

Nonlife Actuarial Models

Chapter 15

Applications of Monte Carlo Methods

Learning Objectives

1. Monte Carlo estimation of critical values and p -values
2. Bootstrap estimation of p -values
3. Bootstrap estimation of bias and mean squared error.
4. Simulation of lognormally distributed asset prices
5. Simulation of asset prices with discrete jumps

15.1 Monte Carlo Simulation for Hypotheses Test

15.1.1 Kolmogorov-Smirnov Test

- For the Kolmogorov-Smirnov D statistic, David and Johnson (1948) show that if the parameters estimated for the null distribution are parameters of *scale* or *location*, and the estimators satisfy certain general conditions, then the joint distribution of the probability-integral transformed observations of the sample will not depend on the *true* parameter values.
- The Kolmogorov-Smirnov test is based on the distribution function under the null, which is the probability integral transform of the sample observations.

- Many commonly used distributions involve parameters of scale and location. For example, for the $\mathcal{N}(\mu, \sigma^2)$ distribution, μ is the location parameter and σ is the scale parameter. The parameter λ in the $\mathcal{E}(\lambda)$ distribution is a location-and-scale parameter.
- In these cases the exact distributions of the D statistics under the null do not depend on the true parameter values, as long as the null distribution functions are computed using the MLE.
- As the null distribution of the D statistic for the normal distribution does not depend on the true parameter values, we may assume *any* convenient values of the parameters without affecting the null distribution.
- This gives rise to the following Monte Carlo procedure to estimate the critical value of D for a given sample size n

1. Generate a random sample of n (call this the **estimation sample size**) standard normal variates $x_1, \dots, x_n$. Calculate the sample mean $\bar{x}$ and sample variance s^2 , and use these values to compute the estimated distribution function $F^*(x_i)$, where $F^*(\cdot)$ is the df of $\mathcal{N}(\bar{x}, s^2)$. Then use equation (13.4) to compute D .
 2. Repeat Step 1 m times (call this the **Monte Carlo sample size**) to obtain m values of D_j , for $j = 1, \dots, m$.
 3. At the level of significance α , the critical value of the Kolmogorov-Smirnov D statistic is computed as the $(1 - \alpha)$ -quantile of the sample of m values of D , estimated using the method in equations (11.9) and (11.10).
- The following critical values are proposed by Lilliefors (1967) for testing normal distributions with unknown mean and variance

Level of significance α	0.10	0.05	0.01
Critical value	$\frac{0.805}{\sqrt{n}}$	$\frac{0.886}{\sqrt{n}}$	$\frac{1.031}{\sqrt{n}}$

- If the null hypothesis is that the sample observations are distributed as $\mathcal{E}(\lambda)$, where λ is not specified, to estimate the critical values of the Kolmogorov-Smirnov statistic, the following procedure can be used
1. Generate a random sample of n variates $x_1, \dots, x_n$ distributed as $\mathcal{E}(1)$. Calculate the sample mean $\bar{x}$ and compute the estimated distribution function $F^*(x_i)$, where $F^*(\cdot)$ is the df of $\mathcal{E}(1/\bar{x})$. Then use equation (13.4) to compute D .
 2. Repeat Step 1 m times to obtain m values of D_j , for $j = 1, \dots, m$.

3. At the level of significance α , the critical value of the Kolmogorov-Smirnov D statistic is computed as the $(1-\alpha)$ -quantile of the sample of m values of D , estimated using the method in equations (11.9) and (11.10).
- The following critical values are proposed by Lilliefors (1969) for testing exponential distributions with unknown mean

Level of significance α	0.10	0.05	0.01
Critical value	$\frac{0.96}{\sqrt{n}}$	$\frac{1.06}{\sqrt{n}}$	$\frac{1.25}{\sqrt{n}}$

15.1.2 Chi-square Goodness-of-Fit Test

- The asymptotic distribution of the X^2 statistic for the goodness-of-fit test is χ_{k-r-1}^2 , where k is the number of groups and r is the number of parameters estimated using the MMLE method.

- This result holds asymptotically for any null distribution. Yet Monte Carlo simulation can be used to investigate the performance of the test and improve the estimates of the critical values in small samples if required.

Example 15.3: Estimate the critical values of the chi-square goodness-of-fit statistic X^2 using Monte Carlo simulation when the null hypothesis is that the observations are distributed as $\mathcal{E}(\lambda)$, where λ is unknown. Compute the X^2 statistics based on the MLE using individual observations as well as the MMLE using grouped data.

Solution: We group the data into intervals $(c_{i-1}, c_i]$, and use the following 4 intervals: $(0, 0.4]$, $(0.4, 1]$, $(1, 1.5]$ and $(1.5, \infty)$. The MLE of λ using the complete individual data is $1/\bar{x}$. Let $\mathbf{n} = \{n_1, \dots, n_4\}$, where n_i is the number of observations in the i th interval. Using grouped data,

the MMLE is solved by maximizing the log-likelihood function

$$\log L(\lambda; \mathbf{n}) = \sum_{i=1}^4 n_i \log [\exp(-\lambda c_{i-1}) - \exp(-\lambda c_i)]$$

with respect to λ . The X^2 statistic is then computed using equation (13.8). Using a Monte Carlo simulation with 10,000 samples, we obtain the estimated critical values of the X^2 statistic summarized in Table 15.3.

Table 15.3: Results of Example 15.3

α	$n = 50$		$n = 100$		$n = 200$		$n = 300$		$\chi^2_{2,1-\alpha}$
	MLE	MMLE	MLE	MMLE	MLE	MMLE	MLE	MMLE	
0.10	4.95	4.70	4.93	4.70	4.91	4.61	4.91	4.66	4.61
0.05	6.31	6.07	6.30	6.07	6.38	6.05	6.30	6.04	5.99
0.01	9.45	9.25	9.48	9.39	9.60	9.37	9.41	9.14	9.21

The asymptotic critical values $\chi^2_{2,1-\alpha}$ are shown in the last column. Two points can be observed from the Monte Carlo results. First, the asymptotic

results are very reliable even for samples of size 50, if the correct MMLE is used to compute X^2 . Second, if MLE is used to compute X^2 , the use of $\chi^2_{2,1-\alpha}$ as the critical value will *over-reject* the null hypothesis. $\square$

15.2 Bootstrap Estimation of p -Value

- There are situations for which the distribution of the test statistic under the null hypothesis depends on some nuisance parameters not specified under the null. For such problems, tabulation of the critical values is not viable.
- As an alternative, we may use **bootstrap method** to estimate the p -value of the test statistic.
- Consider a sample of n observations $\mathbf{x} = (x_1, \dots, x_n)$ and a test statistic $T(\mathbf{x})$ for testing a null hypothesis H_0 .
- Let the computed value of the test statistic for the sample $\mathbf{x}$ be t . Suppose the decision rule of the test is to reject H_0 when t is too large (i.e., on the right-hand extreme tail).

- Assume H_0 contains a nuisance parameter θ , which is not specified. We now consider the estimation the p -value of the test statistic, which is the probability that $T(\mathbf{x})$ is larger than t if the null hypothesis is true, i.e.,

$$p = \Pr(T(\mathbf{x}) > t \mid H_0). \quad (15.1)$$

- As H_0 contains the nuisance parameter θ , we replace the above problem by

$$p = \Pr(T(\mathbf{x}) > t \mid H_0(\hat{\theta})), \quad (15.2)$$

where $\hat{\theta}$ is an estimator of θ . The bootstrap estimate of p can be computed as follows

1. Let the computed value of $T(\mathbf{x})$ based on the sample $\mathbf{x}$ be t , and let the estimated value of θ be $\hat{\theta}$, which may be any appropriate estimator, such as the MLE.

2. Generate a sample of observations from the distributional assumption of $H_0(\hat{\theta})$, call this $\mathbf{x}^*$. Compute the test statistic using data $\mathbf{x}^*$ and call this t^* .
3. Repeat Step 2 m times, which is the bootstrap sample size, to obtain m values of the test statistic t_j^* , for $j = 1, \dots, m$.
4. The estimated p -value of t is computed as

$$\frac{1 + \text{number of } \{t_j^* \geq t\}}{m + 1}. \quad (15.3)$$

- The above is a **parametric bootstrap** procedure, in which the samples $\mathbf{x}^*$ are generated from a parametric distribution.
- At level of significance α , the null hypothesis is rejected if the estimated p -value is less than α .

Example 15.4: You are given the following 20 observations of losses
0.114, 0.147, 0.203, 0.378, 0.410, 0.488, 0.576, 0.868, 0.901, 0.983,
1.049, 1.555, 2.060, 2.274, 4.235, 5.400, 5.513, 5.817, 8.901, 12.699.

- (a) Compute the Kolmogorov-Smirnov D statistic, assuming the data are distributed as $\mathcal{P}(\alpha, 5)$. Estimate the p -value of the test statistic using bootstrap.
- (b) Repeat (a), assuming the null distribution is $\mathcal{P}(\alpha, 40)$.
- (c) Repeat (a), assuming the null distribution is $\mathcal{E}(\lambda)$.

Solution: For (a), we estimate α using MLE, which, from Example 12.9, is given by

$$\hat{\alpha} = \frac{20}{\sum_{i=1}^{20} \log(x_i + 5) - 20 \log(5)},$$

and we obtain $\hat{\alpha} = 2.7447$. The computed D statistic is 0.1424.

To estimate the p -value, we generate 10,000 bootstrap samples of size 20 each from $\mathcal{P}(2.7447, 5)$, estimate α and compute the D statistic for each sample. The proportion of the D values larger than 0.1424 calculated using equation (15.3) is 0.5775, which is the estimated p -value. Thus, the $\mathcal{P}(\alpha, 5)$ assumption cannot be rejected at any conventional level of significance.

For (b), the MLE of α is

$$\hat{\alpha} = \frac{20}{\sum_{i=1}^{20} \log(x_i + 40) - 20 \log(40)} = 15.8233.$$

The computed D statistic is 0.2138. We generate 10,000 samples of size 20 each from the $\mathcal{P}(15.8233, 40)$ distribution and compute the D statistic of each sample. The estimated p -value is 0.0996. Thus, at the level of

significance of 10%, the null hypothesis $\mathcal{P}(\alpha, 40)$ is rejected, but not at the level of significance of 5%.

For (c), the MLE of λ is

$$\hat{\lambda} = \frac{1}{\bar{x}} = 0.3665,$$

and the computed D value is 0.2307. We generate 10,000 samples of size 20 each from the $\mathcal{E}(0.3665)$ distribution using the inversion method. The estimated p -value of the D statistic is 0.0603. Thus, the assumption of $\mathcal{E}(\lambda)$ is rejected at the 10% level, but not at the 5% level.

To conclude, the Kolmogorov-Smirnov test supports the $\mathcal{P}(\alpha, 5)$ distribution assumption for the loss data, but not the $\mathcal{P}(\alpha, 40)$ and $\mathcal{E}(\lambda)$ distributions. □

15.3 Bootstrap Estimation of Bias and Mean Squared Error

- Bootstrap method can also be used to estimate the bias and mean squared error of the parameter estimates of a distribution.
- Consider the estimation of the parameter θ (or a function of the parameter $g(\theta)$) of a distribution using an estimator $\hat{\theta}$ (or $g(\hat{\theta})$), given a random sample of n observations $\mathbf{x} = (x_1, \dots, x_n)$ of X .
- In situations where theoretical results about the bias and mean squared error of $\hat{\theta}$ (or $g(\hat{\theta})$) are intractable, we may use bootstrap method to estimate these quantities.
- When no additional assumption about the distribution of X is made,

we may use the empirical distribution defined by $\mathbf{x}$ as the assumed distribution. We generate a sample of n observations $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ by re-sampling from $\mathbf{x}$ with replacement, and compute the estimate $\hat{\theta}^*$ (or $g(\hat{\theta}^*)$) based on $\mathbf{x}^*$.

- We do this m times to obtain m estimates $\hat{\theta}_j^*$ (or $g(\hat{\theta}_j^*)$), for $j = 1, \dots, m$.
- Based on these bootstrap estimates we can compute the bias and the mean squared error of the estimator $\hat{\theta}$ (or $g(\hat{\theta})$). As $\mathbf{x}^*$ are generated from the empirical distribution defined by $\mathbf{x}$, we call this method **nonparametric bootstrap**.
- To illustrate the idea, we consider the use of the sample mean and the sample variance as estimates of the population mean μ and population variance σ^2 of X , respectively.

- Let μ_E and σ_E^2 be the mean and the variance, respectively, of the empirical distribution defined by $\mathbf{x}$.
 - We note that $\mu_E = \bar{x}$ and $\sigma_E^2 = (n-1)s^2/n$, where $\bar{x}$ and s^2 are the sample mean and the sample variance of $\mathbf{x}$, respectively
 - . To use the bootstrap method to estimate the bias and the mean squared error of $\bar{x}$ and s^2 , we adopt the following procedure
1. Generate a random sample of n observations by re-sampling with replacement from $\mathbf{x}$, call this $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$. Compute the mean $\bar{x}^*$ and variance s^{*2} of $\mathbf{x}^*$.
 2. Repeat Step 1 m times to obtain values $\bar{x}_j^*$ and s_j^{*2} , for $j = 1, \dots, m$.
 3. The bias and the mean squared error of $\bar{x}$ are estimated, respectively,

by

$$\frac{1}{m} \sum_{j=1}^m (\bar{x}_j^* - \mu_E) \quad \text{and} \quad \frac{1}{m} \sum_{j=1}^m (\bar{x}_j^* - \mu_E)^2. \quad (15.4)$$

4. The bias and the mean squared error of s^2 are estimated, respectively, by

$$\frac{1}{m} \sum_{j=1}^m (s_j^{*2} - \sigma_E^2) \quad \text{and} \quad \frac{1}{m} \sum_{j=1}^m (s_j^{*2} - \sigma_E^2)^2. \quad (15.5)$$

- It is theoretically known that $\bar{x}$ and s^2 are unbiased for μ and σ^2 , respectively.
- Furthermore, the expected value of $\bar{x}_j^*$ is μ_E and the expected value of s_j^{*2} is σ_E^2 , so that the bootstrap estimate of the biases should converge to zero when m is large.

- The mean squared error of $\bar{x}$ is

$$\text{MSE}(\bar{x}) = \text{Var}(\bar{x}) = \frac{\sigma^2}{n}, \quad (15.6)$$

which is unknown (as σ^2 is unknown).

- On the other hand, the bootstrap estimate of the MSE of $\bar{x}$ in equation (15.4) converges in probability to σ_{E}^2/n , which is known given $\mathbf{x}$. However, when $\mathbf{x}$ varies $\text{E}(\sigma_{\text{E}}^2/n) = (n-1)\sigma^2/n^2 \neq \text{MSE}(\bar{x})$.

15.4 A General Framework of Bootstrap

- We now provide a framework of the theoretical underpinning of the bootstrap method.
- Let $\mathbf{X} = \{X_1, \dots, X_n\}$ be independently and identically distributed as X with df $F(\cdot)$, which may depend on a parameter θ .
- Suppose $\xi = \xi(F)$ is a quantity of the distribution (e.g., mean, median, a quantile or a population proportion) and $\hat{\xi} = \hat{\xi}(\mathbf{X})$ is an estimate of ξ based on $\mathbf{X}$.
- We define

$$\eta(\mathbf{X}; F) = \hat{\xi}(\mathbf{X}) - \xi(F), \quad (15.8)$$

which is the error in estimating ξ using $\hat{\xi}$. Denoting E_F as the

expectation taken using the df F , the bias of $\hat{\xi}$ is

$$\mathbb{E}_F[\eta(\mathbf{X}; F)] = \mathbb{E}_F[\hat{\xi}(\mathbf{X}) - \xi(F)] \quad (15.9)$$

and the mean squared error of $\hat{\xi}$ is

$$\mathbb{E}_F[\eta(\mathbf{X}; F)^2] = \mathbb{E}_F[(\hat{\xi}(\mathbf{X}) - \xi(F))^2]. \quad (15.10)$$

- For another application, let $T(\mathbf{X})$ be a test statistic for a hypothesis H_0 and its value computed based on a specific sample $\mathbf{x} = (x_1, \dots, x_n)$ be $t = T(\mathbf{x})$. We now define

$$\eta(\mathbf{X}; F) = T(\mathbf{X}) - t. \quad (15.11)$$

- If H_0 is rejected when t is too large, the p -value of the test is

$$\Pr(T(\mathbf{X}) - t > 0 \mid F) = \Pr(\eta(\mathbf{X}; F) > 0 \mid F). \quad (15.12)$$

- In the above cases, we are interested in the expectation or the population proportion of a suitably defined function $\eta(\mathbf{X}; F)$. This set-up includes the evaluation of bias and mean squared error of an estimator and the p -value of a test, as well as many other applications.
- As F is unknown in practice, the quantities in equations (15.9), (15.10) and (15.12) cannot be evaluated.
- However, we may replace F by a known df F^* and consider instead the quantities

$$E_{F^*}[\eta(\mathbf{X}; F^*)] = E_{F^*}[\hat{\xi}(\mathbf{X}) - \xi(F^*)], \quad (15.13)$$

$$E_{F^*}[\eta(\mathbf{X}; F^*)^2] = E_{F^*}[(\hat{\xi}(\mathbf{X}) - \xi(F^*))^2], \quad (15.14)$$

and

$$\Pr(T(\mathbf{X}) - t > 0 \mid F^*) = \Pr(\eta(\mathbf{X}; F^*) > 0 \mid F^*). \quad (15.15)$$

- The above quantities are called the bootstrap approximations.
- The reliability of these approximations depend on how good F^* is as an approximation to F .
- If F^* is taken as the empirical distribution defined by $\mathbf{x}$, we have a nonparametric bootstrap.
- If F^* is taken as $F(\hat{\theta})$ for a suitable estimator $\hat{\theta}$ computed from the sample $\mathbf{x}$, then we have a parametric bootstrap.
- As $\hat{\xi}(\mathbf{X})$ and $T(\mathbf{X})$ may be rather complex functions of $\mathbf{X}$, the evaluation of equations (15.13), (15.14) and (15.15) may remain elusive even with known or given F^* .
- In the case where the sample size n is small and the empirical distribution is used for F^* , we may evaluate these quantities by exhausting

all possible samples of $\mathbf{X}$.

- This approach, however, will not be feasible when n is large or when a parametric df $F(\hat{\theta})$ is used. In such situations the quantities may be estimated using Monte Carlo methods, and we call the solution the Monte Carlo estimate of the bootstrap approximate, or simply the bootstrap estimate.

15.5 Monte Carlo Simulation of Asset Prices

15.5.1 Wiener Process and Generalized Wiener Process

- Let W_t be a stochastic process over time t with the following properties
1. Over a small time interval Δt , the change in W_t , denoted by $\Delta W_t = W_{t+\Delta t} - W_t$, satisfies the property

$$\Delta W_t = \epsilon \sqrt{\Delta t}, \quad (15.16)$$

where $\epsilon \sim \mathcal{N}(0, 1)$.

2. If ΔW_{t_1} and ΔW_{t_2} are changes in the process W_t over two nonoverlapping intervals, then ΔW_{t_1} and ΔW_{t_2} are independent.

- A continuous-time stochastic process satisfying the above two properties is called a **Wiener process** or **standard Brownian motion**.

From the first of these two properties, we can conclude that

$$\mathbb{E}(\Delta W_t) = 0, \quad (15.17)$$

and

$$\text{Var}(\Delta W_t) = \Delta t. \quad (15.18)$$

- For the change over a finite interval $[0, T]$, we can partition the interval into N nonoverlapping small segments of length Δt each, such that

$$T = N(\Delta t) \quad (15.19)$$

and

$$W_T - W_0 = \sum_{i=0}^{N-1} \Delta W_{i(\Delta t)} = \sum_{i=1}^N \epsilon_i \sqrt{\Delta t}, \quad (15.20)$$

where ϵ_i , for $i = 1, \dots, n$, are iid $\mathcal{N}(0, 1)$.

- Thus, given the information at time 0 we have

$$\mathbb{E}(W_T) = W_0 + \sum_{i=1}^N \mathbb{E}(\epsilon_i) \sqrt{\Delta t} = W_0, \quad (15.21)$$

and

$$\text{Var}(W_T) = \sum_{i=1}^N \text{Var}(\epsilon_i) \Delta t = \sum_{i=1}^N \Delta t = T. \quad (15.22)$$

- Hence, W_T is the sum of W_0 and N iid normal variates, which implies, given W_0 ,

$$W_T \sim \mathcal{N}(W_0, T). \quad (15.23)$$

- The Wiener process can be extended to allow for a *drift* in the process and a constant volatility parameter.

- A **generalized Wiener process** or **Brownian motion** X_t has the following change over a small time interval Δt

$$\Delta X_t = a \Delta t + b \Delta W_t, \quad (15.24)$$

where a is the drift rate and b is the volatility rate (a and b are constants), and W_t is a Wiener process.

- It can be verified that, given X_0 , we have

$$X_T \sim \mathcal{N}(X_0 + aT, b^2T), \quad (15.25)$$

for any finite T .

- The Wiener and generalized Wiener processes are continuous-time processes when $\Delta t \rightarrow 0$ in equations (15.16) and (15.24), respectively. Thus, we shall write the differentials of these processes as dW_t and

$$dX_t = a dt + b dW_t, \quad (15.26)$$

respectively.

15.5.2 Diffusion Process and Lognormal Distribution

- A further extension of equation (15.26) is to allow the drift and volatility rates to depend on time t and the process value X_t . Thus, we consider the process

$$dX_t = a(X_t, t) dt + b(X_t, t) dW_t, \quad (15.27)$$

which is called an **Ito process** or **diffusion process**.

- The terms $a(X_t, t)$ and $b(X_t, t)$ are called the **drift rate** and the **diffusion coefficient**, respectively.
- X_t is in general no longer normally distributed.
- We consider a specific member of diffusion processes, called the **geometric Brownian motion**.
- Let S_t be the price of an asset at time t . S_t is said to follow a geometric Brownian motion if

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (15.28)$$

where μ , called the **instantaneous rate of return**, and σ , called the **volatility rate**, are constants.

- The above equation can also be written as

$$\frac{1}{S_t} dS_t = \mu dt + \sigma dW_t. \quad (15.29)$$

- Further analysis (using Ito's lemma) shows that

$$d \log S_t = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t, \quad (15.30)$$

so that $\log S_t$ follows a generalized Wiener process and hence is normally distributed. Thus, following equation (15.25), we conclude

$$\log S_t \sim \mathcal{N} \left(\log S_0 + \left(\mu - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right), \quad (15.31)$$

so that S_t is lognormally distributed with mean

$$\mathbb{E}(S_t) = \exp \left[\log S_0 + \left(\mu - \frac{\sigma^2}{2} \right) t + \frac{\sigma^2 t}{2} \right] = S_0 \exp(\mu t). \quad (15.32)$$

Equation (15.31) can also be written as

$$\log S_t - \log S_0 = \log \left(\frac{S_t}{S_0} \right) \sim \mathcal{N} \left(\left(\mu - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right), \quad (15.33)$$

so that

$$\log \left(\frac{S_t}{S_0} \right) = \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t} Z, \quad (15.34)$$

where Z is standard normal.

- Note that

$$R \equiv \frac{1}{t} \log \left(\frac{S_t}{S_0} \right) \quad (15.36)$$

is the **continuously compounded rate of return** over the interval $[0, t]$.

- Thus, from equation (15.34), the expected continuously compounded

rate of return over the finite interval $[0, t]$ is

$$\mathbb{E} \left[\frac{1}{t} \log \left(\frac{S_t}{S_0} \right) \right] = \mu - \frac{\sigma^2}{2}, \quad (15.37)$$

which is less than the instantaneous rate of return μ .

- The total return of an asset consists of two components: capital gain and dividend yield. As S_t is the price of the asset, μ as defined in equation (15.28) captures the instantaneous capital gain only.
- If the dividend yield is assumed to be continuous at the rate δ , then the total instantaneous return, denoted by μ^* , is given by

$$\mu^* = \mu + \delta. \quad (15.38)$$

- Hence, expressed in terms of the total return and the dividend yield,

the expected continuously compounded rate of capital gain (asset-price appreciation) is

$$\mathbb{E} \left[\frac{1}{t} \log \left(\frac{S_t}{S_0} \right) \right] = \mu - \frac{\sigma^2}{2} = \mu^* - \delta - \frac{\sigma^2}{2}. \quad (15.39)$$

- We now consider the simulation of asset prices that follow the geometric Brownian motion given in equation (15.28), in which the parameter μ captures the return due to asset-price appreciation.
- From equation (15.34), we obtain

$$S_t = S_0 \exp \left[\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t} Z \right], \quad (15.40)$$

which can be used to simulate price paths of the asset. In practical applications, we need the values of the parameters σ and

$$\mu_R = \mu - \frac{\sigma^2}{2}. \quad (15.41)$$

- Suppose we sample return data of the asset over intervals of length h . Let there be n return observations (computed as differences of logarithmic asset prices) with mean $\bar{x}_R$ and sample variance s_R^2 .
- The required parameter estimates are then given by

$$\hat{\mu}_R = \frac{\bar{x}_R}{h} \quad \text{and} \quad \hat{\sigma} = \frac{s_R}{\sqrt{h}}. \quad (15.42)$$

- The asset prices at intervals of h can be simulated recursively using the equation

$$S_{t+(i+1)h} = S_{t+ih} \exp [\bar{x}_R + s_R Z_i], \quad \text{for } i = 0, 1, 2, \dots, \quad (15.43)$$

where Z_i are iid $\mathcal{N}(0, 1)$.

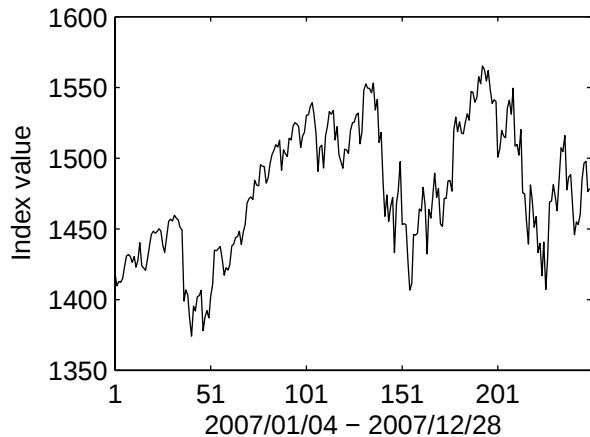
- We use end-of-day S&P500 index values in the period January 3, 2007, through December 28, 2007.

- There are in total 250 index values and we compute 249 daily returns, which are the logarithmic price differences.
- The price index graph and the return graph are plotted in Figure 15.1.
- We estimate the parameters of the price process and obtain $\bar{x}_R = 0.0068\%$ and $s_R = 1.0476\%$.
- These values are in percent per day. If we take $h = 1/250$, the annualized estimate of σ is $1.0476\sqrt{250}\% = 16.5640\%$ per annum. The estimate of μ is

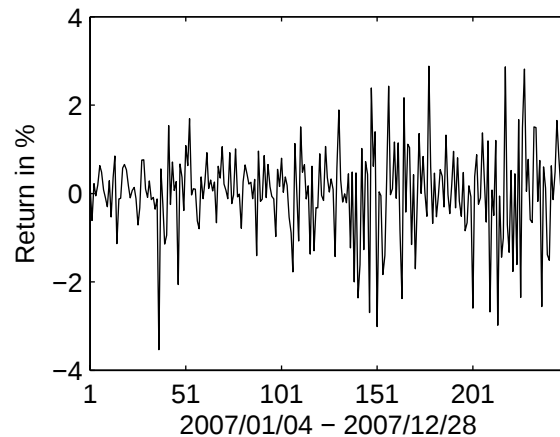
$$\hat{\mu} = \hat{\mu}_R + \frac{\hat{\sigma}^2}{2} = \frac{\bar{x}_R}{h} + \frac{s_R^2}{2h} = 3.0718\% \text{ per annum.} \quad (15.44)$$

- We use these values to simulate the price paths, an example of which is presented in Figure 15.2.

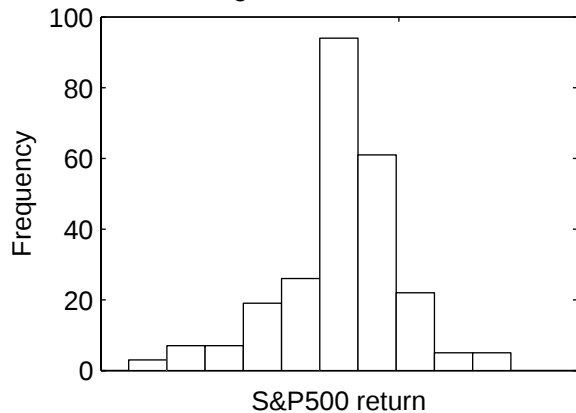
S&P500 index



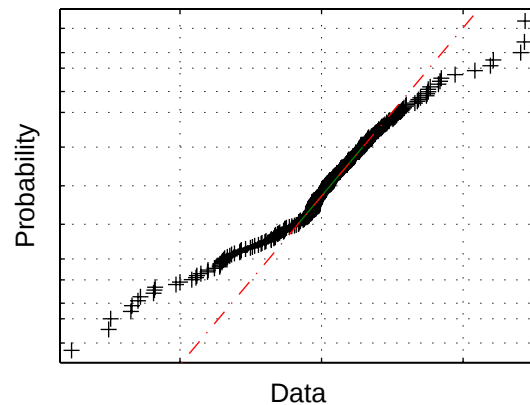
S&P500 return



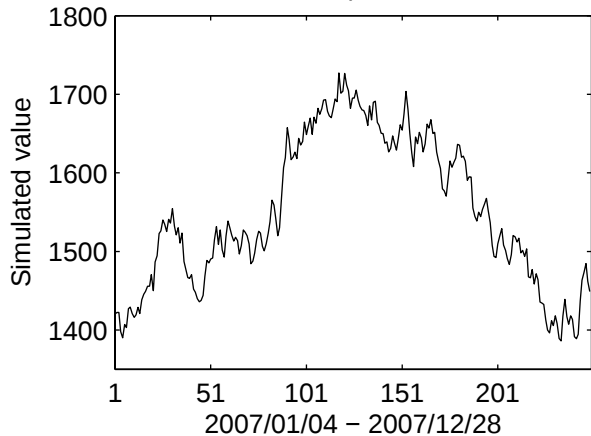
Histogram of S&P500 return



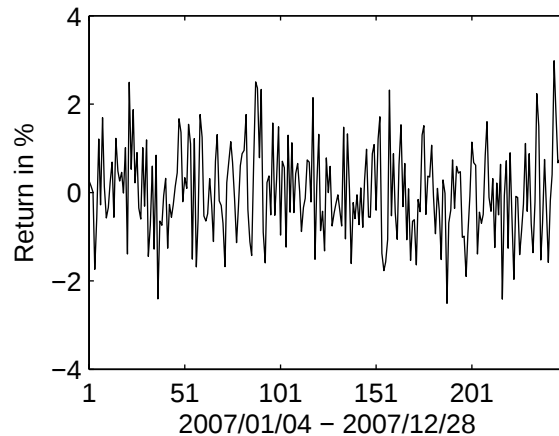
Normal probability plot



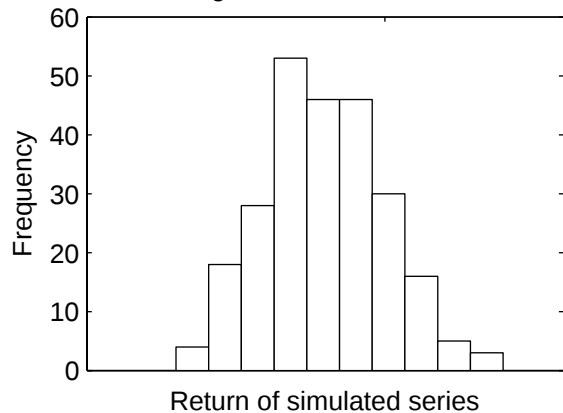
Simulated price series



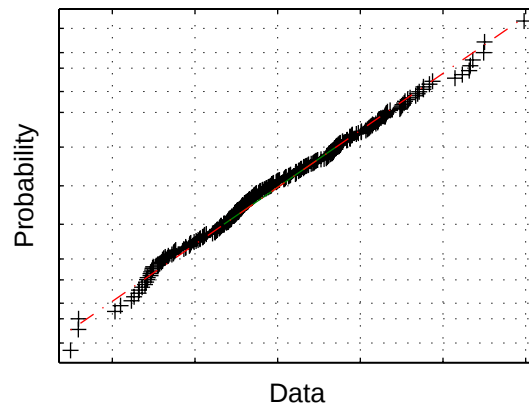
Return of simulated series



Histogram of simulated return



Normal probability plot



15.5.3 Jump-Diffusion Process

- Asset prices following a diffusion process are characterized by paths that are *continuous* in time.
- Anecdotal evidence, however, often suggests that stock prices are more *jumpy* than what would be expected of a diffusion process.
- To allow for discrete jumps in asset prices, we introduce a jump component into the diffusion process and consider asset prices following a **jump-diffusion process**.
- We consider augmenting the geometric Brownian motion with a jump component. We assume the occurrence of a jump in an interval has a Poisson distribution, and when a jump occurs, the jump size is distributed normally.

- We define N_t as a Poisson process with intensity (mean per unit time) λ .
- ΔN_t is the number of jump events occurring in the interval $(t, t + \Delta t]$. We use the notation dN_t when $\Delta t \rightarrow 0$.
- We now augment the geometric Brownian motion in equation (15.30) with a jump component as follows

$$d \log S_t = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t + J_t dN_t - \lambda \mu_J dt, \quad (15.45)$$

where $J_t \sim \mathcal{N}(\mu_J, \sigma_J^2)$ and is distributed independently of N_t .

- Note that the mean of $J_t dN_t$ is $\lambda \mu_J dt$, so that the mean of the augmented component $J_t dN_t - \lambda \mu_J dt$ is zero. Thus, the addition of the term $-\lambda \mu_J dt$ is to center the jump component so that its mean is equal to zero.

- This property is of important significance because jumps are often assumed to be idiosyncratic and does not affect the expected return of the stock.
- We re-write equation (15.45) as

$$d \log S_t = \left(\mu - \lambda \mu_J - \frac{\sigma^2}{2} \right) dt + \sigma dW_t + J_t dN_t. \quad (15.46)$$

- If $\mu_J > 0$, the jump component induces price appreciation on average, and the diffusion part of the price will have a drift term adjusted downwards. On the other hand, if $\mu_J < 0$, investors will be compensated by a higher drift rate to produce the same expected return.
- To simulate the jump-diffusion process defined in equation (15.46) we first consider the jump component.

- Suppose the time interval of the prices simulated is h , to simulate the jump component $J_t dN_t$ we generate a number m from the $\mathcal{PN}(\lambda h)$ distribution, and then simulate m independent variates Z_i from the $\mathcal{N}(0, 1)$ distribution. The jump component is then given by

$$m\mu_J + \sigma_J \sum_{i=1}^m Z_i. \quad (15.47)$$

- The diffusion component is computed as

$$\left(\mu - \lambda\mu_J - \frac{\sigma^2}{2} \right) h + \sigma\sqrt{h}Z, \quad (15.48)$$

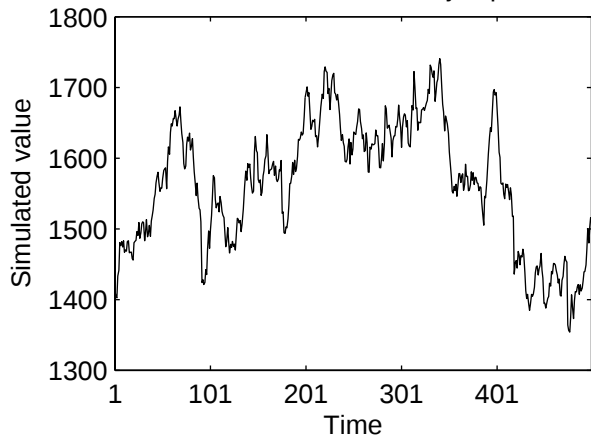
where Z is a standard normal variate independent of Z_i .

- To generate the value of $S_{t+(i+1)h}$ given S_{t+ih} we use the equation

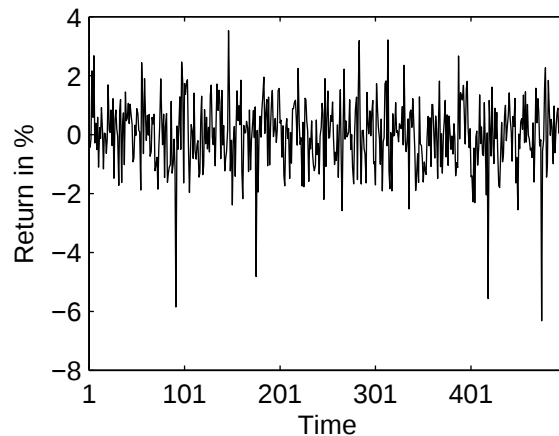
$$S_{t+(i+1)h} = S_{t+ih} \exp \left[\left(\mu - \lambda\mu_J - \frac{\sigma^2}{2} \right) h + \sigma\sqrt{h}Z \right] \exp \left[m\mu_J + \sigma_J \sum_{i=1}^m Z_i \right]. \quad (15.49)$$

- For illustration we simulate a jump-diffusion process using the following parameters: $\mu = 3.0718\%$, $\sigma = 16.5640\%$, $\lambda = 3$, $\mu_J = -2\%$ and $\sigma_J = 3\%$ (the first 3 quantities are per annum).
- Thus, the jumps occur on average 3 times per year, and each jump is normally distributed with mean jump size of 2% down and standard deviation of 3%. To observe more jumps in the simulated process, we simulate 500 daily observations (about 2 years) and an example is presented in Figure 15.3.

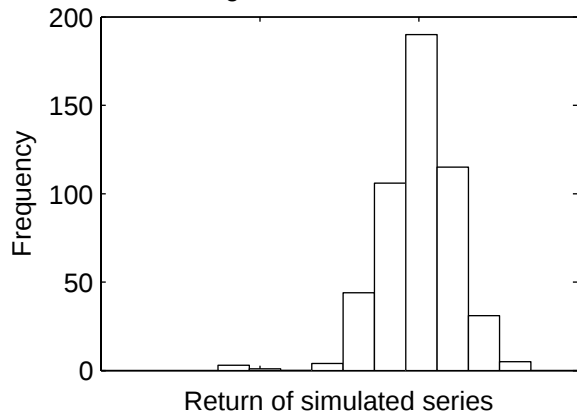
Simulated series with jumps



Return of simulated series



Histogram of simulated return



Normal probability plot

