Nonlife Actuarial Models

Chapter 5

Ruin Theory

Learning Objectives

1. Surplus function, premium rate and loss process

2. Probability of ultimate ruin

3. Probability of ruin before a finite time

4. Adjustment coefficient and Lundberg's inequality

5. Poisson process and continuous-time ruin theory

5.1 Discrete-Time Surplus and Ruin

- An insurance company establishes its business with a start-up capital of u at time 0, called the **initial surplus**.

- It receives premiums of one unit per period at the end of each period. Loss claim of amount X_i is paid out at the end of period i for $i = 1, 2, \cdots$.

- X_i are independently and identically distributed as the loss random variable X.

- The **surplus** at time n with initial capital u, denoted by $U(n; u)$, is given by

$$U(n; u) = u + n - \sum_{i=1}^{n} X_i, \qquad \text{for } n = 1, 2, \cdots. \tag{5.1}$$

- The *numeraire* of the above equation is the amount of premium per period, or the premium rate.All other variables are measured as multiples of the premium rate.

- Thus, the initial surplus u may take values of $0, 1, \cdots$, times the premium rate. Likewise, X_i may take values of j times the premium rate with pf $f_X(j)$ for $j = 0, 1, \cdots$.

- We denote the mean of X by μ_X and its variance by σ_X^2.

- We assume X is of finite support, although in notation we allow j to run to infinity.

- If we denote the premium loading by θ, then we have

$$1 = (1 + \theta)\mu_X, \tag{5.2}$$

which implies

$$\mu_X = \frac{1}{1 + \theta}. \tag{5.3}$$

We shall assume positive loading so that $\mu_X < 1$.

- The business is said to be in **ruin** if the surplus function $U(n; u)$ falls to or below zero sometime after the business started, i.e., at a point $n \geq 1$.

Definition 5.1: Ruin occurs at time n if $U(n; u) \leq 0$ for the first time at n, for $n \geq 1$.

Definition 5.2: The time-of-ruin random variable $T(u)$ is defined as

$$T(u) = \min\{n \geq 1 : U(n; u) \leq 0\}. \tag{5.4}$$

Definition 5.3: Given an initial surplus u, the probability of ultimate ruin, denoted by $\psi(u)$, is

$$\psi(u) = \Pr(T(u) < \infty). \tag{5.5}$$

Definition 5.4: Given an initial surplus u, the probability of ruin by time t, denoted by $\psi(t; u)$, is

$$\psi(t; u) = \Pr(T(u) \leq t), \qquad \text{for } t = 1, 2, \cdots. \tag{5.6}$$

5.2 Discrete-Time Ruin Theory

5.2.1 Ultimate Ruin in Discrete Time

- We now derive recursive formulas for $\psi(u)$.

- First, for $u = 0$, we have

$$\psi(0) = f_X(0)\psi(1) + S_X(0). \tag{5.7}$$

- Similarly, for $u = 1$, we have

$$\psi(1) = f_X(0)\psi(2) + f_X(1)\psi(1) + S_X(1). \tag{5.8}$$

- The above equations can be generalized to larger values of u as follows

$$\psi(u) = f_X(0)\psi(u+1) + \sum_{j=1}^{u} f_X(j)\psi(u+1-j) + S_X(u), \qquad \text{for } u \geq 1. \tag{5.9}$$

- Re-arranging equation (5.9), we obtain the following recursive formula for the probability of ultimate ruin

$$\psi(u+1) = \frac{1}{f_X(0)}\left[\psi(u) - \sum_{j=1}^{u} f_X(j)\psi(u+1-j) - S_X(u)\right], \qquad \text{for } u \geq 1. \tag{5.10}$$

- To apply the above equation we need the starting value $\psi(0)$, which is given by the following theorem.

Theorem 5.1: For the discrete-time surplus model, $\psi(0) = \mu_X$.

Proof: See NAM.

Example 5.1: The claim variable X has the following distribution: $f_X(0) = 0.5$, $f_X(1) = f_X(2) = 0.2$ and $f_X(3) = 0.1$. Calculate the probability of ultimate ruin $\psi(u)$ for $u \geq 0$.

Solution: The survival function of X is $S_X(0) = 0.2 + 0.2 + 0.1 = 0.5$, $S_X(1) = 0.2 + 0.1 = 0.3$, $S_X(2) = 0.1$ and $S_X(u) = 0$ for $u \geq 3$. The mean of X is

$$\mu_X = (0)(0.5) + (1)(0.2) + (2)(0.2) + (3)(0.1) = 0.9,$$

which can also be calculated as

$$\mu_X = \sum_{u=0}^{\infty} S_X(u) = 0.5 + 0.3 + 0.1 = 0.9.$$

Thus, from Theorem 5.1 $\psi(0) = 0.9$, and from equation (5.7), $\psi(1)$ is given by

$$\psi(1) = \frac{\psi(0) - S_X(0)}{f_X(0)} = \frac{0.9 - 0.5}{0.5} = 0.8.$$

From equation (5.8), we have

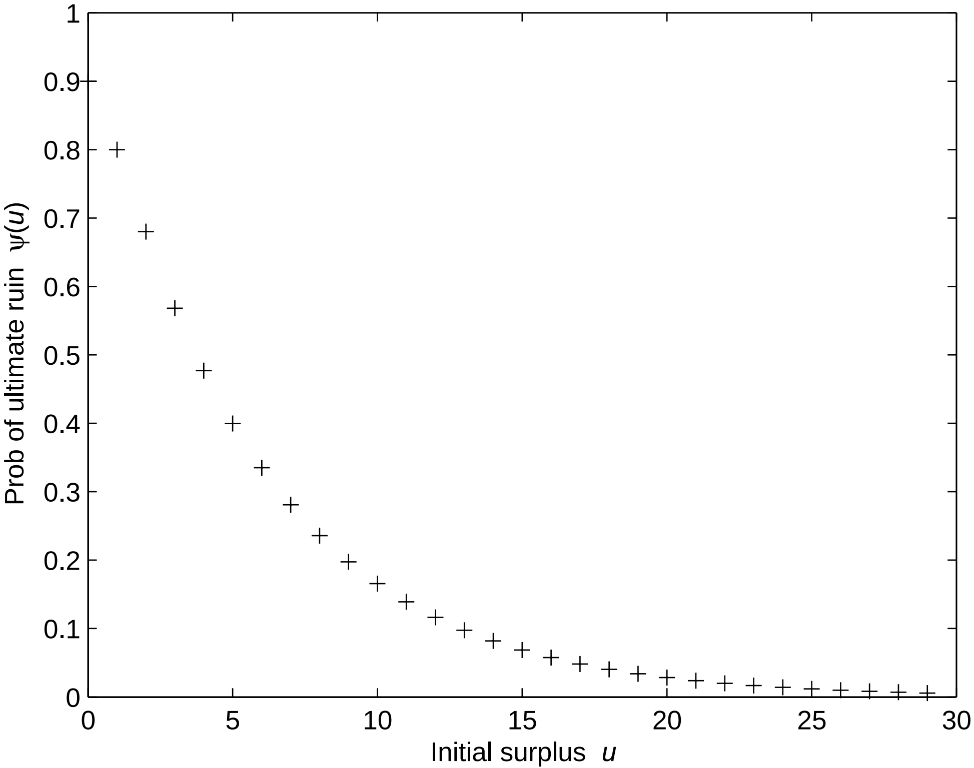
$$\psi(2) = \frac{\psi(1) - f_X(1)\psi(1) - S_X(1)}{f_X(0)} = \frac{0.8 - (0.2)(0.8) - 0.3}{0.5} = 0.68,$$

and applying equation (5.10) for $u = 3$, we have

$$\psi(3) = \frac{\psi(2) - f_X(1)\psi(2) - f_X(2)\psi(1) - S_X(2)}{f_X(0)} = 0.568.$$

As $S_X(u) = 0$ for $u \geq 3$, using equation (5.10) we have, for $u \geq 4$,

$$\psi(u) = \frac{\psi(u) - f_X(1)\psi(u) - f_X(2)\psi(u-1) - f_X(3)\psi(u-2)}{f_X(0)}.$$



Prob of ultimate ruin ψ(u)
Initial surplus u
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
5
10
15
20
25
30

5.2.2 Finite-Time Ruin in Discrete Time

- We now consider the probability of ruin at or before a finite time point t given an initial surplus u.

- First we consider $t = 1$ given initial surplus u.

- As defined in equation (5.6), $\psi(t; u) = \Pr(T(u) \leq t)$. If $u = 0$, the ruin event occurs at time $t = 1$ when $X_1 \geq 1$. Thus,

$$\psi(1; 0) = 1 - f_X(0) = S_X(0). \tag{5.20}$$

- Likewise, for $u > 0$, we have

$$\psi(1; u) = \Pr(X_1 > u) = S_X(u). \tag{5.21}$$

- We now consider $\psi(t; u)$ for $t \geq 2$ and $u \geq 0$.

- The event of ruin occurring at or before time $t \geq 2$ may be due to (a) ruin at time 1, or (b) loss of j at time 1 for $j = 0, 1, \cdots, u$, followed by ruin occurring within the next $t - 1$ periods.

- When there is a loss of j at time 1, the surplus becomes $u + 1 - j$, so that the probability of ruin within the next $t - 1$ periods is $\psi(t - 1; u + 1 - j)$.

- Thus, we conclude that

$$\psi(t; u) = \psi(1; u) + \sum_{j=0}^{u} f_X(j)\psi(t - 1; u + 1 - j). \tag{5.22}$$

Hence, $\psi(t; u)$ can be computed as follows.

1. Construct a table with time t running down the rows for $t = 1, 2, \cdots,$ and u running across the columns for $u = 0, 1, \cdots$.

2. Initialize the first row of the table for $t = 1$ with $\psi(1; u) = S_X(u)$. Note that if M is the maximum loss in each period, then $\psi(1; u) = 0$ for $u \geq M$.

3. Increase the value of t by 1 and calculate $\psi(t; u)$ for $u = 0, 1, \cdots$, using equation (5.22). Note that the computation requires the corresponding entry in the first row of the table, i.e., $\psi(1; u)$, as well as some entries in the $(t-1)$th row. In particular, the $u + 1$ entries $\psi(t-1; 1), \cdots, \psi(t-1; u+1)$ in the $(t-1)$th row are required.

4. Re-do Step 3 until the desired time point.

Example 5.3: As in Example 5.1, the claim variable X has the following distribution: $f_X(0) = 0.5$, $f_X(1) = f_X(2) = 0.2$ and $f_X(3) = 0.1$. Calculate the probability of ruin at or before a finite time t given initial surplus u, $\psi(t; u)$, for $u \geq 0$.

Solution: The results are summarized in Table 5.1 for $t = 1, 2$ and 3, and $u = 0, 1, \cdots, 6$.

Table 5.1: Results of Example 5.3

	Initial surplus u						
Time t	0	1	2	3	4	5	6
1	0.500	0.300	0.100	0.000	0.000	0.000	0.000
2	0.650	0.410	0.180	0.050	0.010	0.000	0.000
3	0.705	0.472	0.243	0.092	0.030	0.007	0.001

The first row of the table is $S_X(u)$. Note that $\psi(1; u) = 0$ for $u \geq 3$, as the maximum loss in each period is 3. For the second row, the details of the computation is as follows. First, $\psi(2; 0)$ is computed as

$$\psi(2; 0) = \psi(1; 0) + f_X(0)\psi(1; 1) = 0.5 + (0.5)(0.3) = 0.65.$$

Similarly,

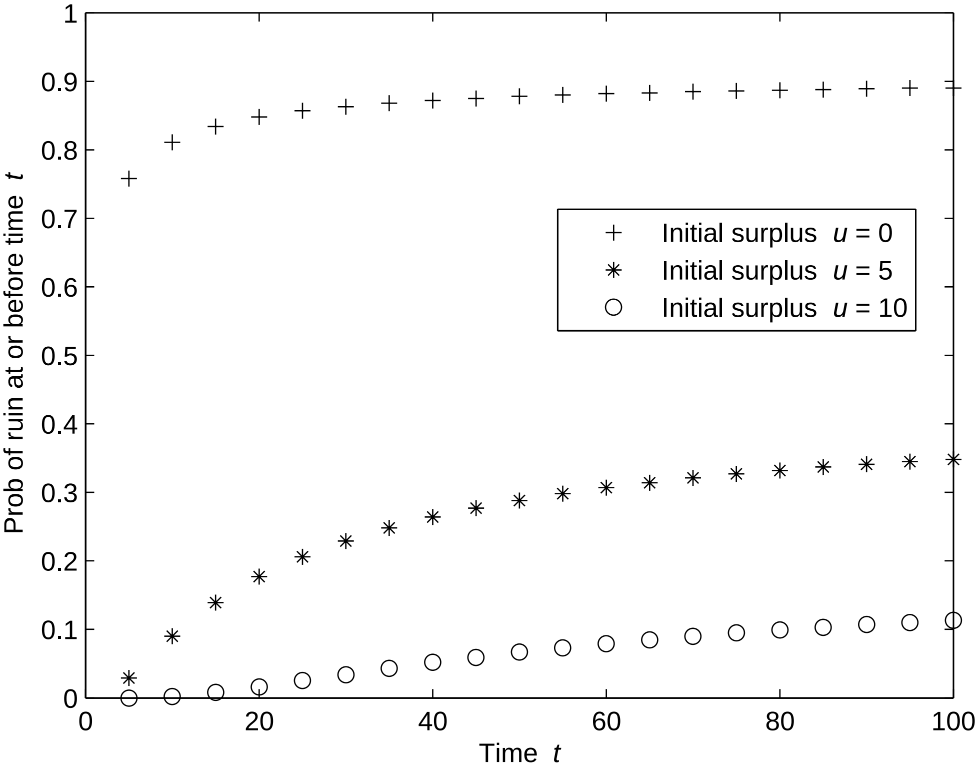
$$\psi(2; 1) = \psi(1; 1) + f_X(0)\psi(1; 2) + f_X(1)\psi(1; 1) = 0.3 + (0.5)(0.1) + (0.2)(0.3) = 0.41,$$

and

$$\psi(2;2) = \psi(1;2) + f_X(0)\psi(1;3) + f_X(1)\psi(1;2) + f_X(2)\psi(1;1) = 0.18.$$

We use $\psi(3;3)$ to illustrate the computation of the third row as follows

$$\begin{aligned}\psi(3;3) &= \psi(1;3) + f_X(0)\psi(2;4) + f_X(1)\psi(2;3) + f_X(2)\psi(2;2) + f_X(3)\psi(2;1) \\ &= 0 + (0.5)(0.01) + (0.2)(0.05) + (0.2)(0.18) + (0.1)(0.41) \\ &= 0.092.\end{aligned}$$


Prob of ruin at or before time t
Time t
0
20
40
60
80
100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Initial surplus u = 0
Initial surplus u = 5
Initial surplus u = 10

5.2.3 Lundberg's inequality in Discrete Time

Definition 5.5: Suppose X is the loss random variable. The adjustment coefficient, denoted by r^*, is the value of r that satisfies the following equation

$$\mathrm{E}\left[\exp\left\{r(X-1)\right\}\right] = 1. \tag{5.23}$$

Example 5.4: Assume the loss random variable X follows the distribution given in Examples 5.1 and 5.3. Calculate the adjustment coefficient r^*.

Solution: Equation (5.23) is set up as follows

$$0.5e^{-r} + 0.2 + 0.2e^{r} + 0.1e^{2r} = 1,$$

which is equivalent to

$$0.1w^3 + 0.2w^2 - 0.8w + 0.5 = 0,$$

for $w = e^r$. We solve the above equation numerically to obtain $w = 1.1901$, so that $r^* = \log(1.1901) = 0.1740$.

Theorem 5.2 (Lundberg's Theorem): For the discrete-time surplus function, the probability of ultimate ruin satisfies the following inequality

$$\psi(u) \leq \exp(-r^* u), \tag{5.28}$$

where r^* is the adjustment coefficient.

Proof: By induction, see NAM.

Example 5.5: Assume the loss random variable X follows the distribution given in Examples 5.1 and 5.4. Calculate the Lundberg upper bound for the probability of ultimate ruin for $u = 0, 1, 2$ and 3.

Solution: From Example 5.4, the adjustment coefficient is $r^* = 0.1740$. The Lundberg upper bound for $u = 0$ is 1, and for $u = 1, 2$ and 3, we have $e^{-0.174} = 0.8403$, $e^{-(2)(0.174)} = 0.7061$ and $e^{-(3)(0.174)} = 0.5933$, respectively. These figures may be compared against the exact values computed in Example 5.1, namely, 0.8, 0.68 and 0.568, respectively.

5.3 Continuous-Time Surplus Function

- In a continuous-time model the insurance company receives premiums continuously, while claim losses may occur at any time.

- We assume that the initial surplus of the insurance company is u and the amount of premium received per unit time is c.

- We denote the number of claims (described as the number of occurrences of events) in the interval $(0, t]$ by $N(t)$, with claim amounts $X_1, \cdots, X_{N(t)}$, which are assumed to be independently and identically distributed as X.

- We denote the aggregate losses up to (and including) time t by $S(t)$, which is given by

$$S(t) = \sum_{i=1}^{N(t)} X_i, \tag{5.39}$$

with the convention that if $N(t) = 0$, $S(t) = 0$.

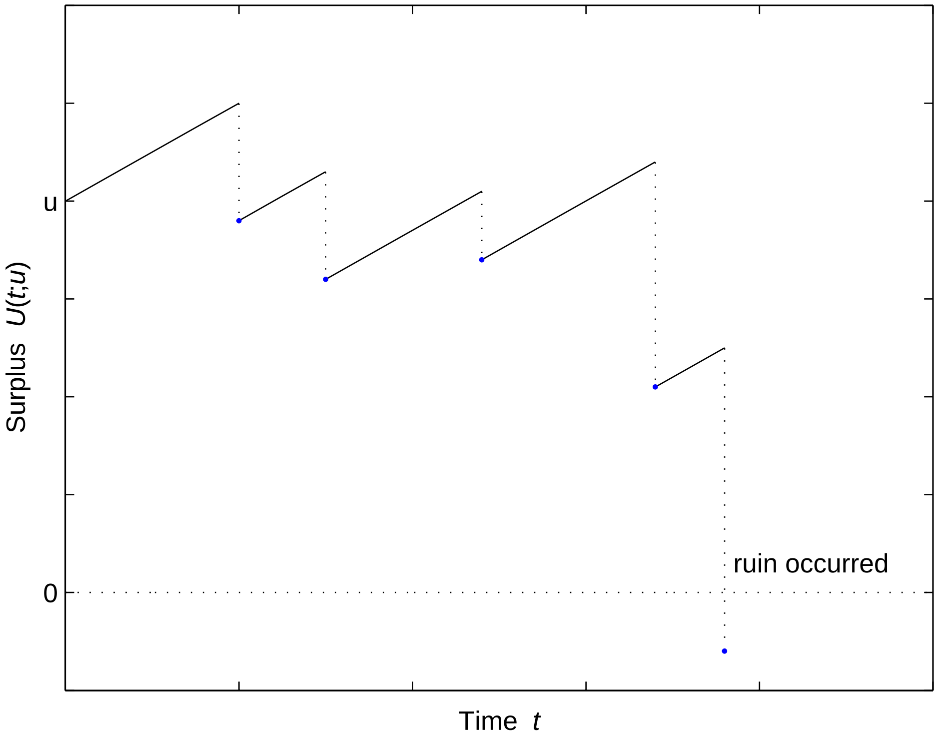
- Thus, the surplus at time t, denoted by $U(t; u)$, is defined as

$$U(t; u) = u + ct - S(t). \tag{5.40}$$

- Figure 5.4 illustrates an example of a realization of the surplus function $U(t; u)$.

- To analyze the behavior of $U(t; u)$ we make some assumptions about the claim process $S(t)$.

- In particular, we assume that the number of occurrences of (claim) events up to (and including) time t, $N(t)$, follows a **Poisson process**.


u
0
Surplus U(t;u)
ruin occurred
Time t

Definition 5.6: $N(t)$ is a Poisson process with parameter λ, which is the rate of occurrences of events per unit time, if (a) in any interval $(t_1, t_2]$, the number of occurrences of events, i.e., $N(t_2) - N(t_1)$, has a Poisson distribution with mean $\lambda(t_2 - t_1)$, and (b) over any non-overlapping intervals, the numbers of occurrences of events are independently distributed.

- For a *fixed* t, $N(t)$ is distributed as a Poisson variable with parameter λt, i.e., $N(t) \sim \mathcal{PN}(\lambda t)$, and $S(t)$ follows a compound Poisson distribution.

- As a function of time t, $S(t)$ is a **compound Poisson process** and the corresponding surplus process $U(t; u)$ is a **compound Poisson surplus process**. We assume that the claim random variable X has a mgf $M_X(r)$ for $r \in [0, \gamma)$.

5.4 Continuous-Time Ruin Theory

5.4.1 Lundberg's Inequality in Continuous Time

- We first define the adjustment coefficient in continuous time. Analogous to the discrete-time case, in which the adjustment coefficient is the solution of

$$1 + (1 + \theta)\, r\mu_X = M_X(r). \tag{5.47}$$

Theorem 5.3: If the surplus function follows a compound Poisson process defined in equation (5.40), the probability of ultimate ruin given initial surplus u, $\psi(u)$, satisfies the inequality

$$\psi(u) \leq \exp(-r^* u), \tag{5.48}$$

where r^* is the adjustment coefficient satisfying equation (5.47).

Example 5.6: Let $U(t;u)$ be a compound Poisson surplus function with $X \sim \mathcal{G}(3, 0.5)$. Compute the adjustment coefficient and its approximate value using equation (5.52), for $\theta = 0.1$ and 0.2. Calculate the upper bounds for the probability of ultimate ruin for $u = 5$ and $u = 10$.

Solution: The mgf of X is, from equation (2.32),

$$M_X(r) = \frac{1}{(1-\beta r)^\alpha} = \frac{1}{(1-0.5r)^3},$$

and its mean and variance are, respectively, $\mu_X = \alpha\beta = 1.5$ and $\sigma_X^2 = \alpha\beta^2 = 0.75$. From equation (5.47), the adjustment coefficient is the solution of r in the equation

$$\frac{1}{(1-0.5r)^3} = 1 + (1+\theta)(1.5)r,$$

from which we solve numerically to obtain $r^* = 0.0924$ when $\theta = 0.1$. The upper bounds for the probability of ultimate ruin are

$$\exp(-r^* u) = \begin{cases} 0.6300, & \text{for } u = 5, \\ 0.3969, & \text{for } u = 10. \end{cases}$$

When the loading is increased to 0.2, $r^* = 0.1718$, so that the upper bounds for the probability of ruin are

$$\exp(-r^* u) = \begin{cases} 0.4236, & \text{for } u = 5, \\ 0.1794, & \text{for } u = 10. \end{cases}$$

To compute the approximate values of r^*, we use equation (5.52) to obtain, for $\theta = 0.1$,

$$r^* \simeq \frac{(2)(0.1)(1.5)}{0.75 + (1.1)^2(1.5)^2} = 0.0864,$$

and, for $\theta = 0.2$,

$$r^* \simeq \frac{(2)(0.2)(1.5)}{0.75 + (1.2)^2(1.5)^2} = 0.1504.$$

Based on these approximate values, the upper bounds for the probability of ultimate ruin are, for $\theta = 0.1$,

$$\exp(-r^* u) = \begin{cases} 0.6492, & \text{for } u = 5, \\ 0.4215, & \text{for } u = 10. \end{cases}$$

and, for $\theta = 0.2$,

$$\exp(-r^* u) = \begin{cases} 0.4714, & \text{for } u = 5, \\ 0.2222, & \text{for } u = 10. \end{cases}$$

Thus, we can see that the adjustment coefficient increases with the premium loading θ. Also, the upper bound for the probability of ultimate ruin decreases with θ and u. We also observe that the approximation of r^* works reasonably well.