Financial Mathematics
for Actuaries

Chapter 4

Rates of Return

Learning Objectives

- Internal rate of return (yield rate)

- One-period rate of return of a fund: time-weighted rate of return and dollar-weighted (money-weighted) rate of return

- Rate of return over longer periods: geometric mean rate of return and arithmetic mean rate of return

- Portfolio return and return of a short-selling strategy

- Crediting interest: investment-year method and portfolio method

- Inflation: real rate of return

- Capital budgeting and project appraisal

4.1 Internal Rate of Return

- Consider a project with initial investment C_0. We assume the cash flows occur at regular intervals.

- The project lasts for n years and the future cash flows are denoted by $C_1, \cdots, C_n$.

- We adopt the convention that cash inflows to the project (investments) are positive and cash outflows from the project (withdrawals) are negative.

- We define the **internal rate of return (IRR)** (also called the **yield rate**) as the rate of interest such that the sum of the present values of the cash flows is equated to zero.

- Denoting the internal rate of return by y, we have

$$\sum_{j=0}^{n} \frac{C_j}{(1+y)^j} = 0, \tag{4.1}$$

  where j is the time at which the cash flow C_j occurs.

- This equation can also be written as

$$C_0 = -\sum_{j=1}^{n} \frac{C_j}{(1+y)^j}. \tag{4.2}$$

- Thus, the net present value of all future withdrawals (injections are negative withdrawals) evaluated at the IRR is equal to the initial investment.

Example 4.1:   A project requires an initial cash outlay of \$2,000 and is expected to generate \$800 at the end of year 1 and \$1,600 at the end

of year 2, at which time the project will terminate. Calculate the IRR of the project.

Solution: If we denote $v = 1/(1 + y)$, we have, from (4.2)

$$2{,}000 = 800v + 1{,}600v^2,$$

or

$$5 = 2v + 4v^2.$$

Dropping the negative answer from the quadratic equation, we have

$$v = \frac{-2 + \sqrt{4 + 4 \times 4 \times 5}}{2 \times 4} = 0.8956.$$

Thus, $y = (1/v) - 1 = 11.66\%$. Note that $v < 0$ implies $y < -1$, i.e., the loss is larger than 100%, which is precluded from consideration. $\square$

- There is generally no analytic solution for y in (4.1) when $n > 2$, and numerical methods have to be used. The Excel function `IRR` enables us to compute the answer easily. Its usage is described as follows:

> **Excel function: IRR(values,guess)**
>
> **values** = an array of values containing the cash flows
> guess = starting value, set to 0.1 if omitted
> Output = IRR of cash flows

Example 4.2: An investor pays \$5 million for a 5-year lease of a shopping mall. He will receive \$1.2 million rental income at the end of each year. Calculate the IRR of his investment.

Solution: See Exhibit 4.1. □

	A	B	C	D
	A7 ▼	f_x =IRR(A1:A6,0.1)		
1	5			
2	-1.2			
3	-1.2			
4	-1.2			
5	-1.2			
6	-1.2			
7	6.4022%			
8				

Exhibit 4.1: IRR computation in Example 4.2

- If the cash flows occur more frequently than once a year, such as monthly or quarterly, y computed from (4.1) is the IRR for the payment interval.

- Suppose cash flows occur m times a year, the nominal IRR in annualized term is $m \cdot y$, while the annual effective rate of return is $(1+y)^m - 1$.

Example 4.3: A cash outlay of \$100 generates incomes of \$20 after 4 months and 8 months, and \$80 after 2 years. Calculate the IRR of the investment.

Solution: If we treat one month as the interest conversion period, the equation of value can be written as

$$100 = \frac{20}{(1+y_1)^4} + \frac{20}{(1+y_1)^8} + \frac{80}{(1+y_1)^{24}},$$

where y_1 is the IRR on monthly interval. The nominal rate of return on monthly compounding is $12y_1$. Alternatively, we can use the 4-month interest conversion interval, and the equation of value is

$$100 = \frac{20}{1 + y_4} + \frac{20}{(1 + y_4)^2} + \frac{80}{(1 + y_4)^6},$$

where y_4 is the IRR on 4-month interval. The nominal rate of return on 4-monthly compounding is $3y_4$. The effective annual rate of return is $y = (1 + y_1)^{12} - 1 = (1 + y_4)^3 - 1$.

The above equations of value have to be solved numerically for y_1 or y_4. We obtain 1.0406% as the IRR per month, namely, y_1. The effective annualized rate of return is then $(1.010406)^{12} - 1 = 13.23\%$. Solving for y_4 with Excel, we obtain the answer 4.2277%. Hence, the annualized effective rate is $(1.042277)^3 - 1 = 13.23\%$, which is equal to the effective rate computed using y_1. $\qquad\square$

- When cash flows occur irregularly, we can define y as the annualized rate and express all time of occurrence of cash flows in years.

- Suppose there are $n+1$ cash flows occurring at time 0, t_1, $\cdots$, t_n, with cash amounts C_0, C_1, $\cdots$, C_n. Equation (4.2) is rewritten as

$$C_0 = -\sum_{j=1}^{n} \frac{C_j}{(1+y)^{t_j}}, \qquad (4.3)$$

  which requires numerical methods for the solution of y.

- We may also use the Excel function $\texttt{XIRR}$ to solve for y. Unlike $\texttt{IRR}$, $\texttt{XIRR}$ allows the cash flows to occur at *irregular* time intervals. The specification of $\texttt{XIRR}$ is as follows:

Excel function: XIRR(**values**,**dates**,guess)
values $= C_j$, an array of values containing the cash flows in (4.3)
dates $= t_j$, an array of values containing the timing of the cash flows in (4.3)
guess = guess starting value, set to 0.1 if omitted
Output = IRR, value of annualized y satisfying (4.3)

Example 4.4: A project requires an outlay of $2.35 million in return for $0.8 million after 9 months, $1 million after 15 months and $1 million after 2 years. What is the IRR of the project?

Solution: Returns of the project occur at time (in years) 0.75, 1.25 and 2. We solve for v numerically from the following equation using Excel Solver (see Exhibit 4.2)
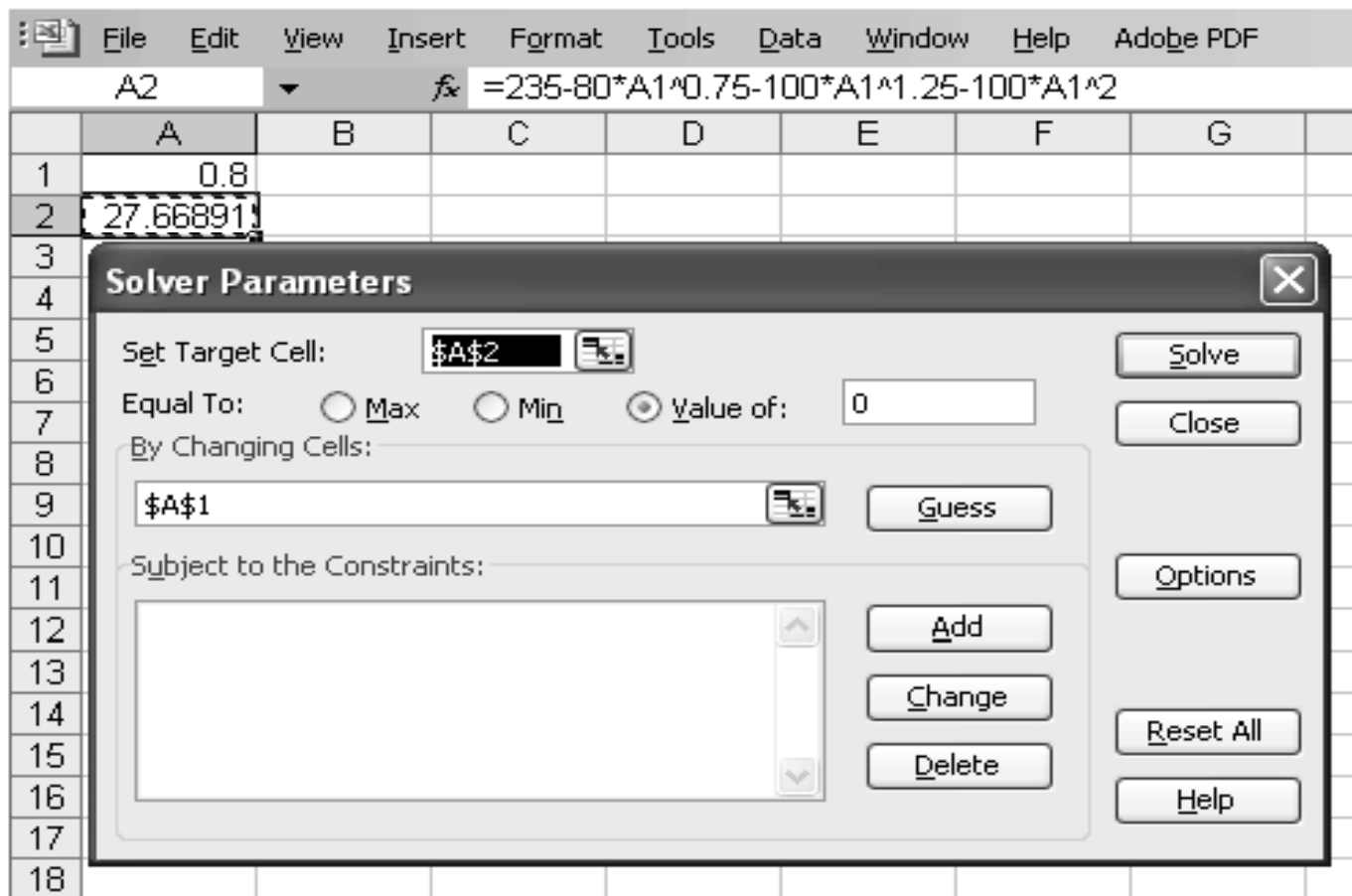
$$235 = 80v^{0.75} + 100v^{1.25} + 100v^2$$

Figure 4.2: Solution of Example 4.4 using Solver

	File	Edit	View	Insert	Format	Tools	Data	Window

A5 ▼ f_x =XIRR(A1:A4,C1:C4)

	A	B	C	D	E
1	-235	0	0		
2	80	0.75	273.75		
3	100	1.25	456.25		
4	100	2	730		
5	0.137751				
6					

Figure 4.3: Solution of Example 4.4 using XIRR

to obtain $v = 0.879$, so that

$$y = \frac{1}{0.879} - 1 = 13.77\%,$$

which is the effective annualized rate of return of the project. Alternatively, we may use the Excel function `XIRR` as shown in Exhibit 4.3. □

- A project with no subsequent investment apart from the initial capital is called a **simple project**.

- For simple projects, (4.1) has a unique solution with $y > -1$, so that IRR is well defined.

Example 4.5: A project requires an initial outlay of \$8 million, generates returns of \$50 million 1 year later, and requires \$50 million to terminate at the end of year 2. Solve for y in (4.1).

Solution: We are required to solve

$$8 = 50v - 50v^2,$$

which has $v = 0.8$ and 0.2 as solutions. This implies y has multiple solutions of 25% and 400%.   □

4.2 One-Period Rate of Return

- We consider methods of calculating the return of a fund over a 1-period interval. The methodology adopted depends on the data available.

- We start with the situation where the exact amounts of fund withdrawals and injections are known, as well as the time of their occurrence.

- Consider a 1-year period with initial fund amount B_0 (equal to C_0). Cash flow of amount C_j occurs at time t_j (in fraction of a year) for $j = 1, \cdots, n$, where $0 < t_1 < \cdots < t_n < 1$.

- Note that C_j are usually fund redemptions and new investments, and do not include investment incomes such as dividends and coupon

payments.

- Denoting the fund value before and after the transaction at time t_j by B_j^B and B_j^A, respectively, we have $B_j^A = B_j^B + C_j$ for $j = 1, \cdots, n$.

- The difference between B_j^B and B_{j-1}^A, i.e., the balance before the transaction at time t_j and after the transaction at time t_{j-1}, is due to investment incomes, as well as capital gains and losses.

- Let the fund balance at time 1 be B_1, and define $B_{n+1}^B = B_1$ and $B_0^A = B_0$ (this notation will allow us to express the gross return as (4.4) below). Figure 4.1 illustrates the time diagram.

- We now introduce two methods to calculate the 1-year rate of return: the **time-weighted rate of return** (TWRR) and the **dollar-weighted rate of return** (DWRR).
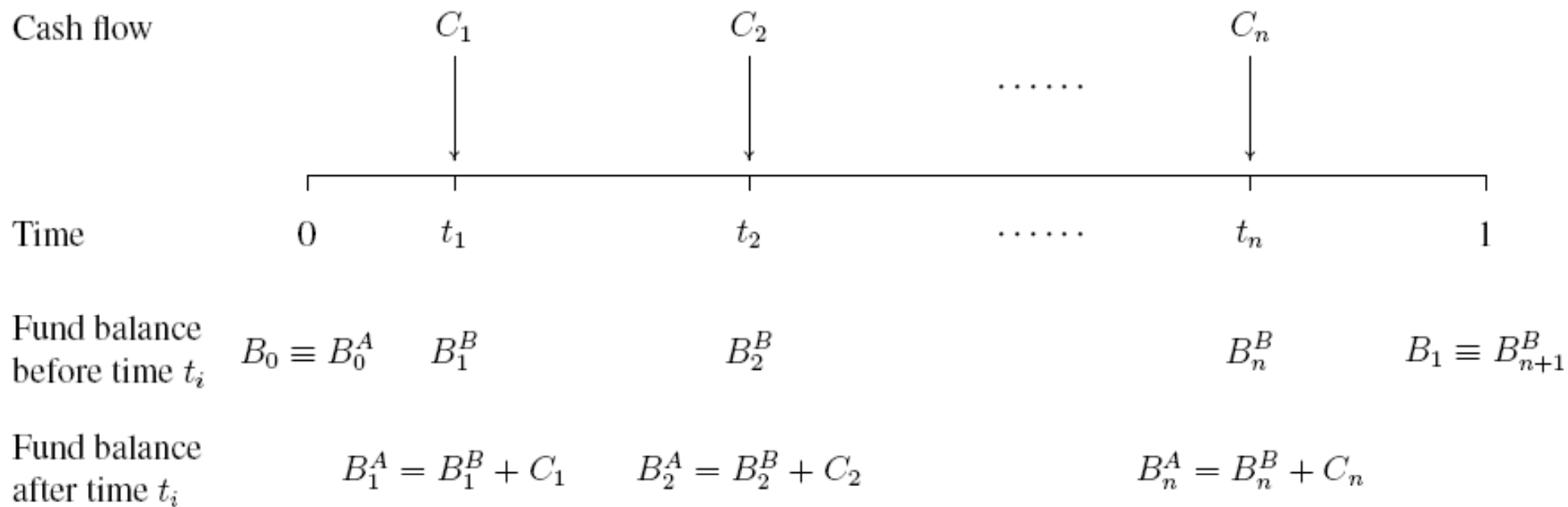
Figure 4.1:   Cash flows and fund values in period (0, 1)

- To compute the TWRR we first calculate the return over each subinterval between the occurrences of transactions by comparing the fund balances just before the new transaction to the fund balance just after the last transaction.

- If we denote R_j as the rate of return over the subinterval t_{j-1} to t_j, we have

$$1 + R_j = \frac{B_j^B}{B_{j-1}^A}, \quad \text{for } j = 1, \cdots, n+1. \tag{4.4}$$

- Then TWRR over the year, denoted by R_T, is

$$R_T = \left[\prod_{j=1}^{n+1} (1 + R_j) \right] - 1. \tag{4.5}$$

- The TWRR requires data of the fund balance prior to each withdrawal or injection. In contrast, the DWRR does not require this

15

information. It only uses the information of the amounts of the withdrawals and injections, as well as their time of occurrence.

- In principle, when cash of amount C_j is injected (withdrawn) at time t_j, there is a gain (loss) of capital of amount $C_j(1 - t_j)$ for the remaining period of the year.

- Thus, the *effective capital* of the fund over the 1-year period, denoted by B, is given by

$$B = B_0 + \sum_{j=1}^{n} C_j(1 - t_j).$$

- Denoting $C = \sum_{j=1}^{n} C_j$ as the net injection of cash (withdrawal if negative) over the year and I as the interest income earned over the year, we have $B_1 = B_0 + I + C$, so that

$$I = B_1 - B_0 - C. \tag{4.6}$$

Hence the DWRR over the 1-year period, denoted by R_D, is

$$R_D = \frac{I}{B} = \frac{B_1 - B_0 - C}{B_0 + \sum_{j=1}^{n} C_j (1 - t_j)}.$$ (4.7)

Example 4.6: On January 1, a fund was valued at 100k (1k = 1,000). On May 1, the fund increased in value to 112k and 30k of new principal was injected. On November 1, the fund value dropped to 125k, and 42k was withdrawn. At the end of the year, the fund was worth 100k. Calculate the DWRR and the TWRR.

Solution: As $C = 30 - 42 = -12$, there is a net withdrawal. From (4.6), the interest income earned over the year is

$$I = 100 - 100 - (-12) = 12.$$

Cash flow		30		42	

Time t, mm/dd   1/1   5/1   11/1   1/1

Value before t 112   125

Value after t 142   83
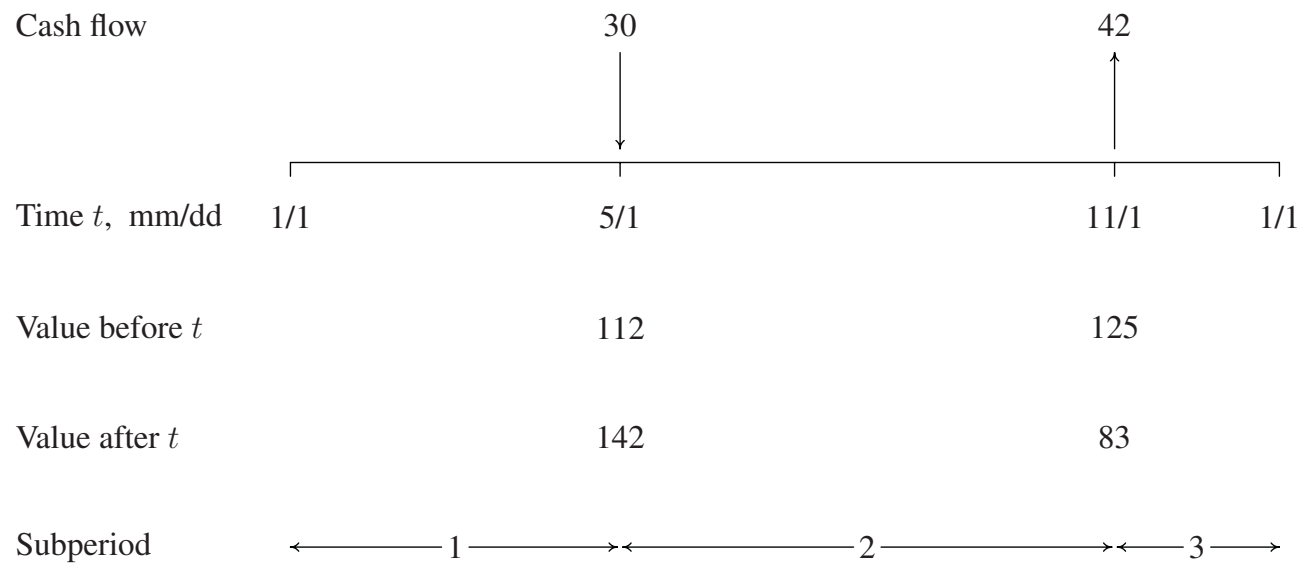
Subperiod   1   2   3

Figure 4.2:   Illustration for Example 4.6

Hence, from (4.7), the DWRR is

$$R_D = \frac{12}{100 + \dfrac{2}{3} \times 30 - \dfrac{1}{6} \times 42} = 10.62\%.$$

The fund balance just after the injection on May 1 is $112 + 30 = 142$k, and its value just after the withdrawal on November 1 is $125 - 42 = 83$k. From (4.4), the fund-value relatives over the three subperiods are

$$1 + R_1 = \frac{112}{100} = 1.120,$$

$$1 + R_2 = \frac{125}{142} = 0.880,$$

$$1 + R_3 = \frac{100}{83} = 1.205.$$

Hence, from (4.5), the TWRR is

$$R_T = 1.120 \times 0.880 \times 1.205 - 1 = 18.76\%.$$

- TWRR compounds the returns of the fund over subperiods after purging the effects of the timing and amount of cash injections and withdrawals.

- As fund managers have no control over the timing of fund injection and withdrawal, the TWRR appropriately measures the *performance of the fund manager.*

- DWRR is sensitive to the timing and amount of the cash flows.

- If the purpose is to measure the *performance of the fund,* the DWRR is appropriate.

- It allows superior market timing to impact the return of the fund.

- For funds with frequent cash injections and withdrawals, the computation of the TWRR may not be feasible. The difficulty lies in the evaluation of the fund value B_j^B, which requires the fund to be constantly **marked to market**.

- In some situations the exact timing of the cash flows may be difficult to identify.

- In this situation, we may approximate (4.7) by assuming the cash flows to be evenly distributed throughout the 1-year evaluation period.

- Hence, we replace $1 - t_j$ by its mean value of 0.5 so that $\sum_{j=1}^{n} C_j (1 - t_j) = 0.5C$, and (4.7) can be written as

$$R_D \simeq \frac{I}{B_0 + 0.5C}$$

$$= \frac{I}{B_0 + 0.5(B_1 - B_0 - I))}$$

$$= \frac{I}{0.5(B_1 + B_0 - I)}. \tag{4.8}$$

Example 4.7:   For the data in Example 4.6, calculate the approximate value of the DWRR using (4.8).

Solution: With $B_0 = B_1 = 100$, and $I = 12$, the approximate R_D is

$$R_D = \frac{12}{0.5(100 + 100 - 12)} = 12.76\%.$$

$\square$

4.3 Rate of Return over Multiple Periods

- We now consider the rate of return of a fund over a m-year period.

- We first consider the case where only annual data of returns are available. Suppose the annual rates of return of the fund have been computed as $R_1, \cdots, R_m$. Note that $-1 \leq R_j < \infty$ for all j.

- The average return of the fund over the m-year period can be calculated as the mean of the sample values. We call this measure the **arithmetic mean rate of return**, denoted by R_A, which is given by

$$R_A = \frac{1}{m} \sum_{j=1}^{m} R_j. \tag{4.9}$$

- An alternative is to use the geometric mean to calculate the average, called the **geometric mean rate of return**, denoted by R_G, which

is given by

$$R_G = \left[\prod_{j=1}^{m} (1 + R_j) \right]^{\frac{1}{m}} - 1$$

$$= [(1 + R_1)(1 + R_2) \cdots (1 + R_m)]^{\frac{1}{m}} - 1. \qquad (4.10)$$

Example 4.8: The annual rates of return of a bond fund over the last 5 years are (in %) as follows:

$$6.4 \quad 8.9 \quad 2.5 \quad -2.1 \quad 7.2$$

Calculate the arithmetic mean rate of return and the geometric mean rate of return of the fund.

Solution: The arithmetic mean rate of return is

$$R_A = (6.4 + 8.9 + 2.5 - 2.1 + 7.2)/5$$

$$= 22.9/5$$

$$= 4.58\%,$$

and the geometric mean rate of return is

$$
\begin{aligned}
R_G &= (1.064 \times 1.089 \times 1.025 \times 0.979 \times 1.072)^{\frac{1}{5}} - 1 \\
&= (1.246)^{\frac{1}{5}} - 1 \\
&= 4.50\%.
\end{aligned}
$$

□

Example 4.9: The annual rates of return of a stock fund over the last 8 years are (in %) as follows:

$$15.2 \quad 18.7 \quad -6.9 \quad -8.2 \quad 23.2 \quad -3.9 \quad 16.9 \quad 1.8$$

Calculate the arithmetic mean rate of return and the geometric mean rate of return of the fund.

Solution: The arithmetic mean rate of return is

$$R_A = (15.2 + 18.7 + \cdots + 1.8)/8 = 7.10\%$$

and the geometric mean rate of return is

$$R_G = (1.152 \times 1.187 \times \cdots \times 1.018)^{\frac{1}{8}} - 1 = 6.43\%.$$

$\square$

- Given any sample of data, the arithmetic mean is always larger than the geometric mean.

- If the purpose is to measure the return of the fund over the *holding period* of m years, the geometric mean rate of return is the appropriate measure.

- The arithmetic mean rate of return describes the average performance of the fund for *one* year taken randomly from the sample period.

- If there are more data about the history of the fund, alternative measures of the performance of the fund can be used. The methodology of the time-weighted rate of return in Section 4.2 can be extended to beyond 1 period (year).

- Suppose there are n subperiods, with returns denoted by $R_1, \cdots, R_n$, over a period of m years. Then we can measure the m-year return by compounding the returns over each subperiod to form the TWRR using the formula

$$R_T = \left[\prod_{j=1}^{n} (1 + R_j) \right]^{\frac{1}{m}} - 1. \tag{4.11}$$

- We can also compute the return over a m-year period using the IRR. We extend the notations for cash flows in Section 4.2 to the m-year period.

- Suppose cash flows of amount C_j occur at time t_j for $j = 1, \cdots, n$, where $0 < t_1 < \cdots < t_n < m$. Let the fund value at time 0 and time m be B_0 and B_1, respectively. We treat $-B_1$ as the last transaction, i.e., fund withdrawal of amount B_1.

- The rate of return of the fund is calculated as the IRR which equates the discounted values of B_0, $C_1, \cdots, C_n$, and $-B_1$ to zero.

- This is referred to as the DWRR over the m-year period. We denote it as R_D, which solves the following equation

$$B_0 + \sum_{j=1}^{n} \frac{C_j}{(1 + R_D)^{t_j}} - \frac{B_1}{(1 + R_D)^m} = 0. \qquad (4.12)$$

27

- The example below concerns the returns of a bond fund. When a bond makes the periodic coupon payments, the bond values drop and the coupons are cash amounts to be withdrawn from the fund.

Example 4.10: A bond fund has an initial value of $20 million. The fund records coupon payments in six-month periods. Coupons received from January 1 through June 30 are regarded as paid on April 1. Likewise, coupons received from July 1 through December 31 are regarded as paid on October 1. For the 2-year period 2008 and 2009, the fund values and coupon payments were recorded in Table 4.1.

Table 4.1:   Cash flows of fund

Time mm/dd/yy	Coupon received ($ millions)	Fund value before date ($ millions)
01/01/08		20.0
04/01/08	0.80	22.0
10/01/08	1.02	22.8
04/01/09	0.97	21.9
10/01/09	0.85	23.5
12/31/09		25.0

Calculate the TWRR and the DWRR of the fund.

Solution: As the coupon payments are withdrawals from the fund (the portfolio of bonds), the fund drops in value after the coupon payments. For example, the bond value drops to $22.0 - 0.80 = 21.2$ million on April 1, 2008 after the coupon payments. Thus, the TWRR is calculated as

29

$$R_T = \left[\frac{22}{20} \times \frac{22.80}{22.0 - 0.8} \times \frac{21.90}{22.80 - 1.02} \times \frac{23.50}{21.90 - 0.97} \times \frac{25.0}{23.50 - 0.85} \right]^{0.5} - 1 = 21.42\%.$$

To calculate the DWRR we solve R_D from the following equation

$$
\begin{aligned}
20 &= \frac{0.8}{(1 + R_D)^{0.25}} + \frac{1.02}{(1 + R_D)^{0.75}} + \frac{0.97}{(1 + R_D)^{1.25}} + \frac{0.85}{(1 + R_D)^{1.75}} + \frac{25}{(1 + R_D)^2} \\
&= 0.8v + 1.02v^3 + 0.97v^5 + 0.85v^7 + 25v^8,
\end{aligned}
$$

where $v = (1 + R_D)^{-0.25}$. We let $(1 + y)^{-1} = v$ and use Excel to obtain $y = 4.949\%$, so that the annual effective rate of return is $R_D = (1.04949)^4 - 1 = 21.31\%$. $\square$

Figure 4.4: DWRR computation in Example 4.10

4.4 Portfolio Return

- We now consider the return of a portfolio of assets.

- Suppose a portfolio consists of N assets denoted by $A_1, \cdots, A_N$. Let the value of asset A_j in the portfolio at time 0 be A_{0j}, for $j = 1, \cdots, N$.

- We allow A_{0j} to be negative for some j, so that asset A_j is sold short in the portfolio.

- The portfolio value at time 0 is $B_0 = \sum_{j=1}^{N} A_{0j}$. Let the asset values at time 1 be A_{1j}, so that the portfolio value is $B_1 = \sum_{j=1}^{N} A_{1j}$.

- Denote R_P as the return of the portfolio in the period from time 0

to time 1. Thus,

$$R_P = \frac{B_1 - B_0}{B_0} = \frac{B_1}{B_0} - 1.$$

- We define
$$w_j = \frac{A_{0j}}{B_0},$$
which is the proportion of the value of asset A_j in the initial portfolio, so that
$$\sum_{j=1}^{N} w_j = 1,$$
and $w_j < 0$ if asset j is short sold in the portfolio.

- We also denote
$$R_j = \frac{A_{1j} - A_{0j}}{A_{0j}} = \frac{A_{1j}}{A_{0j}} - 1,$$

which is the rate of return of asset j. Thus,

$$
\begin{aligned}
1 + R_P &= \frac{B_1}{B_0} \\
&= \frac{1}{B_0} \sum_{j=1}^{N} A_{1j} \\
&= \sum_{j=1}^{N} \frac{A_{0j}}{B_0} \times \frac{A_{1j}}{A_{0j}} \\
&= \sum_{j=1}^{N} w_j (1 + R_j),
\end{aligned}
$$

which implies

$$
R_P = \sum_{j=1}^{N} w_j R_j, \tag{4.13}
$$

so that the return of the portfolio is the weighted average of the returns of the individual assets.

- Eq (4.13) is an identity, and applies to realized returns as well as returns as random variables. If we take the expectations of (4.13), we obtain

$$\mathrm{E}(R_P) = \sum_{j=1}^{N} w_j \mathrm{E}(R_j), \qquad (4.14)$$

so that the expected return of the portfolio is equal to the weighted average of the expected returns of the component assets.

- The variance of the portfolio return is given by

$$\mathrm{Var}(R_P) = \sum_{j=1}^{N} w_j^2 \mathrm{Var}(R_j) + \underbrace{\sum_{h=1}^{N}\sum_{j=1}^{N}}_{h \neq j} w_h w_j \mathrm{Cov}(R_h, R_j). \qquad (4.15)$$

- For example, consider a portfolio consisting of two funds, a stock fund and a bond fund, with returns denoted by R_S and R_B, respec-

tively. Likewise, we use w_S and w_B to denote their weights in the portfolio.

- Then, we have

$$E(R_P) = w_S E(R_S) + w_B E(R_B), \qquad (4.16)$$

and

$$\text{Var}(R_P) = w_S^2 \text{Var}(R_S) + w_B^2 \text{Var}(R_B) + 2 w_S w_B \text{Cov}(R_S, R_B), \quad (4.17)$$

where $w_S + w_B = 1$.

Example 4.11: A stock fund has an expected return of 0.15 and variance of 0.0625. A bond fund has an expected return of 0.05 and variance of 0.0016. The correlation coefficient between the two funds is -0.2.

(a) What is the expected return and variance of the portfolio with 80% in the stock fund and 20% in the bond fund?

(b) What is the expected return and variance of the portfolio with 20% in the stock fund and 80% in the bond fund?

(c) How would you weight the two funds in your portfolio so that your portfolio has the lowest possible variance?

Solution: For (a), we use (4.16) and (4.17), with $w_S = 0.8$ and $w_B = 0.2$, to obtain

$$
\begin{aligned}
\mathrm{E}(R_P) &= (0.8)(0.15) + (0.2)(0.05) = 13\% \\
\mathrm{Var}(R_P) &= (0.8)^2(0.0625) + (0.2)^2(0.0016) + 2(0.8)(0.2)(-0.2)\sqrt{(0.0016)(0.0625)} \\
&= 0.03942.
\end{aligned}
$$

Thus, the portfolio has a standard deviation of $\sqrt{0.03942} = 19.86\%$. For (b), we do similar calculations, with $w_S = 0.2$ and $w_B = 0.8$, to obtain $\mathrm{E}(R_P) = 7\%$, $\mathrm{Var}(R_P) = 0.002884$ and a standard deviation of $\sqrt{0.002884} = 5.37\%$.

Hence, we observe that the portfolio with a higher weightage in stock has a higher expected return but also a higher standard deviation, i.e., higher risk.

For (c) we rewrite (4.17) as

$$\mathrm{Var}(R_P) = w_S^2 \mathrm{Var}(R_S) + (1 - w_S)^2 \mathrm{Var}(R_B) + 2w_S(1 - w_S)\mathrm{Cov}(R_S,\ R_B).$$

To minimize the variance, we differentiate $\mathrm{Var}(R_P)$ with respect to w_S to obtain

$$2w_S \mathrm{Var}(R_S) - 2(1 - w_S)\mathrm{Var}(R_B) + 2(1 - 2w_S)\mathrm{Cov}(R_S,\ R_B).$$

Equating the above to zero, we solve for w_S to obtain

$$
\begin{aligned}
w_S &= \frac{\text{Var}(R_B) - \text{Cov}(R_S,\, R_B)}{\text{Var}(R_B) + \text{Var}(R_S) - 2\text{Cov}(R_S,\, R_B)} \\
&= 5.29\%.
\end{aligned}
$$

The expected return of this portfolio is 5.53%, its variance is 0.001410, and its standard deviation is 3.75%, which is lower than the standard deviation of the bond fund of 4%. Hence, this portfolio *dominates* the bond fund, in the sense that it has a higher expected return and a lower standard deviation.

Note that the fact that the above portfolio indeed minimizes the variance can be verified by examining the second-order condition. $\square$

4.5 Short sales

- A **short sale** is the sale of a security that the seller does not own. It can be executed through a **margin account** with a brokerage firm.

- The seller *borrows* the security from the brokerage firm to deliver to the buyer.

- The sale is based on the belief that the security price will go down in the future so that the seller will be able to buy back the security at a lower price, thus keeping the difference in price as profit.

- Proceeds from the short sale are kept in the margin account, and cannot be invested to earn income.

- The seller is required to place cash or securities into the margin account. The **initial percentage margin** m is the percentage of

the proceeds of the short sold security that the seller must place into the margin account.

- If P_0 is the price of the security when it is sold short, the initial deposit D is $P_0 m$. The seller will earn interest from the deposit.

- At any point in time, there is a **maintenance margin** m^*, which is the minimum percentage of the seller's equity in relation to the current value of the security sold short.

- If the current security price is P_1, the equity E is $P_0 + D - P_1$ and E/P_1 must be larger than m^*.

- If E/P_1 falls below m^*, the seller will get a **margin call** from the broker instructing him to top up his margin account.

Example 4.12: A person sold 1,000 shares of a stock short at $20. If the initial margin is 50%, how much should he deposit in his margin account? If the maintenance margin is 30%, how high can the price go up before there is a margin call?

Solution: The initial deposit is

$$1{,}000 \times 20 \times 0.5 = \$10{,}000.$$

At price P_1, the percent of equity is

$$\frac{10{,}000 + 1{,}000(20 - P_1)}{1{,}000P_1},$$

which must be more than 0.3. Thus, the maximum P_1 is

$$\frac{30}{1.3} = \$23.08.$$

$\square$

- To calculate the rate of return of a short sale strategy we note that the capital is the deposit D in the margin account. The return includes the interest earned in the deposit. The seller, however, pays the dividend to the buyer if there is any dividend payout. The net rate of return will thus take account of the interest earned and the dividend paid.

Example 4.13: A person sells a stock short at $30. The stock pays a dividend of $1 at the end of the year, after which the man covers his short position by buying the stock back at $27. The initial margin is 50% and interest rate is 4%. What is his rate of return over the year?

Solution: The capital investment per share is $15. The gross return after one year is $(30 - 27) - 1 + 15 \times 0.04 = \$2.6$. Hence, the return over

the 1-year period is
$$\frac{2.6}{15} = 17.33\%.$$

Note that in the above calculation we have assumed that there was no margin call throughout the year. $\qquad\square$

4.6 Crediting Interest: Investment-Year Method and Portfolio Method

- A fund pools the investments of individual investors. The investors may invest new money into the fund at any time.

- While the fund invests the aggregate of the investments, there is an issue of how to credit interest to the individual investors' accounts.

- A simple method is to credit the average return of the fund to all investors. This is called the **portfolio method**.

- The portfolio method may not be equitable when the individual investments are made at different times.

- For example, at a time when returns to securities are going up, new investments are likely to achieve higher returns compared to old

investments. In this case, crediting the average return will not be attractive to new investments.

- When security returns are declining, the portfolio method may be crediting higher returns to new investors at the expense of the old ones.

- Another method is to credit the accounts according to their year of investment. This method is called the **investment-year method.**

- Under this method, new investments are credited the investment-year rates of interest over a period of time, after which the investors are credited the portfolio rate of interest.

- Table 4.2 gives an example. For simplicity of exposition, we assume that investments are made at the beginning of the calendar year and interests are credited at the end of the year.

Table 4.2:   An example of investment-year rates and portfolio rates

Calendar year of investment (Y)	Investment-year rates (%)			Portfolio rates (%)
	i_1^Y	i_2^Y	i_3^Y	i^{Y+3}
2004	5.6	5.6	5.7	5.9
2005	5.6	5.7	5.8	6.2
2006	5.8	5.9	6.0	6.3
2007	6.2	6.3	6.6	
2008	6.7	6.4		
2009	7.1			

- In Table 4.2, investments are credited the investment-year rates in the first 3 years. The calendar year of the investment is denoted by Y, the rate of interest credited for the tth year of the investment made in calendar year Y is i_t^Y for $t = 1, 2, 3$, and i^Y is the portfolio rate credited for calendar year Y.

- Using these notations, $i_t^Y = i^{Y+t-1}$ for $t > 3$.

Example 4.14: A fund credits interest according to Table 4.2. Find the total interest credited in the period 2007 through 2009 for (a) an investment in 2002, (b) an investment in 2006, (c) an investment in 2007, (d) an investment in 2007, withdrawn every year and reinvested in the fund as new money. All investments, including reinvestments, are made on January 1.

Solution: The last column of the table gives the portfolio rate of interest in 2007 through 2009. For (a), the investment made in 2002 earns the portfolio rate of interest in 2007 through 2009. Thus, the total return over the 3-year period is

$$1.059 \times 1.062 \times 1.063 - 1 = 19.55\%.$$

For (b), the investment made in 2006 earns the investment-year rates in 2007 and 2008 of 5.9% and 6.0%, respectively, and the portfolio rate in 2009 of 6.3%. The total return is

$$1.059 \times 1.06 \times 1.063 - 1 = 19.33\%.$$

For (c), the investment made in 2007 earns the investment-year rates in 2007 through 2009 to obtain the total return

$$1.062 \times 1.063 \times 1.066 - 1 = 20.34\%.$$

Finally, for (d) if an investment made in 2007 is withdrawn every year and reinvested at the new money rate, the total return is

$$1.062 \times 1.067 \times 1.071 - 1 = 21.36\%.$$

□

4.7 Inflation and Real Rate of Interest

- Inflation refers to the increase in the general price level of goods and services.

- The **rate of inflation** is usually measured by the rate of increase of a general price index such as the consumer price index, whole-sale price index or gross domestic product deflator.

- We define the **real rate of interest** as the interest earned by an investment after taking account of the erosion of purchasing power due to inflation.

- We denote r_I as the inflation rate, and r_N as the nominal rate of interest.

- The real rate of interest r_R is then defined by the equation

$$1 + r_R = \frac{1 + r_N}{1 + r_I}, \tag{4.18}$$

from which we obtain

$$r_R = \frac{1 + r_N}{1 + r_I} - 1 = \frac{r_N - r_I}{1 + r_I}. \tag{4.19}$$

- If the rate of inflation r_I is low, we conclude from (4.19) that

$$r_R \simeq r_N - r_I, \tag{4.20}$$

which says that the real rate of interest is approximately equal to the nominal rate of interest minus the rate of inflation.

- Note that (4.18) and (4.19) are *ex post* relationships. They hold empirically and do not refer to any theoretical relationship between the three quantities.

- The economist Irving Fisher argued that the nominal rate of interest ought to increase one for one with the *expected* rate of inflation.

- The so-called *Fisher equation* states that

$$r_N = r_R + \mathrm{E}(r_I), \qquad (4.21)$$

  where $\mathrm{E}(r_I)$ is the expected rate of inflation.

Example 4.15:   A stock will pay a dividend of $0.50 two months from now and the annual dividend is expected to grow at a rate of 4% per annum indefinitely. If the stock is traded at $7.5 and inflation is expected to be 2% per annum, what is the expected effective annual real rate of return for an investor who purchases the stock?

Solution: Let $v = 1/(1 + r_N)$, where r_N is the nominal rate of return prior to adjustment for inflation. The dividends are $0.5, 1.04(0.5), 1.04^2(0.5), \cdots$

at time of 2 months, 14 months, 26 months, $\cdots$, respectively from now. Thus, the equation of value is

$$
\begin{aligned}
7.5 &= 0.5v^{\frac{1}{6}} + 1.04(0.5)v^{\frac{14}{12}} + 1.04^2(0.5)v^{\frac{26}{12}} + \cdots \\
&= 0.5v^{\frac{1}{6}} \left[1 + 1.04v + (1.04v)^2 + \cdots\right] \\
&= \frac{0.5v^{\frac{1}{6}}}{1 - 1.04v}.
\end{aligned}
$$

The above equation has no analytic solution, and we solve it using Excel to obtain $v = 0.898568$ and $r_N = (1/0.898568) - 1 = 0.112882$. Thus, the real rate of return is

$$
r_R = \frac{1 + r_N}{1 + r_I} - 1 = \frac{1.112882}{1.02} - 1 = 9.1061\%.
$$

$\square$

4.8 Capital Budgeting and Project Appraisal

- Capital budgeting and project appraisal refer to the managerial decision of investing in a project. We consider some methods of making such decisions.

- One approach is called the **IRR rule**, which requires the manager to calculate the IRR of the project. The IRR is then compared against the **required rate of return** of the project, which is the minimum rate of return on an investment needed to make it acceptable to a business.

- We denote the required rate of return by R_R. The IRR rule accepts the project if IRR $\geq R_R$, .

- However, R_R is often a difficult quantity to identify. It depends

on factors such as the market rate of interest, the horizon of the investment and the risk of the project. In what follows, we shall assume R_R to be given and focus on the application of the budgeting rules.

- An alternative approach is the **net present value (NPV) rule**.

- Consider a project with annual cash flows and adopt the notations in Section 4.1. Discounting the future cash flows by R_R, we rewrite (4.2) and define the NPV of the project as

$$\text{NPV} = -\sum_{j=1}^{n} \frac{C_j}{(1 + R_R)^j} - C_0. \tag{4.22}$$

- A positive NPV implies that the sum of the present values of cash outflows created by the project exceeds the sum of the present values

of all investments. Thus, managers should invest in projects with positive NPV.

- As we have seen in Section 4.1, the IRR may not be unique. When there are multiple IRRs, the decision is ambiguous.

Example 4.16: Consider a 2-year project with an initial investment of $1,000. A cash amount of $2,230 will be generated after 1 year, and the project will be terminated with fund injection of $1,242 at the end of year 2. What conclusion can be drawn regarding the acceptance of this project?

Solution: The cash flow diagram is given in Figure 4.3. To calculate the IRR we solve for the equation

$$1,000 - 2,230v + 1,242v^2 = 0,$$

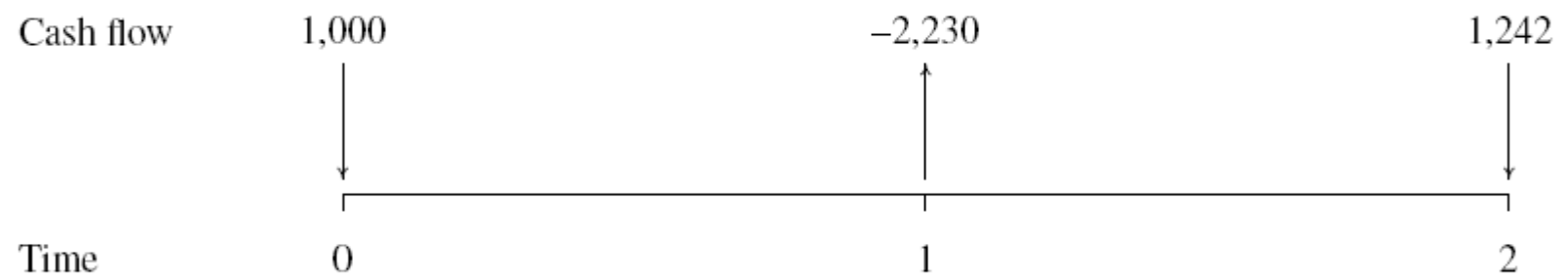| Cash flow	1,000	−2,230	1,242

Figure 4.3:   Time diagram of the project in Example 4.16

where $v = 1/(1+y)$ and y is the IRR. Note that we have adopted the convention of cash inflow being positive and outflow being negative. Solving for the above equation we obtain $y = 8\%$ or 15%. For R_R lying between the two roots of y, the IRR rule cannot be applied. However, if R_R is below the minimum of the two solutions of y, say 6%, can we conclude the project be accepted?

Evaluating the NPV at $R_R = 6\%$, we obtain

$$\begin{aligned} \text{NPV} &= -1{,}000 + \frac{2{,}230}{1.06} - \frac{1{,}242}{(1.06)^2} \\ &= -\$1.60. \end{aligned}$$

Thus, the project has a negative NPV and should be rejected. Figure 4.4 plots the NPV of the project as a function of the interest rate. It shows that the NPV is positive for $0.08 < R_R < 0.15$, which is the region of R_R for which the project should be accepted. Outside this region, the project
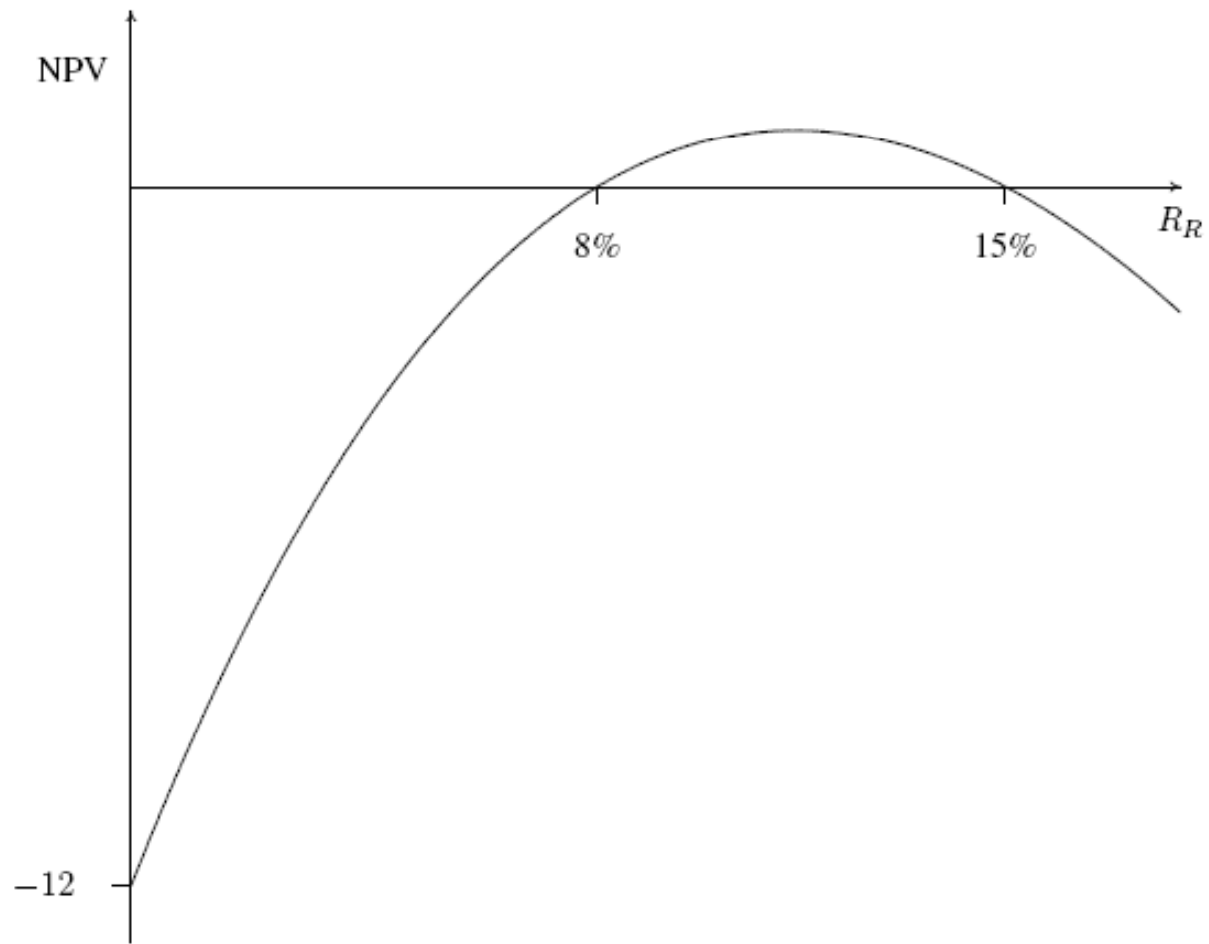
Figure 4.4: NPV versus R_R for project in Example 4.16

has a negative NPV and should be rejected. □

- The NPV rule can be applied with varying interest rates that reflects a term structure which is not flat.

- We can modify (4.22) to calculate the NPV, where the cash flow at time j is discounted by the rate of interest R_j, such as

$$\text{NPV} = -\sum_{j=1}^{n} \frac{C_j}{(1 + R_j)^j} - C_0. \qquad (4.23)$$

- If the IRR method is adopted, the term structure must be assumed to be flat.

- While (4.22) computes the NPV of the project upon its completion, it can be modified to compute the NPV of the project at a time prior to its completion.

- Thus, for $t \leq n$, we define the NPV of the project up to time t as

$$\text{NPV}(t) = -\sum_{j=1}^{t} \frac{C_j}{(1+R_R)^j} - C_0. \qquad (4.25)$$

- If $\text{NPV}(t)$ changes sign only once in the duration of the project, we can calculate the value of t at which $\text{NPV}(t)$ first becomes positive.

- This value of t is called the **discounted payback period** (DPP). Formally, DPP is defined as

$$\text{DPP} = \min\{t : \text{NPV}(t) > 0\}. \qquad (4.26)$$

- A capital budgeting criterion is to choose the project with the lowest DPP.

Example 4.19: Consider the two projects in Example 4.17. Discuss the choice of the two projects based on the DPP criterion.

Solution: For Project B, as the most substantial cash flow generated is at the end of the project duration, the DPP of the project is 15 years. For Project A, Table 4.3 summarizes the results for several required rates of interest R_R.

Table 4.3: NPV of Project A

R_R	t	NPV(t)	DPP
0.03	12	–4.5996	13
	13	63.4955	
0.04	13	–1.4352	14
	14	56.3123	
0.05	14	–10.1359	15
	15	37.9658	
0.06	15	–28.7751	16
	16	10.5895	

For required rates of interest of 3%, 4%, 5% and 6%, the DPP of Project A are, respectively, 13, 14, 15 and 16 years. When the required rate of interest is 6%, Project B has a negative NPV and the DPP is not defined. When the required interest rate is 5%, Project B has the same

DPP as Project A, namely, 15 years. For other required rates of interest considered, the DPP of Project A is shorter. □

- If R_R in (4.25) is set equal to zero, the resulting value of t at which NPV(t) first becomes positive is called the **payback period**. Although a project appraisal method may be based on the minimum payback period, this method lacks theoretical rigor as the time value of money is not taken into account.