Econ 623 Econometrics II
Topic 1: Asymptotic Distribution Theory I

1 Convergence Theory

- **Converge in probability**: \( \forall \varepsilon > 0, \lim_{n \to \infty} P(|X_n - X| < \varepsilon) = 1. \) Usually we denote it by \( \text{plim} X_n = X \) or \( X_n \xrightarrow{p} X. \)

- **Markov inequality**: suppose \( X \) is a nonnegative random variable, \( \forall \delta > 0, P(X \geq \delta) \leq E(X)/\delta. \)

- **Chebychev’s inequality**: suppose \( X \) is a random variable, \( \forall c, \varepsilon > 0, P(|X - c| > \varepsilon) \leq E((X - c)^2)/\varepsilon^2. \)

- Let \( E(X) = 0, \text{Var}(X) = \sigma^2, \) then \( \forall \varepsilon > 0, P(|X| > \varepsilon) \leq \sigma^2/\varepsilon^2. \)

- Suppose \( X_n \) has mean \( \mu_n \) and \( \sigma_n^2 \) such that the ordinary limit of them are \( c \) and 0, respectively. Then we say \( X_n \) converges in mean square to \( c \) or \( X_n \xrightarrow{r^2} c. \)

- If \( X_n \xrightarrow{r^2} c \) then \( X_n \xrightarrow{p} c. \)
• **Weak LLN:** Suppose $X_1, \ldots, X_n$ are a sequence of independent and identically distributed (iid) random variables with mean $\mu < \infty$ and variance $\sigma^2 < \infty$. Then $\bar{X} \xrightarrow{p} \mu$ as $n \to \infty$.

• **Khinchine’s Weak LLN:** Suppose $X_1, \ldots, X_n$ are a sequence of iid random variables with mean $\mu < \infty$. Then $\bar{X} \xrightarrow{p} \mu$.

• **Chebychev’s Weak LLN:** Suppose $X_1, \ldots, X_n$ are a sequence of independent random variables with $E(X_i) = \mu_i < \infty$ and $Var(X_i) = \sigma_i^2 < \infty$ such that $\frac{1}{n} \frac{\sigma^2}{\mu} = \frac{1}{n} \left( \frac{1}{\mu} \sum \sigma_i^2 \right) \to 0$ as $n \to \infty$. Then $\bar{X} - \frac{1}{n} \sum \mu_i \xrightarrow{p} 0$. 
• **Almost Sure Convergence**: $P(\lim_{n \to \infty} X_n = c) = 1$. Usually we denote it by $X_n \xrightarrow{a.s.} c$.

• $X_n \xrightarrow{a.s.} c \iff \lim P(\cup_{n=k}^{\infty} \{X_n - c > \varepsilon\}) = 0, \forall \varepsilon > 0$.

• **Borel-Cantelli Lemma**: if $\sum_{n=1}^{\infty} P(\{X_n - c > \varepsilon\}) < +\infty, \forall \varepsilon > 0$, then $X_n \xrightarrow{a.s.} c$.

• If $X_n \xrightarrow{a.s.} c$ then $X_n \xrightarrow{p} c$.

• **Kolmogorov’s Strong LLN** (White, P32): Suppose $X_1, \ldots, X_n$ are a sequence of iid random variables, then $\bar{X} - \mu \xrightarrow{a.s.} 0$ iff $E(X_i) = \mu < \infty$.

• **Markov Strong LLN** (White, P35): Suppose $X_1, \ldots, X_n$ are a sequence of independent random variables with $E(X_i) = \mu_i < \infty$. If $\exists \delta > 0$, $\sum E(|X_i - \mu_i|^{1+\delta})/i^{1+\delta} < \infty$ as $n \to \infty$. Then $\bar{X} - \frac{1}{n} \sum \mu_i \xrightarrow{a.s.} 0$.

• **Liapounov Strong LLN** (White, P35): Suppose $X_1, \ldots, X_n$ are a sequence of independent random variables with $E(X_i) = \mu_i < \infty$. If $\exists \delta, \Delta > 0$, $E|X_i|^{1+\delta} < \Delta < \infty$, then $\bar{X} - \frac{1}{n} \sum \mu_i \xrightarrow{a.s.} 0$.

• **Slutsky Theorem**: $\text{plim} g(X_n) = g(\text{plim} X_n)$ where $g$ is a continuous function.
2 Asymptotic Distribution Theory for Independent Sequence

- **Convergence in Distribution**: Suppose $X_1, \ldots, X_n$ are a sequence of random variables with cdf $F_n(x)$. Suppose $X$ is a random variable with cdf $F(x)$. If $F_n(x) \rightarrow F(x)$ for all continuity point of $F$, we say $X_n \xrightarrow{d} X$. The mean (variance) of $X$ is called the limiting mean (variance).

  - If $X_n \xrightarrow{d} X$, then $g(X_n) \xrightarrow{d} g(X)$ where $g$ is a continuous function.

  - $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$. The reverse is not necessarily true. However, $X_n \xrightarrow{d} c$ then $X_n \xrightarrow{p} c$.

  - If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{p} a$, $Z_n \xrightarrow{p} b$, then $X_nY_n + Z_n \xrightarrow{d} aX + b$

- **Cramer-Wold Device** (White, P114): If $X_n \xrightarrow{d} X$, then $c^tX_n \xrightarrow{d} c^tX$

- $X_n \xrightarrow{d} X$ does not necessarily lead to $E(X_n) \rightarrow E(X)$
• **Lindberg-Levy CLT** (White, P114): Suppose $X_1, \ldots, X_n$ are a sequence of iid random variables with mean $\mu < \infty$ and variance $\sigma^2 \in (0, \infty)$. Then $\sqrt{n}(\overline{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$.

• **Lindberg-Feller CLT** (White, P117): Suppose $X_1, \ldots, X_n$ are a sequence of independent random variables with $E(X_i) = \mu_i < \infty$ and $Var(X_i) = \sigma_i^2 \in (0, \infty)$. Then

$$
\lim_{n \to \infty} n^{-1} \sigma_n^{-2} \sum_{i=1}^{n} E((X_i - \mu_i)^2 1((X_i - \mu_i)^2 > n\varepsilon \sigma_n^2)) = 0, \ \forall \varepsilon > 0
$$

(1)

iff

$$
\lim_{n \to \infty} \max_{i \in [1,n]} (\sigma_i^2)/(n \sigma_n^2) = 0
$$

and $\sqrt{n}(\overline{X} - \overline{\mu}_n) \xrightarrow{d} N(0, \lim \sigma_n^2)$, where $\sigma_n^2 = \frac{1}{n} \sum \sigma_i^2$. Condition (1) is known as the Lindberg condition.

• **Liapounov CLT** (White, P118): Suppose $X_1, \ldots, X_n$ are a sequence of independent random variables with $E(X_i) = \mu_i < \infty$ and $\exists \Delta > 0, \delta > 0$ such that $E[|X_i - \mu_i|^{2+\delta}] < \Delta < \infty$. Then $\sqrt{n}((\overline{X} - \overline{\mu}_n) \xrightarrow{d} N(0, \lim \sigma_n^2)$ if $\lim \sigma_n^2 > 0$. 


• A random variable $X$ is said to be $O_p(1)$ (or bounded in probability) if for every $\varepsilon > 0$ there exists a finite $M(\varepsilon) > 0$ such that $P(|X| > M(\varepsilon)) < \varepsilon$.

• If $E(X) = 0$, $Var(X) = \sigma^2 < \infty$, then $X \sim O_p(1)$. **Proof:** By Chebychev’s inequality, $\forall M > 0$, $P(|X| > M) \leq \sigma^2/M^2$. Let $M(\varepsilon) > \sqrt{\sigma^2/\varepsilon}$, then $P(|X| > M(\varepsilon)) < \varepsilon$.

• The random sequence $\{b_n\}$ is at most of order $n^\lambda$ in probability, denoted $b_n = O_p(n^\lambda)$, if for every $\varepsilon > 0$ there exists a finite $\Delta(\varepsilon) > 0$ and $N(\varepsilon) \in N$, such that $P(|n^{-\lambda}b_n| > \Delta(\varepsilon)) < \varepsilon$ for all $n \geq N(\varepsilon)$.

• The random sequence $\{b_n\}$ is of smaller order $n^\lambda$ in probability, denoted $b_n = o_p(n^\lambda)$, if $n^{-\lambda}b_n \xrightarrow{p} 0$. A random sequence is $o_p(1)$ means that it converges to 0 in probability.

• **Delta method:** if $\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$ and if $g$ is a differentiable function, then $\sqrt{n}(g(\bar{X}) - g(\mu)) \xrightarrow{d} N \left(0, \frac{d g(\mu)}{d \mu} \sigma^2 \frac{d g(\mu)}{d \mu} \right)$.
3 Asymptotic Distribution Theory for Dependent Sequence

3.1 Stationarity

- Strict stationarity: \( \{X_n\}_{1}^{\infty} \) is strictly stationary if finite dimensional distributions are translation invariant, ie,

\[
(X_{t_1+h}, \cdots, X_{t_p+h}) = (X_{t_1}, \cdots, X_{t_p}), \forall h, p, t_1, \cdots, t_p
\]

- Covariance stationarity. \( \{X_n\}_{1}^{\infty} \) is covariance stationary if (1) \( E(X_t) = \mu < 0 \); (2) \( \text{Var}(X_t) = \sigma^2 < 0 \); (3) \( E(X_t E_{t+h}) = \sigma h \) for all \( t \).

- Theorem: If (1) \( X = \{X_n\}_{-\infty}^{\infty} \) is strictly stationary; (2) \( \varphi : R_{\infty} \rightarrow R \) is measurable (ie \( \varphi^{-1}B \in R_{\infty}, \forall B \in \mathcal{B} \)); (3) \( Y_n = \varphi(\cdots, X_{n-1}, X_n, X_{n+1}, \cdots) \). Then \( \{Y_n\} \) is strictly stationary.

- Can we generalize Kolmogorov LLN to temporally dependent sequences?
  A counterexample: \( X_t = u_t + Z \), where \( U_t \) is an iid uniform \([0, 1]\), \( Z \) is \( N(0,1) \) and independent of \( u_t \). Obviously \( E(X_t) = 1/2 \), but \( \bar{X} = \bar{u} + Z \overset{a.s.}{\rightarrow} 1/2 + Z \). Why LLN breaks down? Too much dependence in the sequence of \( X_t \).
3.2 Ergodicity

- **Theorem (necessary and sufficient condition for ergodicity):** Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(S : \Omega \rightarrow \Omega\) be a measure preserving map on \(\Omega\). \(S\) is ergodic iff
  \[
  \frac{1}{n} \sum_{k=0}^{n-1} P(F \cap S^{-k}G) \rightarrow P(F)P(G), \forall F, G \in \mathcal{F}.
  \]

- The stationary process \(X_t\) is ergodic iff \(\lim_{n \to \infty} \sum_{t=1}^{n} P(X_1 \leq a, X_{t+1} \leq b) = P(X_1 \leq a)P(X_1 \leq b)\) for all \(a, b \in \mathbb{R}\).

- Like the iid sequence, if \(X_t\) is ergodic then \(f(X_t)\) is ergodic.

- Unlike the iid sequence, if \(X_t\) is ergodic then \(f(X_{t+1}, \ldots, X_{t+k})\) is ergodic.

- **Ergodic theorem (Ergodic LLN)** (White, P44): If \(\{X_t\}\) is a stationary and ergodic sequence with \(E|X_t| < \infty\). Then \(\bar{X}_n \xrightarrow{a.s.} E(X)\). If \(EX_t^2 < \infty\). Then \(\frac{1}{n} \sum X_tX_{t+m} \xrightarrow{a.s.} E(X_tX_{t+m})\).

- **Theorem:** If (1) \(X = \{X_t\}_{t=1}^{\infty}\) is strictly stationary and ergodic; (2) \(\varphi : R_\infty \rightarrow R\) is measurable (ie \(\varphi^{-1}B \in R_\infty, \forall B \in \mathfrak{B}\)); (3) \(Y_n = \varphi(\cdots, x_{n-1}, x_n, x_{n+1}, \cdots)\). Then \(\{Y_t\}\) is strictly stationary and ergodic.
If \( \{X_1, \cdots, X_T\} \) is a stationary process with \( E(X_t) = \mu, \ E(X_t - \mu)(X_{t-j} - \mu) = \gamma_j < \infty, \ \sum_{j=0}^{\infty} |\gamma_j| < \infty \). What is the limit of \( \mathbf{X}_T = \frac{1}{T} \sum_{t=1}^{T} X \)? Note that \( E(\mathbf{X}_T) = \mu \) and

\[
T \times E(\mathbf{X}_T - \mu)^2 = \gamma_0 + \frac{T-1}{T} 2 \gamma_1 + \cdots + \frac{1}{T^2} 2 \gamma_{T-1}
\]

\[\leq |\gamma_0| + 2 |\gamma_1| + \cdots + 2 |\gamma_{T-1}| < \infty.\]

Hence, \( \mathbf{X}_T \xrightarrow{r^2} \mu \Rightarrow \mathbf{X}_T \xrightarrow{p} \mu. \)

- **In fact** \( T \times E(\mathbf{X}_T - \mu)^2 \rightarrow \sum_{-\infty}^{\infty} \gamma_j. \)

- **CLT for a stationary process** (Hamilton, P195). Let \( X_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \), where \( \{\varepsilon_t\} \) is an iid sequence of random variables with \( E(\varepsilon_t) = 0, \ E(\varepsilon_t^2) < \infty \) and \( \sum_{j=0}^{\infty} |\psi_j| < \infty \). Then \( \sqrt{T}(\mathbf{X}_T - \mu) \xrightarrow{d} \mathcal{N}(0, \sum_{j=0}^{\infty} \gamma_j). \)

- Let \( X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \), where \( \{\varepsilon_t\} \) is an iid sequence and \( \sum_{j=0}^{\infty} \psi_j^2 < \infty \), then \( X_t \) is strictly stationary and ergodic.
• Some measures of dependence.

\[ \phi(m) = \sup_{A \in \mathcal{F}_{-\infty}^t, B \in \mathcal{F}_{t+m}^\infty} |P(A|B) - P(A)|; \]

\[ \alpha(m) = \sup_{A \in \mathcal{F}_{-\infty}^t, B \in \mathcal{F}_{t+m}^\infty} |P(A \cap B) - P(A)P(B)|. \]

• **Mixing Sequence:** \( \{X_t, \mathcal{F}_t\} \) satisfies a uniform mixing condition or is called \( \phi \)-mixing of order \(-a\) if \( \phi(m) \to 0 \) as \( m \to \infty \) and \( \phi(m) = O(m^{-a-\varepsilon}) \) for some \( \varepsilon > 0 \). \( \{X_t, \mathcal{F}_t\} \) satisfies a strong mixing condition or is called \( \alpha \)-mixing of order \(-a\) if \( \alpha(m) \to 0 \) as \( m \to \infty \) and \( \alpha(m) = O(m^{-a-\varepsilon}) \) for some \( \varepsilon > 0 \). Obviously, \( \phi \)-mixing \( \Rightarrow \) \( \alpha \)-mixing. The bigger the \( a \), the less the dependence.

• Theorem (White P48): For a stationary process, \( \alpha \)-mixing implies ergodicity.

• Theorem (White P50): A measurable function, \( g(X_t, \cdots, X_{t+\tau}) \) for a finite \( \tau \), of a \( \phi \)-mixing (\( \alpha \)-mixing) sequence of order \(-a\) is \( \phi \)-mixing (\( \alpha \)-mixing) of order \(-a\).
• **Asymptotic uncorrelation** (White P52): \( \{ X_t \} \) is asymptotic uncorrelated if
\[
\text{Cov}(X_t, X_{t+r}) \leq \rho_r \left( \text{Var}(X_t) \text{Var}(X_{t+r}) \right)^{1/2}, \quad \rho_r \in [0, 1], \sum_0^\infty \rho_r < \infty.
\]

• **LLN for a mixing sequence** (White P49): Let \( \{ X_t, \mathcal{F}_t \} \) be a sequence with \( \mu_t = E(X_t) < \infty \) and \( \exists \delta \in (0, r), r \geq 1, \) such that
\[
\sum_1^\infty (E|X_t - \mu_t|^{r+\delta} / t^{r+\delta})^{1/r} < \infty.
\] If \( X_t \) is \( \phi \)-mixing of size \(-r/(2r-1)\) or \( \alpha \)-mixing of size \(-r/(r-1)\), then \( \bar{X} - \frac{1}{n} \sum \mu_i \xrightarrow{a.s.} 0 \).

• **LLN for a mixing sequence** (White P49): Let \( \{ X_t, \mathcal{F}_t \} \) be a \( \phi \)-mixing of size \(-r/(2r-1)\) or \( \alpha \)-mixing of size \(-r/(r-1)\). If \( \exists \Delta > 0, \delta > 0 \) such that \( E|X_t|^{r+\delta} < \Delta < \infty \), then \( \bar{X} - \frac{1}{n} \sum \mu_i \xrightarrow{a.s.} 0 \).

• **LLN for an asymptotic uncorrelated sequence** (White P53): \( \{ X_t \} \) is asymptotic uncorrelated with \( \mu_t = E(X_t), \sigma_t^2 = \text{Var}(X_t) < \Delta < \infty \). then \( \bar{X} - \frac{1}{n} \sum \mu_i \xrightarrow{a.s.} 0 \).
- **Martingale difference sequence** (MDS). Let \( \{\mathfrak{F}_t\}_{t=1}^{\infty} \) be an increasing sequence of \( \sigma \)-fields with \( \mathfrak{F}_t \) being generated from \( \{X_1, \ldots, X_t\} \) (namely \( \mathfrak{F}_t \) is adapted to \( X_t \)). If \( E(X_t) = 0, \ E(X_t|\mathfrak{F}_{t-1}) = 0 \) for all \( t \), then \( \{X_t, \mathfrak{F}_t\} \) is a MDS.

- **Martingale** (MG). \( \{S_t, \mathfrak{F}_t\} \) is a MG if \( E(S_t|\mathfrak{F}_{t-1}) = S_{t-1} \) for all \( t \).

- Let \( S_n = \sum_{t=1}^{n} X_t \). If \( X_t \) is a MDS then \( S_t \) is a MG; if \( S_t \) is a MG then \( X_t \) is a MDS.

- Let \( \{Z_t, \mathfrak{F}_t\} \) be an adapted sequence with \( E(Z_t^2) < \infty \). \( Z_t \) is a **mixingale** of size \(-a\) iff \( \exists c_t, \gamma_m \) such that \( \gamma_m \to 0 \) as \( m \to \infty \), \( (E(E(Z_t|\mathfrak{F}_{t-m})^2))^{1/2} \leq c_t \gamma_m \) and \( \gamma_m = O(m^{-a-\varepsilon}) \) for some \( \varepsilon > 0 \).

- **LLN for a mixingale** (Hamilton, P191): Let \( \{Z_t\} \) be a mixingale. If (a) \( \{Z_t\} \) is uniformly integrable (namely \( \forall \varepsilon > 0, \exists a(\varepsilon) > 0 \) such that \( E(|Z_t|1(|Z_t| > a(\varepsilon))) < \varepsilon \) for all \( t \)); (b) \( \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} c_t < \infty \). Then \( Z \overset{p}{\to} 0 \).

- **LLN for MDS** (White P60): \( \{X_t, \mathfrak{F}_t\} \) is a MDS. If for some \( r \geq 1 \), \( \sum_{t=1}^{\infty} (E|X_t|^{2r})/t^{1+r} < \infty \), then \( \overline{X}_T \overset{a.s.}{\to} 0 \).

- **Martingale convergence Theorem**: If \( \{S_t, \mathfrak{F}_t\} \) is a MG such that \( \sup_{n \geq 1} E|S_n| < \infty \), then \( \exists S \) such that \( E|S| < \infty \) and \( S_t \overset{a.s.}{\to} S \).
• **CLT for mixingale** (White P125): Let \( \{Z_t, \mathcal{F}_t\} \) be a stationary ergodic adapted mixingale of order \(-1\), then \( \text{Var}(n^{-1/2} \sum Z_t) \to \sigma^2 < \infty \). If \( \sigma^2 > 0 \), then \( n^{-1/2} \sum Z_t \xrightarrow{d} N(0, \sigma^2) \).

• **CLT for \( \phi \)-mixing** (Ibragimov and Linnik or Taniguchi and Kakizawa): If \( X_t \) is stationary and \( \phi \)-mixing with \( \sum \phi_m^{1/2} < \infty \). If \( E(X_t) = 0, E(X_0^2) + 2 \sum_1^\infty E(X_0 X_t) \) converges to \( \sigma^2 \in (0, \infty) \). Then \( n^{-1/2} \sum X_t \xrightarrow{d} N(0, \sigma^2) \).

• **CLT for \( \alpha \)-mixing** (Ibragimov and Linnik or Taniguchi and Kakizawa): \( X_t \) is \( \alpha \)-mixing with zero mean. Let \( S_t = \sum X_t, \sigma_n^2 = \text{Var}(S_n) \) and \( \sigma_n^2/n \to \sigma^2 > 0 \). Then \( Z_t = S_t/\sigma_t \xrightarrow{d} N(0, 1) \) iff

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^n E(X_n^2 I(X_n > n \sigma_n^2)) = 0,
\]

\[ (2) \]

• **CLT for MDS** (White P133): \( \{X_{nt}, \mathcal{F}_{nt}\} \) is a MDS with \( \sigma_{nt}^2 \in (0, \infty) \). Let \( \bar{\sigma}_n^2 = \text{Var}(n^{-1/2} \sum X_{nt}) \). If \( \forall \varepsilon > 0 \),

\[
\lim_{n \to \infty} n^{-1} \bar{\sigma}_n^{-2} \sum_{t=1}^n E(X_{nt}^2 I(X_{nt}^2 > n \varepsilon \bar{\sigma}_n^2)) = 0,
\]

and \( n^{-1} \sum X_{nt}^2 \xrightarrow{p} \bar{\sigma}_n^2 \), then \( n^{-1/2} \sum X_{nt} \xrightarrow{d} N(0, \lim \bar{\sigma}_n^2) \).

• **CLT for MDS** (White P135): Suppose \( \{X_{nt}, \mathcal{F}_{nt}\} \) is a MDS. If \( \exists \Delta > 0, \delta > 0 \) such that \( E|X_{nt}|^{2+\delta} < \Delta < \infty \) and \( n^{-1} \sum X_{nt}^2 - \bar{\sigma}_n^2 \xrightarrow{p} 0 \). Then \( \sqrt{n}(\bar{X} - \mu_n) \xrightarrow{d} N(0, \lim \bar{\sigma}_n^2) \) if \( \lim \bar{\sigma}_n^2 > 0 \).
• **CLT for MDS:** \( \{X_{nt}, \overline{F}_{nt}\} \) is a MDS. If \( \forall \varepsilon > 0, \)

\[
\lim_{n \to \infty} n^{-1} \sum E(X_{nt}^2 1(X_{nt}^2 > \varepsilon)) = 0,
\]

and \( n^{-1} \sum E(X_{nt}^2 | \overline{F}_{n,t-1}) \xrightarrow{p} \sigma^2 \in (0, \infty). \) then \( n^{-1/2} \sum X_{nt} \xrightarrow{d} N(0, \sigma^2). \)

• **CLT for MDS:** \( \{X_{nt}, \overline{F}_{nt}\} \) is a MDS. If \( \forall \varepsilon > 0, \)

\[
\lim_{n \to \infty} n^{-1} \sum E(X_{nt}^2 1(X_{nt}^2 > \varepsilon)) = 0,
\]

and \( n^{-1} \sum X_{nt}^2 \xrightarrow{p} \sigma^2 \in (0, \infty). \) then \( n^{-1/2} \sum X_{nt} \xrightarrow{d} N(0, \sigma^2). \)

• **CLT for MDS** (Hamilton, P193): \( \{X_t, \overline{F}_t\} \) is a MDS. If (a) \( E(X_t^2) = \sigma_t^2 > 0 \) with \( 1/T \sum_{t=1}^T \sigma_t^2 \to \sigma^2 > 0; \) (b) \( E|X_t|^r < \infty \) for some \( r > 2 \) and all \( t; \) (c) \( 1/T \sum_{t=1}^T X_t^2 \xrightarrow{p} \sigma^2, \) then \( \sqrt{T}X_T \xrightarrow{d} N(0, \sigma^2). \)