Robust Information Cascade with Endogenous Ordering and Competition

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Revised: January 2016

Abstract

We analyze a sequential decision model with endogenous ordering in which decision makers are allowed to choose the time of acting (exercising a risky investment option) or waiting. We show the existence of a unique symmetric equilibrium and characterize information cascade under endogenous ordering. Further, if there are two or more risky investment options, individuals tend to wait longer with competition. Hence, we could end up with a dilemma: more options might be worse.

JEL classification: C73, D62, D71, D81, D83
Keywords: Information Cascade, Endogenous Ordering, Competition

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1 Introduction

How do people make sequential decisions under imperfect information? One may learn from his own experiences or from other people's choices. For instance, individuals currently using a particular software package may also have the choice of upgrading to some new software packages. They may have some knowledge about the new software packages. But if the new software packages are brand new and private information is limited, individuals may be inclined to wait for other people to discourse more information about the newly released software packages before they take any action. If the information previously aggregated dominates their own private information, individuals ignore their own private information and follow their predecessors – **information cascade** occurs.[1] Information cascade prevents the aggregation of information. Therefore, the initial realization of signals can have long-term consequences and information cascade is often error prone. The decisions of the first few individuals' can have a disproportional effect.

Bikhchandani, Hirshleifer, and Welch (1992), hereafter BHW, and Banerjee (1992) investigate information cascade under **exogenous ordering**, in which the decision ordering is exogenously given and only one individual moves in each period. The restaurant example in Banerjee (1992) may fit the exogenous ordering setting.[2] But in many other cases, **endogenous ordering** which allows individuals to choose the time of acting or waiting may be more appropriate. For instance, when individuals decide to buy a new car or computer, they have the option to buy immediately or to wait. With endogenous ordering, there exist strategic interactions among decision makers. Due to the free-rider problem, some decision makers may have incentives to delay their decisions and learn from other decision makers, while others make decisions immediately if they feel confident that their decisions will produce desirable results. Furthermore, more than one individual can act or wait during the same period and consequently their decisions can be clustered together. Thus, under the endogenous ordering setting, the insight will be completely different from that under the exogenous ordering setting.

Continuing with the software upgrading example, suppose there are a number of new software packages A_j, $j \in \{1, 2, ..., M\}$, available for upgrading, where M is some finite integer. Individuals are currently using a software package B. It is known that with some prior probability A_j, $j \in \{1, 2, ..., M\}$, is better than B. Each individual also gets private signals indicating whether A_j, $j \in 1, 2, ..., M$, is better than B or

[1] Çelen and Kariv (2004) attempt to make the distinction between herding and information cascades. They point out that in a herd, individuals choose the same action; but they may have acted differently if the realization of private signals had been different. In an information cascade, individuals ignore their own private information and follow their predecessors.

[2] In the restaurant example in Banerjee (1992), there are two restaurants next to each other. Individuals arrive at the restaurants in sequence. Observing the choices made by people before them, they decide on either one of the two restaurants.

not. Upgrading to A_j, $j \in \{1, 2, ..., M\}$, is an irreversible choice. Once they upgrade to A_j, they are committed to their decisions.[3] They cannot switch back to B, or upgrade to A_m, for $m \neq j$. But there is no commitment to continuing using B. If individuals have not upgraded, they continue to have the option of doing so.[4] As there are a number of new software packages, naturally there exists "competition" among them. The further question of inquiry is: will information cascade be more or less error prone with competition?

We analyze an endogenous ordering sequential decision model in which decision makers are allowed to choose the time of upgrading to some new software package A_j, $j \in \{1, 2, ..., M\}$, or waiting (continuing using the current software package B). To emphasize the information aspect, we focus on pure information externalities: each decision maker's payoff only depends on his own action and the state of nature.

We show the existence of a unique symmetric perfect Bayesian equilibrium (PBE) with the following monotonicity property: in each period there exists critical types of individuals who upgrade with probability less than one; individuals with private signals indicating higher values upgrade with probability one; others wait. In this equilibrium, typically there is a strategic phase, followed by a cascading phase. In the **strategic phase**, depending on their own private signals, some individuals upgrade, while others wait. In the **cascading phase**, all the remaining individuals either upgrade immediately or wait forever regardless of their own private signals. Consequently, there exists either an investment surge or a collapse. Further, if there are two or more risky investment options, individuals tend to wait longer with competition. Hence, we could end up with a dilemma: more options might be worse.[5]

There are some papers which investigate the decision problem with endogenous ordering. For example, Chamley and Gale (1994) investigate a discrete time investment model which assumes the timing of decisions is endogenous, that is, individuals try to find the best place in the decision-making queue to undertake a profitable but risky investment. In their model, there are only two types of individuals: those with investment option and those without. Those individuals without investment option are assumed to be passive. Further, there is only one risky investment. In contrast, in our model we allow for a finite number types of individuals and two or more investment alternatives. Given one's own signals, each individual decides either to act immediately or to wait and learn the true value of the investment options by observing other individuals' actions.

The rest of the paper is organized as follows. Section 2 provides the model with one

[3]There exists extremely high "disruption costs" involved in upgrading. In other words, we could see this upgrade as a perpetual American call option. Individuals are free to exercise the option at any time they want. But once they exercise the option, they cannot reverse their decision.

[4]Throughout the paper, we use the software upgrading example to illustrate our model.

[5]Schwartz (2004) argues that eliminating choices can greatly reduce anxiety of our lives, which focuses on the psychological aspects of connection between choice and anxiety.

risky investment option and shows the existence of a unique symmetric equilibrium with the monotonicity property. Then we characterize information cascade under endogenous ordering. Section 3 extends the model with two or more risky investment options. Section 4 investigates the impact of competition with two or more investment options and discuss our main results. Several extensions are discussed in section 5.

2 One Risky Investment Option

As a benchmark, we start with the case with one risky investment option. There are N individuals. All are rational and risk neutral. There is a new software package A_1 available for upgrading. Individuals currently use software package B. Assume that the true value of A_1, denoted by V_1, is chosen by nature at the beginning of the game, and is unknown to the individuals.[6] Individuals know V_1 follows some prior distribution $F_1(V_1)$, with density $f_1(V_1)$. To emphasize the information aspect, we concentrate on pure information externalities: each individual's payoff only depends on his own action and the state of nature.

We focus on the case that upgrading to A_1 is an irreversible binary choice.[7] Once they upgrade to A_1, they are committed to their decisions. They cannot switch back to B. But there is no commitment to continuing using B. If individuals have not upgraded, they continue to have the option of doing so. The indivisibility of the action space is important. As in Banerjee (1992), since the choices made by individuals are not sufficient statistics for the information they have, the error prone information cascade can occur.[8]

At the beginning of the game, individual i in the market observes some conditionally independent private signal $\theta_i^1 \in \{s_1^1, s_2^1, \ldots, s_{D_1}^1\}$, where $s_1^1 < s_2^1 < \ldots < s_{D_1}^1$

[6] Rosenberg (1976) points out that there exist two types of technological uncertainty. First, when an innovation is introduced, it may have some imperfections: "Innumerable 'bugs' may need to be worked out. The first user often takes considerable risk." In addition, current innovation could be improved further in the future. There are two possible situations for the future possible improvement: expected or unexpected. If it is expected, then it only increases the benefit from waiting by some constant amount. If it is unexpected, it will not affect the strategic interactions of the current game until it happens. Thus, we ignore the second type of technological uncertainty here. When we investigate the switch from one innovation to another, the future improvement, either expected or unexpected, could be incorporated.

[7] Similar to Grenadier (1999), we could see this upgrade as a perpetual American call option. In Grenadier (1999), decisions are made in continuous time and there is a state variable, which follows some exogenous continuous time stochastic process. In this paper, we assume discrete time decision and no exogenous state variable.

[8] Banerjee (1992) assumes a continuous action space and gets similar information cascade results as BHW. This is due to the degenerate payoff function as pointed out by BHW. Park (2001) assumes perfect observability, and hence in his model players share the same information and hidden information is not an issue.

and D_1 is some finite integer. θ_i^1 follows some distribution $F_1(\theta_i^1|V_1)$, with density $f_1(\theta_i^1|V_1)$. Assume $F_1(\theta_i^1|V_1)$ satisfies the *Monotone Likelihood Ratio Property* (MLRP)[9] with respect to V_1: $\frac{f_1(\theta_i^1|\widetilde{V_1})}{f_1(\theta_i^1|V_1)}$ increasing in θ_i^1, for $\widetilde{V_1} > V_1$. That is, individuals are more likely to get a higher private signal (indicating higher value of A_1) if the underlying V_1 is higher.

With endogenous ordering, individuals are allowed to choose the time of upgrading to the new software package A_1 or waiting (continuing using the current software package B). There exist strategic interactions among the individuals. The timing of endogenous ordering is as follows:

> In period 1, each individual decides to upgrade to A_1 or to wait. If he waits in period 1, he gets reservation utility V_0, normalized to zero, and has the option of upgrading later.
>
> In period 2, all the remaining individuals decide to upgrade to A_1 or to wait after observing others' actions in period 1.
>
> The subsequent periods are the same as period 2. The game continues until everyone upgrades to A_1. The time period is denoted by t, $t = 1, 2, 3, ...$

The benefit from waiting is the information revealed about A_1 by other individuals. The cost of waiting is the difference between the gain from A_1 and the reservation utility. Denote individual i's action at date t by $x_{i,t}$, where $x_{i,t} = 1$ if he upgrades to A_1 at date t and $x_{i,t} = 0$ if he waits. Let $x_t \equiv (x_{1,t}, ..., x_{N,t})$ denote the outcome at date t. If individual i upgrades to A_1 in period t, then in the following periods, everyone knows individual i upgrades to A_1 in period t. The history of the game at the beginning of period t is a sequence of outcomes $h_t = (x_1, ..., x_{t-1})$.[10] The common discount factor is $\delta \in (0, 1)$.

Let $\mu_{i,t}^1(V_1|\theta_i^1; h_t)$ denote individual i's posterior belief about the true value of A_1 given the history of the game h_t and his own information θ_i^1 at the beginning of period t. The probability that individual i who has not yet upgraded does so after observing the history h_t is denoted by $\sigma_{i,t}^1(\theta_i^1; h_t)$. Then a behavioral strategy is a function $\sigma_{i,t}^1 : H \to [0, 1]$.

Our solution concept is perfect Bayesian equilibrium (PBE). A PBE consists of a strategy σ and a probability assessment μ, such that (i) each individual's strategy is a best response at every information set and (ii) the probability assessments are consistent with Bayes' rule at every information set that is reached with positive probability.

[9] Landsberger and Meilijson (1990) point out that MLRP holds for exponential type families (binomial with the same number of trials, normal with equal variances, etc.) as well as for some non-exponential cases such as uniform with the same left endpoint.

[10] As the prior, $x_0 = \mathbf{0}$ and hence $h_1 = x_0 = \mathbf{0}$.

Let's see the incentive for upgrading and waiting respectively. The expected benefit from upgrading to A_1 in period t for individual i is:

$$U^{A_1}(\theta_i^1; h_t) = E_{V_1|\theta_i^1;h_t} V_1 \tag{1}$$

The expected benefit from waiting in period t for individual i is:

$$U^W(\theta_i^1; h_t; \sigma_{-i,t}^1) = \delta E_{H_{t+1}(\theta_i^1;h_t;\sigma_{-i,t}^1)} \left[\max\{U^{A_1}(\theta_i^1; h_{t+1}); U^W(\theta_i^1; h_{t+1}; \sigma_{-i,t+1}^1)\}\right] \tag{2}$$

where $\sigma_{-i,t}^1$ represents the strategy profile of all other individuals except for individual i in period t; $H_{t+1}(\theta_i^1; h_t; \sigma_{-i,t}^1)$ represents the set of histories at the beginning of period $t+1$ given θ_i^1, h_t and $\sigma_{-i,t}^1$. From equation 2, we can solve the benefit from waiting forever $\underline{U}^W = 0$. The following lemma shows that under the worst news (no one is willing to upgrade), the waiting information cascade starts in the following period.

Lemma 1 *If $\sigma_{i,t}^1 = 0$ for all i, then $\sigma_{i,\tau}^1 = 0$ for any $\tau > t$ and i.*

Proof. See the Appendix. ■

Intuitively, for sufficiently pessimistic beliefs, no one is willing to upgrade. This is the worst news and also an absorbing state. Under this worst news, individuals will never upgrade to A_1 and the waiting information cascade starts in the following period. This implies, to keep upgrade going, at least one individual must upgrade to A_1 in each period. Thus, with a finite number of N individuals, the game lasts at most N periods before a cascading phase starts.[11]

In this paper, we focus on symmetric equilibrium. The following proposition shows that for any symmetric PBE, it must be monotone with respect to personal private signals. That is, individuals with higher private signals have higher incentive to upgrade.

Proposition 1 *For individual i, $U^{A_1}(\theta_i^1; h_t) - U^W(\theta_i^1; h_t; \sigma_{-i,t}^1)$ is increasing in θ_i^1, for any h_t and $\sigma_{-i,t}^1$.*

Proof. See the Appendix. ■

Intuitively, as assumed, the conditionally independent private signals satisfies MLRP with respect to the true value of V_1. By Bayes' rule, the posterior distribution of V_1 satisfies MLRP with respect to private signals. That is, if an individual gets a higher private signal, given the same history, he believes that V_1 will be higher. So, $U^{A_1}(\theta_i^1; h_t) > U^{A_1}(\widetilde{\theta}_i^1; h_t)$ for $\theta_i^1 > \widetilde{\theta}_i^1$ and any h_t. Further, by lemma 1, with a finite number of N individuals, the game lasts at most N periods before a cascading phase

[11] We may end up with no cascading phase at all. In this case, only the strategic phase exists. Still, the game lasts at most N periods before the game ends.

starts. Let $\mathcal{T}$ denote the period when a cascading phase starts. Lemma 1 implies $\mathcal{T} \leq N+1$. With a cascading phase starting in period $\mathcal{T}$, no more new information is disclosed thereafter. That is, $U^W(\theta_i^1; h_\mathcal{T}; \sigma_{-i,\mathcal{T}}^1) = U^W(\widetilde{\theta}_i^1; h_\mathcal{T}; \sigma_{-i,\mathcal{T}}^1) = 0$. Thus, in period $\mathcal{T}$, $U^{A_1}(\theta_i^1; h_\mathcal{T}) - U^W(\theta_i^1; h_\mathcal{T}; \sigma_{-i,\mathcal{T}}^1) > U^{A_1}(\widetilde{\theta}_i^1; h_\mathcal{T}) - U^W(\widetilde{\theta}_i^1; h_\mathcal{T}; \sigma_{-i,\mathcal{T}}^1)$, for any $h_\mathcal{T}$ and $\sigma_{-i,\mathcal{T}}^1$. By backward induction, for any $t \leq \mathcal{T}$, $U^{A_1}(\theta_i^1; h_t) - U^W(\theta_i^1; h_t; \sigma_{-i,t}^1) > U^{A_1}(\widetilde{\theta}_i^1; h_t) - U^W(\widetilde{\theta}_i^1; h_t; \sigma_{-i,t}^1)$, which is true for any h_t and $\sigma_{-i,t}^1$.

Based on the proposition above, we create an ordering "index" about private signals: $\nu(\theta^1; h_t; \sigma_{-,t}^1) = U^{A_1}(\theta^1; h_t) - U^W(\theta^1; h_t; \sigma_{-,t}^1)$, which is strictly increasing in θ^1 for any h_t and $\sigma_{-,t}^1$, where $\sigma_{-,t}^1$ is the behavior strategy profile for all other remaining individuals in period t.[12] In addition, let $\mathcal{W}_t^1(h_t)$ denote the set of possible remaining types in period t.[13] The following proposition shows the existence of a unique symmetric PBE, in which individuals with higher index of private signals upgrading earlier till a cascading phase starts.

Proposition 2 *There exists a unique symmetric PBE, in which typically there is a strategic phase, followed by a cascading phase. In the strategic phase, the equilibrium strategy profile is a sequence of decreasing critical types $\{\overline{\theta_t^1}(h_t)\}_t$ and a sequence of probability of critical types $\{\sigma_{\overline{\theta_t^1}(h_t)}\}_t$: in period t, individuals with $\nu(\theta; h_t; \sigma_{-,t}^1) > \nu(\overline{\theta_t^1}(h_t); h_t; \sigma_{-,t}^1)$ upgrade;*[14] *the critical type individuals upgrade with probability $\sigma_{\overline{\theta_t^1}(h_t)} \in [0,1)$; others wait.*[15] *If a cascading phase starts, all the remaining individuals in the game either upgrade immediately or wait forever.*

Proof. See the Appendix. ■

Intuitively, by lemma 1, with a finite number of N individuals, the game lasts at most N periods before a cascading phase starts. Further, by Proposition 1, for any symmetric PBE, it must be monotone with respect to personal private signals. Consider $\mathcal{G}_n(h_t)$, the subgame starting from period t with history h_t, where n is the number of individuals remaining. There are three possible cases as follows.

(1) If beliefs are sufficiently optimistic about A_1, all the remaining individuals upgrade to A_1 and A_1-upgrading information cascade occurs: $\mathcal{W}_\tau^1(h_\tau) = \emptyset$ for all $\tau > t$. To illustrate, A_1-upgrading information cascade occurs in period t when $\overline{\theta_t^1}(h_t) = s_1^1$ and $\sigma_{\overline{\theta_t^1}(h_t)} = 1$. All the remaining individuals upgrade to A_1 in period t

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[12] With a slight abuse of notation, from now on we drop the subscript i to indicate the term applied to any individual in the game.

[13] As $s_{D_1}^1$ is the highest possible remaining type in period 1, $\mathcal{W}_1^1(h_1) = \{\, \theta \mid s_1^1 \leq \theta \leq s_{D_1}^1 \}$.

[14] Since $\nu(\theta^1; h_t; \sigma_{-,t}^1)$ is strictly increasing in θ^1, $\nu(\theta; h_t; \sigma_{-,t}^1) > \nu(\overline{\theta_t^1}(h_t); h_t; \sigma_{-,t}^1)$ is equivalent to saying $\theta > \overline{\theta_t^1}(h_t)$.

[15] Note, $\sigma_{\overline{\theta_t^1}(h_t)} < 1$ so that $\overline{\theta_t^1}(h_t)$ is the highest possible remaining type in period $t+1$. Hence, $\mathcal{W}_{t+1}^1(h_{t+1}) = \{\, \theta \mid s_1^1 \leq \theta \leq \overline{\theta_t^1}(h_t)\}$.

regardless of their own private signal and the game ends (see figure 1). Consequently, there exists an investment surge.

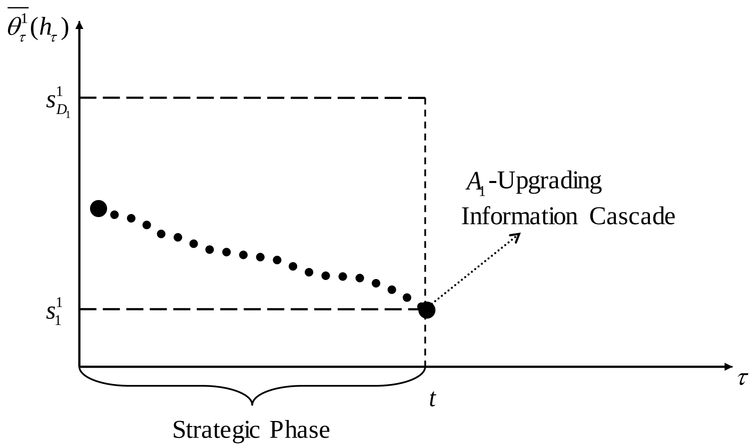

Figure 1: A_1-Upgrading Information Cascade

(2) If beliefs are sufficiently pessimistic about A_1, all the remaining individuals wait and waiting information cascade starts. To illustrate, waiting information cascade starts in period t when $\overline{\theta^1_t}(h_t) = \overline{\theta^1_{t-1}}(h_{t-1})$ and $\sigma_{\overline{\theta^1_t}(h_t)} = 0$. Since no more new information is disclosed thereafter, the game reaches an absorbing state: $\overline{\theta^1_\tau}(h_\tau) = \overline{\theta^1_{\tau-1}}(h_{\tau-1})$, $\sigma_{\overline{\theta^1_\tau}(h_\tau)} = 0$, and $\mathcal{W}^1_\tau(h_\tau) = \mathcal{W}^1_{\tau-1}(h_{\tau-1})$ for all $\tau \geq t$ (see figure 2). Consequently, there exists an investment collapse.

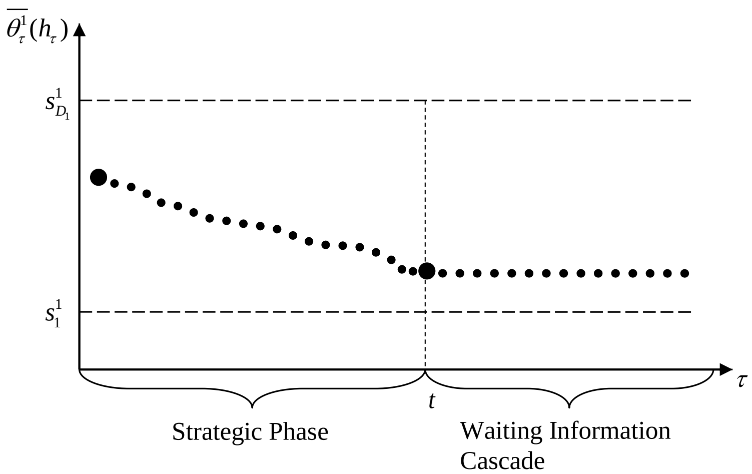

Figure 2: Waiting information cascade

(3) Otherwise, strategic phase continues, in which continuity implies there exists

a critical type $\overline{\theta_t^1}(h_t) \leq \overline{\theta_{t-1}^1}(h_{t-1})$, such that

$$\begin{cases} \nu(\theta^1; h_t; \sigma_{-,t}^1) = U^{A_1}(\theta^1; h_t) - U^W(\theta^1; h_t; \sigma_{-,t}^1) > 0 & \text{if } \theta^1 > \overline{\theta_t^1}(h_t) \\ \nu(\theta^1; h_t; \sigma_{-,t}^1) = U^{A_1}(\theta^1; h_t) - U^W(\theta^1; h_t; \sigma_{-,t}^1) < 0 & \text{if } \theta^1 < \overline{\theta_t^1}(h_t) \end{cases}$$

As for the realization, there are three scenarios depending on the realization of the private signals of the n individuals remaining in the game. If all private signals are greater than $\overline{\theta_t^1}(h_t)$, all the remaining individuals upgrade and the game ends. If all private signals are less than $\overline{\theta_t^1}(h_t)$, all the remaining individuals will wait forever. Otherwise, there are two sub-scenarios: (i) some upgrade and others wait; (ii) some are indifferent between upgrading and waiting, while others upgrade or wait for sure.

For the latter sub-scenario, from $\nu(\overline{\theta_t^1}(h_t); h_t; \sigma_{-,t}^1) = U^{A_1}(\overline{\theta_t^1}(h_t); h_t) - U^W(\overline{\theta_t^1}(h_t); h_t; \sigma_{-,t}^1) = 0$, we can identify $\sigma_{\overline{\theta_t^1}(h_t)}$. Consequently, we may end up with the following outcomes: if all upgrade, the game ends; if a number of individuals, say $n_t < n$, upgrade and others wait, the game continues to the subgame $\mathcal{G}_{n-n_t}(h_{t+1})$; if all wait, waiting information cascade starts. By backward induction, starting with the subgame with one individual remaining, we can construct the equilibrium strategy profile of the original game, $\mathcal{G}_N(h_1)$.

3 Two or More Risky Investment Options

Suppose now there are two or more risky investment options. The setting is the same as the one in the previous section, except that there are a number of new software packages A_j, $j \in \{1, 2, ..., M\}$, available for upgrading, where M is some finite integer greater than 1. Assume that the true value of A_j, denoted by V_j, is chosen by nature at the beginning of the game, and is unknown to the individuals. Individuals know V_j follows some prior distribution $F_j(V_j)$, with density $f_j(V_j)$. Upgrading to A_j, $j \in \{1, 2, ..., M\}$, is an irreversible choice. Once they upgrade to A_j, they are committed to their decisions. They cannot switch back to B, or upgrade to A_m, for $m \neq j$. But there is no commitment to continuing using B. If individuals have not upgraded, they continue to have the option of doing so.

At the beginning of the game, for the new software package A_j, $j \in \{1, 2, ..., M\}$, individual i in the market observes some conditionally independent private signal $\theta_i^j \in \{s_1^j, s_2^j, \ldots, s_{D_j}^j\}$, where $s_1^j < s_2^j < \ldots < s_{D_j}^j$ and D_j is some finite integer. θ_i^j follows some distribution $F_j(\theta_i^j|V_j)$, with density $f_j(\theta_i^j|V_j)$.[16] Assume $F_j(\theta_i^j|V_j)$ satisfies the *Monotone Likelihood Ratio Property* (MLRP) with respect to V_j: $\frac{f_j(\theta_i^j|\widetilde{V_j})}{f_j(\theta_i^j|V_j)}$ increasing in θ_i^j, for $\widetilde{V_j} > V_j$. That is, individuals are more likely to get a higher

[16] For simplicity, assume the distributions of the private signals, including the prior, are independent. In section 5, some extension with correlated signals is discussed.

private signal (indicating higher value of A_j) if the underlying V_j is higher. The signal vector observed by individual i is denoted by $\theta_i = (\theta_i^1, \theta_i^2, ..., \theta_i^M)$.

With endogenous ordering, individuals are allowed to choose the time of upgrading to the new software package A_j, $j \in \{1, 2, ..., M\}$, or waiting (continuing using the current software package B). There exist strategic interactions among the individuals. The timing is the same as the one in the previous section, except that there are a number of new software packages A_j, $j \in \{1, 2, ..., M\}$, available for upgrading. In each period, each individual decides to upgrade to A_j, $j \in \{1, 2, ..., M\}$, or to wait, if he has not yet upgraded. The game continues until everyone upgrades to A_j, $j \in \{1, 2, ..., M\}$. The benefit from waiting is the information revealed about the new software package A_j, $j \in \{1, 2, ..., M\}$, by other individuals. The cost of waiting is the difference between the gain from A_j, $j \in \{1, 2, ..., M\}$, and the reservation utility.

Denote individual i's action at date t by $x_{i,t}$, where $x_{i,t} = j$ if he upgrades to A_j, $j \in \{1, 2, ..., M\}$, at date t and $x_{i,t} = 0$ if he waits. Let $x_t \equiv (x_{1,t}, ..., x_{N,t})$ denote the outcome at date t. If individual i upgrades to A_j in period t, then in the following periods, everyone knows individual i upgrades to A_j in period t. The history of the game at the beginning of period t is a sequence of outcomes $h_t = (x_1, ..., x_{t-1})$.[17] The common discount factor is $\delta \in (0, 1)$.

Let $\mu_{i,t}^j(V_j|\theta_i; h_t)$ denote individual i's posterior belief about the true value of A_j given the history of the game h_t and his own information θ_i at the beginning of period t. The probability that individual i, who has not yet upgraded, chooses to upgrade to A_j after observing the history h_t is denoted by $\sigma_{i,t}^j(\theta_i; h_t)$. Then a behavioral strategy is a function $\sigma_{i,t}^j : H \to [0, 1]$. The vector of the behavioral strategy for individual i in period t is denoted by $\sigma_{i,t} = (\sigma_{i,t}^1, \sigma_{i,t}^2, ..., \sigma_{i,t}^M)$, where $\sum_{j=1}^{M} \sigma_{i,t}^j = 1$.

Our solution concept is perfect Bayesian equilibrium (PBE). A PBE consists of a strategy σ and a probability assessment μ, such that (i) each individual's strategy is a best response at every information set and (ii) the probability assessments are consistent with Bayes' rule at every information set that is reached with positive probability.

Let's see the incentive for upgrading and waiting respectively. The expected benefit from upgrading to A_j, $j \in \{1, 2, ..., M\}$, in period t for individual i is:

$$U^{A_j}(\theta_i; h_t) = E_{V_j|\theta_i;h_t} V_j \tag{3}$$

The expected benefit from waiting in period t for individual i is:

$$U^W(\theta_i; h_t; \sigma_{-i,t}) = \delta E_{H_{t+1}(\theta_i;h_t;\sigma_{-i,t})} \left[\max \left\{ \max_j U^{A_j}(\theta_i; h_{t+1}); U^W(\theta_i; h_{t+1}; \sigma_{-i,t+1}) \right\} \right] \tag{4}$$

[17] As the prior, $x_0 = \mathbf{0}$ and hence $h_1 = x_0 = \mathbf{0}$.

where $\sigma_{-i,t}$ represents the strategy profile of all other individuals except for individual i in period t; $H_{t+1}(\theta_i; h_t; \sigma_{-i,t})$ represents the set of histories at the beginning of period $t+1$ given θ_i, h_t and $\sigma_{-i,t}$. From equation 4, we can solve the benefit from waiting forever $\underline{U}^W = 0$.

For A_j, $j \in \{1, 2, ..., M\}$, we say it is "inactive" or "dormant" in period t, if no one intends to upgrade to A_j in period t. Formally, A_j is ***dormant*** in period t if $\sigma^j_{i,t} = 0$ for all i remaining in the game in period t; otherwise it is ***active***. As there are two or more risky investment options, some investment option may be dormant for a while and back to be active later. In contrast, with only one investment option, if it is dormant, it will remain dormant and never be active again, as showed in lemma 1.

Consider in some period t only A_j is active while all others investment options are dormant. The following lemma shows that if no one upgrades to A_j in period t, then A_j will be dormant in period $t+1$.

Lemma 2 *Suppose $\sigma^{\hat{j}}_{i,t} = 0$ for all i and $\hat{j} \neq j$. If $x_t = \mathbf{0}$, then $\sigma^j_{i,t+1} = 0$ for all i.*

Similar to the proof of lemma 1, for sufficiently pessimistic beliefs about A_j in period t, no one is willing to upgrade to A_j, and this is the worst news for A_j. Under this worst news for A_j in period t, individuals will not upgrade to A_j in the following period $t+1$. Otherwise, suppose there were some individual upgrading to A_j in period $t+1$ under this worst news for A_j in period t. He would for sure upgrade to A_j in period $t+1$. Then, he should have upgraded to A_j in period t, other than waiting for one more period, which is a contradiction.

Lemma 2 says that if only A_j is active in some period and it turns out no one upgrades to A_j, then A_j must be "cooling off" for at least one period. With two or more investment options, it is the turn of other investment options to be active. This implies "information flow" must continue till either all individuals upgrade or all investment options are dormant. In the latter case, waiting information cascade starts. Further implication is that with a finite number of N individuals, M investment options and bounded private signals, the game lasts for a finite number of periods before a cascading phase starts.[18]

The next proposition shows that for any symmetric PBE, it must be monotone with respect to personal private signals. That is, individuals with higher private signals about A_j, $j \in \{1, 2, ..., M\}$, have higher incentive to upgrade to A_j.

[18] Consider $M = 1$. Once the only investment option is dormant, it will remain dormant and never be active again, as showed in lemma 1. Thus, with a finite number of N individuals, the game lasts at most N periods before a cascading phase starts. Comparatively, with two or more investment options, some investment option may be dormant for a while and back to be active later. The game could last much longer period of time before a cascading phase starts.

Proposition 3 *For individual i, consider θ_i and $\widetilde{\theta}_i$, where $\theta_i^j > \widetilde{\theta}_i^j$ and $\theta_i^{\hat{j}} = \widetilde{\theta}_i^{\hat{j}}$, for all $\hat{j} \neq j$. For any h_t and $\sigma_{-i,t}$,*

$$U^{A_j}(\theta_i; h_t) - \max_{-j} U^{A_{-j}}(\theta_i; h_t) > U^{A_j}(\widetilde{\theta}_i; h_t) - \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_t)$$
$$U^{A_j}(\theta_i; h_t) - U^W(\theta_i; h_t; \sigma_{-i,t}) > U^{A_j}(\widetilde{\theta}_i; h_t) - U^W(\widetilde{\theta}_i; h_t; \sigma_{-i,t})$$

Proof. See the Appendix. ■

Intuitively, for A_j, $j \in \{1, 2, ..., M\}$, as assumed, the conditionally independent private signals satisfies MLRP with respect to the true value V_j. By Bayes' rule, the posterior distribution of V_j satisfies MLRP with respect to private signals. That is, if an individual gets a higher private signal about A_j, given the same history, he believes that V_j will be higher. Formally, for individual i, consider θ_i and $\widetilde{\theta}_i$, where $\theta_i^j > \widetilde{\theta}_i^j$ and $\theta_i^{\hat{j}} = \widetilde{\theta}_i^{\hat{j}}$, for all $\hat{j} \neq j$. We have $U^{A_j}(\theta_i; h_t) > U^{A_j}(\widetilde{\theta}_i; h_t)$ for any h_t.

Further, by lemma 2, with a finite number of N individuals, M investment options and bounded private signals, the game lasts for a finite number of periods before a cascading phase starts. Suppose either an upgrading or waiting information cascade starts in period $\mathcal{T}$. With a cascading phase starting in period $\mathcal{T}$, no more new information is disclosed thereafter. That is, $U^W(\theta_i; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}) = U^W(\widetilde{\theta}_i; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}) = 0$. Thus, in period $\mathcal{T}$, for any $h_{\mathcal{T}}$ and $\sigma_{-i,\mathcal{T}}$

$$U^{A_j}(\theta_i; h_{\mathcal{T}}) - \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}}) > U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}}) - \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}})$$
$$U^{A_j}(\theta_i; h_{\mathcal{T}}) - U^W(\theta_i; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}) > U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}}) - U^W(\widetilde{\theta}_i; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}})$$

By backward induction, for any $t \leq \mathcal{T}$

$$U^{A_j}(\theta_i; h_t) - \max_{-j} U^{A_{-j}}(\theta_i; h_t) > U^{A_j}(\widetilde{\theta}_i; h_t) - \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_t)$$
$$U^{A_j}(\theta_i; h_t) - U^W(\theta_i; h_t; \sigma_{-i,t}) > U^{A_j}(\widetilde{\theta}_i; h_t) - U^W(\widetilde{\theta}_i; h_t; \sigma_{-i,t})$$

which is true for any h_t and $\sigma_{-i,t}$.

Similar to the case of one investment option, we create an ordering "index" about private signals: $\nu(\theta; h_t; \sigma_{-,t}) = \max_j U^{A_j}(\theta; h_t) - U^W(\theta; h_t; \sigma_{-,t})$, where $\sigma_{-,t}$ is the behavior strategy profile for all other remaining individuals in period t. In addition, let $\mathcal{W}_t(h_t)$ denote the set of possible remaining types in period t. The following proposition shows the existence of a unique symmetric PBE, in which individuals with higher index of private signals upgrading earlier till a cascading phase starts.

Proposition 4 *There exists a unique symmetric PBE, in which typically there is a strategic phase, followed by a cascading phase. In the strategic phase, the equilibrium strategy profile is a sequence of critical types $\{\overline{\theta}_t(h_t)\}_t$ and a sequence of probability of* *critical types* $\{\sigma_{\overline{\theta_t}(h_t)}\}_t$*: in period* t*, individuals with* $\nu(\theta; h_t; \sigma_{-,t}) > \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t})$ *upgrade;*[19] *the critical type individuals upgrade with probability* $\sum_{j=1}^{M} \sigma_{\overline{\theta_t^j}(h_t)} \in [0, 1)$*; others wait. If a cascading phase starts, all the remaining individuals in the game either upgrade immediately or wait forever.*

Proof. See the Appendix. ■

Intuitively, by lemma 2, with a finite number of N individuals, M investment options and bounded private signals, the game lasts for a finite number of periods before a cascading phase starts. Further, by Proposition 3, for any symmetric PBE, it must be monotone with respect to personal private signals. Consider $\mathcal{G}_n(h_t)$, the subgame starting from period t with history h_t, where n is the number of individuals remaining. There are three possible cases: (1) if sufficiently optimistic about A_j, all the remaining individuals upgrade and A_j-upgrading information cascade occurs: $\mathcal{W}_\tau(h_\tau) = \emptyset$ for all $\tau > t$; (2) if sufficiently pessimistic about all investment options, all the remaining individuals wait and waiting information cascade starts: $\mathcal{W}_\tau(h_\tau) = \mathcal{W}_{\tau-1}(h_{\tau-1})$ for all $\tau \geq t$; (3) otherwise, strategic phase continues, in which continuity implies there exists a critical type $\overline{\theta_t}(h_t)$ such that[20]

$$\begin{cases} \nu(\theta; h_t; \sigma_{-,t}) > 0 & \text{if} \quad \nu(\theta; h_t; \sigma_{-,t}) > \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) \\ \nu(\theta; h_t; \sigma_{-,t}) < 0 & \text{if} \quad \nu(\theta; h_t; \sigma_{-,t}) < \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) \end{cases}$$

As for the realization, there are three scenarios depending on the realization of the private signals of the n individuals remaining in the game. If the realization of the private signals is sufficiently optimistic such that for all the remaining individuals $\nu(\theta; h_t; \sigma_{-,t}) > \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t})$, they will upgrade to some investment option and the game ends. To the contrary, if the realization of the private signals is sufficiently pessimistic such that for all the remaining individuals $\nu(\theta; h_t; \sigma_{-,t}) < \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t})$, they will wait forever. Otherwise, there are two sub-scenarios: (i) some upgrade and others wait; (ii) some are indifferent between upgrading and waiting, while others upgrade or wait for sure.

For the latter sub-scenario, from $\max_j U^{A_j}(\overline{\theta_t}(h_t); h_t) - U^W(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) = 0$, we can identify $\sigma_{\overline{\theta_t}(h_t)}$. Consequently, we may end up with the following outcomes: if all upgrade, the game ends; if a number of individuals, say $n_t < n$, upgrade and others wait, the game continues to the subgame $\mathcal{G}_{n-n_t}(h_{t+1})$; if all wait, waiting

[19]As there are two or more investment options, naturally there exists "competition" among them. Consider some individual i in period t with $\nu(\theta_i; h_t; \sigma_{-i,t}) > \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-i,t})$. He upgrades to the new software package with the highest expected benefit: $\max_j U^{A_j}(\theta_i; h_t)$. If there is a tie at the top, he will follow some tie-breaking rule to upgrade.

[20]As there are two or more investment options, the critical type may not be unique. In case of multiple, there is a tie with the same value of index ν.

information cascade starts. By backward induction, starting with the subgame with one individual remaining, we can construct the equilibrium strategy profile of the original game, $\mathcal{G}_N(h_1)$.

Further, from the proposition above, we have the following corollary, which says that for the sets of possible remaining types in each period, their corresponding convex hulls contains none of the types with positive "index." This implies there is no "break" in between for the "alien" types.

Corollary 1 *For θ such that $\nu(\theta; h_t; \sigma_{-,t}) > 0$, $\theta \notin conv\,(\mathcal{W}_t(h_t))$, where $conv\,(\mathcal{W}_t(h_t))$ is the convex hull of $\mathcal{W}_t(h_t)$.*

Intuitively, the sets of possible remaining types in each period "shrinks" over time till a cascading phase starts. $\mathcal{W}_t(h_t)$ only contains θ such that $\nu(\theta; h_t; \sigma_{-,t}) \leq \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) \leq 0$. By Proposition 3, for any symmetric PBE, it must be monotone with respect to personal private signals. Similar to Leontief preferences, *conv* $(\mathcal{W}_t(h_t))$ exclude θ such that $\nu(\theta; h_t; \sigma_{-,t}) > 0$.

4 The Impact of Competition

As there are two or more risky investment options, naturally there exists "competition" among them. With more investment options, the benefit from waiting is increasing. Given the same public information available, the sets of possible remaining types are enlarged for the existing investment options. This implies individuals tend to wait longer with more investment options.

Delay is costly, but there is potential gain from information revealed by others to make a better choice. However, overall the welfare impact is ambiguous, which is closely tied to the discount factor. We discuss the effect of discount factor on PBE in the following section, and then move on to the welfare properties of PBE.

4.1 The Effect of Discount Factor

In this section, we study the properties of PBE if the discount factor is sufficiently small or large. By equation 4, the smaller the discount factor, the lower the benefit from waiting. The following proposition shows that there exists a lower bound $\underline{\delta}$, such that if $\delta < \underline{\delta}$, individuals with positive expected benefit from upgrading have no incentive to wait in period 1. This resembles self-decision in the one-period static game. The difference is that in our dynamic setting those who have not upgraded in period 1 still have the option to upgrade in the following periods. Formally, the *self-decision rule* in period t is denoted by σ_t^s: individuals with θ, such that $\max_j U^{A_j}(\theta; h_t) > 0$, upgrade; individuals with θ, such that $\max_j U^{A_j}(\theta; h_t) < 0$, wait; others will follow some tie-breaking rule to upgrade.

Proposition 5 *There exists a $\underline{\delta} \in (0,1)$ such that individuals will follow the self-decision rule σ_1^s in period 1 if $\delta < \underline{\delta}$.*

Proof. Suppose in period 1 individuals follow the self-decision rule σ_1^s. Since the private signal space is discrete, we can always find a $j \in \{1, 2, ..., M\}$ and θ^j, such that $U^{A_j}(\theta; h_1)|_{\theta^j} = \min_{U^{A_{\hat{j}}}(\theta;h_1)>0} U^{A_{\hat{j}}}(\theta; h_1)$. That is, for the expected benefit from upgrading, $U^{A_j}(\theta; h_1)|_{\theta^j}$ is the smallest one among those greater than zero. Without loss of generality, consider individual i with private signal θ_i, where $\theta_i^j = \theta^j$ and $\theta_i^{\hat{j}} = s_{D_{\hat{j}}}^{\hat{j}}$ for $\hat{j} \neq j$. Then, set $\underline{\delta} \in (0,1)$, such that $U^{A_j}(\theta_i; h_1) = U^W(\theta_i; h_1; \sigma_{-i,1}^s)$, where the strategy profile $\sigma_{-i,1}^s$ indicates that all other individuals except for i adopt the self-decision rule in period 1. ■

Conversely, by equation 4, the larger the discount factor, the larger the benefit from waiting. Further, by lemma 2, with a finite number of N individuals, M investment options and bounded private signals, the game lasts for a finite number of periods before a cascading phase starts. Consequently, if δ goes to one, we are more likely to end up with the waiting information cascade.

To the extreme case, if $\delta = 1$, most individuals will wait until they are almost certain the return to some investment option is the highest. The story is completely different. The most optimistic individuals upgrade earlier, followed by others. Conceivably, there is a descending order of move based on the private signals. The game could last $\prod_j D_j$ periods, even though N is small.

4.2 Welfare Properties

From proposition 5, with M investment options, there exists a lower bound $\underline{\delta}$, such that if $\delta < \underline{\delta}$, individuals will follow the self-decision rule σ_1^s in period 1. Without loss of generality, suppose there is one more investment option. Again, from proposition 5, with $M+1$ investment options, there exists a lower bound $\underline{\delta}'$, such that if $\delta < \underline{\delta}'$, in period 1 individuals will follow the self-decision rule σ_1^s. With one more investment option, the benefit from waiting is increasing and individuals tend to wait longer, which implies $\underline{\delta}' \leq \underline{\delta}$. Consider the case that the discount factor is small such that $\delta < \underline{\delta}' \leq \underline{\delta}$. In period 1, individuals still follow the self-decision rule σ_1^s even though there is one more investment option. In this case, ex ante, with the additional investment option everyone will be either better off or at least as well off.

However, the story will change if the discount factor is large. By lemma 2, with a finite number of N individuals, M investment options and bounded private signals, the game lasts for a finite number of periods before a cascading phase starts. Consequently, if the waiting costs are low enough, we are more likely to end up with the waiting information cascade, even though it might be a wrong one ex post. Further, with one more investment option, individuals tend to wait longer. This even exaggerates the likelihood to be stuck with the waiting information cascade.

4.2.1 An Example

To illustrate, we study an example with three individuals ($N = 3$) and a binary private signal system to investigate the welfare properties of the PBE. Suppose there is a new software package A_1 available for upgrading. The true value of A_1, V_1, is either $1/2$ or $-1/2$, with equal prior probability. At the beginning of the game, for the new software package A_1, individual i in the market observes some conditionally independent private signal $\theta_i^1 \in \{s_1^1, s_2^1\}$, where $s_1^1 < s_2^1$. Without loss of generality, we label $s_1^1 = L_1$ and $s_2^1 = H_1$. H_1 is observed with probability $p_1 > 1/2$ if $V_1 = 1/2$ and with probability $1 - p_1$ if $V_1 = -1/2$, as described in table 1. By proposition 2

Table 1: Signal Probabilities for A_1

	$Pr(\theta_i^1 = L_1 \mid V_1)$	$Pr(\theta_i^1 = H_1 \mid V_1)$
$V_1 = -1/2$	p_1	$1 - p_1$
$V_1 = 1/2$	$1 - p_1$	p_1

and 5, we have the equilibrium decision rule described in table 2.

Table 2: Equilibrium Decision Rule with A_1

	$t = 1$	$t = 2$	$t = 3$
$\delta < \underline{\delta}$	$H_1 \rightarrow A_1$ $L_1 \rightarrow W$	$2A_1$ in $t = 1 \rightarrow A_1$ otherwise $\rightarrow W$	—
$\delta \geq \underline{\delta}$	$H_1 \rightarrow A_1$ with prob. $\sigma_{H_1} < 1$ $L_1 \rightarrow W$	$2A_1$ in $t = 1 \rightarrow A_1$ $1A_1$ in $t = 1$ & $H_1 \rightarrow A_1$ otherwise $\longrightarrow W$	$1A_1$ in $t = 1$ & $1A_1$ in $t = 2 \rightarrow A_1$ otherwise $\rightarrow W$

Suppose there is one more new software package A_2 available for upgrading, which has the similar binary signal system as A_1. The true value of A_2, V_2, is either $1/2$ or $-1/2$, with equal prior probability. At the beginning of the game, for the new software package A_2, individual i in the market observes some conditionally independent private signal $\theta_i^2 \in \{s_1^2, s_2^2\}$, where $s_1^2 < s_2^2$. Without loss of generality, we label $s_1^2 = L_2$ and $s_2^2 = H_2$. H_2 is observed with probability $p_2 > 1/2$ if $V_2 = 1/2$

and with probability $1 - p_2$ if $V_2 = -1/2$.[21] In addition, we assume $p_1 > p_2 > 1/2$, which indicates signals of A_1 are more accurate. By proposition 4 and 5, we have the equilibrium decision rule described in table 3.[22]

By the previous arguments and proposition 5, in our example with one more investment option A_2, individuals tend to wait longer. Hence, $\underline{\delta}' \leq \underline{\delta}$. Consider the case that the discount factor is in between $\underline{\delta}'$ and $\underline{\delta}$. Without A_2, individuals with positive expected benefit from upgrading have no incentive to wait in period 1. In contrast, with one more investment option A_2, some with positive expected benefit from upgrading may have incentive to wait in period 1. Waiting is costly and even worse we might be stuck with the wrong information cascade. Overall the welfare impact is ambiguous.

Specifically, without A_2, in period 1 individuals with H_1 will upgrade to A_1 and those with L_1 will wait as in table 2. With both A_1 and A_2, individuals with H_1H_2 may wait in period 1 as in table 3. That is to say, with one more investment option, individuals with H_1H_2 may wait instead of upgrading to A_1 even though they have the signal H_1 about A_1. Hence, we could end up with a dilemma: more options might be worse.[23]

5 Discussion and Extension

5.1 Large Number of Individuals

Consider the "large" game as the number of individuals becomes unboundedly large. By the Law of Large Numbers, if information about some investment option A_j disclosed is *non-negligible* in some period, the true value of A_j will be revealed in the following period. Conceivably, individuals whoever upgrade must be among the most optimistic ones. Therefore, all the relevant information is transmitted by the highest signals, and it is irrelevant how good information might be available at lower signal values.[24]

Further, the information disclosed by these most optimistic ones must be *negligible* in each given period. Same spirit as proposition 7 and 8 in Chamley and Gale (1994), in the limit the number of individuals upgrading is given by the Poisson approximation to the binomial distribution with some parameter β, so that the number of individuals

[21] To get the signal probabilities for A_2, simply replace θ_i^1 with θ_i^2 and change the subscript 1 to 2 in table 1.

[22] There are some minor variations depending on the values of parameters. For more details, please refer to the online appendix of the paper.

[23] It is controversial about the welfare comparison criterion, which is rendered to the online appendix of the paper.

[24] See Murto and Välimäki (2013) for the technical analysis of large games.

Table 3: Equilibrium Decision Rule with A_1 and A_2

	$t=1$	$t=2$	$t=3$	$t=4$
$\delta < \underline{\delta}'$	$H_1L_2 \to A_1$ $H_1H_2 \to A_1$ $L_1H_2 \to A_2$ $L_1L_2 \to W$	$2A_1$ in $t=1 \to A_1$ $2A_2$ in $t=1 \to A_2$ $1A_1$ & $1A_2$ in $t=1 \to A_2/W$[25] otherwise $\to W$	—	—
$\delta = \underline{\delta}'$	$H_1L_2 \to A_1$ $H_1H_2 \to A_1$ $L_1H_2 \to A_2$ with prob. $\sigma_{L_1H_2} < 1$ $L_1L_2 \to W$	$2A_1$ in $t=1 \to A_1$ $2A_2$ in $t=1 \to A_2$ $1A_1$ & $1A_2$ in $t=1$ & L_1L_2 $\to A_2/W$ $1A_1$ or $1A_2$ in $t=1$ & $L_1H_2 \to A_2$ otherwise $\to W$	$1A_2$ in $t=1$ & $1A_2$ in $t=2 \to A_2$ $1A_1$ in $t=1$ & $1A_2$ in $t=2 \to A_2/W$ otherwise $\to W$	—
$\underline{\delta}' < \delta < \underline{\delta}''$	$H_1L_2 \to A_1$ $H_1H_2 \to A_1$ otherwise $\to W$	$2A_1$ in $t=1 \to A_1$ $1A_1$ or $0A_1$ in $t=1$ & $L_1H_2 \to A_2$ otherwise $\to W$	$1A_1$ in $t=1$ & $1A_2$ in $t=2 \to A_2/W$ $2A_2$ in $t=2 \to A_2$ otherwise $\to W$	—
$\delta = \underline{\delta}''$	$H_1L_2 \to A_1$ $H_1H_2 \to A_1$ with prob. $\sigma_{H_1H_2} < 1$ otherwise $\to W$	$2A_1$ in $t=1 \to A_1$ $1A_1$ in $t=1$ & $H_1H_2 \to A_1$ $0A_1$ in $t=1$ & $L_1H_2 \to A_2$ otherwise $\to W$	$1A_1$ in $t=1$ & $1A_1$ in $t=2 \to A_1$ $2A_2$ in $t=2 \to A_2$ $1A_2$ in $t=2$ & $H_1H_2 \to A_2$ otherwise $\to W$	$1A_2$ in $t=2$ & $1A_2$ in $t=3 \to A_2$
$\underline{\delta}'' < \delta < \underline{\delta}'''$	$H_1L_2 \to A_1$ otherwise $\to W$	$2A_1$ in $t=1 \to A_1$ $1A_1$ in $t=1$ & $H_1H_2 \to A_1$ $0A_1$ in $t=1$ & $L_1H_2 \to A_2$ otherwise $\to W$	$1A_1$ in $t=1$ & $1A_1$ in $t=2 \to A_1$ $2A_2$ in $t=2 \to A_2$ $1A_2$ in $t=2$ & $H_1H_2 \to A_2$ otherwise $\to W$	$1A_2$ in $t=2$ & $1A_2$ in $t=3 \to A_2$
$\delta \geq \underline{\delta}'''$	$H_1L_2 \to A_1$ with prob. $\sigma_{H_1L_2} < 1$ otherwise $\to W$	$2A_1$ in $t=1 \to A_1$ $1A_1$ in $t=1$ & $(H_1H_2$ or $H_1L_2)$ $\to A_1$ $0A_1$ in $t=1$ & $L_1H_2 \to A_2$ otherwise $\to W$	$1A_1$ in $t=1$ & $1A_1$ in $t=2 \to A_1$ $1A_1$ in $t=1$ & $0A_1$ in $t=2$ & $L_1H_2 \to A_2$ $2A_2$ in $t=2 \to A_2$ $1A_2$ in $t=2$ & $H_1H_2 \to A_2$ otherwise $\to W$	$1A_1$ in $t=1$ & $0A_1$ in $t=2$ & $1A_2$ in $t=3 \to A_2/W$ $1A_2$ in $t=2$ & $1A_2$ in $t=3 \to A_2$ otherwise $\to W$

[25] In this case, there is a tie. The individual remaining in the game will upgrade to A_2 or wait according to some tie breaking rue, which applies to the latter cases with only individual remaining.

upgrading in any given period is almost surely finite.

5.2 Correlated Private Signal Systems

For the previous analysis, we assume the distributions of the private signals, including the prior, are independent. What if the signals are correlated? To illustrate, consider the restaurant example in Banerjee (1992). There are two restaurants A and B next to each other. Individuals arrive at the restaurants in sequence. Observing the choices made by people before them, they decide on either one of the two restaurants. It is known that only one restaurant is good, either A or B. Apart from the prior, each individual also gets a signal says either that A is better or that B is better. Comparing with the independent private signal systems, in the restaurant example there are two investment options and the signals are perfectly negatively correlated. That is, if a signal says that A is better, then it indicates that B is worse, and vice versa.

In the setting of Banerjee (1992), the order of choice is exogenously fixed. In contrast, if the order of choice is endogenized, individuals could wait before deciding on either one of the two restaurants. As signals are perfectly negatively correlated, waiting information cascade never happens, which is completely different from our analysis with the independent signal system. However, the overall analysis is rather complicated, and we do not yet have very precise ideas about what happens in the general case.

5.3 Continuous Private Signal Systems

In our setting, the signal space is discrete, which is compatible with the setting of BHW. Instead, if the signal space is continuous, our conjecture is that the main result still holds as long as the signal space is bounded.[26] Further, with continuous signal space, we may focus on the pure strategy equilibrium, whereas we need to deal with the possible mixed strategy if there is a tie with the discrete signal space. In addition, back to corollary 1, with continuous signal space, $\mathcal{W}_t(h_t)$, the set of possible remaining types in period t, itself will be convex. Now that, $\mathcal{W}_t(h_t)$ "shrinks" over time till a cascading phase starts, which resembles onion-peeling in the convex hull algorithm.

[26]If the signal space is unbounded, we may end up with the trivial case that all others will wait till those "fully informed" moves.

5.4 Disclosure of Public Information and Fragility of Cascades

With endogenous ordering, once a cascading phase starts, all the remaining individuals either act immediately or wait forever regardless of their own private signals. Thus, there exists either an investment surge or a collapse when information cascade starts. Disclosure of public information could have an influence on the strategic and cascading behavior of individuals.

Since the public information disclosed only needs to offset the information from individuals' actions in the last period before the waiting information cascade starts, the waiting information cascade is not robust to the public disclosure of information, which is similar to the fragility result of information cascade in BHW with exogenous ordering. However, if the game falls into the upgrading information cascade, further disclosure of public information will not have any effect since the game ends once the upgrading information cascade occurs.

Appendix

Proof of Lemma 1

If in period t individual i chooses to wait, then $U^{A_1}(\theta_i^1; h_t) \leq U^W(\theta_i^1; h_t; \sigma_{-i,t}^1)$. By the Martingale property,

$$U^{A_1}(\theta_i^1; h_t) = E_{H_{t+1}(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\theta_i^1; h_{t+1})$$

The set of histories $H_{t+1}(\theta_i^1; h_t; \sigma_{-i,t}^1)$ can be decomposed into two disjoint sets: $H_{t+1}^{A_1}(\theta_i^1; h_t; \sigma_{-i,t}^1)$ and $H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1)$, where $H_{t+1}^{A_1}(\theta_i^1; h_t; \sigma_{-i,t}^1)$ is the set of histories in period $t+1$ in which individual i will upgrade to A_1 according to some strategy of individual i; $H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1)$ is the set of histories in period $t+1$ in which individual i will wait according to some strategy of individual i. Then we have

$$U^{A_1}(\theta_i^1; h_t) = E_{H_{t+1}^{A_1}(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\theta_i^1; h_{t+1}) + E_{H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\theta_i^1; h_{t+1})$$

By equation 2,

$$\begin{aligned} U^W(\theta_i^1; h_t; \sigma_{-i,t}^1) &= \delta E_{H_{t+1}(\theta_i^1; h_t; \sigma_{-i,t}^1)}[\max\{U^{A_1}(\theta_i^1; h_{t+1}); U^W(\theta_i^1; h_{t+1}; \sigma_{-i,t+1}^1)\}] \\ &= \delta[E_{H_{t+1}^{A_1}(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\theta_i^1; h_{t+1}) + E_{H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^W(\theta_i^1; h_{t+1}; \sigma_{-i,t+1}^1)] \end{aligned}$$

Consider the case $\sigma_{i,t}^1 = 0$ for all i. No one upgrades in t. This is the worst news from period t. Suppose under this worst news from period t individual i still upgrades in period $t+1$. Then he will for sure upgrade in period $t+1$, which means $H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1) = \emptyset$.

Back to the equations above, we have

$$U^{A_1}(\theta_i^1; h_t) = E_{H_{t+1}^{A_1}(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\theta_i^1; h_{t+1})$$

$$U^W(\theta_i^1; h_t; \sigma_{-i,t}^1) = \delta E_{H_{t+1}^{A_1}(\theta_i^1; h_t; \sigma_{-i,t})} U^{A_1}(\theta_i^1; h_{t+1})$$

Since $H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1) = \emptyset$, $U^W(\theta_i^1; h_t; \sigma_{-i,t}^1) > \underline{U}^W = 0$.[27] We have $U^{A_1}(\theta_i^1; h_t) > U^W(\theta_i^1; h_t; \sigma_{-i,t}^1)$, which is a contradiction. Therefore, under this worst news, the waiting information cascade starts in period $t+1$: $\sigma_{i,\tau}^1 = 0$ for any $\tau > t$ and i. ■

Proof of Proposition 1

As assumed, $F_1(\theta_i^1|V_1)$ satisfies MLRP with respect to V_1. By Bayes' rule, the posterior distribution of V_1 satisfies MLRP with respect to private signal θ_i^1: $\frac{f_1(V_1|\theta_i^1, h_t)}{f_1(V_1|\widetilde{\theta}_i^1, h_t)}$ is increasing in V_1, for $\theta_i^1 > \widetilde{\theta}_i^1$ and any h_t. That is, if an individual gets a higher private signal, given the same history, he believes that V_1 will be higher. According to Landsberger and Meilijson (1990), $F_1(V_1|\theta_i^1, h_t)$ first order stochastic dominates (FOSD) $F_1(V_1|\widetilde{\theta}_i^1, h_t)$ for any $\theta_i^1 > \widetilde{\theta}_i^1$ and h_t. So, $U^{A_1}(\theta_i^1; h_t) > U^{A_1}(\widetilde{\theta}_i^1; h_t)$, for $\theta_i^1 > \widetilde{\theta}_i^1$ and any h_t.

Similar to the proof of lemma 1, by the Martingale property,

$$U^{A_1}(\theta_i^1; h_t) = E_{H_{t+1}^{A_1}(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\theta_i^1; h_{t+1}) + E_{H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\theta_i^1; h_{t+1})$$

$$U^W(\theta_i^1; h_t; \sigma_{-i,t}^1) = \delta[E_{H_{t+1}^{A_1}(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\theta_i^1; h_{t+1}) + E_{H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^W(\theta_i^1; h_{t+1}; \sigma_{-i,t+1}^1)]$$

Thus, for any non-negative integer k

$$\begin{aligned} U^{A_1}(\theta_i^1; h_t) - \delta^k U^W(\theta_i^1; h_t; \sigma_{-i,t}^1) &= (1-\delta^{k+1}) E_{H_{t+1}^{A_1}(\theta_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\theta_i^1; h_{t+1}) \\ &+ E_{H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1)} [U^{A_1}(\theta_i^1; h_{t+1}) - \delta^{k+1} U^W(\theta_i^1; h_{t+1}; \sigma_{-i,t+1}^1)] \end{aligned} \tag{5}$$

Let us check the incentives of waiting and upgrading for individual i who has a lower private signal $\widetilde{\theta}_i^1 < \theta_i^1$. Similarly, we have

$$\begin{aligned} U^{A_1}(\widetilde{\theta}_i^1; h_t) - \delta^k U^W(\widetilde{\theta}_i^1; h_t; \sigma_{-i,t}^1) &= (1-\delta^{k+1}) E_{H_{t+1}^{A_1}(\widetilde{\theta}_i^1; h_t; \sigma_{-i,t}^1)} U^{A_1}(\widetilde{\theta}_i^1; h_{t+1}) \\ &+ E_{H_{t+1}^W(\widetilde{\theta}_i^1; h_t; \sigma_{-i,t}^1)} [U^{A_1}(\widetilde{\theta}_i^1; h_{t+1}) - \delta^{k+1} U^W(\widetilde{\theta}_i^1; h_{t+1}; \sigma_{-i,t+1}^1)] \end{aligned} \tag{6}$$

By lemma 1, with a finite number of N individuals, the game lasts at most N periods before a cascading phase starts. Let $\mathcal{T}$ denote the period when a cascading phase starts. Lemma 1 implies $\mathcal{T} \leq N+1$. With a cascading phase starting in period $\mathcal{T}$, no more new

[27] Otherwise, $U^W(\theta_i^1; h_t; \sigma_{-i,t}^1) = \underline{U}^W = 0$, in which individual i will be indifferent between upgrading or waiting. Since waiting leaves the upgrading option open, it is convenient to set the tie-breaking rule such that individual i will wait with some positive probability, which implies $H_{t+1}^W(\theta_i^1; h_t; \sigma_{-i,t}^1)$ is not empty.

information is disclosed thereafter. That is, $U^W(\theta_i^1; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}^1) = U^W(\widetilde{\theta}_i^1; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}^1) = 0$. Thus, in period $\mathcal{T}$, for any $h_{\mathcal{T}}$ and $\sigma_{-i,\mathcal{T}}^1$

$$U^{A_1}(\theta_i^1; h_{\mathcal{T}}) - \delta^k U^W(\theta_i^1; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}^1) > U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}}) - \delta^k U^W(\widetilde{\theta}_i^1; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}^1)$$

Back to period $\mathcal{T}-1$, since information cascade starts in period $\mathcal{T}$, $H_{\mathcal{T}}^{A_1}(\theta_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1) = H_{\mathcal{T}}^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1)$ and $H_{\mathcal{T}}^{W}(\theta_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1) = H_{\mathcal{T}}^{W}(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1)$. By equation 5 and 6, for any $h_{\mathcal{T}-1}$ and $\sigma_{-i,\mathcal{T}-1}^1$

$$U^{A_1}(\theta_i^1; h_{\mathcal{T}-1}) - \delta^k U^W(\theta_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1) > U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}) - \delta^k U^W(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1)$$

Considering $k = 0$, the equation above indicates that individuals with private signals indicating higher value of A_1 have a higher incentive to upgrade to A_1 given the same public history in period $\mathcal{T} - 1$. That is to say,

$$\begin{aligned} H_{\mathcal{T}-1}^{A_1}(\theta_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1) &\supseteq H_{\mathcal{T}-1}^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1) \\ H_{T-1}^{W}(\theta_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1) &\subseteq H_{\mathcal{T}-1}^{W}(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1) \end{aligned}$$

Back to period $\mathcal{T} - 2$, by equation 5 and 6,

$$\begin{aligned} U^{A_1}(\theta_i^1; h_{\mathcal{T}-2}) - \delta^k U^W(\theta_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1) &= (1-\delta^{k+1}) E_{H_{\mathcal{T}-1}^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1)} U^{A_1}(\theta_i^1; h_{\mathcal{T}-1}) \\ &+ (1-\delta^{k+1}) E_{H_{\mathcal{T}-1}^{A_1}(\theta_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1) \cap H_{\mathcal{T}-1}^{W}(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1)} U^{A_1}(\theta_i^1; h_{\mathcal{T}-1}) \\ &+ E_{H_{\mathcal{T}-1}^{W}(\theta_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1)} [U^{A_1}(\theta_i^1; h_{\mathcal{T}-1}) - \delta^{k+1} U^W(\theta_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1)] \end{aligned}$$

$$\begin{aligned} U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}) - \delta^k U^W(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1) &= (1-\delta^{k+1}) E_{H_{\mathcal{T}-1}^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1)} U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}) \\ &+ E_{H_{\mathcal{T}-1}^{A_1}(\theta_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1) \cap H_{\mathcal{T}-1}^{W}(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1)} [U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}) - \delta^{k+1} U^W(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1)] \\ &+ E_{H_{\mathcal{T}-1}^{W}(\theta_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1)} [U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}) - \delta^{k+1} U^W(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1)] \end{aligned}$$

For $h_{\mathcal{T}-1} \in [H_{\mathcal{T}-1}^{A_1}(\theta_i^1; h_{T-2}; \sigma_{-i,\mathcal{T}-2}^1) \cap H_{T-1}^{W}(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1)]$,

$$U^W(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1) \geq U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-1})$$

which implies $U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}) - \delta^{k+1} U^W(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}^1) \leq (1-\delta^{k+1}) U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-1}) < (1-\delta^{k+1}) U^{A_1}(\theta_i^1; h_{\mathcal{T}-1})$. Thus, for any $h_{\mathcal{T}-2}$ and $\sigma_{-i,\mathcal{T}-2}^1$

$$U^{A_1}(\theta_i^1; h_{\mathcal{T}-2}) - \delta^k U^W(\theta_i^1; h_{\mathcal{T}-2}; \sigma_{-i,T-2}^1) > U^{A_1}(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}) - \delta^k U^W(\widetilde{\theta}_i^1; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}^1)$$

And so on, backwards further, for any $t \leq \mathcal{T}$

$$U^{A_1}(\theta_i^1; h_t) - \delta^k U^W(\theta_i^1; h_t; \sigma_{-i,t}^1) > U^{A_1}(\widetilde{\theta}_i^1; h_t) - \delta^k U^W(\widetilde{\theta}_i^1; h_t; \sigma_{-i,t}^1)$$

which is true for any h_t and $\sigma_{-i,t}^1$. Set $k = 0$. We are done. ■

Proof of Proposition 2

By lemma 1, with a finite number of N individuals, the game lasts at most N periods before a cascading phase starts. As defined earlier, $\mathcal{G}_n(h_t)$ denotes the subgame starting from period t with history h_t, where n is the number of individuals remaining. In addition, $\mathcal{W}_t^1(h_t)$ denotes the set of possible remaining types in period t.

By backward induction, consider the subgame with only one individual i_1^1 remaining, $\mathcal{G}_1(h_t)$. There are three possible cases: (1) if $U^{A_1}(\theta^1; h_t) > 0$ for all $\theta^1 \in \mathcal{W}_t^1(h_t)$, upgrading information cascade occurs; (2) if $U^{A_1}(\theta^1; h_t) \leq 0$ for all $\theta^1 \in \mathcal{W}_t^1(h_t)$, waiting information cascade starts; (3) otherwise, strategic phase continues, in which there exists a critical type $\overline{\theta_t^1}(h_t) \in \mathcal{W}_t^1(h_t)$, such that

$$\begin{cases} U^{A_1}(\theta^1; h_t) > 0 & \text{if } \theta^1 > \overline{\theta_t^1}(h_t) \\ U^{A_1}(\theta^1; h_t) < 0 & \text{if } \theta^1 < \overline{\theta_t^1}(h_t) \end{cases}$$

As for the realization, there are three scenarios depending on $\theta_{i_1^1}$. If $U^{A_1}(\theta^1; h_t) > 0$, $\sigma_{i_1^1,t} = 1$ and the game ends. If $U^{A_1}(\theta^1; h_t) < 0$, $\sigma_{i_1^1,t} = 0$ and individual i_1^1 will wait forever. If $U^{A_1}(\theta^1; h_t) = U^{A_1}(\overline{\theta_t^1}(h_t); h_t) = 0$, individual i_1^1 is indifferent between upgrading and waiting and will follow some tie-breaking rule to upgrade with probability $\sigma_{i_1^1,t} = \sigma_{\overline{\theta_t^1}(h_t)} \in [0, 1)$.

Now consider the subgame with two individuals i_1^1, i_2^1 remaining, $\mathcal{G}_2(h_t)$. By Proposition 1, for any symmetric PBE, it must be monotone with respect to personal private signals. That is, individuals with higher private signals have higher incentive to upgrade. Similar to $\mathcal{G}_1(h_t)$, there are three possible cases: (1) if $U^{A_1}(\theta^1; h_t) > 0$ for all $\theta^1 \in \mathcal{W}_t^1(h_t)$, upgrading information cascade occurs; (2) if $U^{A_1}(\theta^1; h_t) \leq 0$ for all $\theta^1 \in \mathcal{W}_t^1(h_t)$, waiting information cascade starts; (3) otherwise, strategic phase continues, in which continuity implies there exists a critical type $\overline{\theta_t^1}(h_t) \in \mathcal{W}_t^1(h_t)$, such that

$$\begin{cases} \nu(\theta^1; h_t; \sigma_{-,t}^1) = U^{A_1}(\theta^1; h_t) - U^W(\theta^1; h_t; \sigma_{-,t}^1) > 0 & \text{if } \theta^1 > \overline{\theta_t^1}(h_t) \\ \nu(\theta^1; h_t; \sigma_{-,t}^1) = U^{A_1}(\theta^1; h_t) - U^W(\theta^1; h_t; \sigma_{-,t}^1) < 0 & \text{if } \theta^1 < \overline{\theta_t^1}(h_t) \end{cases}$$

As for the realization, there are three scenarios depending on $\theta_{i_1^1}$ and $\theta_{i_2^1}$. If $\min\{\theta_{i_1^1}, \theta_{i_2^1}\} > \overline{\theta_t^1}(h_t)$, $\sigma_{i_1^1,t} = \sigma_{i_2^1,t} = 1$ and the game ends. If $\max\{\theta_{i_1^1}, \theta_{i_2^1}\} < \overline{\theta_t^1}(h_t)$, $\sigma_{i_1^1,t} = \sigma_{i_2^1,t} = 0$ and individual i_1^1, i_2^1 will wait forever. Otherwise, there are two sub-scenarios: (i) one upgrades and the other waits; (ii) one is indifferent between upgrading and waiting, while the other upgrades or waits for sure. For the latter sub-scenario, from $\nu(\overline{\theta_t^1}(h_t); h_t; \sigma_{-,t}^1) = U^{A_1}(\overline{\theta_t^1}(h_t); h_t) - U^W(\overline{\theta_t^1}(h_t); h_t; \sigma_{-,t}^1) = 0$, we can identify $\sigma_{\overline{\theta_t^1}(h_t)}$.[28] Consequently, we may end up with the following outcomes: if both upgrade, the game ends; if one upgrades and the other waits, the game continues to $\mathcal{G}_1(h_{t+1})$; if both wait, waiting information cascade starts.

[28]Note, $U^W(\theta^1; h_t; \sigma_{-,t}^1) = 0$ for $\sigma_{-,t}^1 = \mathbf{0}$ and $\sigma_{-,t}^1 = \mathbf{1}$. Therefore, $U^W(\theta^1; h_t; \sigma_{-,t}^1)$ may not be monotonic with respect to $\sigma_{-,t}^1$.

Continue backwards to the subgame with n individuals $i_1^1, i_2^1, ..., i_n^1$ remaining, $\mathcal{G}_n(h_t)$. Similarly, there are three possible cases: (1) if $U^{A_1}(\theta^1; h_t) > 0$ for all $\theta^1 \in \mathcal{W}_t^1(h_t)$, upgrading information cascade occurs; (2) if $U^{A_1}(\theta^1; h_t) \le 0$ for all $\theta^1 \in \mathcal{W}_t^1(h_t)$, waiting information cascade starts; (3) otherwise, strategic phase continues, in which continuity implies there exists a critical type $\overline{\theta_t^1}(h_t) \in \mathcal{W}_t^1(h_t)$, such that

$$\begin{cases} \nu(\theta^1; h_t; \sigma_{-,t}^1) = U^{A_1}(\theta^1; h_t) - U^W(\theta^1; h_t; \sigma_{-,t}^1) > 0 & \text{if } \theta^1 > \overline{\theta_t^1}(h_t) \\ \nu(\theta^1; h_t; \sigma_{-,t}^1) = U^{A_1}(\theta^1; h_t) - U^W(\theta^1; h_t; \sigma_{-,t}^1) < 0 & \text{if } \theta^1 < \overline{\theta_t^1}(h_t) \end{cases}$$

As for the realization, there are three scenarios depending on $\theta_{i_1^1}, \theta_{i_2^1}, ..., \theta_{i_n^1}$. If $\min\{\theta_{i_1^1}, \theta_{i_2^1}, ..., \theta_{i_n^1}\} > \overline{\theta_t^1}(h_t)$, $\sigma_{i_1^1,t} = \sigma_{i_2^1,t} = ... = \sigma_{i_n^1,t} = 1$ and the game ends. If $\max\{\theta_{i_1^1}, \theta_{i_2^1}, ..., \theta_{i_n^1}\} < \overline{\theta_t^1}(h_t)$, $\sigma_{i_1^1,t} = \sigma_{i_2^1,t} = ... = \sigma_{i_n^1,t} = 0$ and individual $i_1^1, i_2^1, ..., i_n^1$ will wait forever. Otherwise, there are two sub-scenarios: (i) some upgrade and others wait; (ii) some are indifferent between upgrading and waiting, while others upgrade or wait for sure. For the latter sub-scenario, similarly from $\nu(\overline{\theta_t^1}(h_t); h_t; \sigma_{-,t}^1) = U^{A_1}(\overline{\theta_t^1}(h_t); h_t) - U^W(\overline{\theta_t^1}(h_t); h_t; \sigma_{-,t}^1) = 0$, we can identify $\sigma_{\overline{\theta_t^1}(h_t)}$. Consequently, we may end up with the following outcomes: if all upgrade, the game ends; if a number of individuals, say $n_t < n$, upgrade and others wait, the game continues to $\mathcal{G}_{n-n_t}(h_{t+1})$; if all wait, waiting information cascade starts.

Set $n = N$ and $h_t = h_1$. Then $\mathcal{G}_n(h_t)$ becomes $\mathcal{G}_N(h_1)$, which is the original game. ■

Proof of Proposition 3

For the new software package A_j, $j \in \{1, 2, ..., M\}$, as assumed, $F_j(\theta_i^j | V_j)$ satisfies MLRP with respect to V_j. By Bayes' rule, the posterior distribution of V_j satisfies MLRP with respect to private signal θ_i^j: $\frac{f_j(V_j|\theta_i^j, h_t)}{f_j(V_j|\widetilde{\theta}_i^j, h_t)}$ increasing in V_j, for $\theta_i^j > \widetilde{\theta}_i^j$. That is, if an individual gets a higher private signal about A_j, given the same history, he believes that V_j will be higher. According to Landsberger and Meilijson (1990), $F_j(V_j|\theta_i^j, h_t)$ first order stochastic dominates (FOSD) $F_j(V_j|\widetilde{\theta}_i^j, h_t)$ for any $\theta_i^j > \widetilde{\theta}_i^j$. For individual i, consider θ_i and $\widetilde{\theta}_i$, where $\theta_i^j > \widetilde{\theta}_i^j$ and $\theta_i^{\hat{j}} = \widetilde{\theta}_i^{\hat{j}}$ for all $\hat{j} \neq j$. We have $U^{A_j}(\theta_i; h_t) > U^{A_j}(\widetilde{\theta}_i; h_t)$ for any h_t.

Similar to the proof of Proposition 1, by the Martingale property,

$$\begin{aligned} U^{A_j}(\theta_i; h_t) = & E_{H_{t+1}^{A_j}(\theta_i; h_t; \sigma_{-i,t})} U^{A_j}(\theta_i; h_{t+1}) + E_{H_{t+1}^{A_{-j}}(\theta_i; h_t; \sigma_{-i,t})} U^{A_j}(\theta_i; h_{t+1}) \\ & + E_{H_{t+1}^{W}(\theta_i; h_t; \sigma_{-i,t})} U^{A_j}(\theta_i; h_{t+1}) \\ \max_{-j} U^{A_{-j}}(\theta_i; h_t) = & E_{H_{t+1}^{A_j}(\theta_i; h_t; \sigma_{-i,t})} \max_{-j} U^{A_{-j}}(\theta_i; h_{t+1}) + E_{H_{t+1}^{A_{-j}}(\theta_i; h_t; \sigma_{-i,t})} \max_{-j} U^{A_{-j}}(\theta_i; h_{t+1}) \\ & + E_{H_{t+1}^{W}(\theta_i; h_t; \sigma_{-i,t})} \max_{-j} U^{A_{-j}}(\theta_i; h_{t+1}) \end{aligned}$$

$$\begin{aligned} U^W(\theta_i; h_t; \sigma_{-i,t}) = & \delta E_{H_{t+1}^{A_j}(\theta_i; h_t; \sigma_{-i,t})} U^{A_j}(\theta_i; h_{t+1}) + \delta E_{H_{t+1}^{A_{-j}}(\theta_i; h_t; \sigma_{-i,t})} \left[\max_{-j} U^{A_{-j}}(\theta_i; h_{t+1}) \right] \\ & + \delta E_{H_{t+1}^{W}(\theta_i; h_t; \sigma_{-i,t})} U^W(\theta_i; h_{t+1}; \sigma_{-i,t+1}) \end{aligned}$$

where $H_{t+1}^{A_j}(\theta_i; h_t; \sigma_{-i,t})$ is the set of histories in period $t+1$ in which individual i will upgrade to A_j according to some strategy of individual i; $H_{t+1}^{A_{-j}}(\theta_i; h_t; \sigma_{-i,t})$ is the set of histories in period $t+1$ in which individual i will upgrade to a new software package other than A_j according to some strategy of individual i; $H_{t+1}^{W}(\theta_i; h_t; \sigma_{-i,t})$ is the set of histories in period $t+1$ in which individual i will wait according to some strategy of individual i.

Thus, for any non-negative integer k

$$\begin{aligned} U^{A_j}(\theta_i; h_t) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_t) = {} & E_{H_{t+1}^{A_j}(\theta_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\theta_i; h_{t+1}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{t+1})\right] \\ & + E_{H_{t+1}^{A_{-j}}(\theta_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\theta_i; h_{t+1}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{t+1})\right] \\ & + E_{H_{t+1}^{W}(\theta_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\theta_i; h_{t+1}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{t+1})\right] \end{aligned} \tag{7}$$

$$\begin{aligned} U^{A_j}(\theta_i; h_t) - \delta^k U^{W}(\theta_i; h_t; \sigma_{-i,t}) = {} & (1-\delta^{k+1}) E_{H_{t+1}^{A_j}(\theta_i; h_t; \sigma_{-i,t})} U^{A_j}(\theta_i; h_{t+1}) \\ & + E_{H_{t+1}^{A_{-j}}(\theta_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\theta_i; h_{t+1}) - \delta^{k+1} \max_{-j} U^{A_{-j}}(\theta_i; h_{t+1})\right] \\ & + E_{H_{t+1}^{W}(\theta_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\theta_i; h_{t+1}) - \delta^{k+1} U^{W}(\theta_i; h_{t+1}; \sigma_{-i,t+1})\right] \end{aligned} \tag{8}$$

Let us check the incentives of waiting and upgrading for individual i who has a lower private signal about A_j. Again, consider θ_i and $\widetilde{\theta}_i$, where $\theta_i^j > \widetilde{\theta}_i^j$ and $\theta_i^{\hat{j}} = \widetilde{\theta}_i^{\hat{j}}$ for all $\hat{j} \neq j$. Similarly, we have

$$\begin{aligned} U^{A_j}(\widetilde{\theta}_i; h_t) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_t) = {} & E_{H_{t+1}^{A_j}(\widetilde{\theta}_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\widetilde{\theta}_i; h_{t+1}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{t+1})\right] \\ & + E_{H_{t+1}^{A_{-j}}(\widetilde{\theta}_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\widetilde{\theta}_i; h_{t+1}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{t+1})\right] \\ & + E_{H_{t+1}^{W}(\widetilde{\theta}_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\widetilde{\theta}_i; h_{t+1}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{t+1})\right] \end{aligned} \tag{9}$$

$$\begin{aligned} U^{A_j}(\widetilde{\theta}_i; h_t) - \delta^k U^{W}(\widetilde{\theta}_i; h_t; \sigma_{-i,t}) = {} & (1-\delta^{k+1}) E_{H_{t+1}^{A_j}(\widetilde{\theta}_i; h_t; \sigma_{-i,t})} U^{A_j}(\widetilde{\theta}_i; h_{t+1}) \\ & + E_{H_{t+1}^{A_{-j}}(\widetilde{\theta}_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\widetilde{\theta}_i; h_{t+1}) - \delta^{k+1} \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{t+1})\right] \\ & + E_{H_{t+1}^{W}(\widetilde{\theta}_i; h_t; \sigma_{-i,t})}\left[U^{A_j}(\widetilde{\theta}_i; h_{t+1}) - \delta^{k+1} U^{W}(\widetilde{\theta}_i; h_{t+1}; \sigma_{-i,t+1})\right] \end{aligned} \tag{10}$$

Further, by lemma 2, with a finite number of N individuals, M investment options and bounded private signals, the game lasts for a finite number of periods before a cascading phase starts. Suppose either an upgrading or waiting information cascade starts in period $\mathcal{T}$, which means no one will upgrade after period $\mathcal{T}$ given history $h_{\mathcal{T}}$ and strategy profile $(\sigma_{i,t}, \sigma_{-i,t})$. With a cascading phase starting in period $\mathcal{T}$, no more new information is disclosed thereafter. That is, $U^W(\theta_i; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}) = U^W(\widetilde{\theta}_i; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}) = 0$. Thus, in period $\mathcal{T}$, for any $h_{\mathcal{T}}$ and $\sigma_{-i,\mathcal{T}}$

$$U^{A_j}(\theta_i; h_{\mathcal{T}}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}}) > U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}})$$

$$U^{A_j}(\theta_i; h_{\mathcal{T}}) - \delta^k U^{W}(\theta_i; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}}) > U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}}) - \delta^k U^{W}(\widetilde{\theta}_i; h_{\mathcal{T}}; \sigma_{-i,\mathcal{T}})$$

Back to period $\mathcal{T}-1$, since information cascade starts in period $\mathcal{T}$,

$$H_{\mathcal{T}}^{A_j}(\theta_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}) = H_{\mathcal{T}}^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1})$$
$$H_{\mathcal{T}}^{A_{-j}}(\theta_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}) = H_{\mathcal{T}}^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1})$$
$$H_{\mathcal{T}}^{W}(\theta_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}) = H_{\mathcal{T}}^{W}(\widetilde{\theta}_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1})$$

By equation $7-10$, for any $h_{\mathcal{T}-1}$ and $\sigma_{-i,\mathcal{T}-1}$

$$U^{A_j}(\theta_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}-1}) > U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-1})$$
$$U^{A_j}(\theta_i; h_{\mathcal{T}-1}) - \delta^k U^{W}(\theta_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}) > U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^k U^{W}(\widetilde{\theta}_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1})$$

When $j=0$, the equation above implies that individuals with private signals indicating higher value of A_j have a higher incentive to upgrade to A_j given the same public history in period $\mathcal{T}-1$. That is to say,

$$H_{\mathcal{T}-1}^{A_j}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) \supseteq H_{\mathcal{T}-1}^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})$$
$$H_{\mathcal{T}-1}^{A_{-j}}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) \subseteq H_{\mathcal{T}-1}^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})$$
$$H_{\mathcal{T}-1}^{W}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) \subseteq H_{\mathcal{T}-1}^{W}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})$$

Back to period $\mathcal{T}-2$, by equation 7 and 9, we have

$$\begin{aligned}
U^{A_j}(\theta_i; h_{\mathcal{T}-2}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}-2}) &= E_{H_{\mathcal{T}-1}^{A_j}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,t})}\left[U^{A_j}(\theta_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}-1})\right] \\
&+ E_{H_{\mathcal{T}-1}^{A_{-j}}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,t})}\left[U^{A_j}(\theta_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}-1})\right] \\
&+ E_{H_{\mathcal{T}-1}^{W}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,t})}\left[U^{A_j}(\theta_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}-1})\right]
\end{aligned}$$

$$\begin{aligned}
U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-2}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-2}) &= E_{H_{\mathcal{T}-1}^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,t})}\left[U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-1})\right] \\
&+ E_{H_{\mathcal{T}-1}^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,t})}\left[U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-1})\right] \\
&+ E_{H_{\mathcal{T}-1}^{W}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,t})}\left[U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-1})\right]
\end{aligned}$$

As $U^{A_j}(\theta_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}-1}) > U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-1})$ for any $h_{\mathcal{T}-1}$ and $\sigma_{-i,\mathcal{T}-1}$, we have

$$U^{A_j}(\theta_i; h_{\mathcal{T}-2}) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}-2}) > U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-2}) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-2})$$

Similarly, by equation 8 and 10, we have

$$\begin{aligned}
U^{A_j}(\theta_i; h_{\mathcal{T}-2}) - \delta^k U^{W}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) &= (1-\delta^{k+1}) E_{H_{\mathcal{T}-1}^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})} U^{A_j}(\theta_i; h_{\mathcal{T}-1}) \\
&+ (1-\delta^{k+1}) E_{H_{\mathcal{T}-1}^{A_j}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) \cap H_{\mathcal{T}-1}^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})} U^{A_j}(\theta_i; h_{\mathcal{T}-1}) \\
&+ (1-\delta^{k+1}) E_{H_{\mathcal{T}-1}^{A_j}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) \cap H_{\mathcal{T}-1}^{W}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})} U^{A_j}(\theta_i; h_{\mathcal{T}-1}) \\
&+ E_{H_{\mathcal{T}-1}^{A_{-j}}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,t})}\left[U^{A_j}(\theta_i; h_{\mathcal{T}-1}) - \delta^{k+1} \max_{-j} U^{A_{-j}}(\theta_i; h_{\mathcal{T}-1})\right] \\
&+ E_{H_{\mathcal{T}-1}^{W}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})}\left[U^{A_j}(\theta_i; h_{\mathcal{T}-1}) - \delta^{k+1} U^{W}(\theta_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1})\right]
\end{aligned}$$

$$\begin{aligned}
U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-2}) - \delta^k U^W(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) &= (1-\delta^{k+1}) E_{H^{A_j}_{\mathcal{T}-1}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})} U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) \\
&+ (1-\delta^{k+1}) E_{H^{A_j}_{\mathcal{T}-1}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) \cap H^{A_{-j}}_{\mathcal{T}-1}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})} U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) \\
&+ E_{H^{A_j}_{\mathcal{T}-1}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) \cap H^{W}_{\mathcal{T}-1}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})} \left[U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^{k+1} U^W(\widetilde{\theta}_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}) \right] \\
&+ E_{H^{A_{-j}}_{\mathcal{T}-1}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,t})} \left[U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^{k+1} \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) \right] \\
&+ E_{H^{W}_{\mathcal{T}-1}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})} \left[U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^{k+1} U^W(\widetilde{\theta}_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}) \right]
\end{aligned}$$

For $h_{\mathcal{T}-1} \in [H^{A_j}_{\mathcal{T}-1}(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) \cap H^{W}_{\mathcal{T}-1}(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})]$, $U^W(\widetilde{\theta}_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}) \geq U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1})$, which implies $U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) - \delta^{k+1} U^W(\widetilde{\theta}_i; h_{\mathcal{T}-1}; \sigma_{-i,\mathcal{T}-1}) \leq (1-\delta^{k+1}) U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-1}) < (1-\delta^{k+1}) U^{A_j}(\theta_i; h_{\mathcal{T}-1})$. Thus,

$$U^{A_j}(\theta_i; h_{\mathcal{T}-2}) - \delta^k U^W(\theta_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2}) > U^{A_j}(\widetilde{\theta}_i; h_{\mathcal{T}-2}) - \delta^k U^W(\widetilde{\theta}_i; h_{\mathcal{T}-2}; \sigma_{-i,\mathcal{T}-2})$$

And so on, backwards further, for any $t \leq \mathcal{T}$

$$\begin{aligned}
U^{A_j}(\theta_i; h_t) - \delta^k \max_{-j} U^{A_{-j}}(\theta_i; h_t) &> U^{A_j}(\widetilde{\theta}_i; h_t) - \delta^k \max_{-j} U^{A_{-j}}(\widetilde{\theta}_i; h_t) \\
U^{A_j}(\theta_i; h_t) - \delta^k U^W(\theta_i; h_t; \sigma_{-i,t}) &> U^{A_j}(\widetilde{\theta}_i; h_t) - \delta^k U^W(\widetilde{\theta}_i; h_t; \sigma_{-i,t})
\end{aligned}$$

which is true for any h_t and $\sigma_{-i,t}$. Set $k = 0$. We are done. ■

Proof of Proposition 4

By lemma 2, with a finite number of N individuals, M investment options and bounded private signals, the game lasts for a finite number of periods before a cascading phase starts. As defined earlier, $\mathcal{G}_n(h_t)$ denotes the subgame starting from period t with history h_t, where n is the number of individuals remaining. In addition, $\mathcal{W}_t(h_t)$ denotes the set of possible remaining types in period t.

By backward induction, consider the subgame with only one individual i_1 remaining, $\mathcal{G}_1(h_t)$. There are three possible cases: (1) if there exists $j \in \{1, 2, ..., M\}$, such that $U^{A_j}(\theta; h_t) > \max \left\{ \max_{\hat{j} \neq j} U^{A_{\hat{j}}}(\theta; h_t); 0 \right\}$ for all $\theta \in \mathcal{W}_t(h_t)$, A_j-upgrading information cascade occurs; (2) if $\max_j U^{A_j}(\theta; h_t) \leq 0$ for all $\theta \in \mathcal{W}_t(h_t)$, waiting information cascade starts; (3) otherwise, strategic phase continues, in which there exists a critical type $\overline{\theta_t}(h_t) \in \mathcal{W}_t(h_t)$, such that[29]

$$\begin{cases} \max_j U^{A_j}(\theta; h_t) > 0 & \text{if } \max_j U^{A_j}(\theta; h_t) > \max_j U^{A_j}(\overline{\theta_t}(h_t); h_t) \\ \max_j U^{A_j}(\theta; h_t) < 0 & \text{if } \max_j U^{A_j}(\theta; h_t) < \max_j U^{A_j}(\overline{\theta_t}(h_t); h_t) \end{cases}$$

[29]For the case that $U^{A_j}(\theta; h_t) \leq 0$ for all $\theta \in \mathcal{W}_t(h_t)$, A_j will drop out of the race as no one has incentive to upgrade to A_j anymore.

As for the realization, there are three scenarios depending on θ_{i_1}. If $\max_j U^{A_j}(\theta_{i_1}; h_t) > 0$, individual i_1 will upgrade and the game ends. If $\max_j U^{A_j}(\theta_{i_1}; h_t) < 0$, $\sigma_{i_1,t} = 0$ and individual i_1 will wait forever. Otherwise, $\max_j U^{A_j}(\theta; h_t) = \max_j U^{A_j}(\overline{\theta_t}(h_t); h_t) = 0$. Individual i_1 is indifferent between upgrading and waiting and will follow some tie-breaking rule to upgrade with probability $\sigma_{\overline{\theta_t}(h_t)} \in [0, 1)$.

Now consider the subgame with two individuals i_1, i_2 remaining, $\mathcal{G}_2(h_t)$. By Proposition 3, for any symmetric PBE, it must be monotone with respect to personal private signals. That is, individuals with higher private signals have higher incentive to upgrade. Similar to $\mathcal{G}_1(h_t)$, there are three possible cases: (1) if there exists $j \in \{1, 2, ..., M\}$, such that $U^{A_j}(\theta; h_t) > \max\left\{\max_{\hat{j}\neq j} U^{A_{\hat{j}}}(\theta; h_t); 0\right\}$ for all $\theta \in \mathcal{W}_t(h_t)$, A_j-upgrading information cascade occurs; (2) if $\max_j U^{A_j}(\theta; h_t) \leq 0$ for all $\theta \in \mathcal{W}_t(h_t)$, waiting information cascade starts; (3) otherwise, strategic phase continues, in which continuity implies there exists a critical type $\overline{\theta_t}(h_t) \in \mathcal{W}_t(h_t)$, such that

$$\begin{cases} \nu(\theta; h_t; \sigma_{-,t}) > 0 & \text{if} \quad \nu(\theta; h_t; \sigma_{-,t}) > \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) \\ \nu(\theta; h_t; \sigma_{-,t}) < 0 & \text{if} \quad \nu(\theta; h_t; \sigma_{-,t}) < \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) \end{cases}$$

As for the realization, there are three scenarios depending on θ_{i_1} and θ_{i_2}. If $\min\{\nu(\theta_{i_1}; h_t; \sigma_{-,t}), \nu(\theta_{i_2}; h_t; \sigma_{-,t})\} > \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t})$, they will upgrade to some new software package and the game ends. If $\max\{\nu(\theta_{i_1}; h_t; \sigma_{-,t}), \nu(\theta_{i_2}; h_t; \sigma_{-,t})\} > \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t})$, they will wait forever. Otherwise, there are two sub-scenarios: (i) one upgrades and the other waits; (ii) one is indifferent between upgrading and waiting, while the other upgrades or waits for sure. For the latter sub-scenario, from $\nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) = \max_j U^{A_j}(\overline{\theta_t}(h_t); h_t) - U^W(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) = 0$, we can identify $\sigma_{\overline{\theta_t}(h_t)}$. Consequently, we may end up with the following outcomes: if both upgrade, the game ends; if one upgrades and the other waits, the game continues to $\mathcal{G}_1(h_{t+1})$; if both wait, waiting information cascade starts.

Continue backwards to the subgame with n individuals $i_1, i_2, ..., i_n$ remaining, $\mathcal{G}_n(h_t)$. Similarly, there are three possible cases: (1) if there exists $j \in \{1, 2, ..., M\}$, such that $U^{A_j}(\theta; h_t) > \max\left\{\max_{\hat{j}\neq j} U^{A_{\hat{j}}}(\theta; h_t); 0\right\}$ for all $\theta \in \mathcal{W}_t(h_t)$, A_j-upgrading information cascade occurs; (2) if $\max_j U^{A_j}(\theta; h_t) \leq 0$ for all $\theta \in \mathcal{W}_t(h_t)$, waiting information cascade starts; (3) otherwise, strategic phase continues, in which continuity implies there exists a critical type $\overline{\theta_t}(h_t) \in \mathcal{W}_t(h_t)$, such that

$$\begin{cases} \nu(\theta; h_t; \sigma_{-,t}) > 0 & \text{if} \quad \nu(\theta; h_t; \sigma_{-,t}) > \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) \\ \nu(\theta; h_t; \sigma_{-,t}) < 0 & \text{if} \quad \nu(\theta; h_t; \sigma_{-,t}) < \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) \end{cases}$$

As for the realization, there are three scenarios depending on $\theta_{i_1}, \theta_{i_2}, ..., \theta_{i_n}$. If $\min\{\nu(\theta_{i_1}; h_t; \sigma_{-,t}), \nu(\theta_{i_2}; h_t; \sigma_{-,t}), ..., \nu(\theta_{i_n}; h_t; \sigma_{-,t})\} > \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t})$, all individuals will upgrade to some new software package and the game ends. If

$\max\{\nu(\theta_{i_1}; h_t; \sigma_{-,t}), \nu(\theta_{i_2}; h_t; \sigma_{-,t}), ..., \nu(\theta_{i_n}; h_t; \sigma_{-,t})\} < \nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t})$, all individuals will wait forever. Otherwise, there are two sub-scenarios: (i) some upgrade and others wait; (ii) some are indifferent between upgrading and waiting, while others upgrade or wait for sure. For the latter sub-scenario, similarly from $\nu(\overline{\theta_t}(h_t); h_t; \sigma_{-,t}) = 0$, we can identify $\sigma_{\overline{\theta_t}(h_t)}$. Consequently, we may end up with the following outcomes: if all upgrade, the game ends; if a number of individuals, say $n_t < n$, upgrade and others wait, the game continues to $\mathcal{G}_{n-n_t}(h_{t+1})$; if all wait, waiting information cascade starts.

Set $n = N$ and $h_t = h_1$. Then $\mathcal{G}_n(h_t)$ becomes $\mathcal{G}_N(h_1)$, which is the original game. ■

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