Nonlife Actuarial Models

Chapter 4

Risk Measures

Learning Objectives

1. Risk measures based on premium principles
2. Risk measures based on capital requirements
3. Value-at-Risk and conditional tail expectation
4. Distortion functions
5. Proportional hazard transform

4.1 Uses of Risk Measures

Types of risks

1. Market risk
2. Credit risk
3. Operational risk

Uses of risk measures

1. Determination of economic capital
2. Determination of insurance premium
3. Internal risk management
4. External regulatory reporting

- **Definition 4.1:** A risk measure of the random loss X, denoted by $\varrho(X)$, is a real-valued function $\varrho : X \rightarrow R$, where R is the set of real numbers.

- As a loss random variable, X is nonnegative. Thus, the risk measure $\varrho(X)$ may be imposed to be nonnegative for the purpose of measuring insurance risks.

- However, if the purpose is to measure the risks of a portfolio of assets, X may stand for the *change* in portfolio value, which may be positive or negative. In such cases, the risk measure $\varrho(X)$ may be positive or negative.

4.2 Some Premium-Based Risk Measures

- Let X be a random loss. Denote $\mathrm{E}(X) = \mu_X$ and $\mathrm{Var}(X) = \sigma_X^2$. Denote $\varrho(X)$ as a risk measure of the loss X.

- Expected-value principle premium risk measure: premium with a loading on the expected loss, i.e., $\varrho(X) = (1+\theta)\mu_X$, where $\theta \geq 0$ is the premium loading factor.

- Pure premium risk measure: no loading, i.e., $\theta = 0$, so that $\varrho(X) = \mu_X$.

- Variance premium risk measure: loading on variance, i.e., $\varrho(X) = \mu_X + \alpha\sigma_X^2, \alpha \geq 0$.

- Standard-deviation risk measure: loading on standard deviation, i.e., $\varrho(X) = \mu_X + \alpha\sigma_X, \alpha \geq 0$.

4.3 Coherent Risk Measures

Four axioms for coherent risk measures

Axiom 4.1 Translational invariance (T): For any loss variable X and any nonnegative constant $a \geq 0$, $\varrho(X + a) = \varrho(X) + a$.

Axiom 4.2 Subadditivity (S): For any loss variables X and Y, $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$.

Axiom 4.3 Positive homogeneity (PH): For any loss variable X and any nonnegative constant a, $\varrho(aX) = a\varrho(X)$.

Axiom 4.4 Monotonicity (M): For any loss variables X and Y such that $X \leq Y$ under all states of nature, $\varrho(X) \leq \varrho(Y)$.

Example 4.2: Show that the expected-value premium risk measure satisfies Axioms S, PH and M, but not T.

Solution: For any risks X and Y, we have

$$\begin{aligned}
\varrho(X+Y) &= (1+\theta)\mathrm{E}(X+Y) \\
&= (1+\theta)\mathrm{E}(X) + (1+\theta)\mathrm{E}(Y) \\
&= \varrho(X) + \varrho(Y).
\end{aligned}$$

Thus, Axiom S holds. Now for $Y = aX$ with $a \geq 0$, we have

$$\varrho(Y) = (1+\theta)\mathrm{E}(Y) = (1+\theta)\mathrm{E}(aX) = a(1+\theta)\mathrm{E}(X) = a\varrho(X),$$

which proves Axiom PH. For two risks X and Y, $X \geq Y$ implies $\mu_X \geq \mu_Y$. Thus,

$$\varrho(X) = (1+\theta)\mu_X \geq (1+\theta)\mu_Y = \varrho(Y),$$

and Axiom M holds. To examine Axiom T, we consider an arbitrary constant $a > 0$. Note that, if $\theta > 0$,

$$\varrho(X + a) = (1 + \theta)\mathrm{E}(X + a) > (1 + \theta)\mathrm{E}(X) + a = \varrho(X) + a.$$

Thus, Axiom T is not satisfied if $\theta > 0$, which implies the expected-value premium is in general not a coherent risk measure. However, when $\theta = 0$, Axiom T holds. Thus, the pure premium risk measure is coherent. $\square$

- It can be shown that the variance premium risk measure satisfies Axiom T, but not Axioms S, M and PH.

- On the other hand, the standard-deviation premium risk measure satisfies Axioms S, T and PH, but not Axiom M. Readers are invited to prove these results (see Exercises 4.2 and 4.3).

- The axioms of coherent risk narrow down the set of risk measures to be considered for management and regulation. However, they do not specify a unique risk measure to be used in practice.

- Some risk measures (such as the pure premium risk measure) that are coherent may not be a suitable risk measure for some reasons. Thus, the choice of which measure to use depends on additional considerations.

4.4 Some Capital-Based Risk Measures

4.4.1 Value-at-Risk

- One of the most widely used measures of risk

- $\mathrm{VaR}_\delta(X)$ is the δ-quantile of X, i.e.,

$$\begin{aligned} \mathrm{VaR}_\delta(X) &= F_X^{-1}(\delta) \\ &= \inf\{x \in [0, \infty) \,:\, F_X(x) \geq \delta\}. \end{aligned} \tag{4.5}$$

- **Example 4.3:** Find VaR_δ of the following loss distributions X: (a) $\mathcal{E}(\lambda)$, (b) $\mathcal{L}(\mu, \sigma^2)$, and (c) $\mathcal{P}(\alpha, \gamma)$.

- **Solution:** For (a), from Example 2.8, we have

$$\text{VaR}_\delta = -\frac{\log(1-\delta)}{\lambda}.$$

For (b), from Example 2.8, the VaR is

$$\text{VaR}_\delta = \exp\left[\mu + \sigma\Phi^{-1}(\delta)\right].$$

For (c), from equation (2.38), the df of $\mathcal{P}(\alpha, \gamma)$ is

$$F_X(x) = 1 - \left(\frac{\gamma}{x+\gamma}\right)^\alpha,$$

so that its quantile function is

$$F_X^{-1}(\delta) = \gamma(1-\delta)^{-\frac{1}{\alpha}} - \gamma,$$

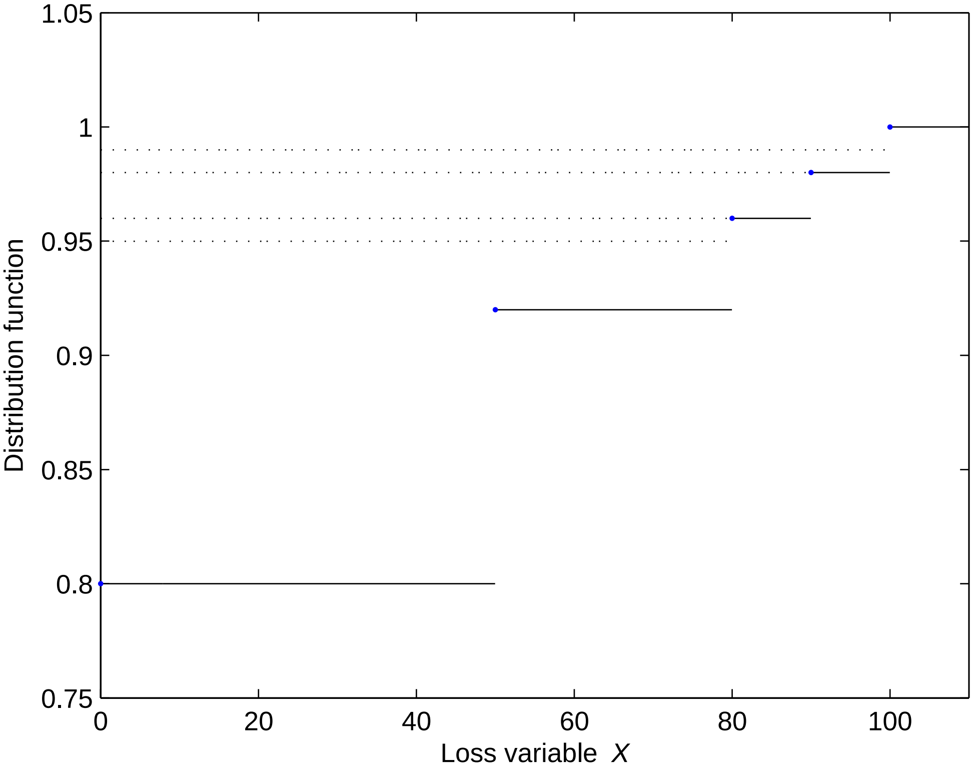
and

$$\text{VaR}_\delta = F_X^{-1}(\delta) = \gamma\left[(1-\delta)^{-\frac{1}{\alpha}} - 1\right].$$

- **Example 4.4:** Find VaR_δ, for $\delta = 0.95, 0.96, 0.98$ and 0.99, of the following discrete loss distribution

$$X = \begin{cases} 100, & \text{with prob } 0.02, \\ 90, & \text{with prob } 0.02, \\ 80, & \text{with prob } 0.04, \\ 50, & \text{with prob } 0.12, \\ 0, & \text{with prob } 0.80. \end{cases}$$

- **Solution:** As X is discrete, we use the definition of VaR in equation (4.5). The df of X is plotted in Figure 4.1. The dotted horizontal lines correspond to the probability levels 0.95, 0.96, 0.98 and 0.99. Note that the df of X is a step function. For VaR_δ we require the value of X corresponding to the probability level equal to or next-step higher than δ. Thus, VaR_δ for $\delta = 0.95, 0.96, 0.98$ and 0.99, are, respectively, 80, 80, 90 and 100.


1.05
1
0.95
0.9
0.85
0.8
0.75
Distribution function
0
20
40
60
80
100
Loss variable X

4.4.2 Conditional Tail Expectation

- VaR does not use any information about the loss distribution beyond the cut-off point.

- Conditional tail expectation, denoted by $\mathrm{CTE}_\delta(X)$, rectifies this.

- Defintion:

$$\mathrm{CTE}_\delta(X) = \mathrm{E}\left[X \mid X > \mathrm{VaR}_\delta(X)\right] \tag{4.10}$$

- When X is continuous, we have

$$\mathrm{CTE}_\delta = \mathrm{E}(X \mid X > \mathrm{VaR}_\delta) = \frac{1}{1-\delta}\int_{\mathrm{VaR}_\delta}^{\infty} x f_X(x)\,dx. \tag{4.17}$$

- Using change of variable $\xi = F_X(x)$, the integral above can be written as

$$\int_{\mathrm{VaR}_\delta}^{\infty} x f_X(x)\,dx \;=\; \int_{\mathrm{VaR}_\delta}^{\infty} x\,dF_X(x)$$

$$= \int_{\delta}^{1} \mathrm{VaR}_{\xi}\, d\xi, \tag{4.18}$$

which implies

$$\mathrm{CTE}_{\delta} = \frac{1}{1-\delta} \int_{\delta}^{1} \mathrm{VaR}_{\xi}\, d\xi. \tag{4.19}$$

- Thus, CTE_{δ} can be interpreted as the *average* of the VaRs exceeding VaR_{δ}. We call the expression on the RHS of (4.19) the **Tail VaR**, denoted as TVaR_{δ}.

- When X is not continuous, we use the following formula

$$\mathrm{CTE}_{\delta} = \frac{(\bar{\delta} - \delta)\mathrm{VaR}_{\delta} + (1 - \bar{\delta})\mathrm{E}(X \,|\, X > \mathrm{VaR}_{\delta})}{1-\delta}, \tag{4.22}$$

where

$$\bar{\delta} = \Pr(X \leq \mathrm{VaR}_{\delta}). \tag{4.21}$$

Remarks

- VaR is not a coherent risk measure. It violates the subadditivity axiom.

- CTE is a coherent risk measure.

Example 4.6: Calculate CTE_δ for the loss distribution given in Example 4.4, for $\delta = 0.95$, 0.96, 0.98 and 0.99. Also, calculate TVaR corresponding to these values of δ.

Solution: As X is not continuous we use equation (4.22) to calculate CTE_δ. Note that $\mathrm{VaR}_{0.95} = \mathrm{VaR}_{0.96} = 80$. For $\delta = 0.95$, we have $\bar{\delta} = 0.96$. Thus,

$$\mathrm{E}(X \mid X > \mathrm{VaR}_{0.95} = 80) = \frac{90(0.02) + 100(0.02)}{1 - 0.96} = 95,$$

so that from equation (4.22) we obtain

$$\text{CTE}_{0.95} = \frac{(0.96 - 0.95)80 + (1 - 0.96)95}{1 - 0.95} = 92.$$

For $\delta = 0.96$, we have $\bar{\delta} = 0.96$, so that

$$\text{CTE}_{0.96} = \text{E}(X \mid X > \text{VaR}_{0.96} = 80) = 95.$$

For TVaR_δ, we use equation (4.23) to obtain

$$\begin{aligned}
\text{TVaR}_{0.95} &= \frac{1}{1 - 0.95} \int_{0.95}^{1} \text{VaR}_\xi \, d\xi \\
&= \frac{1}{0.05} \left[(80)(0.01) + (90)(0.02) + (100)(0.02)\right] \\
&= 92,
\end{aligned}$$

and

$$\text{TVaR}_{0.96} = \frac{1}{1 - 0.96} \int_{0.96}^{1} \text{VaR}_\xi \, d\xi = \frac{1}{0.04} \left[(90)(0.02) + (100)(0.02)\right] = 95.$$

For $\delta = 0.98$, we have $\bar{\delta} = 0.98$, so that

$$\mathrm{CTE}_{0.98} = \mathrm{E}(X \mid X > \mathrm{VaR}_{0.98} = 90) = \frac{(100)(0.02)}{1 - 0.98} = 100,$$

which is also the value of $\mathrm{TVaR}_{0.98}$. Finally, for $\delta = 0.99$, we have $\bar{\delta} = 1$ and $\mathrm{VaR}_{0.99} = 100$, so that $\mathrm{CTE}_{0.99} = \mathrm{VaR}_{0.99} = 100$. On the other hand, we have

$$\mathrm{TVaR}_{0.99} = \frac{1}{1 - 0.99} \int_{0.99}^{1} \mathrm{VaR}_{\xi} \, d\xi = \frac{(100)(0.01)}{0.01} = 100.$$

4.5 More Premium-Based Risk Measure

4.5.1 Proportional hazard transform and risk-adjusted premium

- The premium-based risk measures define risk based on a loading of the expected loss. The expected loss μ_X of a continuous random loss X can be written as

$$\mu_X = \int_0^\infty S_X(x)\, dx. \tag{4.27}$$

- Thus, instead of adding a *loading* to μ_X to obtain a premium we may *re-define* the distribution of the loss.

- Suppose $\tilde{X}$ is distributed with sf $S_{\tilde{X}}(x) = [S_X(x)]^{\frac{1}{\rho}}$, where $\rho \geq 1$, then the mean of $\tilde{X}$ is

$$\mathrm{E}(\tilde{X}) = \mu_{\tilde{X}} = \int_0^\infty S_{\tilde{X}}(x)\, dx = \int_0^\infty [S_X(x)]^{\frac{1}{\rho}}\, dx. \tag{4.28}$$

- The parameter ρ is called the **risk-aversion index**.
- The distribution of $\tilde{X}$ is called the **proportional hazard (PH) transform** of the distribution of X with parameter ρ.
- If we denote $h_X(x)$ and $h_{\tilde{X}}(x)$ as the hf of X and $\tilde{X}$, respectively, then from equations (2.2) and (2.3), we have

$$\begin{aligned} h_{\tilde{X}}(x) &= -\frac{1}{S_{\tilde{X}}(x)}\left(\frac{dS_{\tilde{X}}(x)}{dx}\right) \\ &= -\frac{1}{\rho}\left(\frac{[S_X(x)]^{\frac{1}{\rho}-1}S'_X(x)}{[S_X(x)]^{\frac{1}{\rho}}}\right) \\ &= -\frac{1}{\rho}\left(\frac{S'_X(x)}{S_X(x)}\right) \\ &= \frac{1}{\rho}h_X(x), \end{aligned} \tag{4.30}$$

so that the hf of $\tilde{X}$ is *proportional to* the hf of X.

- As $\rho \geq 1$, the hf of $\tilde{X}$ is less than that of X, implying that $\tilde{X}$ has a thicker tail than that of X. Also, $S_{\tilde{X}}(x) = [S_X(x)]^{\frac{1}{\rho}}$ declines slower than $S_X(x)$ so that $\mu_{\tilde{X}} > \mu_X$, the difference of which represents the loading.

Example 4.7: If $X \sim \mathcal{E}(\lambda)$, find the PH transform of X with parameter ρ and the risk-adjusted premium.

Solution: The sf of X is $S_X(x) = e^{-\lambda x}$. Thus, the sf of the PH transform is $S_{\tilde{X}}(x) = \left(e^{-\lambda x}\right)^{\frac{1}{\rho}} = e^{-\frac{\lambda}{\rho}x}$, which implies $\tilde{X} \sim \mathcal{E}(\lambda/\rho)$. Hence, the risk-adjusted premium is $\mathrm{E}(\tilde{X}) = \rho/\lambda \geq 1/\lambda = \mathrm{E}(X)$.

Example 4.8: If $X \sim \mathcal{P}(\alpha, \gamma)$ with $\alpha > 1$, find the PH transform of X with parameter $\rho \in [1, \alpha)$ and the risk-adjusted premium.

Solution: The sf of X is

$$S_X(x) = \left(\frac{\gamma}{\gamma + x}\right)^{\alpha},$$

with mean

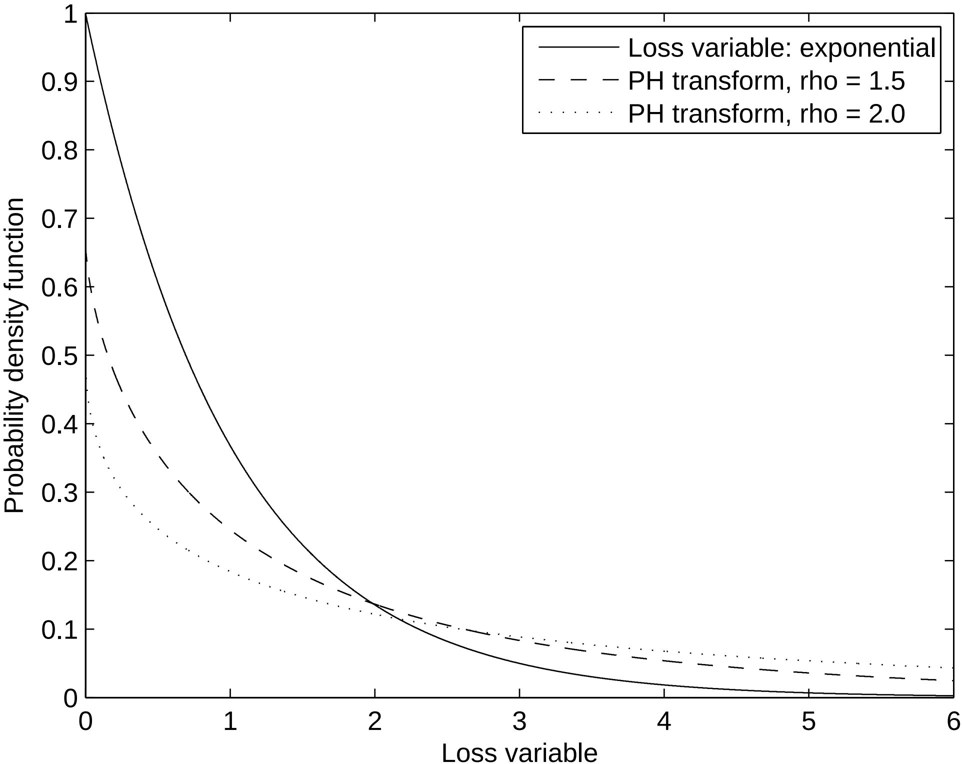
$$\mu_X = \frac{\gamma}{\alpha - 1}.$$

The sf of $\tilde{X}$ is

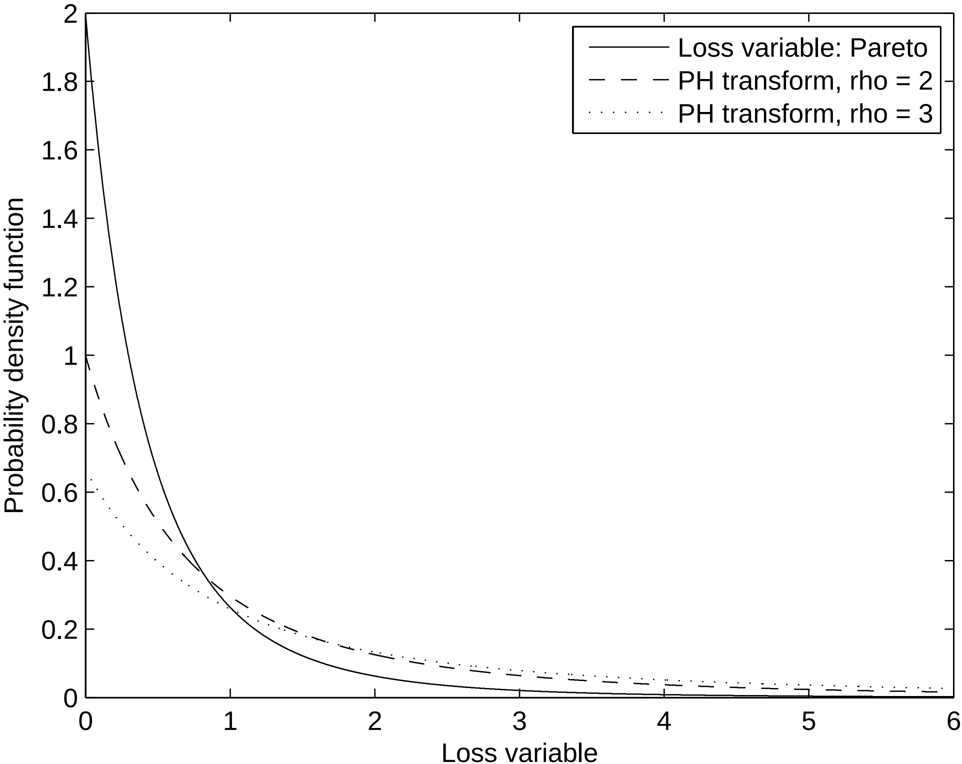
$$S_{\tilde{X}}(x) = [S_X(x)]^{\frac{1}{\rho}} = \left(\frac{\gamma}{\gamma + x}\right)^{\frac{\alpha}{\rho}},$$

so that $\tilde{X} \sim \mathcal{P}(\alpha/\rho, \gamma)$. Hence, the mean of $\tilde{X}$ (the risk-adjusted premium) is

$$\mu_{\tilde{X}} = \frac{\gamma}{\dfrac{\alpha}{\rho} - 1} = \frac{\rho\gamma}{\alpha - \rho} > \frac{\gamma}{\alpha - 1} = \mu_X.$$



Loss variable: exponential
PH transform, rho = 1.5
PH transform, rho = 2.0
Probability density function
Loss variable
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5
6



Loss variable: Pareto
PH transform, rho = 2
PH transform, rho = 3
Probability density function
Loss variable
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
1
2
3
4
5
6

4.6 The Distortion-Function Approach

The **distortion function** is a mathematical device to construct risk measures.

Definition 4.2: A distortion function is a nondecreasing function $g(\cdot)$ satisfying $g(1) = 1$ and $g(0) = 0$.

- Suppose X is a loss random variable with sf $S_X(x)$. As the distortion function $g(\cdot)$ is nondecreasing and $S_X(x)$ is a nonincreasing function of x, $g(S_X(x))$ is a nonincreasing function of x.

- Together with the property that $g(S_X(0)) = g(1) = 1$ and $g(S_X(\infty)) = g(0) = 0$, $g(S_X(x))$ is a well defined sf over the support $[0, \infty)$.

- We denote the random variable with this sf as $\tilde{X}$, which may be interpreted as a risk-adjusted loss random variable, and $g(S_X(x))$ is the risk adjusted sf.

- We further assume that $g(\cdot)$ is concave down (i.e., $g''(x) \leq 0$ if the derivative exists).

Definition 4.3: Let X be a nonnegative loss random variable. The distortion risk measure based on the distortion function $g(\cdot)$, denoted by $\varrho(X)$, is defined as

$$\varrho(X) = \int_0^\infty g(S_X(x))\, dx. \tag{4.42}$$

Thus, the distortion risk measure $\varrho(X)$ is the mean of the risk-adjusted loss $\tilde{X}$. The class of distortion risk measures include the following measures we have discussed.

Pure premium risk measure

It can be seen easily by defining

$$g(u) = u, \tag{4.43}$$

which satisfies the conditions $g(0) = 0$ and $g(1) = 1$, and $g(\cdot)$ is nondecreasing. Now

$$\varrho(X) = \int_0^\infty g(S_X(x))\, dx = \int_0^\infty S_X(x)\, dx = \mu_X, \tag{4.44}$$

which is the pure premium risk measure.

Proportional hazard risk-adjusted premium risk measure

This can be seen by defining

$$g(u) = u^{\frac{1}{\rho}}, \qquad \rho \geq 1. \tag{4.45}$$

VaR risk measure

For VaR_δ we define the distortion function as

$$g(S_X(x)) = \begin{cases} 0, & \text{if } 0 \leq S_X(x) < 1-\delta, \\ 1, & \text{if } 1-\delta \leq S_X(x) \leq 1, \end{cases} \tag{4.46}$$

which is equivalent to

$$g(S_X(x)) = \begin{cases} 0, & \text{if } x > \mathrm{VaR}_\delta, \\ 1, & \text{if } 0 \leq x \leq \mathrm{VaR}_\delta. \end{cases} \tag{4.47}$$

Hence,

$$\varrho(X) = \int_0^\infty g(S_X(x))\, dx = \int_0^{\mathrm{VaR}_\delta} dx = \mathrm{VaR}_\delta. \tag{4.48}$$

CTE risk measure

For CTE_δ we define the distortion function as (subject to the condition

X is continuous)

$$g(S_X(x)) = \begin{cases} \dfrac{S_X(x)}{1-\delta}, & \text{if } 0 \le S_X(x) < 1-\delta, \\ \\ 1, & \text{if } 1-\delta \le S_X(x) \le 1, \end{cases} \tag{4.49}$$

which is equivalent to

$$g(S_X(x)) = \begin{cases} \dfrac{S_X(x)}{1-\delta}, & \text{if } x > x_\delta, \\ \\ 1, & \text{if } 0 \le x \le x_\delta. \end{cases} \tag{4.50}$$

Theorem 4.1: Let $g(\cdot)$ be a concave-down distortion function. The risk measure of the loss X defined in equation (4.42) is translational invariant, monotonic, positively homogeneous and subadditive, and is thus coherent.