$$\begin{aligned} 0.033973 &= 0.017418 + 0.005877 + 0.004750 + 0.003589 \\ &\quad + 0.001511 + 0.000828 \end{aligned}$$

- This sum equals the total volatility of all 6 excess returns as given by the sum of their variances

$$\begin{aligned} 0.033973 &= 0.001913 + 0.005358 + 0.000887 + 0.008771 \\ &\quad + 0.011307 + 0.005737 \end{aligned}$$

A Latent Factor CAPM

- The proportionate contributions of the first three eigenvalues to total volatility are

$$\begin{aligned}\widehat{\lambda}_1 &= \frac{0.017418}{0.033973} = 0.5127\\ \widehat{\lambda}_2 &= \frac{0.005877}{0.033973} = 0.1730\\ \widehat{\lambda}_3 &= \frac{0.004750}{0.033973} = 0.1398\end{aligned}$$

  - The first factor explains 51.27% of total volatility (equal to 0.033973).
  - The second factor explains 17.30% of total volatility.
  - The third factor explains 13.98% of volatility.

- So the first three factors explain jointly

$$0.5127 + 0.1730 + 0.1398 = 0.8255$$

or 82.55% of total volatility. This suggests that a 1-factor CAPM is potentially inappropriate and there is a need for a 3-factor model (maybe even higher).

A Latent Factor CAPM

- To estimate the multi-factor CAPM, the intercepts (α_i) are estimated using the sample means as given by the EViews output.

	E_EXXON	E_GE	E_GOLD	E_IBM	E_MSOFT	E_WMART
Mean	0.013836	0.019874	-0.003052	0.008706	0.017464	0.010084
Median	0.016349	0.011553	-0.006375	0.008115	0.017968	0.014281
Maximum	0.159486	0.270685	0.156079	0.301615	0.337298	0.230868
Minimum	-0.108606	-0.195864	-0.065345	-0.306635	-0.425440	-0.237609
Std. Dev.	0.043870	0.073412	0.029877	0.093927	0.106647	0.075968
Skewness	0.039693	0.125928	1.118160	-0.078041	-0.177437	-0.177876
Kurtosis	3.484690	3.501747	7.281451	3.764451	4.563176	3.243565
Jarque-Bera	1.718736	2.245671	166.2401	4.337322	18.30736	1.324417
Probability	0.423430	0.325356	0.000000	0.114331	0.000106	0.515711
Sum	2.365874	3.398516	-0.521932	1.488642	2.986310	1.724319
Sum Sq. Dev.	0.327176	0.916176	0.151744	1.499796	1.933503	0.981098
Observations	171	171	171	171	171	171

Source: EViews file capm.wf1

A Latent Factor CAPM

- Using the first eigen vector, the $K = 1$ factor CAPM is estimated as

$$\widehat{\beta}_1 = \sqrt{\widehat{\lambda}_1}\widehat{P}_1 = \sqrt{0.017418}\begin{bmatrix} 0.122268 \\ 0.330706 \\ -0.056703 \\ 0.535891 \\ 0.720721 \\ 0.256614 \end{bmatrix} = \begin{bmatrix} 1.6137 \times 10^{-2} \\ 4.3646 \times 10^{-2} \\ -7.4835 \times 10^{-3} \\ 7.0725 \times 10^{-2} \\ 9.5119 \times 10^{-2} \\ 3.3867 \times 10^{-2} \end{bmatrix}$$

- These results show that
  - (i) Gold moves in the opposite direction to the other assets, which is consistent with the asset representing a hedge stock.
  - (ii) Microsoft has the highest loading, equal to 9.5119×10^{-2}, showing that this asset responds the most to changes in the factor $s_{1,t}$, compared to the other stocks. This result is consistent with Microsoft being an aggressive stock at least relative to the other stocks.

A Latent Factor CAPM

- Using the sample means and the loadings on the first factor, the estimated $K = 1$ factor model is then

$$
\begin{array}{lcl}
\text{Exxon} & : & r_{1,t} = 0.013836 + 1.6137 \times 10^{-2} s_{1,t} + \widehat{u}_{1,t} \\
\text{GE} & : & r_{2,t} = 0.019874 + 4.3646 \times 10^{-2} s_{1,t} + \widehat{u}_{2,t} \\
\text{Gold} & : & r_{3,t} = -0.003052 - 7.4835 \times 10^{-3} s_{1,t} + \widehat{u}_{3,t} \\
\text{IBM} & : & r_{4,t} = 0.008706 + 7.0725 \times 10^{-2} s_{1,t} + \widehat{u}_{4,t} \\
\text{Microsoft} & : & r_{5,t} = 0.017464 + 9.5119 \times 10^{-2} s_{1,t} + \widehat{u}_{5,t} \\
\text{Walmart} & : & r_{6,t} = 0.010084 + 3.3867 \times 10^{-2} s_{1,t} + \widehat{u}_{6,t}
\end{array}
$$

- For comparison the OLS estimates of the CAPM with the excess return on the market as the (observable) factor are given below.
- The beta-risk estimates are very different from the two models. Part of the reason for this is that the variance of $s_{1,t}$ by construction is normalized to be unity, whereas the variance of the excess return on the market $r_{m,t} - r_{f,t}$, is not.

A Latent Factor CAPM

Dependent Variable: E_EXXON
Method: Least Squares
Date: 03/02/14 Time: 06:35
Sample (adjusted): 1990M05 2004M07
Included observations: 171 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.012018	0.002945	4.080186	0.0001
E_MARKET	0.501768	0.068830	7.289990	0.0000

R-squared	0.239232	Mean dependent var	0.013836
Adjusted R-squared	0.234730	S.D. dependent var	0.043870
S.E. of regression	0.038377	Akaike info criterion	-3.671078
Sum squared resid	0.248905	Schwarz criterion	-3.634333
Log likelihood	315.8771	Hannan-Quinn criter.	-3.656168
F-statistic	53.14396	Durbin-Watson stat	2.094047
Prob(F-statistic)	0.000000		

Dependent Variable: E_GE
Method: Least Squares
Date: 03/02/14 Time: 06:36
Sample (adjusted): 1990M05 2004M07
Included observations: 171 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.015731	0.004214	3.732752	0.0003
E_MARKET	1.143629	0.098484	11.61238	0.0000

R-squared	0.443800	Mean dependent var	0.019874
Adjusted R-squared	0.440509	S.D. dependent var	0.073412
S.E. of regression	0.054911	Akaike info criterion	-2.954568
Sum squared resid	0.509577	Schwarz criterion	-2.917824
Log likelihood	254.6156	Hannan-Quinn criter.	-2.939659
F-statistic	134.8474	Durbin-Watson stat	2.348651
Prob(F-statistic)	0.000000		

Dependent Variable: E_GOLD
Method: Least Squares
Date: 03/02/14 Time: 06:37
Sample (adjusted): 1990M05 2004M07
Included observations: 171 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.002695	0.002277	-1.183889	0.2381
E_MARKET	-0.098481	0.053205	-1.850954	0.0659

R-squared	0.019870	Mean dependent var	-0.003052
Adjusted R-squared	0.014070	S.D. dependent var	0.029877
S.E. of regression	0.029666	Akaike info criterion	-4.186027
Sum squared resid	0.148729	Schwarz criterion	-4.149282
Log likelihood	359.9053	Hannan-Quinn criter.	-4.171117
F-statistic	3.426033	Durbin-Watson stat	1.814118
Prob(F-statistic)	0.065921		

Dependent Variable: E_IBM
Method: Least Squares
Date: 03/02/14 Time: 06:37
Sample (adjusted): 1990M05 2004M07
Included observations: 171 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.004340	0.006045	0.717930	0.4738
E_MARKET	1.205004	0.141260	8.530396	0.0000

R-squared	0.300982	Mean dependent var	0.008706
Adjusted R-squared	0.296846	S.D. dependent var	0.093927
S.E. of regression	0.078762	Akaike info criterion	-2.233144
Sum squared resid	1.048385	Schwarz criterion	-2.196399
Log likelihood	192.9338	Hannan-Quinn criter.	-2.218235
F-statistic	72.76766	Durbin-Watson stat	2.127968
Prob(F-statistic)	0.000000		

Dependent Variable: E_MSOFT
Method: Least Squares
Date: 03/02/14 Time: 06:38
Sample (adjusted): 1990M05 2004M07
Included observations: 171 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.012220	0.006685	1.827943	0.0693
E_MARKET	1.447368	0.156223	9.264771	0.0000

R-squared	0.336828	Mean dependent var	0.017464
Adjusted R-squared	0.332904	S.D. dependent var	0.106647
S.E. of regression	0.087105	Akaike info criterion	-2.031783
Sum squared resid	1.282245	Schwarz criterion	-1.995038
Log likelihood	175.7174	Hannan-Quinn criter.	-2.016873
F-statistic	85.83598	Durbin-Watson stat	2.268318
Prob(F-statistic)	0.000000		

Dependent Variable: E_WMART
Method: Least Squares
Date: 03/02/14 Time: 06:39
Sample (adjusted): 1990M05 2004M07
Included observations: 171 after adjustments

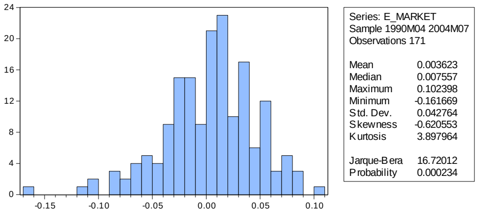
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.006937	0.005101	1.359933	0.1757
E_MARKET	0.868455	0.119209	7.285115	0.0000

R-squared	0.238989	Mean dependent var	0.010084
Adjusted R-squared	0.234486	S.D. dependent var	0.075968
S.E. of regression	0.066467	Akaike info criterion	-2.572584
Sum squared resid	0.746627	Schwarz criterion	-2.535840
Log likelihood	221.9560	Hannan-Quinn criter.	-2.557675
F-statistic	53.07290	Durbin-Watson stat	2.168152
Prob(F-statistic)	0.000000		

A Latent Factor CAPM

- To make the beta-risk estimates commensurate across the two estimated models the approach is to rescale $s_{1,t}$ to make it equivalent to the variance of $r_{m,t} - r_{f,t}$.
- The EViews output of the descriptive statistics of $r_{m,t} - r_{f,t}$ shows that the variance is

$$var\left(r_{m,t} - r_{f,t}\right) = 0.042764^2$$



Source: EViews file capm.wf1

A Latent Factor CAPM

- Reconsider the 1-factor CAPM

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,1} s_{1,t} + u_{i,t}$$

- Defining σ_m as the standard deviation of the excess return on the market, then the model is rewritten as

$$\begin{aligned} r_{i,t} - r_{f,t} &= \alpha_i + \beta_{i,1} \frac{\sigma_m}{\sigma_m} s_{1,t} + u_{i,t} \\ &= \alpha_i + \frac{\beta_{i,1}}{\sigma_m} \left(\sigma_m s_{1,t}\right) + u_{i,t} \end{aligned}$$

  Here the factor know has a variance equal to the variance of the excess return on the market as

$$E\left[\left(\sigma_m s_{1,t}\right)^2\right] = \sigma_m^2 E\left[s_{1,t}^2\right] = \sigma_m^2 \times 1 = \sigma_m^2$$

- Thus, the rescaled beta-risk estimates are obtained by dividing the loading vector $\beta_{i,1}$, by the standard deviation of $\sigma_m = r_{m,t} - r_{f,t}$.

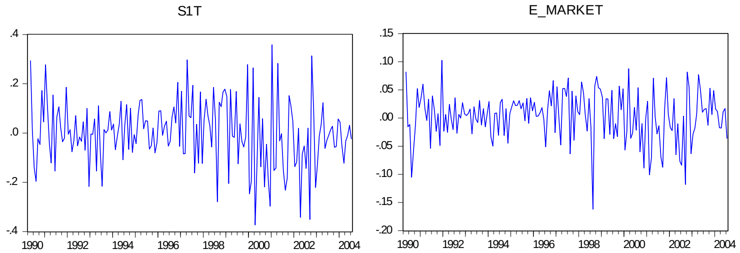
- These rescaled beta estimates are

$$\widetilde{\beta}_1 = \frac{\widehat{\beta}_1}{0.042764} = \begin{bmatrix} 1.6137 \times 10^{-2}/0.042764 \\ 4.3646 \times 10^{-2}/0.042764 \\ -7.4835 \times 10^{-3}/0.042764 \\ 7.0725 \times 10^{-2}/0.042764 \\ 9.5119 \times 10^{-2}/0.042764 \\ 3.3867 \times 10^{-2}/0.042764 \end{bmatrix} = \begin{bmatrix} 0.3773 \\ 1.0206 \\ -0.1750 \\ 1.6538 \\ 2.2243 \\ 0.7919 \end{bmatrix}$$

- Interpretation:
  (i) GE tracks the market with a beta-risk of 1.0206.
  (ii) Exxon and Walmart are conservative stocks with estimates between 0 and 1.
  (iii) The tech-stocks of IBM and Microsoft are aggressive stocks with estimates greater than 1.
  (iv) Gold is a hedge stock with a beta-risk of -0.1750.

A Latent Factor CAPM

- The estimate of the first factor $(\widehat{s}_{1,t})$ is plotted over time (note that the two factors are not scaled to have the same variances).



Source: EViews file capm.wf1

- These results show that there are some similarities in the two factors as well as some differences.
- The correlation between the two factors is 0.7614 showing that the first factor is highly correlated with the excess returns on the market.

End of Lecture