Nonlife Actuarial Models

Chapter 2

Claim-Severity Distribution

Learning Objectives

- Continuous and mixed distributions
- Exponential, gamma, Weibull and Pareto distributions
- Mixture distributions
- Tail weights, limiting ratios and conditional tail expectation
- Coverage modification and claim-severity distribution

2.1 Review of Statistics

2.1.1 Survival function and hazard function

- **Survival function:** The survival function of a random variable X, also called **decumulative function**, denoted by $S_X(x)$, is the complement of the df, i.e.,

$$S_X(x) = 1 - F_X(x) = \Pr(X > x). \tag{2.1}$$

- The following properties hold:

$$f_X(x) = \frac{dF_X(x)}{dx} = -\frac{dS_X(x)}{dx}. \tag{2.2}$$

- The sf $S_X(x)$ is monotonic nonincreasing.

- Also, we have $F_X(-\infty) = S_X(\infty) = 0$ and $F_X(\infty) = S_X(-\infty) = 1$.
- If X is nonnegative, then $F_X(0) = 0$ and $S_X(0) = 1$.
- The **hazard function** of a nonnegative random variable X, denoted by $h_X(x)$, is defined as

$$h_X(x) = \frac{f_X(x)}{S_X(x)}. \tag{2.3}$$

- We have

$$\begin{aligned} h_X(x)\,dx &= \frac{f_X(x)\,dx}{S_X(x)} \\ &= \frac{\Pr(x \le X < x + dx)}{\Pr(X > x)} \\ &= \frac{\Pr(x \le X < x + dx \text{ and } X > x)}{\Pr(X > x)} \\ &= \Pr(x < X < x + dx \,|\, X > x), \end{aligned} \tag{2.4}$$

- Thus, $h_X(x)\,dx$ can be interpreted as the conditional probability of X taking value in the infinitesimal interval $(x, x+dx)$ given $X > x$.

- To derive the sf given the hf, we note that

$$h_X(x) = -\frac{1}{S_X(x)}\left(\frac{dS_X(x)}{dx}\right) = -\frac{d\log S_X(x)}{dx}, \tag{2.5}$$

so that

$$h_X(x)\,dx = -d\log S_X(x). \tag{2.6}$$

Integrating both sides of the equation, we obtain

$$\int_0^x h_X(s)\,ds = -\int_0^x d\log S_X(s) = -\log S_X(s)\big]_0^x = -\log S_X(x), \tag{2.7}$$

as $\log S_X(0) = \log(1) = 0$. Thus, we have

$$S_X(x) = \exp\left(-\int_0^x h_X(s)\,ds\right), \tag{2.8}$$

- **Example 2.1:** Let X be a uniformly distributed random variable in the interval $[0,\ 100]$, denoted by $\mathcal{U}(0, 100)$. Compute the pdf, df, sf and hf of X.

- **Solution:** The pdf, df and sf of X are, for $x \in [0,\ 100]$,

$$f_X(x) = 0.01,$$

$$F_X(x) = 0.01x,$$

and

$$S_X(x) = 1 - 0.01x.$$

From equation (2.3) we obtain the hf as

$$h_X(x) = \frac{f_X(x)}{S_X(x)} = \frac{0.01}{1 - 0.01x},$$

which increases with x. In particular, $h_X(x) \to \infty$ as $x \to 100$.

2.1.2 Mixed distribution

- A random variable X is said to be of the **mixed type** if its df $F_X(x)$ is continuous and differentiable except for some values x belonging to a countable set Ω_X.

- Thus, if X has a mixed distribution, there exists a function $f_X(x)$ such that

$$F_X(x) = \Pr(X \leq x) = \int_{-\infty}^{x} f_X(x)\, dx + \sum_{x_i \in \Omega_X,\, x_i \leq x} \Pr(X = x_i). \tag{2.9}$$

- Using **Stieltjes integral** we write, for any constants a and b,

$$\Pr(a \leq X \leq b) = \int_a^b dF_X(x), \tag{2.10}$$

which is equal to

$$\int_a^b f_X(x)\, dx, \qquad \text{if } X \text{ is continuous,} \tag{2.11}$$

$$\sum_{x_i \in \Omega_X,\, a \leq x_i \leq b} \Pr(X = x_i), \qquad \text{if } X \text{ is discrete with support } \Omega_X, \tag{2.12}$$

and

$$\int_a^b f_X(x)\, dx + \sum_{x_i \in \Omega_X,\, a \leq x_i \leq b} \Pr(X = x_i), \qquad \text{if } X \text{ is mixed.} \tag{2.13}$$

- **Expectation of a function of** X: The expected value of $g(X)$, denoted by $\mathrm{E}[g(X)]$, is defined as the Stieltjes integral

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, dF_X(x), \tag{2.14}$$

- If X is continuous and nonnegative, and $g(\cdot)$ is a nonnegative, monotonic and differentiable function, the following result holds

$$\mathrm{E}[g(X)] = \int_0^\infty g(x)\, dF_X(x) = g(0) + \int_0^\infty g'(x)[1 - F_X(x)]\, dx, \quad (2.17)$$

where $g'(x)$ is the derivative of $g(x)$ with respect to x.

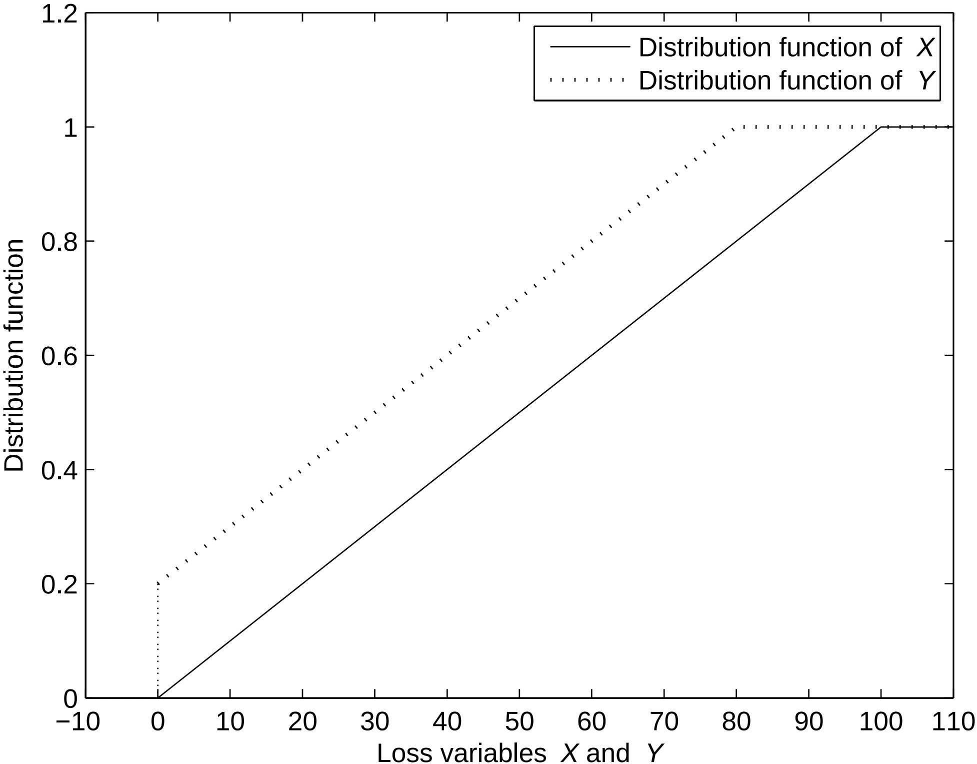
- Defining $g(x) = x$, so that $g(0) = 0$ and $g'(x) = 1$, the mean of X can be evaluated by

$$\mathrm{E}(X) = \int_0^\infty [1 - F_X(x)]\, dx = \int_0^\infty S_X(x)\, dx. \quad (2.18)$$

Example 2.2: Let $X \sim \mathcal{U}(0, 100)$. Define a random variable Y as follows

$$Y = \begin{cases} 0, & \text{for } X \leq 20, \\ X - 20, & \text{for } X > 20. \end{cases}$$

Determine the df of Y, and its density and mass function.



Distribution function of X
Distribution function of Y
Distribution function
Loss variables X and Y
1.2
1
0.8
0.6
0.4
0.2
0
−10
0
10
20
30
40
50
60
70
80
90
100
110

2.1.3 Distribution of functions of random variables

- Let $g(\cdot)$ be a continuous and differentiable function, and X be a continuous random variable with pdf $f_X(x)$. We define $Y = g(X)$.

- **Theorem 2.1:** Let X be a continuous random variable taking values in $[a, b]$ with pdf $f_X(x)$, and let $g(\cdot)$ be a continuous and differentiable one-to-one transformation. Denote $\alpha = g(a)$ and $\beta = g(b)$. The pdf of $Y = g(X)$ is

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \dfrac{dg^{-1}(y)}{dy} \right|, & \text{if } y \in [\alpha, \beta], \\ 0, & \text{otherwise.} \end{cases} \tag{2.19}$$

2.2 Some Continuous Distributions

2.2.1 Exponential Distribution

- A random variable X has an exponential distribution with parameter λ, denoted by $\mathcal{E}(\lambda)$, if its pdf is

$$f_X(x) = \lambda e^{-\lambda x}, \qquad \text{for } x \geq 0. \tag{2.21}$$

- The df and sf of X are

$$F_X(x) = 1 - e^{-\lambda x}, \tag{2.22}$$

and

$$S_X(x) = e^{-\lambda x}. \tag{2.23}$$

Thus, the hf of X is

$$h_X(x) = \frac{f_X(x)}{S_X(x)} = \lambda, \tag{2.24}$$

which is a constant, irrespective of the value of x. The mean and variance of X are

$$\mathrm{E}(X) = \frac{1}{\lambda} \qquad \text{and} \qquad \mathrm{Var}(X) = \frac{1}{\lambda^2}. \tag{2.25}$$

The mgf of X is

$$M_X(t) = \frac{\lambda}{\lambda - t}. \tag{2.26}$$

2.2.2 Gamma distribution

- X is said to have a gamma distribution with parameters α and β ($\alpha > 0$ and $\beta > 0$), denoted by $\mathcal{G}(\alpha, \beta)$, if its pdf is

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, \qquad \text{for } x \geq 0. \tag{2.27}$$

The function $\Gamma(\alpha)$ is called the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy, \tag{2.28}$$

which exists (i.e., the integral converges) for $\alpha > 0$.

- For $\alpha > 1$, $\Gamma(\alpha)$ satisfies the following recursion

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1). \tag{2.29}$$

In addition, if α is a positive integer, we have

$$\Gamma(\alpha) = (\alpha - 1)!. \tag{2.30}$$

- The mean and variance of X are

$$\mathrm{E}(X) = \alpha\beta \qquad \text{and} \qquad \mathrm{Var}(X) = \alpha\beta^2, \tag{2.31}$$

and its mgf is

$$M_X(t) = \frac{1}{(1-\beta t)^{\alpha}}, \qquad \text{for } t < \frac{1}{\beta}. \tag{2.32}$$

2.2.3 Weibull distribution

- A random variable X has a 2-parameter Weibull distribution with parameters α and λ, denoted by $\mathcal{W}(\alpha, \lambda)$, if its pdf is

$$f_X(x) = \left(\frac{\alpha}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{\alpha-1} \exp\left[-\left(\frac{x}{\lambda}\right)^{\alpha}\right], \qquad \text{for } x \geq 0, \tag{2.34}$$

  where α is the shape parameter and λ is the scale parameter.

- The mean and variance of X are

$$\mathrm{E}(X) = \mu = \lambda\,\Gamma\left(1 + \frac{1}{\alpha}\right) \qquad \text{and} \qquad \mathrm{Var}(X) = \lambda^2\,\Gamma\left(1 + \frac{2}{\alpha}\right) - \mu^2. \tag{2.35}$$

- The df of X is

$$F_X(x) = 1 - \exp\left[-\left(\frac{x}{\lambda}\right)^{\alpha}\right], \qquad \text{for } x \geq 0. \tag{2.36}$$

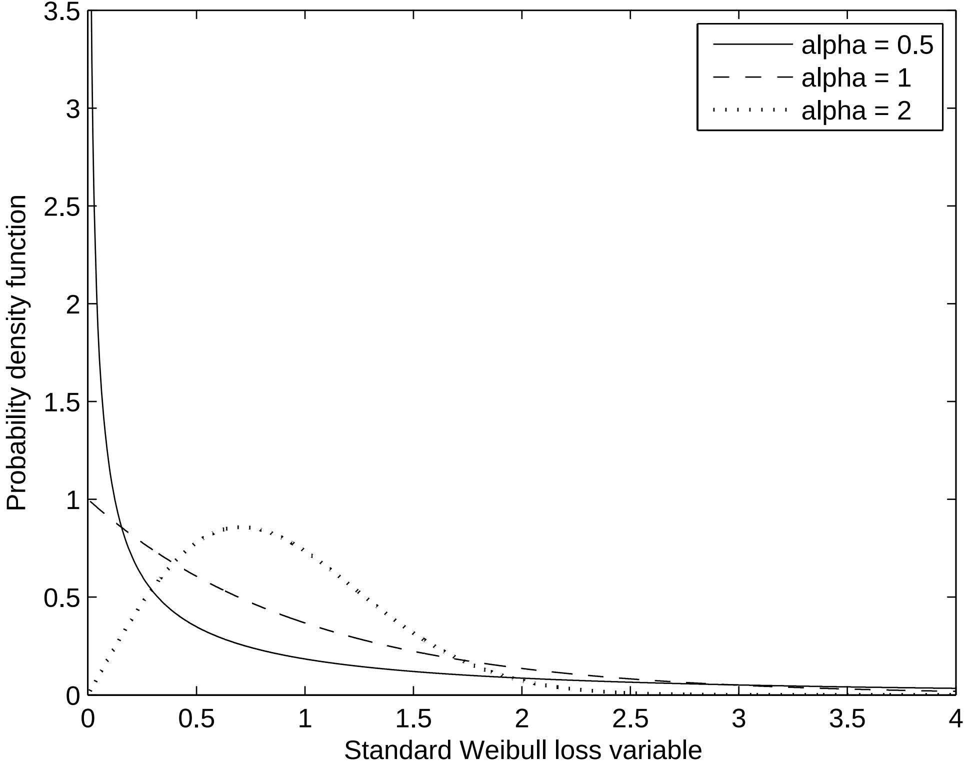
2.2.4 Pareto distribution

- A random variable X has a Pareto distribution with parameters $\alpha > 0$ and $\gamma > 0$, denoted by $\mathcal{P}(\alpha, \gamma)$, if its pdf is

$$f_X(x) = \frac{\alpha\gamma^{\alpha}}{(x+\gamma)^{\alpha+1}}, \qquad \text{for } x \geq 0. \tag{2.37}$$

- The df of X is

$$F_X(x) = 1 - \left(\frac{\gamma}{x+\gamma}\right)^{\alpha}, \qquad \text{for } x \geq 0. \tag{2.38}$$


alpha = 0.5
alpha = 1
alpha = 2
Probability density function
Standard Weibull loss variable
0
0.5
1
1.5
2
2.5
3
3.5
4

- The kth moment of X exists for $k < \alpha$. For $\alpha > 2$, the mean and variance of X are

$$\mathrm{E}(X) = \frac{\gamma}{\alpha - 1} \qquad \text{and} \qquad \mathrm{Var}(X) = \frac{\alpha\gamma^2}{(\alpha-1)^2(\alpha-2)}. \tag{2.40}$$

Table A.2: Some continuous distributions

Distribution, parameters, notation and support	pdf $f_X(x)$	mgf $M_X(t)$	Mean	Variance
Exponential $\mathcal{E}(\lambda)$ $x \in [0,\ \infty)$	$\lambda e^{-\lambda x}$	$\dfrac{\lambda}{\lambda - t}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Gamma $\mathcal{G}(\alpha, \beta)$ $x \in [0,\ \infty)$,	$\dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}$	$\dfrac{1}{(1-\beta t)^\alpha}$	$\alpha\beta$	$\alpha\beta^2$
Pareto $\mathcal{P}(\alpha, \gamma)$ $x \in [0,\ \infty)$,	$\dfrac{\alpha\gamma^\alpha}{(x+\gamma)^{\alpha+1}}$	Does not exist	$\dfrac{\gamma}{\alpha - 1}$	$\dfrac{\alpha\gamma^2}{(\alpha-1)^2(\alpha-2)}$
Weibull $\mathcal{W}(\alpha, \lambda)$ $x \in [0,\ \infty)$,	$\left(\dfrac{\alpha}{\lambda}\right)\left(\dfrac{x}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^\alpha}$	Not presented	$\mu = \lambda\,\Gamma\left(1 + \dfrac{1}{\alpha}\right)$	$\lambda^2\,\Gamma\left(1+\dfrac{2}{\alpha}\right) - \mu^2$

2.3 Creating New Distributions

2.3.1 Transformation of random variable:

- **Scaling:** Let $X \sim \mathcal{W}(\alpha, \lambda)$. Consider the **scaling** of X by the scale parameter λ and define

$$Y = \frac{X}{\lambda}. \tag{2.41}$$

  Then Y has a standard Weibull distribution.

- **Power transformation:** Assume $X \sim \mathcal{E}(\lambda)$ and define $Y = X^{1/\alpha}$ for an arbitrary constant $\alpha > 0$. Then $Y \sim \mathcal{W}(\alpha, \beta) \equiv \mathcal{W}(\alpha, 1/\lambda^{1/\alpha})$.

- **Exponential transformation:** Let X be normally distributed with mean μ and variance σ^2, denoted by $X \sim \mathcal{N}(\mu, \sigma^2)$. A new random variable may be created by taking the exponential of X. Thus, we define $Y = e^X$, so that $x = \log y$.

- The pdf of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]. \tag{2.48}$$

- The pdf of Y as

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma y} \exp\left[-\frac{(\log y-\mu)^2}{2\sigma^2}\right]. \tag{2.50}$$

- A random variable Y with pdf given by equation (2.50) is said to have a **lognormal distribution** with parameters μ and σ^2, denoted by $\mathcal{L}(\mu, \sigma^2)$.

- In other words, if $\log Y \sim \mathcal{N}(\mu, \sigma^2)$, then $Y \sim \mathcal{L}(\mu, \sigma^2)$. The mean and variance of $Y \sim \mathcal{L}(\mu, \sigma^2)$ are given by

$$\mathrm{E}(Y) = \exp\left(\mu + \frac{\sigma^2}{2}\right), \tag{2.51}$$

and

$$\mathrm{Var}(Y) = \left[\exp\left(2\mu + \sigma^2\right)\right]\left[\exp(\sigma^2) - 1\right]. \tag{2.52}$$

2.3.2 Mixture distribution

- Let X be a continuous random variable with pdf $f_X(x \mid \lambda)$, which depends on the parameter λ.

- We allow λ to be the realization of a random variable Λ with support Ω_Λ and pdf $f_\Lambda(\lambda \mid \theta)$, where θ is the parameter determining the distribution of Λ, sometimes called the **hyperparameter**.

- A new random variable Y may then be created by *mixing* the pdf $f_X(x \,|\, \lambda)$ to form the pdf

$$f_Y(y \,|\, \theta) = \int_{\lambda \in \Omega_\Lambda} f_X(x \,|\, \lambda) f_\Lambda(\lambda \,|\, \theta)\, d\lambda. \tag{2.54}$$

- **Example 2.4:** Assume $X \sim \mathcal{E}(\lambda)$, and let the parameter λ be distributed as $\mathcal{G}(\alpha, \beta)$. Determine the mixture distribution.

- **Solution:** We have

$$f_X(x \,|\, \lambda) = \lambda e^{-\lambda x},$$

and

$$f_\Lambda(\lambda \,|\, \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\, \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}.$$

Thus,

$$\int_0^\infty f_X(x \,|\, \lambda) f_\Lambda(\lambda \,|\, \alpha, \beta)\, d\lambda \;=\; \int_0^\infty \lambda e^{-\lambda x} \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\, \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}\right] d\lambda$$

$$
\begin{aligned}
&= \int_0^\infty \frac{\lambda^\alpha \exp\left[-\lambda\left(x+\frac{1}{\beta}\right)\right]}{\Gamma(\alpha)\beta^\alpha}\, d\lambda \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\beta^\alpha}\left[\frac{\beta}{\beta x+1}\right]^{\alpha+1}.
\end{aligned}
$$

If we let $\gamma = 1/\beta$, the above expression can be written as

$$
\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\beta^\alpha}\left[\frac{\beta}{\beta x+1}\right]^{\alpha+1} = \frac{\alpha\gamma^\alpha}{(x+\gamma)^{\alpha+1}},
$$

which is the pdf of $\mathcal{P}(\alpha, \gamma)$. Thus, the gamma–exponential mixture has a Pareto distribution. We also see that the distribution of the mixture distribution depends on α and β (or α and γ).

- Another important result is that the gamma-Poisson mixture has a negative-binomial distribution. See Q2.27 in NAM.

- The example below illustrates the computation of the mean and variance of a continuous mixture using rules for **conditional expectation.** For the mean, we use the following result

$$\mathrm{E}(X) = \mathrm{E}\left[\mathrm{E}\left(X \mid \Lambda\right)\right]. \tag{2.56}$$

For the variance, we use the result

$$\mathrm{Var}(X) = \mathrm{E}\left[\mathrm{Var}(X \mid \Lambda)\right] + \mathrm{Var}\left[\mathrm{E}(X \mid \Lambda)\right]. \tag{2.57}$$

- **Example 2.5:** Assume $X \mid \Lambda \sim \mathcal{E}(\Lambda)$, and let the parameter Λ be distributed as $\mathcal{G}(\alpha, \beta)$. Calculate the unconditional mean and variance of X using rules for conditional expectation.

- **Solution:** As the conditional distribution of X is $\mathcal{E}(\Lambda)$, from Table A.2 we have

$$\mathrm{E}(X \mid \Lambda) = \frac{1}{\Lambda}.$$

Thus, from equation (2.56) we have

$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{E}\left(\frac{1}{\Lambda}\right) \\
&= \int_0^\infty \frac{1}{\lambda}\left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\,\lambda^{\alpha-1}e^{-\frac{\lambda}{\beta}}\right]d\lambda \\
&= \frac{\Gamma(\alpha-1)\beta^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \\
&= \frac{1}{(\alpha-1)\beta}.
\end{aligned}$$

Also, from Table A.2 we have

$$\mathrm{Var}(X\,|\,\Lambda) = \frac{1}{\Lambda^2},$$

so that using equation (2.57) we have

$$\mathrm{Var}(X) = \mathrm{E}\left(\frac{1}{\Lambda^2}\right) + \mathrm{Var}\left(\frac{1}{\Lambda}\right) = 2\mathrm{E}\left(\frac{1}{\Lambda^2}\right) - \left[\mathrm{E}\left(\frac{1}{\Lambda}\right)\right]^2.$$

As

$$\begin{aligned}
\mathrm{E}\left(\frac{1}{\Lambda^2}\right) &= \int_0^\infty \frac{1}{\lambda^2}\left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\,\lambda^{\alpha-1}e^{-\frac{\lambda}{\beta}}\right]d\lambda \\
&= \frac{\Gamma(\alpha-2)\beta^{\alpha-2}}{\Gamma(\alpha)\beta^\alpha} \\
&= \frac{1}{(\alpha-1)(\alpha-2)\beta^2},
\end{aligned}$$

we conclude

$$\mathrm{Var}(X) = \frac{2}{(\alpha-1)(\alpha-2)\beta^2} - \left[\frac{1}{(\alpha-1)\beta}\right]^2 = \frac{\alpha}{(\alpha-1)^2(\alpha-2)\beta^2}.$$

- The above results can be obtained directly from the mean and variance of a Pareto distribution.

2.3.3 Splicing

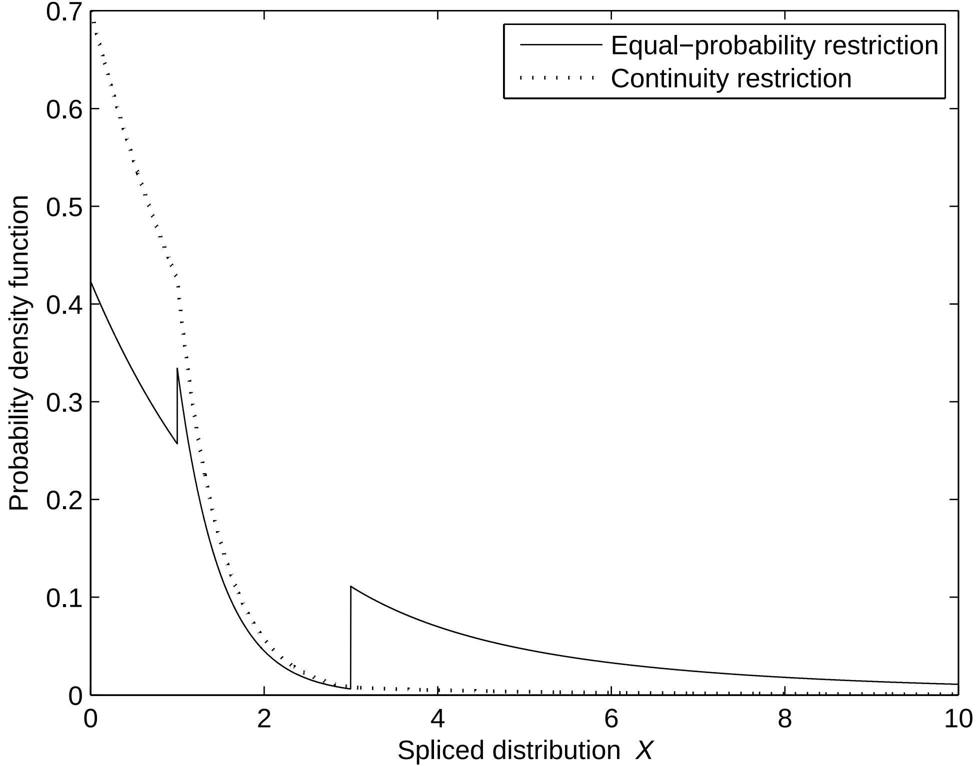
- Splicing is a technique to create a new distribution from standard distributions using different standard pdf in different parts of the support. Suppose there are k pdf, denoted by $f_1(x), \cdots, f_k(x)$ defined on the support $\Omega_X = [0, \infty)$, a new pdf $f_X(x)$ can be defined as follows

$$f_X(x) = \begin{cases} p_1 f_1^*(x), & x \in [0, c_1), \\ p_2 f_2^*(x), & x \in [c_1, c_2), \\ \cdot & \cdot \\ \cdot & \cdot \\ p_k f_k^*(x), & x \in [c_{k-1}, \infty), \end{cases} \tag{2.58}$$

where $p_i \geq 0$ for $i = 1, \cdots, k$ with $\sum_{i=1}^{k} p_i = 1$, $c_0 = 0 < c_1 < c_2 \cdots < c_{k-1} < \infty = c_k$, and $f_i^*(x)$ is a legitimate pdf based on $f_i(x)$ in the interval $[c_{i-1}, c_i)$ for $i = 1, \cdots, k$.

- **Example 2.6:** Let $X_1 \sim \mathcal{E}(0.5)$, $X_2 \sim \mathcal{E}(2)$ and $X_3 \sim \mathcal{P}(2, 3)$, with corresponding pdf $f_i(x)$ for $i = 1, 2$ and 3. Construct a spliced distribution using $f_1(x)$ in the interval $[0, 1)$, $f_2(x)$ in the interval $[1, 3)$ and $f_3(x)$ in the interval $[3, \infty)$, so that each interval has a probability content of one third. Also, determine the spliced distribution so that its pdf is continuous, without imposing equal probabilities for the three segments.

- See Figure 2.4.



Equal−probability restriction
Continuity restriction
Probability density function
Spliced distribution X
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10

2.4 Tail Properties

- A severity distribution with high probability of heavy loss is said to have a **fat tail**, **heavy tail** or **thick tail.**

- To compare the tail behavior of two distributions we may take the **limiting ratio** of their sf. The faster the sf approaches zero, the thinner is the tail.

- If $S_1(x)$ and $S_2(x)$ are the sf of the random variables X_1 and X_2, respectively, with corresponding pdf $f_1(x)$ and $f_2(x)$, we have

$$\lim_{x\to\infty} \frac{S_1(x)}{S_2(x)} = \lim_{x\to\infty} \frac{S_1'(x)}{S_2'(x)} = \lim_{x\to\infty} \frac{f_1(x)}{f_2(x)}. \tag{2.61}$$

- **Example 2.7:** Let $f_1(x)$ be the pdf of a $\mathcal{P}(\alpha,\gamma)$ distribution, and $f_2(x)$ be the pdf of a $\mathcal{G}(\theta,\beta)$ distribution. Determine the limiting ratio of these distributions, and suggest which distribution has a thicker tail.

- **Solution:** The limiting ratio of the Pareto versus the gamma distribution is

$$\begin{aligned}\lim_{x\to\infty}\frac{f_1(x)}{f_2(x)} &= \lim_{x\to\infty}\frac{\dfrac{\alpha\gamma^{\alpha}}{(x+\gamma)^{\alpha+1}}}{\dfrac{1}{\Gamma(\theta)\beta^{\theta}}\,x^{\theta-1}e^{-\frac{x}{\beta}}} \\ &= \alpha\gamma^{\alpha}\Gamma(\theta)\beta^{\theta}\lim_{x\to\infty}\frac{e^{\frac{x}{\beta}}}{(x+\gamma)^{\alpha+1}\,x^{\theta-1}},\end{aligned}$$

which tends to infinity as x tends to infinity.

- Thus, we conclude that the Pareto distribution has a thicker tail than the gamma distribution.

- The **quantile function (qf)** is the inverse of the df. Thus, if

$$F_X(x_\delta) = \delta, \tag{2.64}$$

then

$$x_\delta = F_X^{-1}(\delta). \tag{2.65}$$

- $F_X^{-1}(\cdot)$ is called the quantile function and x_δ is the δ-quantile (or the 100δ-percentile) of X. Equation (2.65) assumes that for any $0 < \delta < 1$ a unique value x_δ exists.

- **Example 2.8:** Let $X \sim \mathcal{E}(\lambda)$ and $Y \sim \mathcal{L}(\mu, \sigma^2)$. Derive the quantile functions of X and Y. If $\lambda = 1$, $\mu = -0.5$ and $\sigma^2 = 1$, compare the quantiles of X and Y for $\delta = 0.95$ and 0.99.

- **Solution:** We have

$$F_X(x_\delta) = 1 - e^{-\lambda x_\delta} = \delta,$$

so that $e^{-\lambda x_\delta} = 1 - \delta$, implying

$$x_\delta = -\frac{\log(1-\delta)}{\lambda}.$$

For Y we have

$$\begin{aligned}
\delta &= \Pr(Y \leq y_\delta) \\
&= \Pr(\log Y \leq \log y_\delta) \\
&= \Pr(\mathcal{N}(\mu, \sigma^2) \leq \log y_\delta) \\
&= \Pr\left(Z \leq \frac{\log y_\delta - \mu}{\sigma}\right),
\end{aligned}$$

where Z follows the standard normal distribution. Thus,

$$\frac{\log y_\delta - \mu}{\sigma} = \Phi^{-1}(\delta),$$

where $\Phi^{-1}(\cdot)$ is the quantile function of the standard normal. Hence, $y_\delta = \exp\left[\mu + \sigma\Phi^{-1}(\delta)\right]$.

For X, given the parameter value $\lambda = 1$, $\mathrm{E}(X) = \mathrm{Var}(X) = 1$ and $x_{0.95} = -\log(0.05) = 2.9957$.

For Y with $\mu = -0.5$ and $\sigma^2 = 1$, from equations (2.51) and (2.52) we have $\mathrm{E}(Y) = 1$ and $\mathrm{Var}(Y) = \exp(1) - 1 = 1.7183$.

Hence, X and Y have the same mean, while Y has a larger variance. For the quantile of Y we have $\Phi^{-1}(0.95) = 1.6449$, so that

$$y_{0.95} = \exp\left[\mu + \sigma\Phi^{-1}(0.95)\right] = \exp(1.6449 - 0.5) = 3.1421.$$

Similarly, we obtain $x_{0.99} = 4.6052$ and $y_{0.99} = 6.2109$.

Thus, Y has larger quantiles for $\delta = 0.95$ and 0.99, indicating it has a thicker upper tail.

- The quantile x_δ indicates the loss which will be exceeded with probability $1 - \delta$. However, it does not provide information about how bad the loss might be if loss exceeds this threshold.

- To address this issue, we may compute the expected loss conditional on the threshold being exceeded. We call this the **conditional tail expectation (CTE)** with tolerance probability $1 - \delta$, denoted by CTE_δ, which is defined as

$$\mathrm{CTE}_\delta = \mathrm{E}(X \,|\, X > x_\delta). \tag{2.66}$$

- CTE_δ is computed by

$$\begin{aligned}
\mathrm{CTE}_\delta &= \int_{x_\delta}^{\infty} x f_{X \,|\, X > x_\delta}(x)\, dx \\
&= \int_{x_\delta}^{\infty} x \left[\frac{f_X(x)}{S_X(x_\delta)} \right] dx \\
&= \frac{\int_{x_\delta}^{\infty} x f_X(x)\, dx}{1 - \delta}.
\end{aligned} \tag{2.68}$$

- **Example 2.9:** For the loss distributions X and Y given in Example 2.8, calculate $\mathrm{CTE}_{0.95}$.

- **Solution:** We first consider X. As $f_X(x) = \lambda e^{-\lambda x}$, the numerator of the last line of equation (2.68) is

$$\begin{aligned}
\int_{x_\delta}^{\infty} \lambda x e^{-\lambda x}\, dx &= -\int_{x_\delta}^{\infty} x\, de^{-\lambda x} \\
&= -\left(x e^{-\lambda x}\Big]_{x_\delta}^{\infty} - \int_{x_\delta}^{\infty} e^{-\lambda x}\, dx \right) \\
&= x_\delta e^{-\lambda x_\delta} + \frac{e^{-\lambda x_\delta}}{\lambda},
\end{aligned}$$

which, for $\delta = 0.95$ and $\lambda = 1$, is equal to

$$3.9957 e^{-2.9957} = 0.1997866.$$

Thus, $\text{CTE}_{0.95}$ of X is

$$\frac{0.1997866}{0.05} = 3.9957.$$

- The pdf of the lognormal distribution is given in equation (2.50). To compute the numerator of (2.68) we need to calculate

$$\int_{y_\delta}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\log x - \mu)^2}{2\sigma^2}\right] dx.$$

- To do this, we define the transformation

$$z = \frac{\log x - \mu}{\sigma} - \sigma.$$

- As

$$\begin{aligned}\exp\left[-\frac{(\log x-\mu)^2}{2\sigma^2}\right] &= \exp\left[-\frac{(z+\sigma)^2}{2}\right]\\ &= \exp\left(-\frac{z^2}{2}\right)\exp\left(-\sigma z-\frac{\sigma^2}{2}\right),\end{aligned}$$

  and

$$dx=\sigma x\,dz=\sigma\exp(\mu+\sigma^2+\sigma z)\,dz,$$

  we have

$$\begin{aligned}\int_{y_\delta}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(\log x-\mu)^2}{2\sigma^2}\right]dx &= \exp\left(\mu+\frac{\sigma^2}{2}\right)\int_{z^*}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz\\ &= \exp\left(\mu+\frac{\sigma^2}{2}\right)[1-\Phi(z^*)],\end{aligned}$$

  where $\Phi(\cdot)$ is the df of the standard normal and

$$z^*=\frac{\log y_\delta-\mu}{\sigma}-\sigma.$$

- Now we substitute $\mu = -0.5$ and $\sigma^2 = 1$ to obtain

$$z^* = \log y_{0.95} - 0.5 = \log(3.1424) - 0.5 = 0.6450,$$

so that the $\text{CTE}_{0.95}$ of Y is

$$\text{CTE}_{0.95} = \frac{e^0 \left[1 - \Phi(0.6450)\right]}{0.05} = 5.1900,$$

which is larger than that of X. Thus, Y gives rise to more extreme losses compared to X, whether we measure the extreme events by the upper quantiles or CTE.

2.5 Coverage Modifications

- To reduce risks and/or control problems of **moral hazard**, insurance companies often modify the policy coverage.

- Examples of such modifications are **deductibles**, **policy limits** and **coinsurance**.

- We need to distinguish between a **loss event** and a **payment event**. A loss event occurs whenever there is a loss, while a payment event occurs only when the insurer is liable to pay for (some or all of) the loss.

- We define the following notations:

1. X = amount paid in a loss event when there is no coverage modification

2. X_L = amount paid in a loss event when there is coverage modification

3. X_P = amount paid in a payment event when there is coverage modification

- Thus, X and X_P are positive and X_L is nonnegative.

2.5.1 Deductibles

- An insurance policy with a per-loss deductible of d will not pay the insured if the loss X is less than or equal to d, and will pay the insured $X - d$ if the loss X exceeds d.

- Thus, the amount paid in a loss event, X_L, is given by

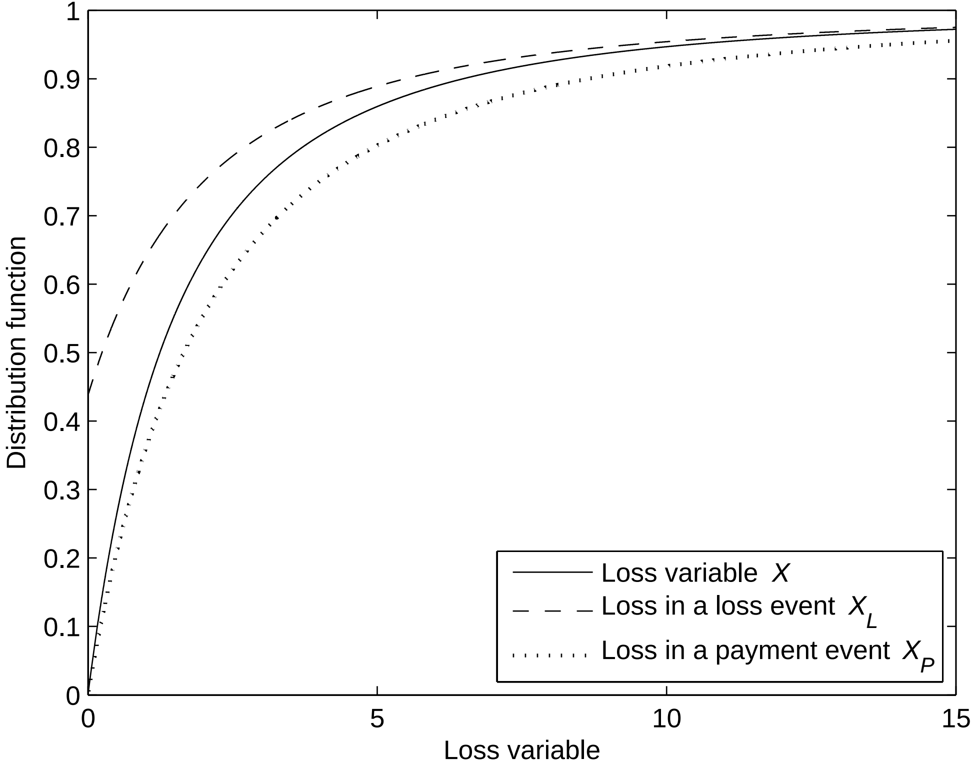
$$X_L = \begin{cases} 0, & \text{if } X \leq d, \\ X - d, & \text{if } X > d. \end{cases} \tag{2.69}$$

- If we adopt the notation

$$x_+ = \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } x > 0, \end{cases} \tag{2.70}$$

  then X_L may also be defined as

$$X_L = (X - d)_+. \tag{2.71}$$



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Distribution function
0
5
10
15
Loss variable
Loss variable X
Loss in a loss event X_L
Loss in a payment event X_P

- Note that $\Pr(X_L = 0) = F_X(d)$. Thus, X_L is a mixed-type random variable. It has a probability mass at point 0 of $F_X(d)$ and a density function of

$$f_{X_L}(x) = f_X(x+d), \qquad \text{for } x > 0. \tag{2.72}$$

- The random variable X_P, called the **excess-loss variable**, is defined only when there is a payment, i.e., when $X > d$. It is a conditional random variable, defined as $X_P = X - d \,|\, X > d$.

- Figure 2.6 plots the df of X, X_L and X_P.

- The mean of X_L can be computed as follows

$$\begin{aligned} \mathrm{E}(X_L) &= \int_0^\infty x f_{X_L}(x)\,dx \\ &= \int_d^\infty (x-d) f_X(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&= -\int_d^\infty (x-d)\, dS_X(x) \\
&= -\left[(x-d)S_X(x)]_d^\infty - \int_d^\infty S_X(x)\, dx \right] \\
&= \int_d^\infty S_X(x)\, dx. \qquad (2.76)
\end{aligned}
$$

- The mean of X_P, called the **mean excess loss**, is given by the following formula

$$
\begin{aligned}
\mathrm{E}(X_P) &= \int_0^\infty x f_{X_P}(x)\, dx \\
&= \int_0^\infty x \left[\frac{f_X(x+d)}{S_X(d)} \right] dx \\
&= \frac{\int_0^\infty x f_X(x+d)\, dx}{S_X(d)} \\
&= \frac{\int_d^\infty (x-d) f_X(x)\, dx}{S_X(d)}
\end{aligned}
$$

$$= \frac{\mathrm{E}(X_L)}{S_X(d)}. \tag{2.77}$$

- Using conditional expectation, we have

$$\begin{aligned}
\mathrm{E}(X_L) &= \mathrm{E}(X_L \,|\, X_L > 0)\,\Pr(X_L > 0) + \mathrm{E}(X_L \,|\, X_L = 0)\,\Pr(X_L = 0) \\
&= \mathrm{E}(X_L \,|\, X_L > 0)\,\Pr(X_L > 0) \\
&= \mathrm{E}(X_P)\,\Pr(X_L > 0), \qquad (2.78)
\end{aligned}$$

which implies

$$\mathrm{E}(X_P) = \frac{\mathrm{E}(X_L)}{\Pr(X_L > 0)} = \frac{\mathrm{E}(X_L)}{S_{X_L}(0)} = \frac{\mathrm{E}(X_L)}{S_X(d)}, \tag{2.79}$$

as proved in equation (2.77).

- Also, from the fourth line of equation (2.77), we have

$$\mathrm{E}(X_P) = \frac{\int_d^\infty x f_X(x)\,dx - d\int_d^\infty f_X(x)\,dx}{S_X(d)}$$

$$
\begin{aligned}
&= \frac{\int_d^\infty x f_X(x)\, dx - d[S_X(d)]}{S_X(d)} \\
&= \mathrm{CTE}_\delta - d, \qquad \text{where } \delta = F_X^{-1}(d). \qquad (2.81)
\end{aligned}
$$

- **Example 2.10:** For the loss distributions X and Y given in Examples 2.8 and 2.9, assume there is a deductible of $d = 0.25$. Calculate $\mathrm{E}(X_L)$, $\mathrm{E}(X_P)$, $\mathrm{E}(Y_L)$ and $\mathrm{E}(Y_P)$.

- **Solution:** For X, we compute $\mathrm{E}(X_L)$ from equation (2.76) as follows

$$
\mathrm{E}(X_L) = \int_{0.25}^\infty e^{-x}\, dx = e^{-0.25} = 0.7788.
$$

Now $S_X(0.25) = e^{-0.25} = 0.7788$. Thus, from equation (2.77), $\mathrm{E}(X_P) = 1$. For Y, we use the results in Example 2.9. First, we have

$$
\mathrm{E}(Y_L) = \int_d^\infty (y - d) f_Y(y)\, dy = \int_d^\infty y f_Y(y)\, dy - d\,[S_Y(d)]\,.
$$

Replacing y_δ in Example 2.9 by d, the first term of the above expression becomes

$$\int_d^\infty y f_Y(y)\, dy = \int_d^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\log y - \mu)^2}{2\sigma^2}\right] dy = \exp\left(\mu + \frac{\sigma^2}{2}\right)\left[1 - \Phi(z^*)\right],$$

where

$$z^* = \frac{\log d - \mu}{\sigma} - \sigma = \log(0.25) - 0.5 = -1.8863.$$

As $\Phi(-1.8663) = 0.0296$, we have

$$\int_d^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\log y - \mu)^2}{2\sigma^2}\right] dy = (e^{-0.5+0.5})[1 - 0.0296] = 0.9704.$$

Now,

$$S_Y(d) = \Pr\left(Z > \frac{\log(d) - \mu}{\sigma}\right) = \Pr(Z > -0.8863) = 0.8123.$$

Hence,

$$\mathrm{E}(Y_L) = 0.9704 - (0.25)(0.8123) = 0.7673,$$

and

$$\mathrm{E}(Y_P) = \frac{0.7673}{0.8123} = 0.9446.$$

- The computation of $\mathrm{E}(Y_L)$ for $Y \sim \mathcal{L}(\mu, \sigma^2)$ is summarized below.

Theorem 2.2: Let $Y \sim \mathcal{L}(\mu, \sigma^2)$, then for a positive constant d,

$$\mathrm{E}\left[(Y - d)_+\right] = \exp\left(\mu + \frac{\sigma^2}{2}\right)[1 - \Phi(z^*)] - d[1 - \Phi(z^* + \sigma)], \tag{2.82}$$

where

$$z^* = \frac{\log d - \mu}{\sigma} - \sigma. \tag{2.83}$$

- The expected reduction in loss due to the deductible is

$$\mathrm{E}(X) - \mathrm{E}\left[(X - d)_+\right] = \mathrm{E}(X) - \mathrm{E}(X_L). \tag{2.87}$$

We define the **loss elimination ratio** with deductible d, denoted by $\mathrm{LER}(d)$, as the ratio of the expected reduction in loss due to the deductible to the expected loss without the deductible, which is given by

$$\mathrm{LER}(d) = \frac{\mathrm{E}(X) - \mathrm{E}(X_L)}{\mathrm{E}(X)}. \tag{2.88}$$

2.5.2 Policy limit

- For an insurance policy with a **policy limit**, the insurer compensates the insured up to a pre-set amount, say, u, called the **maximum covered loss**.

- We denote the amount paid for a policy with a policy limit by X_U.

- If we define the binary operation $\wedge$ as the minimum of two quantities, so that

$$a \wedge b = \min\{a, b\}, \tag{2.91}$$

then

$$X_U = X \wedge u, \tag{2.92}$$

- X_U defined above is called the **limited-loss variable**.

- For any arbitrary constant q, the following identity holds

$$X = (X \wedge q) + (X - q)_+. \tag{2.94}$$

- LER can be written as

$$\mathrm{LER}(d) = \frac{\mathrm{E}(X) - \mathrm{E}\left[(X-d)_+\right]}{\mathrm{E}(X)} = \frac{\mathrm{E}(X) - [\mathrm{E}(X) - \mathrm{E}(X \wedge d)]}{\mathrm{E}(X)} = \frac{\mathrm{E}(X \wedge d)}{\mathrm{E}(X)}. \tag{2.95}$$

- From (2.94) we have

$$(X - q)_+ = X - (X \wedge q),$$

which implies

$$\mathrm{E}[(X - q)_+] = \mathrm{E}(X) - \mathrm{E}[(X \wedge q)].$$

As $\mathrm{E}[(X \wedge q)]$ is tabulated in the Exam C Tables for commonly used distributions of X, the above equation is a convenient way to calculate $\mathrm{E}[(X - q)_+]$.

- The above equation also implies, for any positive rv X,

$$\begin{aligned}
\mathrm{E}[(X \wedge q)] &= \mathrm{E}(X) - \mathrm{E}[(X - q)_+] \\
&= \int_0^\infty S_X(x)dx - \int_q^\infty S_X(x)dx \\
&= \int_0^q S_X(x)dx.
\end{aligned}$$

2.5.3 Coinsurance

- An insurance policy may specify that the insurer and insured share the loss in a loss event, which is called **coinsurance**.

- We consider a simple coinsurance in which the insurer pays the insured a fixed percentage c of the loss in a loss event, where $0 < c < 1$.

- We denote X_C as the payment made by the insurer. Thus,

$$X_C = c\,X, \tag{2.96}$$

where X is the loss without policy modification. The pdf of X_C is

$$f_{X_C}(x) = \frac{1}{c}\,f_X\left(\frac{x}{c}\right) \tag{2.97}$$

- Now we consider a policy which has a deductible of amount d, a policy limit of amount u $(u > d)$ and a coinsurance factor c $(0 < c < 1)$.

- We denote the loss random variable in a loss event by X_T, which is given by

$$X_T = c\left[(X \wedge u) - (X \wedge d)\right] = c\left[(X-d)_+ - (X-u)_+\right]. \quad (2.99)$$

- It can be checked that X_T defined above satisfies

$$X_T = \begin{cases} 0, & \text{for } X < d, \\ c(X-d), & \text{for } d \leq X < u, \\ c(u-d), & \text{for } X \geq u. \end{cases} \quad (2.100)$$

- From equation (2.99) we have

$$\mathrm{E}(X_T) = c\left\{\mathrm{E}\left[(X-d)_+\right] - \mathrm{E}\left[(X-u)_+\right]\right\}, \quad (2.101)$$

which can be computed using equation (2.76).

- **Example 2.12:** For the exponential loss distribution X and lognormal loss distribution Y given in Examples 2.8 through 2.11, assume there is a deductible of $d = 0.25$, maximum covered loss of $u = 4$ and coinsurance factor of $c = 0.8$. Calculate the mean loss in a loss event of these two distributions.

- **Solution:** We use equation (2.101) to calculate $\mathrm{E}(X_T)$ and $\mathrm{E}(Y_T)$. Note that $\mathrm{E}[(X - d)_+]$ and $\mathrm{E}[(Y - d)_+]$ are computed in Example 2.10 as 0.7788 and 0.7673, respectively. We now compute $\mathrm{E}[(X - u)_+]$ and $\mathrm{E}[(Y - u)_+]$ using the method in Example 2.10, with u replacing d. For X, we have

$$\mathrm{E}\left[(X - u)_+\right] = \int_u^\infty e^{-x}\, dx = e^{-4} = 0.0183.$$

For Y, we have $z^* = \log(4) - 0.5 = 0.8863$ so that $\Phi(z^*) = 0.8123$, and

$$S_Y(u) = \Pr\left(Z > \frac{\log(u) - \mu}{\sigma}\right) = \Pr(Z > 1.8863) = 0.0296.$$

Thus,

$$\mathrm{E}\left[(Y - u)_+\right] = (1 - 0.8123) - (4)(0.0296) = 0.0693.$$

Therefore, from equation (2.101), we have

$$\mathrm{E}(X_T) = (0.8)\,(0.7788 - 0.0183) = 0.6084,$$

and $\mathrm{E}(Y_T) = (0.8)(0.7673 - 0.0693) = 0.5584$.

2.5.4 Effects of inflation

- While loss distributions are specified based on current experience and data, inflation may cause increases in the costs. On the other hand, policy specifications will remain unchanged for the policy period.

- We consider a one-period insurance policy and assume the rate of price increase in the period to be r. We use a tilde to denote inflation adjusted losses.

- Thus, the inflation adjusted loss distribution is denoted by $\tilde{X}$, which is equal to $(1 + r)X$. For an insurance policy with deductible d, the loss in a loss event and the loss in a payment event with inflation adjustment are denoted by $\tilde{X}_L$ and $\tilde{X}_P$, respectively.

- As the deductible is not inflation adjusted, we have

$$\tilde{X}_L = \left(\tilde{X} - d\right)_+ = \tilde{X} - (\tilde{X} \wedge d), \tag{2.106}$$

and

$$\tilde{X}_P = \tilde{X} - d \,|\, \tilde{X} - d > 0 = \tilde{X}_L \,|\, \tilde{X}_L > 0. \tag{2.107}$$

- Thus, the mean inflation adjusted loss is given by

$$\begin{aligned} \mathrm{E}(\tilde{X}_L) &= \mathrm{E}\left[\left(\tilde{X} - d\right)_+\right] \\ &= \mathrm{E}\left[(1+r)\left(X - \frac{d}{1+r}\right)_+\right] \\ &= (1+r)\mathrm{E}\left[\left(X - \frac{d}{1+r}\right)_+\right]. \end{aligned} \tag{2.109}$$

- Also,

$$\mathrm{E}(\tilde{X}_P) = \mathrm{E}(\tilde{X}_L \,|\, \tilde{X}_L > 0) = \frac{\mathrm{E}(\tilde{X}_L)}{\Pr(\tilde{X}_L > 0)}. \tag{2.111}$$

As

$$\Pr(\tilde{X}_L > 0) = \Pr(\tilde{X} > d) = \Pr\left(X > \frac{d}{1+r}\right) = S_X\left(\frac{d}{1+r}\right), \tag{2.112}$$

we conclude

$$\mathrm{E}(\tilde{X}_P) = \frac{\mathrm{E}(\tilde{X}_L)}{S_X\left(\frac{d}{1+r}\right)}. \tag{2.113}$$

Table 2.2: Some Excel functions for the computation of the pdf $f_X(x)$ and df $F_X(x)$ of continuous random variable X

X	Excel function	Example input	Example output
$\mathcal{E}(\lambda)$	`EXPONDIST(x1,x2,ind)`	`EXPONDIST(4,0.5,FALSE)`	0.0677
	x1 $= x$	`EXPONDIST(4,0.5,TRUE)`	0.8647
	x2 $= \lambda$		
$\mathcal{G}(\alpha, \beta)$	`GAMMADIST(x1,x2,x3,ind)`	`GAMMADIST(4,1.2,2.5,FALSE)`	0.0966
	x1 $= x$	`GAMMADIST(4,1.2,2.5,TRUE)`	0.7363
	x2 $= \alpha$		
	x3 $= \beta$		
$\mathcal{W}(\alpha, \lambda)$	`WEIBULL(x1,x2,x3,ind)`	`WEIBULL(10,2,10,FALSE)`	0.0736
	x1 $= x$	`WEIBULL(10,2,10,TRUE)`	0.6321
	x2 $= \alpha$		
	x3 $= \lambda$		
$\mathcal{N}(0, 1)$	`NORMSDIST(x1)`	`NORMSDIST(1.96)`	0.9750
	x1 $= x$		
	output is $\Pr(\mathcal{N}(0, 1) \leq x)$		
$\mathcal{N}(\mu, \sigma^2)$	`NORMDIST(x1,x2,x3,ind)`	`NORMDIST(3.92,1.96,1,FALSE)`	0.0584
	x1 $= x$	`NORMDIST(3.92,1.96,1,TRUE)`	0.9750
	x2 $= \mu$		
	x3 $= \sigma$		
$\mathcal{L}(\mu, \sigma^2)$	`LOGNORMDIST(x1,x2,x3)`	`LOGNORMDIST(3.1424,-0.5,1)`	0.9500
	x1 $= x$		
	x2 $= \mu$		
	x3 $= \sigma$		
	output is $\Pr(\mathcal{L}(\mu, \sigma^2) \leq x)$		

Table 2.3: Some Excel functions for the computation of the inverse of the df $F_X^{-1}(\delta)$ of continuous random variable X

		Example	
X	Excel function	input	output
$\mathcal{G}(\alpha,\beta)$	`GAMMAINV(x1,x2,x3)` `x1` $=\delta$ `x2` $=\alpha$ `x3` $=\beta$	`GAMMAINV(0.8,2,2)`	5.9886
$\mathcal{N}(0,1)$	`NORMSINV(x1)` `x1` $=\delta$	`NORMSINV(0.9)`	1.2816
$\mathcal{N}(\mu,\sigma^2)$	`NORMINV(x1,x2,x3)` `x1` $=\delta$ `x2` $=\mu$ `x3` $=\sigma$	`NORMINV(0.99,1.2,2.5)`	7.0159