Financial Mathematics for Actuaries

Chapter 3

Spot Rates, Forward Rates and the Term Structure

Learning Objectives

1. Spot rate of interest
2. Forward rate of interest
3. Yield curve
4. Term structure of interest rates

3.1 Spot and Forward Rates of Interest

- We now allow the rate of interest to vary with the duration of the investment.

- We consider the case where investments over different horizons earn different rates of interest, although the principle of compounding still applies.

- We consider two notions of interest rates, namely, the **spot rate of interest** and the **forward rate of interest**.

- Consider an investment at time 0 earning interest over t periods. We assume that the period of investment is fixed at the time of investment, but the rate of interest earned per period varies according to the investment horizon.

- Thus, we define i_t^S as the spot rate of interest, which is the annualized effective rate of interest for the period from time 0 to t.

- The subscript t in i_t^S highlights that the annual rate of interest varies with the investment horizon.

- Hence, a unit payment at time 0 accumulates to

$$a(t) = (1 + i_t^S)^t \tag{3.1}$$

  at time t.

- The present value of a unit payment due at time t is

$$\frac{1}{a(t)} = \frac{1}{(1 + i_t^S)^t}. \tag{3.2}$$

- We define i_t^F as the rate of interest applicable to the period $t-1$ to t, called the forward rate of interest.

- This rate is determined at time 0, although the payment is due at time $t-1$ (thus, the use of the term *forward*).

- By convention, we have $i_1^S \equiv i_1^F$. However, i_t^S and i_t^F are generally different for $t = 2, 3, \cdots$.

- See Figure 3.1.

- A plot of i_t^S against t is called the **yield curve**, and the mathematical relationship between i_t^S and t is called the **term structure of interest rates**.

- The spot and forward rates are not free to vary independently of each other.

- Consider the case of $t = 2$. If an investor invests a unit amount at time 0 over 2 periods, the investment will accumulate to $(1 + i_2^S)^2$ at

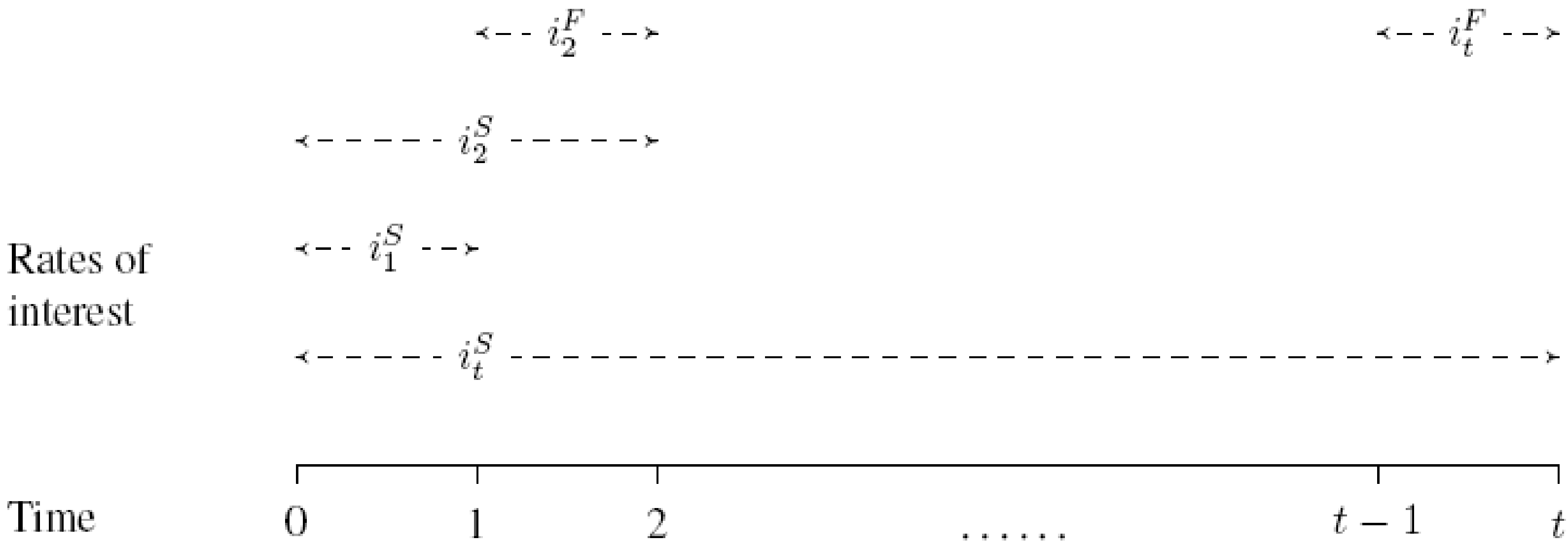

Figure 3.1: Spot and forward rates of interest

time 2.

- Alternatively, she can invest a unit payment at time 0 over 1 period, and enters into a forward agreement to invest $1 + i_1^S$ unit at time 1 to earn the forward rate of i_2^F for 1 period.

- This rollover strategy will accumulate to $(1 + i_1^S)(1 + i_2^F)$ at time 2. The two strategies will accumulate to the same amount at time 2, so that

$$(1 + i_2^S)^2 = (1 + i_1^S)(1 + i_2^F), \tag{3.3}$$

if the capital market is *perfectly competitive*, so that no arbitrage opportunities exist.

- Equation (3.3) can be generalized to the following relationship concerning spot and forward rates of interest

$$(1 + i_t^S)^t = (1 + i_{t-1}^S)^{t-1}(1 + i_t^F), \tag{3.4}$$

for $t = 2, 3, \cdots$.

- We can also conclude that

$$(1 + i_t^S)^t = (1 + i_1^F)(1 + i_2^F) \cdots (1 + i_t^F). \tag{3.5}$$

- Given i_t^S, the forward rates of interest i_t^F satisfying equations (3.4) and (3.5) are called the *implicit* forward rates.

- The *quoted* forward rates in the market may differ from the implicit forward rates in practice, as when the market is noncompetitive.

- Unless otherwise stated we shall assume that equations (3.4) and (3.5) hold, so that it is the implicit forward rates we are referring to in our discussions.

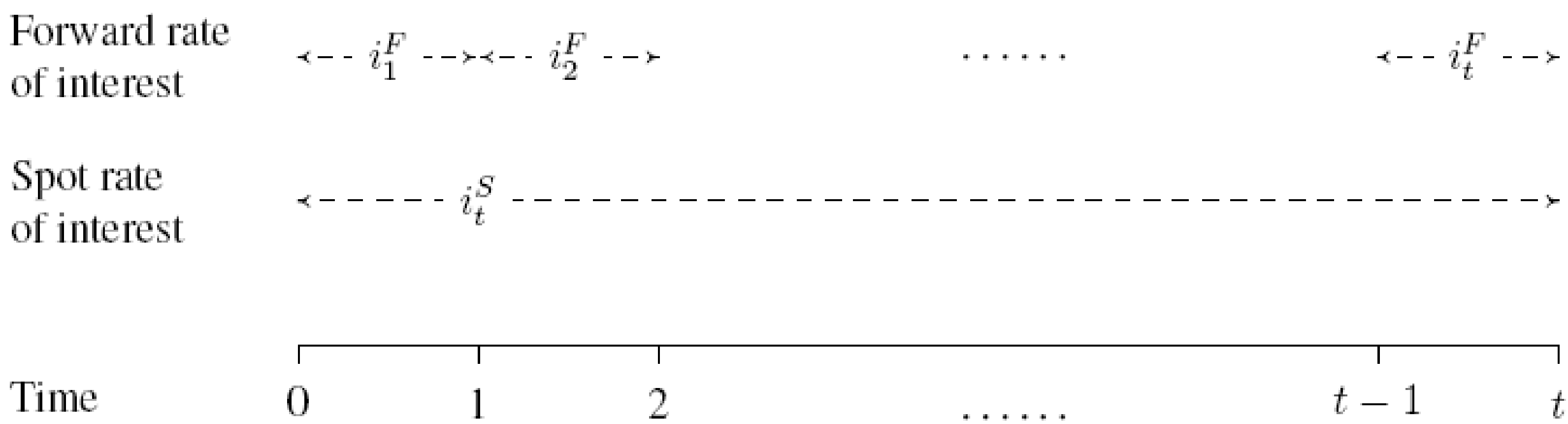

Figure 3.2: Illustration of equation (3.5)

- From equation (3.4),

$$i_t^F = \frac{(1+i_t^S)^t}{(1+i_{t-1}^S)^{t-1}} - 1. \tag{3.6}$$

Example 3.1: Suppose the spot rates of interest for investment horizons of 1, 2, 3 and 4 years are, respectively, 4%, 4.5%, 4.5%, and 5%. Calculate the forward rates of interest for $t = 1$, 2, 3 and 4.

Solution: First, $i_1^F = i_1^S = 4\%$. The rest of the calculation, using (3.6), is as follows

$$i_2^F = \frac{(1+i_2^S)^2}{1+i_1^S} - 1 = \frac{(1.045)^2}{1.04} - 1 = 5.0024\%,$$

$$i_3^F = \frac{(1+i_3^S)^3}{(1+i_2^S)^2} - 1 = \frac{(1.045)^3}{(1.045)^2} - 1 = 4.5\%$$

and

$$i_4^F = \frac{(1+i_4^S)^4}{(1+i_3^S)^3} - 1 = \frac{(1.05)^4}{(1.045)^3} - 1 = 6.5144\%.$$

$\square$

Example 3.2: Suppose the forward rates of interest for investments in year 1, 2, 3 and 4 are, respectively, 4%, 4.8%, 4.8% and 5.2%. Calculate the spot rates of interest for $t = 1$, 2, 3 and 4.

Solution: First, $i_1^S = i_1^F = 4\%$. From (3.5) the rest of the calculation is as follows

$$i_2^S = \left[(1+i_1^F)(1+i_2^F)\right]^{\frac{1}{2}} - 1 = \sqrt{1.04 \times 1.048} - 1 = 4.3992\%,$$

$$i_3^S = \left[(1+i_1^F)(1+i_2^F)(1+i_3^F)\right]^{\frac{1}{3}} - 1 = (1.04 \times 1.048 \times 1.048)^{\frac{1}{3}} - 1 = 4.5327\%$$

and

$$i_4^S = \left[(1+i_1^F)(1+i_2^F)(1+i_3^F)(1+i_4^F)\right]^{\frac{1}{4}} - 1$$

$$
\begin{aligned}
&= (1.04 \times 1.048 \times 1.048 \times 1.052)^{\frac{1}{4}} - 1 \\
&= 4.6991\%.
\end{aligned}
$$

$\square$

- We define the multi-period forward rate $i^F_{t,\tau}$ as the annualized rate of interest applicable over τ periods from time t to $t + \tau$, for $t \geq 1$ and $\tau > 0$, with the rate being determined at time 0.

- The following no-arbitrage relationships hold

$$(1+i^F_{t,\tau})^\tau = (1+i^F_{t+1})(1+i^F_{t+2}) \cdots (1+i^F_{t+\tau}), \quad \text{for } t \geq 1, \tau > 0, \tag{3.7}$$

and

$$(1 + i^S_{t+\tau})^{t+\tau} = (1 + i^S_t)^t (1 + i^F_{t,\tau})^\tau, \quad \text{for } t \geq 1, \tau > 0. \tag{3.8}$$

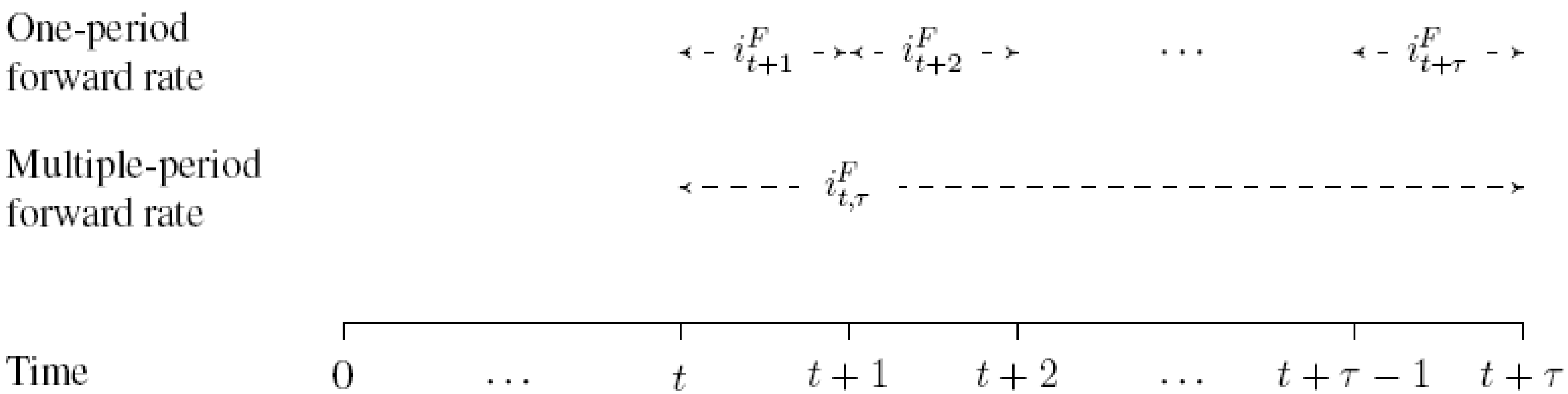

Figure 3.3: Illustration of equation (3.7)

Example 3.3: Based on the spot rates of interest in Example 3.1, calculate the multi-period forward rates of interest $i^F_{1,2}$ and $i^F_{1,3}$.

Solution: Using (3.7) we obtain

$$i^F_{1,2} = [(1 + i^F_2)(1 + i^F_3)]^{\frac{1}{2}} - 1 = (1.050024 \times 1.045)^{\frac{1}{2}} - 1 = 4.7509\%.$$

Similarly, we have

$$i^F_{1,3} = (1.050024 \times 1.045 \times 1.065144)^{\frac{1}{3}} - 1 = 5.3355\%.$$

We may also use (3.8) to compute the multi-period forward rates. Thus,

$$i^F_{1,2} = \left[\frac{(1 + i^S_3)^3}{1 + i^S_1}\right]^{\frac{1}{2}} - 1 = \left[\frac{(1.045)^3}{1.04}\right]^{\frac{1}{2}} - 1 = 4.7509\%,$$

and similarly,

$$i^F_{1,3} = \left[\frac{(1 + i^S_4)^4}{1 + i^S_1}\right]^{\frac{1}{3}} - 1 = \left[\frac{(1.05)^4}{1.04}\right]^{\frac{1}{3}} - 1 = 5.3355\%.$$

3.2 The Term Structure of Interest Rates

- Empirically the term structure can take various shapes. A sample of four yield curves of the US market are presented in Figure 3.4.
- The spot-rate curve on 1998/06/30 is an example of a nearly **flat term structure**.
- On 2000/03/31 we have a **downward sloping term structure**. In this case the forward rate drops below the spot rate.
- We have an **upward sloping term structure** on 2001/08/31.
- Unlike the case of a downward sloping yield curve, the forward rate exceeds the spot rate when the yield curve is upward sloping.

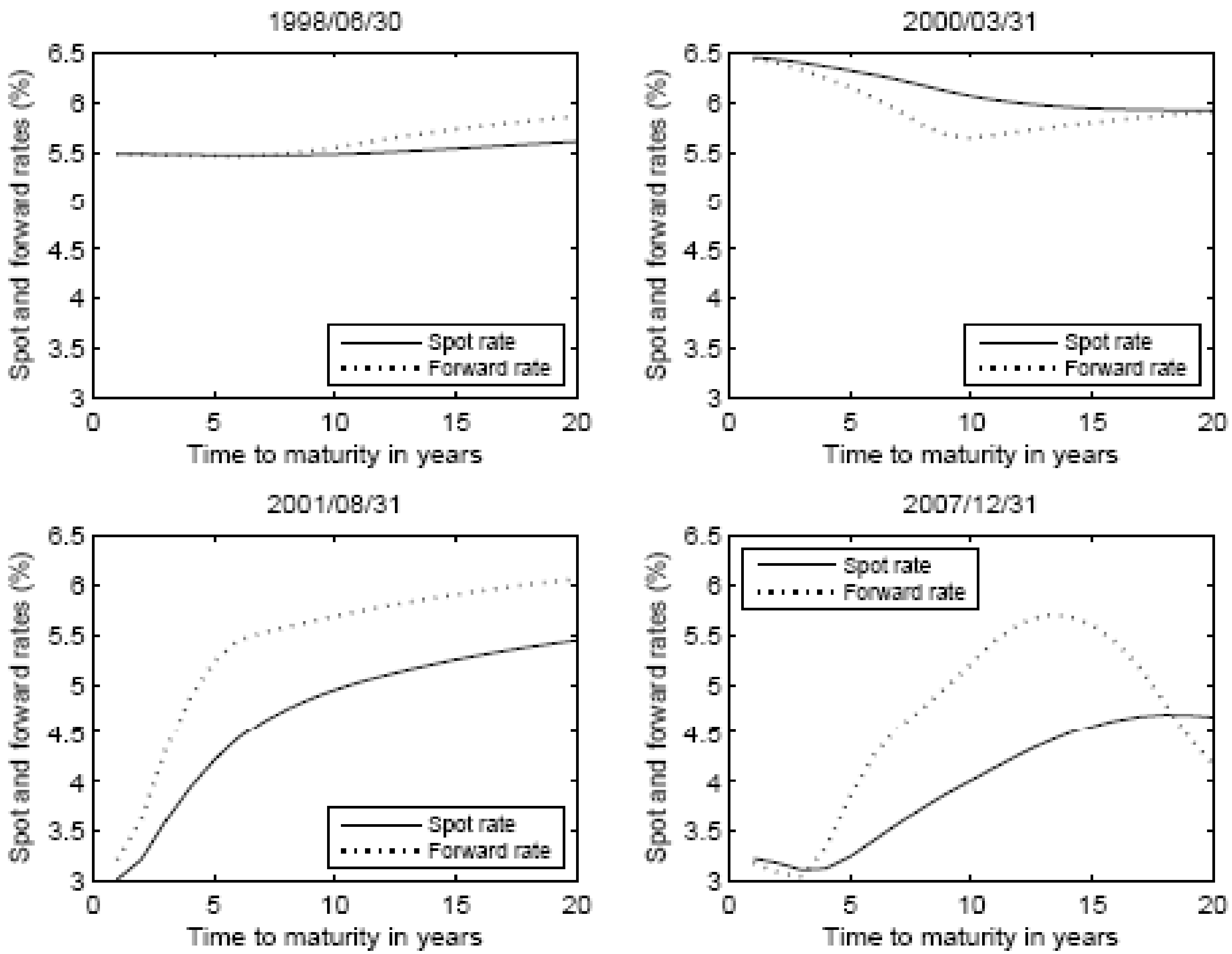

Figure 3.4: Yield curves of the US market

- An upward sloping yield curve is also said to have a **normal term structure** as this is the most commonly observed term structure empirically.
- We have an inverted **humped yield curve** on 2007/12/31.
- Some questions may arise from a cursory examination of this sample of yield curves. For example,
  - How are the yield curves obtained empirically?
  - What determines the shape of the term structure?
  - Why are upward sloping yield curves observed more often?
  - Does the term structure have any useful information about the real economy?

3.3 Present and Future Values Given the Term Structure

- We now consider the present and future values of an annuity given the term structure.
- We continue to use the actuarial notations introduced in Chapter 2.
- The present value of a unit-payment annuity-immediate over n periods is

$$\begin{aligned} a_{\overline{n}|} &= \sum_{t=1}^{n} \frac{1}{a(t)} \\ &= \sum_{t=1}^{n} \frac{1}{(1+i_t^S)^t} \\ &= \frac{1}{(1+i_1^S)} + \frac{1}{(1+i_2^S)^2} + \cdots + \frac{1}{(1+i_n^S)^n}. \end{aligned} \tag{3.9}$$

- The future value at time t of a unit payment at time 0 is

$$a(t) = (1 + i^S_t)^t = \prod_{j=1}^{t}(1 + i^F_j) = (1 + i^F_1)(1 + i^F_2)\cdots(1 + i^F_t). \quad (3.10)$$

- The present value of a unit payment due at time t is

$$\frac{1}{a(t)} = \frac{1}{\prod_{j=1}^{t}(1 + i^F_j)}, \quad (3.11)$$

and the present value of a n-payment annuity-immediate can also be written as

$$\begin{aligned} a_{\overline{n}\rceil} &= \sum_{t=1}^{n} \frac{1}{a(t)} \quad (3.12) \\ &= \sum_{t=1}^{n} \frac{1}{\prod_{j=1}^{t}(1 + i^F_j)} \\ &= \frac{1}{(1 + i^F_1)} + \frac{1}{(1 + i^F_1)(1 + i^F_2)} + \cdots + \frac{1}{(1 + i^F_1)\cdots(1 + i^F_n)}. \end{aligned}$$

- The computation of the future value at time n of a payment due at time t, where $0 < t < n$, requires additional assumptions.

- We consider the assumption that a **payment occurring in the future earns the forward rates of interest**.

- The future value at time n of a unit payment due at time t is

$$(1 + i^F_{t,n-t})^{n-t} = (1 + i^F_{t+1}) \cdots (1 + i^F_n), \tag{3.13}$$

and the future value at time n of a n-period annuity-immediate is

$$\begin{aligned} s_{\overline{n}\rceil} &= \left[\sum_{t=1}^{n-1} (1 + i^F_{t,n-t})^{n-t}\right] + 1 \\ &= \left[\sum_{t=1}^{n-1} \left(\prod_{j=t+1}^{n} (1 + i^F_j)\right)\right] + 1 \\ &= [(1 + i^F_2)(1 + i^F_3) \cdots (1 + i^F_n)] + [(1 + i^F_3)(1 + i^F_4) \cdots (1 + i^F_n)] + \cdots \end{aligned}$$

$$\cdots + (1 + i_n^F) + 1. \tag{3.14}$$

- From (3.14) we can see that

$$s_{\overline{n}|} = \left[\prod_{t=1}^{n}(1 + i_t^F)\right] \times \left[\frac{1}{1 + i_1^F} + \frac{1}{(1 + i_1^F)(1 + i_2^F)} + \cdots + \frac{1}{(1 + i_1^F)\cdots(1 + i_n^F)}\right]. \tag{3.16}$$

- Hence,

$$s_{\overline{n}|} = \left(\prod_{t=1}^{n}(1 + i_t^F)\right) a_{\overline{n}|} = a(n) a_{\overline{n}|}. \tag{3.17}$$

- An alternative formula to calculate $s_{\overline{n}|}$ using the spot rates of interest is

$$\begin{aligned} s_{\overline{n}|} &= (1 + i_n^S)^n \, a_{\overline{n}|} \\ &= (1 + i_n^S)^n \left[\sum_{t=1}^{n} \frac{1}{(1 + i_t^S)^t}\right]. \end{aligned} \tag{3.18}$$

Example 3.4: Suppose the spot rates of interest for investment horizons of 1, 2, 3 and 4 years are, respectively, 4%, 4.5%, 4.5%, and 5%. Calculate $a_{\overline{4}|}$, $s_{\overline{4}|}$, $a_{\overline{3}|}$ and $s_{\overline{3}|}$.

Solution: From (3.9) we have

$$a_{\overline{4}|} = \frac{1}{1.04} + \frac{1}{(1.045)^2} + \frac{1}{(1.045)^3} + \frac{1}{(1.05)^4} = 3.5763.$$

From (3.17) we obtain

$$s_{\overline{4}|} = (1.05)^4 \times 3.5763 = 4.3470.$$

Similarly,

$$a_{\overline{3}|} = \frac{1}{1.04} + \frac{1}{(1.045)^2} + \frac{1}{(1.045)^3} = 2.7536,$$

and

$$s_{\overline{3}|} = (1.045)^3 \times 2.7536 = 3.1423.$$

$\square$

Example 3.5: Suppose the 1-period forward rates of interest for investments due at time 0, 1, 2 and 3 are, respectively, 4%, 4.8%, 4.8% and 5.2%. Calculate $a_{\overline{4}|}$ and $s_{\overline{4}|}$.

Solution: From (3.12) we have

$$\begin{aligned} a_{\overline{4}|} &= \frac{1}{1.04} + \frac{1}{1.04 \times 1.048} + \frac{1}{1.04 \times 1.048 \times 1.048} + \frac{1}{1.04 \times 1.048 \times 1.048 \times 1.052} \\ &= 3.5867. \end{aligned}$$

As $a(4) = 1.04 \times 1.048 \times 1.048 \times 1.052 = 1.2016$, we have

$$s_{\overline{4}|} = 1.2016 \times 3.5867 = 4.3099.$$

Alternatively, from (3.14) we have

$$\begin{aligned} s_{\overline{4}|} &= 1 + 1.052 + 1.048 \times 1.052 + 1.048 \times 1.048 \times 1.052 \\ &= 4.3099. \end{aligned}$$

$\square$

- To further understand (3.17), we write (3.13) as (see (3.8))

$$(1 + i^F_{t,n-t})^{n-t} = \frac{(1 + i^S_n)^n}{(1 + i^S_t)^t} = \frac{a(n)}{a(t)}. \tag{3.19}$$

- Thus, the future value of the annuity is, from (3.14) and (3.19),

$$\begin{aligned} s_{\overline{n}|} &= \left[\sum_{t=1}^{n-1} (1 + i^F_{t,n-t})^{n-t}\right] + 1 \\ &= \sum_{t=1}^{n} \frac{a(n)}{a(t)} \\ &= a(n) \sum_{t=1}^{n} \frac{1}{a(t)} \\ &= a(n) a_{\overline{n}|}. \end{aligned} \tag{3.20}$$

- In Chapter 2 we assume that the current accumulation function $a(t)$ applies to all future payments.

- Under this assumption, if condition (1.35) holds, equation (3.20) is valid. However, for a given general term structure, we note that

$$a(n-t) = (1 + i^S_{n-t})^{n-t} \neq \frac{(1 + i^S_n)^n}{(1 + i^S_t)^t} = \frac{a(n)}{a(t)},$$

  so that condition (1.35) does not hold.

- Thus, if future payments are assumed to earn spot rates of interest based on the current term structure, equation (3.20) does not hold in general.

Example 3.6: Suppose the spot rates of interest for investment horizons of 1 to 5 years are 4%, and for 6 to 10 years are 5%. Calculate the present value of an annuity-due of \$100 over 10 years. Compute the future value of the annuity at the end of year 10, assuming (a) future payments earn forward rates of interest, and (b) future payments earn the spot rates of interest as at time 0.

Solution: We consider the 10-period annuity-due as the sum of an annuity-due for the first 6 years and a deferred annuity-due of 4 payments starting at time 6. The present value of the annuity-due for the first 6 years is

$$\begin{aligned} 100 \times \ddot{a}_{\overline{6}|0.04} &= 100 \times \left[\frac{1-(1.04)^{-6}}{1-(1.04)^{-1}}\right] \\ &= \$545.18. \end{aligned}$$

The present value at time 0 for the deferred annuity-due in the last 4 years is

$$
\begin{aligned}
100 \times (\ddot{a}_{\overline{10}|0.05} - \ddot{a}_{\overline{6}|0.05}) &= 100 \times \left[\frac{1-(1.05)^{-10}}{1-(1.05)^{-1}} - \frac{1-(1.05)^{-6}}{1-(1.05)^{-1}} \right] \\
&= 100 \times (8.1078 - 5.3295) \\
&= \$277.83.
\end{aligned}
$$

Hence, the present value of the 10-period annuity-due is

$$545.18 + 277.83 = \$823.01.$$

We now consider the future value of the annuity at time 10. Under assumption (a) that future payments earn the forward rates of interest, the future value of the annuity at the end of year 10 is, by equation (3.20),

$$(1.05)^{10} \times 823.01 = \$1{,}340.60.$$

Note that using (3.20) we do not need to compute the forward rates of interest to determine the future value of the annuity, as would be required if (3.16) is used.

Based on assumption (b), the payments at time $0, \cdots, 4$ earn interest at 5% per year (the investment horizons are 10 to 6 years), while the payments at time $5, \cdots, 9$ earn interest at 4% per year (the investment horizons are 5 to 1 years). Thus, the future value of the annuity is

$$100 \times (\ddot{s}_{\overline{10}|0.05} - \ddot{s}_{\overline{5}|0.05}) + 100 \times \ddot{s}_{\overline{5}|0.04} = \$1{,}303.78.$$

Thus, equation (3.20) does not hold under assumption (b). $\square$

3.4 Accumulation Function and the Term Structure

- Equation (3.1) can be extended to any $t > 0$, which need not be an integer.

- The annualized spot rate of interest for time to maturity t, i_t^S, is given by

$$i_t^S = [a(t)]^{\frac{1}{t}} - 1. \tag{3.22}$$

- We may also use an accumulation function to define the forward rates of interest.

- Let us consider the forward rate of interest for a payment due at time $t > 0$, and denote the accumulation function of this payment by $a_t(\cdot)$, where $a_t(0) = 1$.

- Now a unit payment at time 0 accumulates to $a(t+\tau)$ at time $t+\tau$, for $\tau > 0$.

- On the other hand, a strategy with an initial investment over t periods and a rollover at the forward rate for the next τ periods will accumulate to $a(t)a_t(\tau)$ at time $t+\tau$.

- By the no-arbitrage argument, we have

$$a(t)a_t(\tau) = a(t+\tau), \tag{3.23}$$

so that

$$a_t(\tau) = \frac{a(t+\tau)}{a(t)}. \tag{3.24}$$

- The annualized forward rate of interest in the period t to $t+\tau$, $i^F_{t,\tau}$, satisfies

$$a_t(\tau) = (1+i^F_{t,\tau})^\tau,$$

so that

$$i^F_{t,\tau} = [a_t(\tau)]^{\frac{1}{\tau}} - 1. \tag{3.25}$$

- If $\tau < 1$, we define the forward rate of interest per unit time (year) for the fraction of a period t to $t + \tau$ as

$$i^F_{t,\tau} = \frac{1}{\tau} \times \frac{a_t(\tau) - a_t(0)}{a_t(0)} = \frac{a_t(\tau) - 1}{\tau}. \tag{3.26}$$

- The **instantaneous forward rate of interest** per unit time at time t is equal to $i^F_{t,\tau}$ for $\tau \to 0$, which is given by

$$\begin{aligned}
\lim_{\tau \to 0} i^F_{t,\tau} &= \lim_{\tau \to 0} \frac{a_t(\tau) - 1}{\tau} \\
&= \lim_{\tau \to 0} \frac{1}{\tau} \times \left[\frac{a(t+\tau)}{a(t)} - 1 \right] \\
&= \frac{1}{a(t)} \lim_{\tau \to 0} \left[\frac{a(t+\tau) - a(t)}{\tau} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a'(t)}{a(t)} \\
&= \delta(t). \qquad (3.27)
\end{aligned}
$$

- Thus, the instantaneous forward rate of interest per unit time is equal to the force of interest.

Example 3.7: Suppose $a(t) = 0.01t^2 + 0.1t + 1$. Compute the spot rates of interest for investments of 1, 2 and 2.5 years. Derive the accumulation function for payments due at time 2, assuming the payments earn the forward rates of interest. Calculate the forward rates of interest for time to maturity of 1, 2 and 2.5 years.

Solution: Using (3.22), we obtain $i_1^S = 11\%$, $i_2^S = 11.36\%$ and $i_{2.5}^S = 11.49\%$. Thus, we have an upward sloping spot-rate curve. To calculate the accumulation function of payments at time 2 we first compute $a(2)$ as

$$a(2) = 0.01(2)^2 + 0.1(2) + 1 = 1.24.$$

Thus, the accumulation function for payments at time 2 is

$$a_2(t) = \frac{a(2+t)}{a(2)} = \frac{0.01(2+t)^2 + 0.1(2+t) + 1}{1.24} = 0.0081t^2 + 0.1129t + 1.$$

Using the above equation, we obtain $a_2(1) = 1.1210$, $a_2(2) = 1.2581$ and $a_2(2.5) = 1.3327$, from which we conclude $i^F_{2,1} = 12.10\%$,

$$i^F_{2,2} = (1.2581)^{\frac{1}{2}} - 1 = 12.16\%$$

and

$$i^F_{2,2.5} = (1.3327)^{\frac{1}{2.5}} - 1 = 12.17\%.$$

Thus, the forward rates exceed the spot rates, which agrees with what might be expected of an upward sloping yield curve. $\square$

- We can further establish the relationship between the force of interest and the forward accumulation function, and thus the forward rates of interest.

- From (3.24), we have

$$a_t(\tau) = \frac{a(t+\tau)}{a(t)} = \frac{\exp\left(\int_0^{t+\tau} \delta(s)\, ds\right)}{\exp\left(\int_0^{t} \delta(s)\, ds\right)} = \exp\left(\int_t^{t+\tau} \delta(s)\, ds\right), \tag{3.28}$$

  so that we can compute the forward accumulation function from the force of interest.

Example 3.8: Suppose $\delta(t) = 0.05t$. Derive the accumulation function for payments due at time 2, assuming the payments earn the forward rates of interest. Calculate the forward rates of interest for time to maturity of 1 and 2 years.

Solution: Using (3.26) we obtain

$$a_2(t) = \exp\left(\int_2^{2+t} 0.05s\,ds\right) = \exp\left[0.025(2+t)^2 - 0.025(2)^2\right] = \exp(0.025t^2 + 0.1t).$$

Thus, we can check that $a_2(0) = 1$. Now,

$$a_2(1) = \exp(0.125) = 1.1331,$$

so that $i^F_{2,1} = 13.31\%$. Also, $a_2(2) = \exp(0.3) = 1.3499$, so that

$$i^F_{2,2} = (1.3499)^{\frac{1}{2}} - 1 = 16.18\%.$$

$\square$

- We now consider payments of $C_1, C_2, \cdots, C_n$ at time $t_1 < t_2 < \cdots < t_n$, respectively.

- We wish to compute the value of these cash flows at any time t (≥ 0).

- For the payment C_j at time $t_j \le t$, its accumulated value at time t is $C_j\, a_{t_j}(t - t_j)$. On the other hand, if $t_j > t$, the discounted value of C_j at time t is $C_j / a_t(t_j - t)$.

- Thus, the value of the cash flows at time t is (see equation (3.24))

$$\begin{aligned}
\sum_{t_j \le t} C_j\, a_{t_j}(t - t_j) + \sum_{t_j > t} C_j \left[\frac{1}{a_t(t_j - t)}\right] &= \sum_{t_j \le t} C_j \left[\frac{a(t)}{a(t_j)}\right] + \sum_{t_j > t} C_j \left[\frac{a(t)}{a(t_j)}\right] \\
&= \sum_{j=1}^{n} C_j \left[\frac{a(t)}{a(t_j)}\right] \\
&= a(t) \sum_{j=1}^{n} C_j\, v(t_j) \\
&= a(t) \times \text{present value of cash flows.}
\end{aligned} \tag{3.29}$$

- An analogous result can be obtained if we consider a continuous cash flow.

- If $C(t)$ is the instantaneous rate of cash flow at time t for $0 \leq t \leq n$, the value of the cash flow at time $\tau \in [0, n]$ is

$$\begin{aligned}
\int_0^\tau C(t)a_t(\tau - t)\,dt + \int_\tau^n \frac{C(t)}{a_\tau(t-\tau)}\,dt &= \int_0^\tau \frac{C(t)a(\tau)}{a(t)}\,dt + \int_\tau^n \frac{C(t)a(\tau)}{a(t)}\,dt \\
&= a(\tau)\int_0^n \frac{C(t)}{a(t)}\,dt \\
&= a(\tau)\int_0^n C(t)v(t)\,dt. \qquad (3.30)
\end{aligned}$$

Example 3.9: Suppose $a(t) = 0.02t^2 + 0.05t + 1$. Calculate the value at time 3 of a 1-period deferred annuity-immediate of 4 payments of \$2 each. You may assume that future payments earn the forward rates of interest.

Solution: We first compute the present value of the annuity. The payments of \$2 are due at time 2, 3, 4 and 5. Thus, the present value of the cash flows is

$$2 \times \left[\frac{1}{a(2)} + \frac{1}{a(3)} + \frac{1}{a(4)} + \frac{1}{a(5)}\right].$$

Now, $a(2) = 0.02(2)^2 + 0.05(2) + 1 = 1.18$, and similarly we have $a(3) = 1.33$, $a(4) = 1.52$ and $a(5) = 1.75$. Thus, the present value of the cash flow is

$$2 \times \left[\frac{1}{1.18} + \frac{1}{1.33} + \frac{1}{1.52} + \frac{1}{1.75}\right] = 2 \times 2.82866 = \$5.6573,$$

and the value of the cash flow at time 3 is $a(3) \times 5.6573 = 1.33 \times 5.6573 = \$7.5242$. $\square$

Example 3.10: Suppose $\delta(t) = 0.02t$. An investor invests in a fund at the rate of $10t$ per period at time t, for $t > 0$. How much would she accumulate in the fund at time 2? You may assume that future payments earn the forward rates of interest.

Solution: The amount she invests in the period $(t, t + \Delta t)$ is $10t\Delta t$, which would accumulate to $(10t\Delta t)a_t(2 - t)$ at time 2. Thus, the total amount accumulated at time 2 is

$$\int_0^2 10ta_t(2-t)dt.$$

From (3.28), we have

$$a_t(2-t) = \exp\left(\int_t^2 \delta(s)\, ds\right) = \exp\left(\int_t^2 0.02s\, ds\right).$$

Now, we have

$$\int_t^2 0.02s\, ds = 0.01s^2]_t^2 = 0.01(2)^2 - 0.01t^2,$$

so that

$$a_t(2-t) = \exp(0.04 - 0.01t^2)$$

and

$$\begin{aligned}
\int_0^2 10ta_t(2-t)dt &= 10\int_0^2 te^{0.04-0.01t^2}\,dt \\
&= 10e^{0.04}\int_t^2 te^{-0.01t^2}\,dt \\
&= \frac{10e^{0.04}}{0.02}\left(-e^{-0.01t^2}]_0^2\right) \\
&= \frac{10e^{0.04}(1-e^{-0.04})}{0.02} \\
&= 20.4054.
\end{aligned}$$

□

Example 3.11: Suppose the principal is C and interest is earned at the force of interest $\delta(t)$, for $t > 0$. What is the present value of the interest earned over n periods.

Solution: As $\delta(t)$ is the instantaneous rate of interest per period at time t, the amount of interest earned in the period $(t, t + \Delta t)$ is $C\delta(t)\Delta t$, and the present value of this interest is $[C\delta(t)\Delta t]v(t)$. Thus, the present value of all the interest earned in the period $(0, n)$ is

$$\int_0^n C\delta(t)v(t)\,dt.$$

Now, we have

$$\begin{aligned}
\int_0^n \delta(t)v(t)\,dt &= \int_0^n \delta(t)\exp\left(-\int_0^t \delta(s)\,ds\right)dt \\
&= \left(-\exp\left(-\int_0^t \delta(s)\,ds\right)\right)\Bigg]_0^n \\
&= \exp\left(-\int_0^0 \delta(s)\,ds\right) - \exp\left(-\int_0^n \delta(s)\,ds\right)
\end{aligned}$$

$$= \quad 1 - v(n).$$

Hence, the present value of the interest earned is

$$C[1 - v(n)] = C - Cv(n),$$

which is the principal minus the present value of the principal redeemed at time n. $\square$